The needle problem approach to non-periodic homogenization

Ben Schweizer [1] and Marco Veneroni[2]

July 5, 2011

Abstract: We introduce a new method to homogenization of non-periodic problems and illustrate the approach with the elliptic equation $-\nabla\cdot(a^\varepsilon\nabla u^\varepsilon) = f$. On the coefficients a^ε we assume that solutions u^ε of homogeneous ε-problems on simplices with average slope $\xi \in \mathbb{R}^n$ have the property that flux-averages $\fint a^\varepsilon\nabla u^\varepsilon \in \mathbb{R}^n$ converge, for $\varepsilon \to 0$, to some limit $a^*(\xi)$, independent of the simplex. Under this assumption, which is comparable to H-convergence, we show the homogenization result for general domains and arbitrary right hand side. The proof uses a new auxiliary problem, the *needle problem*. Solutions of the needle problem depend on a triangulation of the domain, they solve an ε-problem in each simplex and are affine on faces.

1 Introduction

Due to its relevance in many applications, homogenization theory is nowadays an important field of mathematical analysis. To give a very general description, homogenization is concerned with solutions u^ε of partial differential equations $\mathcal{A}^\varepsilon(u^\varepsilon) = f$, where f are given data and $\mathcal{A}^\varepsilon$ is a differential operator with oscillatory coefficients that vary on a scale of order $\varepsilon > 0$. The task is to determine a homogenized operator $\mathcal{A}^*$ such that solutions u^* of $\mathcal{A}^*u^* = f$ are approximations of the oscillatory solutions u^ε in the sense that $u^\varepsilon \to u^*$ for $\varepsilon \to 0$ in some norm.

Let us be more specific and describe the idea in the most simple case of $(\mathcal{A}^\varepsilon u)(x) = -\nabla\cdot(a^\varepsilon(x)\nabla u(x))$ for $u \in H^1_0(Q)$, understood in the weak sense on $Q \subset \mathbb{R}^n$. The homogenized operator turns out to be $\mathcal{A}^*u = -\nabla\cdot(a^*\nabla u(x))$ with a matrix $a^* \in \mathbb{R}^{n\times n}$ that can be characterized as follows. If a solution sequence u^ε of $\nabla\cdot(a^\varepsilon\nabla u^\varepsilon) = 0$ has the average slope $\xi \in \mathbb{R}^n$, then the corresponding fluxes $a^\varepsilon\nabla u^\varepsilon$ have the average value $a^*\xi$,

$$\nabla u^\varepsilon \rightharpoonup \xi \qquad \Rightarrow \qquad a^\varepsilon\nabla u^\varepsilon \rightharpoonup a^*\xi. \tag{1.1}$$

Most often, the effective coefficient $a^* \in \mathbb{R}^{n\times n}$ is described with a cell problem, a periodic problem on a unit cell in the case of periodic homogenization problems, and a problem on $\mathbb{R}^n$ in the case of stochastic homogenization problems. The more

[1]Technische Universität Dortmund, Fakultät für Mathematik, Vogelpothsweg 87, D-44227 Dortmund, Germany.

[2]Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Street West, H3A 2K6 Montreal, Quebec, Canada.

abstract characterization of $a^* \in \mathbb{R}^{n\times n}$ of (1.1) can be considered the defining property of H-convergence, see [21] and [31, Definition 6.4].

The aim of our contribution is to provide a new method of homogenization, which allows one to conclude from property (1.1) of the coefficients that $u^\varepsilon \rightharpoonup u^*$, weakly in $H^1(Q)$, where u^* is a solution of $-\nabla\cdot(a^*\nabla u^*) = f$, for arbitrary Q and f. Our method does not rely on periodicity of the coefficients or a specific stochastic construction.

As we detail below, the theory of H-convergence provides a simple proof of the same result. Our aim is to introduce a new and flexible method. A novelty in our method is the explicit construction of an approximating sequence, in the spirit of multiscale finite elements [4, 17] and of the heterogeneous multiscale method [15, 16].

The above problem was treated and solved for periodic coefficients [5, 6, 14, 26, 30], with the method of two-scale convergence [2, 22], with the periodic unfolding method [10], and in the stochastic case [8, 13, 18, 19]. Regarding homogenization of other equations we mention [1, 27, 29, 32, 33], regarding a further analysis of the homogenization limit or the homogenized problem [20, 34]. In the forthcoming contribution [28], we address the extension of stationary homogenization results to time-dependent parabolic and hysteresis problems. Recent results typically regard large coefficients or singular geometries [3, 7], for more abstract approaches see [23, 24]. Numerical studies are concerned with the construction of fast methods that resolve the fine scale only on small sub-domains.

Homogenization and discretization. The needle problem approach is inspired by numerical methods and, more generally, by the principle of representative volume elements (RVEs). A loose description of such approaches is the following: the macroscopic domain is discretized with a triangulation as if a homogenized problem was available. In order to find the effective coefficients in each volume element of size h, a representative volume element is chosen with diameter large compared to ε, but small compared to h. The solution of an ε-problem on the RVE provides via (1.1) the effective coefficients in the volume element.

The heterogeneous multiscale method follows this idea, convergence results for the elliptic problem are obtained e.g. in [16]. The authors use an error e(HMM) which measures how well the homogenized matrix can be recovered by solving problems on RVEs. Theorem 1.1 of [16] shows that, without any assumptions on the coefficients, e(HMM) and the grid size control the error of the scheme. Further theorems provide the smallness of e(HMM) with appropriate bounds in several cases: in the periodic case, and in a stochastic case with mixing properties in dimensions 1 and 3.

We show a rigorous result in this spirit: we assume that homogeneous solutions on simple domains with affine boundary conditions corresponding to slope ξ have an averaged flux $a^*\xi$, independent of the domain. Our result is that then a^* is the matrix of homogenized coefficients in general boundary value problems. The needle problem introduces intermediate solutions that can be regarded as the analog to discrete solutions in the heterogeneous multiscale method. The method has also similarities with multiscale finite element methods. In equation (23) of [4], a reference problem similar to ours is used, but the further construction uses Kozlov's harmonic coordinates.

Homogenization as a two-step procedure. We regard the homogenization of an equation as a two-step procedure: in a first step one has to understand the behavior of solutions u^ε that approximate an affine function. These are the functions that are usually considered in cell problems. For such functions, the constitutive relation (e.g. between flux $a^\varepsilon \nabla u^\varepsilon$ and gradient ∇u^ε) must be investigated and an averaged constitutive relation for weak limits must be derived. In our case, this averaged relation is given by the matrix a^* in (1.1). In a second step, the data of the concrete problem are incorporated. One considers no longer simple domains and homogeneous solutions, but solutions u^ε to given data Q and f. The aim in this second step is to show that the averaged constitutive relation defines indeed the averaged operator $\mathcal{A}^*$. Our contribution regards entirely the second step, our aim is to assume as little as possible about the first step.

With this aim, we will not even use the weak convergence that was indicated in (1.1), but we impose only a property of averages. Our stabilization result provides (1.1) as a consequence of the weaker assumption of Definition 1.1. The main difficulty in the verification of that assumption is to show that the limit of the averages exists and that it is independent of the simplex. In the context of stochastic coefficients, these properties can be regarded as an ergodicity and stationarity assumption on the coefficients. We emphasize that, in the standard stochastic setting, all our assumptions are satisfied, see Appendix A.

Since our new approach is very general, we believe that it allows furthermore to perform the second step of the homogenization procedure for more complex operators such as e.g. hysteresis operators of plasticity equations.

The technique of the needle problem approach. The usual way to perform step 2 in the above program is to start from solutions of the cell problem and to construct test-functions. Our aim is not to use cell problem solutions, since they might not be available. As a replacement, we use solutions to the needle problem. The needle problem is the original problem with coefficients a^ε, introducing a side condition with a triangulation $\mathscr{T}_h$: we search for functions u_h^ε that are solutions in each simplex of $\mathscr{T}_h$ and that are affine on all faces of the grid $\mathscr{T}_h$. The condition of affine boundary data on each simplex implies that our general assumption on solutions to affine boundary data of Definition 1.1 is applicable. On the other hand, for small h, the side condition is not a severe restriction, and we find that $u^\varepsilon - u_h^\varepsilon$ is small. The combination of these two facts allows to conclude the homogenization result.

