Homogenization of plasticity equations with hardening using Finite-Element approaches

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(joint work with Marco Veneroni)

Plasticity equations describe the deformations e.g. of metal [1, 3]. As in elasticity, one describes the body at rest with a domain $\Omega \subset \mathbb{R}^n$, the deformation of the material point $x \in \Omega$ by $u(x) \in \mathbb{R}^n$, uses the symmetric gradient $\nabla^s u(x) = (\nabla u(x) + \nabla u(x)^T)/2$ to describe local deformations, and the stress tensor $\sigma(x)$ to describe inner forces. The balance of linear momentum is as in elasticity, (1a) with density ϱ and load f.

In contrast to elasticity, the stress tensor is *not* in a linear relation with $\nabla^s u(x)$. Instead, the deformation is decomposed (here: additively) into two parts, an elastic strain and a plastic strain, $\nabla^s u(x) = e(x) + p(x)$, such that with $e(x)$ Hooke's law is satisfied, $\sigma(x) = De(x)$ for some elasticity tensor D. The plastic strain can be considered as a component of a possibly larger vector of interior variables $\xi \in \mathbb{R}^N$; with a linear operator $B : \mathbb{R}^N \to \mathbb{R}^{n\times n}_s$ that maps into the space of symmetric matrices, we write $p = B\xi$ and obtain (1b) as stress-strain relation. Finally, in order to close the system, we have to introduce a *flow-rule* in (1c). It provides an ordinary differential equation for the internal variables $\xi(x,t) \in \mathbb{R}^N$. The nonlinear function $g : \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be monotone and can be multi-valued. The flow rule expresses changes of the internal variables ξ under the influence of the forces σ. Hardening models include $L\xi$ in the argument of g, strictly monotone operators $L : \mathbb{R}^N \to \mathbb{R}^N$ contribute to regularity properties of solutions. In particular, p need not be regarded as a measure, but can be expected to be an element of $L^2(\Omega)$ for every time instance.

In the homogenization analysis, one is interested in oscillatory dependence of the material parameters on $x \in \Omega$, we therefore provide the material variables with a subscript η where $\eta > 0$ stands for the typical length scale in the heterogeneous model. Since the solution depends on the coefficients, we mark also the solution variables u^η, σ^η, and ξ^η with a superscript η. The system under consideration reads

$$\varrho_\eta \partial_t^2 u^\eta = \nabla\cdot\sigma^\eta + f \tag{1a}$$

$$\sigma^\eta = D_\eta(\nabla^s u^\eta - B_\eta \xi^\eta) \tag{1b}$$

$$\partial_t \xi^\eta \in g_\eta(B_\eta^T \sigma^\eta - L_\eta \xi^\eta) \tag{1c}$$

For a time horizon $T > 0$, the equations are posed on $\Omega_T = \Omega \times (0,T)$.

The homogenization of the plasticity system (1) was analyzed recently by several authors. The method of two-scale convergence was used in [9, 10, 11], quasi-stationary evolutions were the underlying concept in [4, 5], and a phase-shift construction was used in [2, 6]. Another method to derive similar results was introduced in [8]: Based on Tartar's original method of oscillating test functions in homogenization, we constructed oscillating test-functions from the two-scale limit problem in order to prove the homogenization result. We note that all the approaches mentioned above have their own strengths. A strength of the approach in [8] is the simplicity and flexibility. In particular, the treatment of the wave equation (inclusion of $\varrho\partial_t^2 u$ in (1a)) is possible without any additional difficulties.

Why is the homogenization of the plasticity problem difficult? It is astonishing that the homogenization of this problem was only recently performed. Difficulties lie in the non-linear character of the equations and, in particular, in the differential inclusion (1c), which is typically formulated by imposing an energy-dissipation inequality. From our view-point, the main problem lies in the fact that solutions (of the two-scale problem) do not have sufficient regularity properties (in (x, y)). We circumvent this problem as follows: we *do not use* the solution of the two-scale problem to construct an oscillatory test-function, but we *do use* a Finite-Element approximation of the two-scale problem. This approximation has all the regularity properties that we need.