The main technical problem in our new method is that we need a div-curl-Lemma in each simplex of the triangulation. Since in the simplices of the triangulation we do not have prescribed boundary conditions for u^ε, the standard div-curl-lemma does not apply. We will provide a div-curl-lemma under the assumption that the grid is adapted to the sequence u^ε. To give a first idea of that property, we observe the following: Since the sequence ∇u^ε is bounded in $L^2(Q)$, on almost every hyperplane E through Q, the sequence $\nabla u^\varepsilon|_E$ is also bounded. This implies that the trace $u^\varepsilon|_E$ is not only controlled in $H^{1/2}(E)$, but also in $H^1(E)$. The corresponding compactness allows to conclude the div-curl-lemma.

The construction of adapted grids is lengthy, we perform it in several steps in

Section 3. The final result is the div-curl-Lemma on fine grids, as presented in Theorem 1.3. We state it in a general and self-contained way. We hope that it turns out to be useful in other non-periodic homogenization problems. We remark that the construction of the adapted grids has some similarities with the constructions of [11, 12].

Main results

Let $Q \subset \mathbb{R}^n$ be bounded, open, with Lipschitz boundary, and let the family of coefficients $(a^\varepsilon)_\varepsilon$, with $a^\varepsilon \in L^\infty(Q;\mathbb{R}^{n\times n})$ for $\varepsilon > 0$, satisfy the uniform ellipticity and boundedness condition

$$\alpha_1|\eta|^2 \le a^\varepsilon(x)\eta\cdot\eta \le \alpha_2|\eta|^2, \qquad \forall\,\eta\in\mathbb{R}^n,\ \text{for a.e.}\ x\in\mathbb{R}^n, \tag{1.2}$$

for constants $0 < \alpha_1 < \alpha_2$. In the next condition we use a simplex $T \subset Q$ and, for $\xi \in \mathbb{R}^n$ and $b \in \mathbb{R}$, the affine function $U_\xi(x) := \xi\cdot x + b$ on T to prescribe boundary conditions. To these data, we study the unique weak solution $u^\varepsilon_{T,\xi} : T \to \mathbb{R}$ of the problem

$$\begin{aligned} -\nabla\cdot(a^\varepsilon\nabla u^\varepsilon_{T,\xi}) &= 0 \qquad &&\text{in } T,\\ u^\varepsilon_{T,\xi} &= U_\xi &&\text{on } \partial T. \end{aligned} \tag{1.3}$$

In the subsequent definition we use the notation $⨍_A f := |A|^{-1}\int_A f$ for averages of an integrable function f on a domain A.

Definition 1.1. *We say that the coefficients a^ε* allow averaging of the constitutive relation with the matrix $a^* \in \mathbb{R}^{n\times n}$ *if the following is satisfied: for every simplex $T \subset Q$ and every $\xi\in\mathbb{R}^n$, $b\in\mathbb{R}$, the solutions $u^\varepsilon_{T,\xi}$ of* (1.3) *satisfy*

$$\lim_{\varepsilon\to 0} ⨍_T a^\varepsilon\nabla u^\varepsilon_{T,\xi} = a^*\xi\,. \tag{1.4}$$

As mentioned before, the property (1.4) is satisfied for periodic coefficients a^ε and for ergodic stochastic coefficients. Regarding the latter, we mention in Appendix A a theorem which is derived in [18] and which implies that ergodic stochastic coefficients allow averaging of the constitutive relation.

It would be slightly more general to write on the right hand side of (1.4) a general function $a^*(\xi)$ with $a^* : \mathbb{R}^n \to \mathbb{R}^n$. Since the problems are linear in ξ, we actually know that the limit (if it exists) must also be linear in ξ. The important assumption is therefore that the limit exists and that it is independent of T.

In order to illustrate our new approach, we prove the following homogenization theorem.

Theorem 1.2. *Let $Q\subset\mathbb{R}^n$ be an n-dimensional bounded domain with Lipschitz boundary and $n = 2$ or $n = 3$. Let $f\in L^2(Q)$ be arbitrary and let $\psi\in H^1(Q)$ be affine. We assume that the coefficients $(a^\varepsilon)_\varepsilon$ satisfy the ellipticity relation* (1.2) *and* *that they allow averaging of the constitutive relation with the matrix* $a^* \in \mathbb{R}^{n\times n}$ *in the sense of Definition 1.1. Then the sequence* $(u^\varepsilon)_\varepsilon$ *of weak solutions of*

$$\begin{aligned} -\nabla\cdot(a^\varepsilon\nabla u^\varepsilon) &= f && \text{in } Q,\\ u^\varepsilon &= \psi && \text{on } \partial Q, \end{aligned} \tag{1.5}$$

satisfies

$$u^\varepsilon \rightharpoonup u^* \qquad \text{weakly in } H^1(Q), \tag{1.6}$$

$$a^\varepsilon\nabla u^\varepsilon \rightharpoonup a^*\nabla u^* \qquad \text{weakly in } L^2(Q), \tag{1.7}$$

where u^* *is the weak solution of*

$$\begin{aligned} -\nabla\cdot(a^*\nabla u^*) &= f && \text{in } Q,\\ u^* &= \psi && \text{on } \partial Q. \end{aligned} \tag{1.8}$$

The theorem is given here only for space dimension $n = 2$ and $n = 3$. The needle-problem approach, used in Section 2, is independent of the dimension, but it uses the adapted grids of Theorem 1.3. The construction of adapted grids is performed only in the lower dimensional cases $n = 2$ and $n = 3$ to avoid involved notation. We expect that Theorem 1.3 holds in arbitrary space dimension.

By an approximation argument, the condition $f \in L^2(Q)$ can easily be relaxed to $f \in H^{-1}(Q)$. The above theorem is stated for an affine boundary condition ψ. A general Dirichlet condition with $\psi \in H^1(Q)$ can also be treated, we restrict to the affine case for ease of notation. We note that the boundary condition $u^* = \psi$ on ∂Q is automatically satisfied for $H^1(Q)$-weak limits u^*. Therefore, we only have to verify the elliptic relation of (1.8) in the interior of Q.

Our method does not exploit the scalar character of the equation and we expect that the proof extends to the vector valued case. Furthermore, the effective coefficient may also depend on the slow variable, $a^* = a^*(x)$. In such a situation we would assume (1.1) with $a^*(x)$ instead of (1.4). The needle problem approach can provide homogenization result also in this case.

Theorem 1.2 in the light of H-convergence. A powerful abstract method for the derivation of non-periodic homogenization results has been developed in [21] with the notion of H-convergence, for which we refer also to the recent monograph [31]. The definition of H-convergence of coefficients a^ε to a matrix field a^* as in [31, Definition 6.4] is closely related to property (1.1). The compactness result of [21] (compare [31, Theorem 6.5]) can be used to show our Theorem 1.2 along the following lines. For a subsequence, the coefficients a^ε H-converge to some matrix field and by assumption (1.4) the limit must be a^*. In particular, the whole sequence a^ε H-converges to a^*. The H-convergence of the coefficients implies Theorem 1.2.

At this point we emphasize once more that our main goal is not to prove Theorem 1.2, but to introduce a new method of homogenization.

The needle problem method

Our method is based on a discretization of Q. The discretization introduces a mesh $\mathscr{T}_h$, the parameter h stands for the mesh-size. Given the triangulation, we consider two auxiliary problems. The first problem is the standard finite element discretization of the homogenized problem (1.8) with a solution U_h, introduced in Subsection 2.1. The solution U_h is used additionally in (2.7) to substitute the given right hand side f with an equivalent jump condition across the interfaces of the mesh.

The second auxiliary problem is the *needle problem* and we refer to Subsection 2.2 for its definition. Solutions are denoted as u_h^ε, these functions are affine on the interfaces introduced by $\mathscr{T}_h$, and they solve $-\nabla \cdot (a^\varepsilon \nabla u_h^\varepsilon) = 0$ in the simplices. These conditions help to conclude $u_h^\varepsilon \rightharpoonup U_h$ weakly in $H^1(Q)$, for $\varepsilon \to 0$. The homogenization program follows the scheme

$$
\begin{tikzcd}
u_h^\varepsilon \arrow[r, "L.~2.10", "\varepsilon"'] \arrow[d, leftrightarrow, "\varepsilon, h"', "P.~2.6"] & U_h \arrow[d, "h"', "L.~2.1"] \\
u^\varepsilon & u^*
\end{tikzcd}
\tag{1.9}
$$

The diagram illustrates the following results: $\lim_{h\to 0}\lim_{\varepsilon\to 0}\|u^\varepsilon - u_h^\varepsilon\|_{H^1(Q)} = 0$ of Proposition 2.6, the weak-$H^1(Q)$ convergence $u_h^\varepsilon \rightharpoonup U_h$ for $\varepsilon \to 0$ of Lemma 2.10, and $U_h \rightharpoonup u^*$ in $H^1(Q)$ for $h \to 0$ of Lemma 2.1. The combination of these results provides, since h is arbitrary, the weak-$H^1(Q)$ convergence $u^\varepsilon \rightharpoonup u^*$. In the diagram, the arrow on the right is a standard result for finite element discretizations. The arrow on the left is done by energy methods and reflects the testing procedure in common homogenization approaches; our new div-curl lemma is used here. The arrow on top is based on the averaging assumption of Definition 1.1. It involves a stabilization result, namely that indeed ∇u^ε and $a^\varepsilon \nabla u^\varepsilon$ converge weakly in $L^2(Q)$ to piece-wise constant functions as in (1.1).

We will prove Theorem 1.2 with the needle problem idea in Section 2. The procedure will be rather elementary, but we use Theorem 1.3 in Proposition 2.6. Theorem 1.3 is shown in Section 3.

Adapted grids and a div-curl Theorem

We consider bounded Lipschitz (not necessarily polygonal) domains $Q \subset \mathbb{R}^n$ in two or three space dimensions. Since our technique is based on the homogeneous solutions on simplices, we want to introduce a triangulation of the domain. To be precise, we use, for arbitrary $h > 0$, a polygonal domain $Q_h \subset Q$ and a triangulation with the properties

$$
\begin{aligned}
&\mathscr{T}_h := \{T_k\}_{k\in\Lambda_h} \quad \text{is a triangulation of } Q_h, \quad \operatorname{diam}(T_k) < h \quad \forall\, T_k \in \mathscr{T}_h, \\
&Q_h \text{ has the property that } x \in Q, \operatorname{dist}(x, \partial Q) \geq h \text{ implies } x \in Q_h\,,
\end{aligned}
\tag{1.10}
$$

where T_k are disjoint open simplices and Λ_h is a finite set of indices. We always assume that the sequence of meshes is regular in the sense of [9], section 3.1.

Much of the effort of this contribution is devoted to the construction of grids as above with additional properties regarding a fixed sequence of functions. The result of Section 3 is the following.

Theorem 1.3. *Let $Q \subset \mathbb{R}^n$, $n = 2$ or $n = 3$ be a bounded Lipschitz domain, $(u^\varepsilon)_\varepsilon$ be a sequence of functions with*

$$u^\varepsilon \rightharpoonup u \text{ weakly in } H^1(Q) \text{ for } \varepsilon \to 0.$$

Let $h > 0$ be arbitrary. Then there exists an adapted grid, i.e. $Q_h \subset Q$ and a triangulation $\mathscr{T}_h$ of Q_h with the properties (1.10)*, such that the following compensated compactness result holds.*

For every sequence $(q^\varepsilon)_\varepsilon$ in $L^2(Q, \mathbb{R}^n)$ satisfying

$$q^\varepsilon \rightharpoonup q \quad \text{weakly in } L^2(Q), \tag{1.11}$$

$$f^\varepsilon := \nabla \cdot q^\varepsilon \to f \quad \text{strongly in } H^{-1}(T), \quad \text{for all } T \in \mathscr{T}_h \tag{1.12}$$

holds

$$\lim_{\varepsilon \to 0} \int_{Q_h} q^\varepsilon \cdot \nabla u^\varepsilon \, dx = \int_{Q_h} q \cdot \nabla u \, dx. \tag{1.13}$$

Let us describe assumption (1.12) more precisely. For a fixed simplex T, we consider the distribution $f^\varepsilon := \nabla \cdot q^\varepsilon : H_0^1(T) \to \mathbb{R}$, a linear form that acts on test-functions with vanishing boundary values. For this reason, assumption (1.12) contains no information on the divergence of q^ε along the boundary of the simplex T. The crucial point in the formulation of the theorem is therefore the choice of an appropriate grid $\mathscr{T}_h$. If $\mathscr{T}_h$ is chosen such that the sequence u^ε has good regularity properties on all boundary pieces ∂T, then this additional information on u^ε can compensate for the lack of information on q^ε.

2 The needle problem approach on adapted grids

2.1 Discretization and the solution U_h

For arbitrary $h > 0$ we want to discretize Q with simplices. Since Q is, in general, not a polygonal domain, we discretize only a smaller, polygonal domain $Q_h \subset Q$ as described in (1.10). For this triangulation, we consider the finite element space of continuous and piecewise linear functions with vanishing boundary values,

$$Y_h := \left\{ \phi \in H_0^1(Q) \,:\, \phi|_{T_k} \text{ is affine } \forall\, T_k \in \mathscr{T}_h,\ \phi \equiv 0 \text{ on } Q \setminus Q_h \right\}.$$

With the matrix $a^* \in \mathbb{R}^{n \times n}$ of Definition 1.1, with $f \in L^2(Q)$ and the affine boundary condition ψ, we consider the following approximate problem.

$$\text{Find} \quad U_h \in \psi + Y_h \text{ with } \int_Q (a^* \nabla U_h) \cdot \nabla \phi = \int_Q f \phi, \qquad \forall\, \phi \in Y_h. \tag{2.1}$$

The following comparison is a standard observation for finite element approximations.

Lemma 2.1 (Comparison of U_h and u^*)**.** *There exists a unique solution U_h of* (2.1)*. For an affine boundary condition ψ there holds*

$$U_h \rightharpoonup u^* \text{ in } H^1(Q) \tag{2.2}$$

for $h \to 0$, where u^* is the solution of (1.8)*.*

Proof. Existence and uniqueness of solutions U_h together with uniform estimates in $H^1(Q)$ follow from the Lax-Milgram theorem, applied in the space Y_h. Weak convergence of a subsequence follows by compactness. The unique characterization of the limit is a consequence of the fact that the L^2-orthogonal projections $P_h : H^1_0(Q) \to Y_h \subset H^1_0(Q)$ satisfy $P_h(\phi) \to \phi$ for $h \to 0$, strongly in $H^1(Q)$, for all $\phi \in H^1_0(Q)$. □

Our next aim is to transform the right hand side f into jump conditions across edges of the grid $\mathscr{T}_h$. We will extract the relevant information on jumps from the finite element solution U_h of system (2.1). We denote the set of interior interfaces by Γ_h and the interface of two simplices T_k and T_j by Γ_{kj},

$$\Gamma_h := \left(\bigcup_k \partial T_k \right) \setminus \partial Q_h = \bigcup_{k<j} \Gamma_{kj}, \qquad \Gamma_{kj} := \overline{T}_k \cap \overline{T}_j\,.$$

We furthermore use the notation $\nu_{(k)}$ for the outer normal to T_k on ∂T_k. For a function $f \in L^2(Q;\mathbb{R}^n)$, such that $f|_{T_k}$ has a trace on ∂T_k for all k, the jump across Γ_{kj} is defined as

$$[\![f]\!]_{kj} := f|_{T_k} \cdot \nu_{(k)} + f|_{T_j} \cdot \nu_{(j)} = \left(f|_{T_k} - f|_{T_j}\right) \cdot \nu_{(k)}.$$

By definition, there holds $[\![f]\!]_{kj} = [\![f]\!]_{jk}$. We consider the jump as a scalar function on Γ_h. With the solution U_h of (2.1), we define $g_h : \Gamma_h \to \mathbb{R}$ as the function

$$g_h|_{\Gamma_{kj}} := [\![a^* \nabla U_h]\!]_{kj}. \tag{2.3}$$

The gradients ∇U_h are constant in each simplex T_k, hence $g_h : \Gamma_h \to \mathbb{R}$ is constant on each interface Γ_{kj}.