Results. We consider periodic homogenization. With the periodicity cell $Y = [0, 1)^n$ we assume that the material parameters are given as

$$D_\eta(x) = D\left(x, \frac{x}{\eta}\right), \quad L_\eta(x) = L\left(x, \frac{x}{\eta}\right), \quad B_\eta(x) = B\left(x, \frac{x}{\eta}\right),$$
$$\varrho_\eta(x) = \varrho\left(x, \frac{x}{\eta}\right), \quad g_\eta(\,\cdot\,; x) = g\left(\,\cdot\,; \frac{x}{\eta}\right).$$

The limit system for plasticity consists in a two-scale problem. In general, this problem cannot be decoupled — we do *not* obtain a single macroscopic plasticity system as an effective equation (except for the one-dimensional case, [7]). The unknowns in the two-scale problem are the macroscopic deformation $u : \Omega_T \to \mathbb{R}^n$, a corrector for the deformation $v : \Omega_T \times Y \to \mathbb{R}^n$, the two-scale internal variables $w : \Omega_T \times Y \to \mathbb{R}^N$, and the two-scale stress $z : \Omega_T \times Y \to \mathbb{R}^{n\times n}_s$. The two-scale system reads, for the averaged density $\bar\varrho(x) > 0$,

$$\bar\varrho\partial_t^2 u = \nabla\cdot\left(\int_Y z\, dy\right) + f \tag{2a}$$

$$z = D(\nabla_x^s u + \nabla_y^s v - Bw) \tag{2b}$$

$$\nabla_y \cdot z = 0 \tag{2c}$$

$$\partial_t w \in g(B^T z - Lw\,; y) \tag{2d}$$

The homogenization result takes the following form. We are currently working on optimal assumptions on the coefficients, the following formulation is meant to indicate work in progress (but we note that precise assumptions for a less general model with an indicatrix map g are given in [8]).

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded polygonal domain, $T > 0$. Let the maps $D(x)$ and $L(x)$ be monotone, uniformly bounded with uniformly bounded inverse operators. Let $g : \mathbb{R}^N \times Y \to \mathbb{R}^N$ be a multi-valued monotone operator. Let the density ϱ be strictly positive, we use $\bar\varrho(x) = \int_Y \varrho(x, y)\, dy$. Let initial data and boundary conditions be such that solutions to the oscillatory system and to the* *two-scale system exist. Let* $(u^\eta, \sigma^\eta, \xi^\eta)$ *and* (u, v, w, z) *be solutions of problems* (1) *and* (2). *Then, as* $\eta \to 0$,

$$\partial_t u^\eta \to \partial_t u \quad \text{strongly in } L^2(\Omega_T),$$
$$\sigma^\eta \rightharpoonup \int_Y z\, dy, \quad \xi^\eta \rightharpoonup \int_Y w\, dy \quad \text{weakly in } L^2(\Omega_T).$$

As indicated, the proof of the theorem is based on solutions of a semi-discrete version of the two-scale problem. More precisely, we define a space of piecewise affine finite elements $U_h \subset H^1(\Omega)$ and search for $u_h(.,t) \in U_h$. The other variables, $v_h(x,y,t)$, $z_h(x,y,t)$, and $w_h(x,y,t)$ are searched for in spaces of piecewise constant functions in x. We solve

$$\text{(3a)} \qquad \int_\Omega \left(\int_Y z_h\, dy \right) : \nabla\psi\, dx = \int_\Omega (f - \bar{\varrho}\partial_t^2 u_h)\cdot \psi\, dx \qquad \forall\, \psi \in U_h$$
$$\text{(3b)} \qquad z_h = D(\nabla^s u_h + \nabla_y^s v_h - B w_h)$$
$$\text{(3c)} \qquad \nabla_y \cdot z_h = 0$$
$$\text{(3d)} \qquad \partial_t w_h \in g(B^T z_h - L w_h;\, y)$$

For any function $\phi : \Omega \times Y \times (0,T) \to \mathbb{R}^m$, we construct an oscillatory test-function by setting $\phi_\eta(x,t) := \phi(x, x/\eta, t)$. Using the solution (u_h, v_h, z_h, w_h) of (3), constructing the oscillatory functions $(u_{h,\eta}, v_{h,\eta}, z_{h,\eta}, w_{h,\eta})$, and using them as test-functions in the original system and in (3), we can derive Theorem 1 with energy methods, using taylored div-curl-lemmas to show smallness of error terms.

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