Remark 2.2. *The finite element solution U_h was defined in* (2.1) *with f. We can equivalently characterize U_h with g_h as the unique solution of*

$$U_h \in \psi + Y_h, \qquad \text{with} \qquad [\![a^* \nabla U_h]\!]_{kj} = g_h|_{\Gamma_{kj}} \ \forall k < j. \tag{2.4}$$

Problem (2.4) *is equivalent to problem* (2.1)*. This is a consequence of the fact that the jump conditions determine piecewise affine functions uniquely: for all $U, V \in Y_h$*

$$[\![\nabla U]\!]_{kj} = [\![\nabla V]\!]_{kj},\ \forall\, k \neq j \qquad \text{implies} \qquad U \equiv V.$$

The remark indicates that the right hand side f has been transformed into the jump condition g_h. This is even more clear with the observation that, for all $\phi \in Y_h$,

$$\int_Q f\phi = \int_Q a^* \nabla U_h \cdot \nabla \phi = \sum_k \int_{\partial T_k} (a^* \nabla U_h \cdot \nu_{(k)})\phi = \sum_{k<j} \int_{\Gamma_{kj}} [\![a^* \nabla U_h]\!]_{kj} \phi = \int_{\Gamma_h} g_h \phi, \tag{2.5}$$

since $a^* \nabla U_h$ is constant in each T_k. Considering only functions $\phi \in Y_h$, we have therefore equivalently replaced $f \in L^2(Q)$ by $g_h\, \mathcal{H}^{n-1}|_{\Gamma_h} \in H^{-1}(Q)$.

2.2 Approximation property of the needle problem

Until now, we considered the original problem with solution u^ε and a discrete problem with solution U_h. The needle problem lies in between: we search for a function u_h^ε which solves the original problem in each simplex, but we demand that it is affine on all interfaces. The above transformation of f into jump conditions g_h is made in order to reduce the problem to harmonic solutions in each simplex. In the subsequent definition we assume that a discretization of $Q_h \subset Q$ is given as in (1.10).

Definition 2.3 (The needle problem)**.** *We are given a Lipschitz domain $Q \subset \mathbb{R}^n$, a triangulation $\mathscr{T}_h$ of $Q_h \subset Q$ with interior interfaces Γ_h, and a piecewise affine function ψ prescribing a boundary condition. We introduce the function space*

$$\mathcal{N}_h := \left\{ \phi \in H_0^1(Q) \, : \, \phi|_{\partial T_k} \text{ is affine for all } T_k \in \mathscr{T}_h, \ \phi \equiv 0 \text{ on } Q \setminus Q_h \right\} .$$

For a given function $g_h : \Gamma_h \to \mathbb{R}$, the needle problem *is to find $u_h^\varepsilon \in \psi + \mathcal{N}_h$ such that*

$$\int_Q a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla \phi = \int_{\Gamma_h} g_h \phi \qquad \forall \phi \in \mathcal{N}_h \, . \tag{2.6}$$

We observe that, for $g_h \in L^2(\Gamma_h, \mathbb{R})$, the trace theorem implies $g_h \mathcal{H}^{n-1}|_{\Gamma_h} \in H^{-1}(Q)$. In particular, in that case, the Lax-Milgram theorem is applicable and yields the unique existence of a solution $u_h^\varepsilon \in \psi + \mathcal{N}_h$ of the needle problem.

A formulation of (2.6) on single simplices is as follows: we search for $u_h^\varepsilon \in \psi + \mathcal{N}_h$ with

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla u_h^\varepsilon) &= 0 \qquad \text{in } T_k, \quad \forall\, T_k \in \mathscr{T}_h \, , \\ \int_{\Gamma_h} \left([\![a^\varepsilon \nabla u_h^\varepsilon]\!] - g_h \right) \phi &= 0 \qquad \forall\, \phi \in \mathcal{N}_h. \end{aligned} \tag{2.7}$$

Indeed, from equation (2.7) we calculate for $\phi \in \mathcal{N}_h$

$$\int_Q a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla \phi = \sum_k \int_{T_k} a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla \phi = \int_{\Gamma_h} [\![a^\varepsilon \nabla u_h^\varepsilon]\!] \phi = \int_{\Gamma_h} g_h \phi.$$

A similar calculation shows that every solution of (2.6) solves (2.7).

The name *needle problem* is chosen for the following reason. We think of a two-dimensional domain Q and of functions $u : Q \to \mathbb{R}$, which we consider as height functions that describe a two-dimensional surface above Q. In the needle problem we search for a surface that minimizes the Dirichlet energy corresponding to a^ε, but we want the surface to contain a straight segment above each Γ_{kj}. We imagine the surface like a soap-film containing thin needles which force the free boundary to follow straight segments at certain places.

Definition 2.4. *We introduce projections $\mathscr{F}_h : \mathcal{N}_h \to Y_h \subset \mathcal{N}_h$ as follows: a function $u \in \mathcal{N}_h$ (which is piecewise affine on edges) is mapped to the piecewise affine extension of the values of u on edges. More precisely, $\mathscr{F}_h(u) : Q \to \mathbb{R}$ is the function*

$$\mathscr{F}_h(u) \in Y_h, \quad \mathscr{F}_h(u)|_{\Gamma_h} = u|_{\Gamma_h} \, . \tag{2.8}$$

We use the construction also in affine spaces and define $\mathscr{F}_h^\psi : \psi + \mathcal{N}_h \to \psi + Y_h$ as $\mathscr{F}_h^\psi(u) := \psi + \mathscr{F}_h(u - \psi)$.

Some useful properties of the projections $\mathscr{F}_h$ are collected in Lemma 2.5 below. At this point, we want to observe the following consequence of the above constructions: for solutions u_h^ε of the needle problem and arbitrary $\phi \in \mathcal{N}_h$ holds

$$\int_Q a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla \phi \overset{(2.6)}{=} \int_{\Gamma_h} g_h \phi \overset{(2.8)}{=} \int_{\Gamma_h} g_h \mathscr{F}_h(\phi) \overset{(2.5)}{=} \int_Q f \mathscr{F}_h(\phi). \tag{2.9}$$

This shows once more that the needle problem (2.6) can be regarded as a variant of the original problem with right hand side f in the space $\mathcal{N}_h$.

Lemma 2.5. *We study the projections $\mathscr{F}_h : \mathcal{N}_h \to Y_h \subset \mathcal{N}_h$ of Definition 2.4. These projections and their affine counterparts $\mathscr{F}_h^\psi$ have the following properties.*

1. $\nabla \mathscr{F}_h(u)(x) = \fint_{T_k} \nabla u$ *for* $x \in T_k$.
2. *Let* $u^\varepsilon \in \mathcal{N}_h$, $u^\varepsilon \rightharpoonup u$ *weakly in* $H^1(Q)$ *for fixed* $h > 0$. *Then*
$$\mathscr{F}_h(u^\varepsilon) \underset{\varepsilon}{\rightharpoonup} \mathscr{F}_h(u), \quad \textit{weakly in } H^1(Q).$$
3. *Let* $u_h \in \mathcal{N}_h$, $u_h \rightharpoonup u$ *weakly in* $H^1(Q)$ *for* $h \to 0$. *Then*
$$\mathscr{F}_h(u_h) \underset{h}{\rightharpoonup} u, \quad \textit{weakly in } H^1(Q).$$

Proof. Concerning property 1, we first note that $\nabla \mathscr{F}_h(u)$ is indeed a constant vector in each simplex. The claim follows from the following calculation for a direction e_j, $j = 1, ..., n$, and a simplex T_k with exterior normal ν,

$$\fint_{T_k} \partial_j \mathscr{F}_h(u) = \frac{1}{|T_k|} \int_{\partial T_k} \mathscr{F}_h(u)\, e_j \cdot \nu = \frac{1}{|T_k|} \int_{\partial T_k} u\, e_j \cdot \nu = \fint_{T_k} \partial_j u\,.$$

For property 2 we note that the projection is bounded in $H^1(Q)$. Indeed, for $u \in \mathcal{N}_h$, by Poincaré's and Jensen's inequalities

$$\|\mathscr{F}_h(u)\|^2_{H^1(Q)} \le C \|\nabla \mathscr{F}_h(u)\|^2_{L^2(Q)} = C \sum_k \int_{T_k} \left| \fint_{T_k} \nabla u \right|^2 \le C \int_Q |\nabla u|^2.$$

In particular, for sequences $u^\varepsilon \in \mathcal{N}_h$, $u^\varepsilon \rightharpoonup u$ weakly in $H^1(Q)$ for $\varepsilon \to 0$, we find a subsequence of $\mathscr{F}_h(u^\varepsilon)$ which converges weakly in $H^1(Q)$ to a limit $F \in Y_h$. We used here that Y_h is weakly closed in $H^1(Q)$. We can identify the limit to be $F = \mathscr{F}_h(u)$ by noting that, for all $T_k \in \mathscr{T}_h$ and all $x \in T_k$

$$\nabla \mathscr{F}_h(u^\varepsilon)(x) = \fint_{T_k} \nabla u^\varepsilon \underset{\varepsilon}{\to} \fint_{T_k} \nabla u = \nabla \mathscr{F}_h(u)(x).$$

In order to show property 3, let $\mathcal{N}_h \ni u_h \rightharpoonup u$ weakly in $H^1(Q)$. As noted above, the sequence $\mathscr{F}_h(u_h)$ is also bounded in $H^1(Q)$. We can thus find a subsequence such that $\mathscr{F}_{h_l}(u_{h_l}) \rightharpoonup F$ in $H^1(Q)$.

In order to identify the limit as $F = u$, we choose an arbitrary test-function $\phi \in C_c^\infty(Q;\mathbb{R}^n)$. By density of the piecewise constant functions in L^2, we find a sequence (ϕ_h) of piecewise constant functions with $\phi_h \to \phi$ strongly in $L^2(Q;\mathbb{R}^n)$. We compute

$$\begin{aligned}
&\left| \int_Q \nabla \mathscr{F}_h(u_h) \cdot \phi - \int_Q \nabla u \cdot \phi \right| \\
&\quad = \left| \int_Q \nabla \mathscr{F}_h(u_h) \cdot \phi_h + \int_Q \nabla \mathscr{F}_h(u_h) \cdot (\phi - \phi_h) - \int_Q \nabla u \cdot \phi \right| \\
&\quad \le \left| \int_Q \nabla u_h \cdot \phi_h - \int_Q \nabla u \cdot \phi \right| + \|\nabla \mathscr{F}_h(u_h)\|_{L^2} \|\phi - \phi_h\|_{L^2}.
\end{aligned}$$

The first term on the right-hand side converges to zero since $\nabla u_h \rightharpoonup \nabla u$ weakly and $\phi_h \to \phi$ strongly in $L^2(Q;\mathbb{R}^n)$, the second term vanishes by boundedness of the first factor and strong convergence of ϕ_h. We can therefore conclude $F = u$.

The definition of $\mathscr{F}_h^\psi$ implies that properties remain valid on affine subspaces. □

Our next aim is to compare the original solution u^ε with the needle problem solution u_h^ε. This comparison is provided with the following Proposition.

Proposition 2.6 (Comparison of u_h^ε and u^ε)**.** *Let coefficients $a^\varepsilon \in L^\infty(Q;\mathbb{R}^{n\times n})$, $n = 2$ or $n = 3$, satisfy the ellipticity* (1.2) *and let ψ be an affine function. Let $u^\varepsilon \in H^1(Q)$ be the weak solution of the original problem* (1.5)*, and let $u_h^\varepsilon \in \psi + \mathcal{N}_h$ be solutions to the needle problem* (2.6) *with g_h of* (2.3)*. Furthermore, we assume that the grids $\mathscr{T}_h$ are adapted grids for $(u^\varepsilon)_\varepsilon$, such that the assertion of Theorem 1.3 holds. Then*

$$\lim_{h\to 0} \lim_{\varepsilon \to 0} \|u_h^\varepsilon - u^\varepsilon\|_{H^1(Q)} = 0. \tag{2.10}$$

The idea of the proof is to use $(u^\varepsilon - u_h^\varepsilon)$ as a test-function for the original problem (1.5) and in the needle problem (2.6), and to take the difference. We note that this test function satisfies a homogeneous Dirichlet condition. By ellipticity of a^ε, the result provides an upper bound for $\|u^\varepsilon - u_h^\varepsilon\|_{H^1(T)}^2$. It remains to show that the upper bound vanishes in the limit as $\varepsilon \to 0$ and then $h \to 0$.

Proof. All solution sequences of the proposition are bounded in $H^1(Q)$. This allows to choose a subsequence and limit functions such that, as $\varepsilon \to 0$,

$$u^\varepsilon \rightharpoonup u, \quad u_h^\varepsilon \rightharpoonup u_h \quad \text{weakly in } H^1(Q), \tag{2.11}$$

$$\nabla u_h^\varepsilon \rightharpoonup \nabla u_h, \quad q_h^\varepsilon := a^\varepsilon \nabla u_h^\varepsilon \rightharpoonup q_h \quad \text{weakly in } L^2(Q). \tag{2.12}$$

We note that the distributional divergence of q_h^ε vanishes in each simplex T_k by (2.7).

Since the needle problem does not allow to use u^ε as a test function, we must apply a projection. We use the $L^2(Q)$-orthogonal projection $P_h : L^2(Q) \to Y_h \subset L^2(Q)$ and the affine counterpart $P_h^\psi : L^2(Q) \to \psi + Y_h$ defined by $P_h^\psi(u) := \psi + P_h(u - \psi)$. As a consequence of (2.11), we have the strong convergence $u^\varepsilon \to u$ in $L^2(Q)$, and hence also $P_h^\psi(u^\varepsilon) \to P_h^\psi(u)$ in $L^2(Q)$. Since P_h^ψ maps into a space of finite dimension, the convergence is in all norms, in particular, as $\varepsilon \to 0$, also

$$P_h^\psi(u^\varepsilon) \to P_h^\psi(u) \qquad \text{in } H^1(Q).$$

We can now start the computations. For some $\alpha_0 > 0$ that combines the ellipticity constant $\alpha_1 > 0$ and the constant from Poincaré's inequality, we find

$$
\begin{aligned}
\alpha_0 \|u^\varepsilon - u_h^\varepsilon\|^2_{H^1(Q)} &\le \int_Q a^\varepsilon \nabla(u^\varepsilon - u_h^\varepsilon) \cdot \nabla(u^\varepsilon - u_h^\varepsilon) \\
&= \int_Q a^\varepsilon \nabla u^\varepsilon \cdot \nabla(u^\varepsilon - u_h^\varepsilon) - \int_Q a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla(u^\varepsilon - u_h^\varepsilon) \\
&\overset{(1.5)}{=} \int_Q f\,(u^\varepsilon - u_h^\varepsilon) - \int_Q a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla(u^\varepsilon - P_h^\psi(u^\varepsilon)) - \int_Q a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla(P_h^\psi(u^\varepsilon) - u_h^\varepsilon) \\
&\overset{(2.9)}{=} \int_Q f\,(u^\varepsilon - u_h^\varepsilon) - \int_Q a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla(u^\varepsilon - P_h^\psi(u^\varepsilon)) - \int_Q f\, \mathscr{F}_h(P_h^\psi(u^\varepsilon) - u_h^\varepsilon) \\
&= \int_Q f\,(u^\varepsilon - P_h^\psi(u^\varepsilon)) + \int_Q f\,(\mathscr{F}_h^\psi(u_h^\varepsilon) - u_h^\varepsilon) - \int_Q q_h^\varepsilon \cdot \nabla(u^\varepsilon - P_h^\psi(u^\varepsilon)).
\end{aligned}
$$

In the last line we only re-ordered terms and used $\mathscr{F}_h^\psi \circ P_h^\psi(u^\varepsilon) = P_h^\psi(u^\varepsilon)$. Our aim is to show that the right hand side vanishes as $\varepsilon \to 0$, and then $h \to 0$. Concerning the first integral we have

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} \int_Q f\,(u^\varepsilon - P_h^\psi(u^\varepsilon)) = \lim_{h\to 0} \int_Q f\,(u - P_h^\psi(u)) = 0.$$

In order to treat the second integral we select a subsequence $h \to 0$ such that $u_h \rightharpoonup \tilde{u}$ for $h \to 0$, weakly in $H^1(Q)$ for some limit $\tilde{u}$. This allows to use Lemma 2.5, first property 2 together with (2.11), and then property 3. We find

$$\lim_{\varepsilon\to 0} \int_Q f\,(\mathscr{F}_h(u_h^\varepsilon) - u_h^\varepsilon) = \int_Q f\,(\mathscr{F}_h(u_h) - u_h) \to 0 \quad \text{for } h \to 0.$$

Concerning the third integral, we must use a div-curl lemma. The integrand is the product of the functions $q_h^\varepsilon = a^\varepsilon \nabla u_h^\varepsilon \rightharpoonup q_h$ in $L^2(Q)$, and of $\nabla(u^\varepsilon - P_h^\psi(u^\varepsilon)) \rightharpoonup \nabla(u - P_h^\psi(u))$ weakly in $L^2(Q)$, both convergences for $\varepsilon \to 0$. On the other hand, we treat the product of a weakly convergent sequence q_h^ε satisfying $\nabla \cdot q_h^\varepsilon = 0$ with a weakly convergent sequence of gradients. The grid is adapted to the sequence u^ε, such that the assertion of the div-curl Theorem 1.3 can be used. Relation (1.13) allows to calculate the limit

$$\lim_{\varepsilon\to 0} \int_Q q_h^\varepsilon \cdot \nabla(u^\varepsilon - P_h^\psi(u^\varepsilon)) = \lim_{\varepsilon\to 0} \int_{Q_h} q_h^\varepsilon \cdot \nabla(u^\varepsilon - P_h^\psi(u^\varepsilon)) = \int_{Q_h} q_h \cdot \nabla(u - P_h^\psi(u)).$$

We now use that q_h is bounded in $L^2(Q)$ and $P_h^\psi(u) \to u$ converges strongly in $H^1(Q)$ to conclude

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} \int_Q q_h^\varepsilon \cdot \nabla(u^\varepsilon - P_h^\psi(u^\varepsilon)) = \lim_{h\to 0} \int_Q q_h \cdot \nabla(u - P_h^\psi(u)) = 0.$$

This implies smallness of the third integral and verifies the claim of the proposition. $\square$

We note that, at this point, we have already verified the smallness conditions regarding vertical arrows in the diagram of (1.9), namely $\lim_{h\to 0}\lim_{\varepsilon\to 0} \|u^\varepsilon - u_h^\varepsilon\|_{H^1(Q)} = 0$ of the above Proposition, and $U_h \underset{h}{\rightharpoonup} u^*$ in $H^1(Q)$ in Lemma 2.1. We emphasize that we used one non-trivial ingredient: the fact that the triangulation can be chosen adapted to the sequence u^ε and the corresponding div-curl Theorem 1.3.

2.3 Stabilization result and proof of Theorem 1.2

To conclude the diagram of (1.9), it remains to check the horizontal arrow. We want to verify for the needle problem solution u_h^ε and the finite elements solution U_h the weak H^1-convergence $u_h^\varepsilon \underset{\varepsilon}{\rightharpoonup} U_h$. This convergence result is quite straightforward once that we know, using the notation of Definition 1.1, the L^2-convergence $\nabla u^\varepsilon \rightharpoonup \xi$ and $a^\varepsilon \nabla u^\varepsilon \rightharpoonup a^* \xi$ for some ξ in each triangle. The important point here is that the weak limits are constant functions; we refer to this fact as stabilization. The verification of the stabilization is the main purpose of this section. The conclusion of Theorem 1.2 is then performed easily with Lemma 2.10.

As a preparation, we observe that the averaging property (1.4) extends to sequences of affine boundary conditions.

Lemma 2.7. *Let the coefficients a^ε allow averaging of the constitutive relation with the matrix a^*. Then, for every simplex $T \subset Q$ and every sequence $U_{\xi^\varepsilon}(x) = \xi^\varepsilon \cdot x + b^\varepsilon \to U_\xi(x) = \xi \cdot x + b$, the solutions $u^\varepsilon_{T,\xi^\varepsilon}$ of*

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla u^\varepsilon_{T,\xi^\varepsilon}) &= 0 \qquad \text{in } T,\\ u^\varepsilon_{T,\xi^\varepsilon} &= U_{\xi^\varepsilon} \quad \text{on } \partial T, \end{aligned} \tag{2.13}$$

satisfy

$$\lim_{\varepsilon\to 0} \fint_T a^\varepsilon \nabla u^\varepsilon_{T,\xi^\varepsilon} = a^* \xi. \tag{2.14}$$

Proof. It suffices to compare $u^\varepsilon_{T,\xi^\varepsilon}$ and $u^\varepsilon_{T,\xi}$. For the solutions $u^\varepsilon_{T,\xi}$, the averages converge as in (2.14) by the averaging property (1.4). On the other hand, the difference $u^\varepsilon - \tilde{u}^\varepsilon$ is small in $H^1(T)$. This smallness follows by linearity and ellipticity of the equation. □

Proposition 2.8 (Stabilization)**.** *Let the coefficients $a^\varepsilon \in L^\infty(Q; \mathbb{R}^{n\times n})$ satisfy* (1.2) *and allow averaging with matrix a^* in the sense of Definition 1.1. Let $T \subset \mathbb{R}^n$ be a simplex, $U_\xi(x) = \xi \cdot x + b$ an affine function, and u^ε a sequence of weak solutions of*

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla u^\varepsilon) &= 0 \quad \text{in } T,\\ u^\varepsilon &= U_\xi \quad \text{on } \partial T. \end{aligned} \tag{2.15}$$

We denote the limits of functions and fluxes by u and q, i.e. we assume

$$\begin{aligned} u^\varepsilon &\rightharpoonup u \qquad \text{weakly in } H^1(T, \mathbb{R}),\\ q^\varepsilon := a^\varepsilon \nabla u^\varepsilon &\rightharpoonup q \qquad \text{weakly in } L^2(T; \mathbb{R}^n). \end{aligned}$$

Then u is affine and q is constant. More precisely, there holds

$$\nabla u \equiv \xi \quad \text{in } T, \tag{2.16}$$

$$q \equiv a^* \xi \quad \text{in } T. \tag{2.17}$$

Proof. In this proof, we consider sequences u^ε on a fixed simplex T. The simplex T now plays the role of the arbitrary domain Q of Subsection 2.2, and our aim is to use the results obtained so far. We fix a sequence $h \searrow 0$. We choose polygonal domains $T_h \subset T$ and triangulations of T_h,

$$\mathcal{S}_h := \{S_k\}_{k\in\Lambda_h} \quad \text{be a triangulation of } T_h,$$

where S_k are simplices such that $\max\{\mathrm{diam}(S_k)|\, k \in \Lambda_h\} < h$ and $T_h \subset T$ as in (1.10). By Theorem 1.3 we may assume that, for all h, the subdivision $\mathcal{S}_h$ is an adapted grid for u^ε and that the div-curl property (1.13) holds.

Let $(u_h^\varepsilon)_\varepsilon$ be a subsequence of solutions of the needle problem (2.6) on T with vanishing jump conditions $g \equiv 0$ and with boundary condition $\psi = U_\xi$. We select a subsequence $\varepsilon \to 0$ and limit functions u_h such that, for all h in the sequence, $u_h^\varepsilon \rightharpoonup u_h$ for $\varepsilon \to 0$, weakly in $H^1(T)$. We note that all functions u_h^ε, and thus also u_h, are affine on all ∂S_k. The needle problem comparison result of Proposition 2.6 yields $\|u - u_h\|_{H^1}^2 \le \limsup_{\varepsilon\to 0} \|u^\varepsilon - u_h^\varepsilon\|_{H^1}^2 \le \eta(h) \to 0$ for $h \to 0$.

Proof of relation (2.16). Corresponding to the needle problem solution u_h^ε, we consider the piecewise affine functions $\bar u_h^\varepsilon := \mathscr{F}_h^\psi(u_h^\varepsilon)$, and (after selection of a weakly convergent subsequence) their weak limits $\bar u_h \in H^1(T)$. We use the abbreviations $\xi_k^\varepsilon := \nabla \bar u_h^\varepsilon|_{S_k} \to \nabla \bar u_h|_{S_k} =: \xi_k$. For fixed h, we consider a test-function ϕ in the corresponding needle space: ϕ is continuous on $\bar T$, vanishes on $T\setminus T_h$, and is piecewise affine on every simplex S_k. We calculate, exploiting that $\nabla\phi$ is constant on each simplex S_k, for $\varepsilon \to 0$,

$$0 \overset{(2.6)}{=} \int_T a^\varepsilon \nabla u_h^\varepsilon \nabla\phi = \sum_k \int_{S_k} a^\varepsilon \nabla u_h^\varepsilon \nabla\phi \overset{(2.14)}{\to} \sum_k \int_{S_k} a^* \xi_k \nabla\phi = \int_T a^* \nabla \bar u_h \nabla\phi\,.$$

We conclude that $\bar u_h$ is a finite element solution of $-\nabla\cdot(a^*\nabla \bar u_h) = 0$ with affine boundary condition U_ξ, which implies $\bar u_h = U_\xi$. Property 2 of Lemma 2.5 implies $\bar u_h^\varepsilon = \mathscr{F}_h^\psi(u_h^\varepsilon) \rightharpoonup \mathscr{F}_h^\psi(u_h)$ in H^1, hence $U_\xi = \bar u_h = \mathscr{F}_h^\psi(u_h)$. The convergence $u_h \to u$ in $H^1(T)$ from the needle problem estimate allows to conclude, using property 3 of Lemma 2.5, $\mathscr{F}_h^\psi(u_h) \rightharpoonup u$ in H^1 for $h \to 0$, and hence $u = U_\xi$. This shows (2.16).

Proof of relation (2.17). We consider, after selection of a subsequence, the limiting fluxes $q^\varepsilon = a^\varepsilon \nabla u^\varepsilon \rightharpoonup q$ and $q_h^\varepsilon := a^\varepsilon \nabla u_h^\varepsilon \rightharpoonup q_h$, with weak convergence in $L^2(T)$ for $\varepsilon \to 0$. Lower semi-continuity of the norm and the estimate for the needle problem of Proposition 2.6 yields $\lim_{h\to 0} \|q - q_h\|_{L^2} \le \lim_{h\to 0} \liminf_{\varepsilon\to 0} \|a^\varepsilon \nabla u^\varepsilon - a^\varepsilon \nabla u_h^\varepsilon\|_{L^2} = 0$. Our aim is to show $q \equiv a^*\xi$.

We use an arbitrary function $\psi \in C_c^1(T)$, which we approximate by functions $\psi_h : T \to \mathbb{R}$ that vanish on $T\setminus T_h$ and are piece-wise constant in each simplex $S_k \subset T$ (for the triangulation corresponding to h), with $\psi_h \to \psi$ strongly in $L^2(T)$ for $h \to 0$. We use once more Lemma 2.7 in each S_k, where u_h^ε satisfies affine boundary conditions with slope $\xi_k^\varepsilon \to \xi$. We calculate, for $\varepsilon \to 0$,

$$\int_T q_h \psi_h \leftarrow \int_T a^\varepsilon \nabla u_h^\varepsilon \psi_h = \sum_k \int_{S_k} (a^\varepsilon \nabla u_h^\varepsilon)\psi_h \to \sum_k \int_{S_k} a^*\xi\psi_h = \int_T a^*\xi\psi_h\,.$$

The strong L^2-convergences $q_h \to q$ and $\psi_h \to \psi$ yield $q \equiv a^*\xi$, since ψ was arbitrary. This concludes the proof of Proposition 2.8. $\square$

The result of the above proposition remains valid for a convergent sequence of affine boundary conditions. We note this direct consequence for later use.

Corollary 2.9. *Let the coefficients a^ε satisfy* (1.2) *and allow averaging with matrix a^* in the sense of Definition 1.1. We study a simplex T and a convergent sequence of affine functions $U_{\xi^\varepsilon}(x) = \xi^\varepsilon \cdot x + b^\varepsilon \to U_\xi(x) = \xi \cdot x + b$. Then, the solutions (w^ε) of*

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla w^\varepsilon) &= 0 && \text{in } T \\ w^\varepsilon &= U_{\xi^\varepsilon} && \text{on } \partial T \end{aligned}$$

satisfy

$$\begin{aligned} \nabla w^\varepsilon &\rightharpoonup \xi && \text{weakly in } L^2(T), \\ a^\varepsilon \nabla w^\varepsilon &\rightharpoonup a^*\xi && \text{weakly in } L^2(T, \mathbb{R}^n). \end{aligned}$$

Proof. We use the solutions u^ε of

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla u^\varepsilon) &= 0 && \text{in } T \\ u^\varepsilon &= U_\xi && \text{on } \partial T \end{aligned}$$

as studied in Proposition 2.8. In view of that proposition, it suffices to derive smallness in $H^1(T)$ of $u^\varepsilon - w^\varepsilon$. We multiply the equation for $u^\varepsilon - w^\varepsilon$ with $(u^\varepsilon - U_\xi) - (w^\varepsilon - U_{\xi^\varepsilon})$, which vanishes on the boundary ∂T. By Hölder's inequality and uniform ellipticity of a^ε, there exists $C > 0$ such that

$$\|u^\varepsilon - w^\varepsilon\|^2_{H^1(T)} \le C \|U_\xi - U_{\xi^\varepsilon}\|^2_{H^1(T)} \to 0.$$

This yields the claim. □

The subsequent lemma shows the missing convergence in the diagram of (1.9). It hence concludes the proof of Theorem 1.2.

Lemma 2.10 (Comparison of needle problem and discretized problem)**.** *Let the domain Q, coefficients a^ε, f and ψ be as in Theorem 1.2. Let $h > 0$ be fixed, U_h the solution of the auxiliary problem* (2.1) *and g_h as in* (2.3)*. Let u_h^ε be the solution of the needle problem* (2.6)*. Then, as $\varepsilon \to 0$,*

$$\begin{aligned} u_h^\varepsilon &\rightharpoonup U_h && \text{weakly in } H^1(Q, \mathbb{R}), \\ a^\varepsilon \nabla u_h^\varepsilon &\rightharpoonup a^* \nabla U_h && \text{weakly in } L^2(Q, \mathbb{R}^n). \end{aligned}$$

Proof. Let u_h^ε be the solution of (2.6) and let u_h be any $H^1(Q)$-weak limit point of $(u_h^\varepsilon)_\varepsilon$, as $\varepsilon \to 0$. As solutions of the needle problem, the functions u_h^ε are affine on the boundaries of each simplex. For fixed h and fixed simplex T_k, we denote the corresponding affine function by $U^{(k)}_{\xi_k^\varepsilon}$, and find further subsequences $\varepsilon \to 0$ such that these functions converge for each simplex to affine functions $U^{(k)}_{\xi_k}$. Corollary 2.9 implies, for all $T_k \in \mathscr{T}_h$, as $\varepsilon \to 0$,

$$\begin{aligned} \nabla u_h^\varepsilon &\rightharpoonup \xi_k && \text{weakly in } L^2(T_k), \\ a^\varepsilon \nabla u_h^\varepsilon &\rightharpoonup a^* \xi_k && \text{weakly in } L^2(T_k). \end{aligned}$$

In particular, $u_h \in Y_h$. We now use an arbitrary test-function $\phi \in Y_h$ and use the needle problem characterization (2.9) to find, for $\varepsilon \to 0$,

$$\int_Q f\phi = \int_Q a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla\phi \to \int_Q a^* \nabla u_h \cdot \nabla\phi.$$

By uniqueness of solutions of the discrete problem (2.1), we find $u_h = U_h$ and have thus verified the claim. □

3 Adapted grids

This section is devoted to the proof of Theorem 1.3. We consider an n-dimensional domain Ω and a fixed family of functions $u_k : \Omega \to \mathbb{R}$, bounded in $H^1(\Omega)$. Since we will treat integrals over objects of different dimensions, we write $\mathcal{L}^m$ and $\mathcal{H}^m$ for the m-dimensional Lebesgue- and Hausdorff-measure. Our boundedness assumption on the sequence u_k is then written as

$$\int_\Omega |u_k(z)|^2 \, d\mathcal{L}^n(z) + \int_\Omega |\nabla u_k(z)|^2 \, d\mathcal{L}^n(z) \le C_0 \qquad \forall\, k \in \mathbb{N}, \tag{3.1}$$

for some $C_0 > 0$. Our interest in this section is to find (many) simplices contained in Ω, such that, loosely speaking, ∇u_k is L^2-bounded on the faces. Such a boundedness implies compactness of the boundary values in $H^{1/2}$ and allows to construct extensions of the boundary values that are strongly convergent in H^1. The fact that on almost all $(n-1)$-dimensional hyperplanes the functions ∇u_k are L^2-bounded is a consequence of Fubini's theorem.

In the construction of strongly convergent extensions we must be careful in the treatment of the $(n-2)$-dimensional edges of the simplices, the boundaries of the $(n-1)$-dimensional faces. In order to treat these boundaries, we demand additionally that the averages of $|\nabla u_k|^2$ over small neighborhoods of edges are bounded. To make such a property precise, we use a sequence of positive numbers $\delta_k \to 0$, these numbers will be radii of small balls or cylinders.

3.1 Adapted grids in two dimensions

This subsection is devoted to the construction of adapted grids for the case $n = 2$. Some concepts are independent of the dimension and are treated here for general dimension as a preparation for $n = 3$. We always assume that we are given a sequence of positive numbers $\delta_k \to 0$ and a sequence of functions $u_k : \Omega \to \mathbb{R}$ satisfying (3.1).

Definition 3.1 (Points of typical average)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open domain, $\delta_k \to 0$, and $(u_k)_k$ be a sequence in $H^1(\Omega)$. We say that $x \in \Omega$ is a* point of typical average *for $(\delta_k)_k$ and $(u_k)_k$ if the following holds. There exists a subsequence $k_j \to \infty$ and* *real numbers* c_x *and* M_x *such that*

$$\fint_{B_{\delta_{k_j}}(x)} |\nabla u_{k_j}(z)|^2 \, d\mathcal{L}^n(z) \le M_x \quad \forall\, k_j \tag{3.2}$$

$$c_x^{k_j} := \fint_{B_{\delta_{k_j}}(x)} u_{k_j}(z) \, d\mathcal{L}^n(z) \ \to c_x \qquad \textit{for } k_j \to \infty. \tag{3.3}$$

We say that $(k_j)_j$ *is a* good subsequence *for the point* x *when* (3.2) *and* (3.3) *are satisfied along this subsequence for some* c_x *and* M_x*.*

Since u_k is not defined outside Ω, we use the convention that integrals $\int_B$ denote integrals $\int_{B\cap\Omega}$. Since Ω is open, the balls $B_{\delta_k}(x)$ are contained in Ω for large k.

We note that a point of typical average is similar to a Lebesgue point — but the point has good properties for a whole sequence of functions.

Lemma 3.2 (Many points of typical average)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be an open domain,* $\delta_k \to 0$*, and* $(u_k)_k$ *be a bounded sequence in* $H^1(\Omega)$*. Then almost every point* $x \in \Omega$ *is a point of typical average for* $(\delta_k)_k$ *and* $(u_k)_k$*.*

Proof. For $x \in \Omega$ and $k \in \mathbb{N}$ we set

$$F(k,x) := \fint_{B_{\delta_k}(x)} |u_k|^2 + |\nabla u_k|^2 \,. \tag{3.4}$$

As a first step of the proof we verify the following. *Claim 1.* For arbitrary $\vartheta > 0$ there exists a small exceptional set $E \subset \Omega$ of Lebesgue measure $|E| \le \vartheta$ with the property

$$\forall x \in \Omega \setminus E \quad \exists \text{ subsequence } (k_j)_j : \quad F(k_j, x) \text{ is bounded.} \tag{3.5}$$

Once that Claim 1 is verified, the assertion of the lemma follows easily. Indeed, since the set E has arbitrarily small measure, for almost every $x \in \Omega$ the boundedness of $F(k,x)$ along a subsequence is satisfied. This shows (3.2) and, because of $|u_k| \le 1 + |u_k|^2$, it proves also the boundedness of the $c_x^{k_j}$ in (3.3) for almost every point along an appropriate subsequence. By boundedness of $c_x^{k_j}$, taking a further subsequence, we find additionally a limit value c_x and the convergence as claimed in (3.3). This shows that almost every point is a point of typical average and concludes the proof of the lemma.

In order to verify Claim 1, we fix an arbitrary $\vartheta > 0$. For a contradiction argument we assume that there exists a (large) exceptional set $E \subset \Omega$ of measure $|E| > \vartheta$, consisting of points x such $F(k_j, x)$ is unbounded along every subsequence $(k_j)_j$. In order to derive a contradiction we fix a constant $M \in \mathbb{R}$ with $M > 3^{n+1} C_0/\vartheta$, where C_0 is the $H^1(\Omega)$-bound of the sequence u_k. Let now $x \in E$ be arbitrary. Since along all subsequences k_j the values $F(k_j, x)$ are unbounded, there exists $K(x) \in \mathbb{N}$ such that

$$F(k,x) \ge M \qquad \text{for all } k \ge K(x). \tag{3.6}$$

We choose with $K(x) := 1 + \max\{k \in \mathbb{N} | F(k,x) < M\}$ the minimal $K(x)$ with this property. With this choice, $K : \Omega \to \mathbb{N}$ is lower semi-continuous, as can be seen with a brief contradiction argument: assume that for a sequence $x_j \to x_0$ there exists $\bar{k} \in \mathbb{N}$ such that $K(x_j) \to \bar{k} < K(x_0)$. Then $F(\bar{k}, x_j) \geq M$ and $F(\bar{k}, x_0) < M$ for all sufficiently large j, a contradiction to continuity of $x \mapsto F(x, \bar{k})$. The lower semi-continuity of K implies, in particular, that K is (Borel-)measurable.

We now consider the measurable sets

$$E_N := \{x \in E : K(x) \leq N\},$$

such that

$$E = \bigcup_{N \in \mathbb{N}} E_N, \quad E_{N+1} \supset E_N, \quad \text{and hence } |E| = \lim_{N \to \infty} |E_N|. \tag{3.7}$$

By hypothesis we have $|E| > \vartheta$, thus we find $N \in \mathbb{N}$ with $|E_N| > \vartheta/2$. By measurability of E_N, there exists a compact set $\tilde{E}_N$ satisfying

$$\tilde{E}_N \subset E_N, \qquad |\tilde{E}_N| > \frac{\vartheta}{3}. \tag{3.8}$$

Corresponding to the covering

$$\tilde{E}_N \subset \bigcup_{x \in \tilde{E}_N} B_{\delta_N}(x)$$

we find a finite sub-covering by compactness of $\tilde{E}_N$. We can apply an elementary covering lemma (see, e.g., [25], Lemma 7.3) to select a finite set of points $(x_m)_m$ such that

$$\tilde{E}_N \subset \bigcup_m B_{3\delta_N}(x_m), \qquad B_{\delta_N}(x_{m_1}) \cap B_{\delta_N}(x_{m_2}) = \emptyset, \quad \text{for all } m_1 \neq m_2\,. \tag{3.9}$$

Up to choosing a new, possibly bigger, value for N, it is not restrictive to assume that $d(\tilde{E}_N, \partial\Omega) > 1/N$, so that $B_{\delta_N}(x_m) \subset \Omega$. Recalling the H^1-boundedness (3.1) of the sequence, we can now calculate with $k = N$

$$\begin{aligned}
C_0 &\geq \int_\Omega \left\{|u_k|^2 + |\nabla u_k|^2\right\} \geq \int_{\bigcup_m B_{\delta_N}(x_m)} \left\{|u_k|^2 + |\nabla u_k|^2\right\} \\
&\overset{(3.9)}{=} \sum_m \int_{B_{\delta_N}(x_m)} \left\{|u_k|^2 + |\nabla u_k|^2\right\} \overset{(3.6)}{\geq} \sum_m |B_{\delta_N}(x_m)| M \\
&\geq \left| \bigcup_m B_{3\delta_N}(x_m) \right| \frac{M}{3^n} \overset{(3.9)}{\geq} |\tilde{E}_N| \frac{M}{3^n} \overset{(3.8)}{\geq} M \frac{\vartheta}{3^{n+1}} > C_0,
\end{aligned}$$

where we used $M > 3^{n+1} C_0/\vartheta$ in the last step. This provides the desired contradiction. We exploited in the above calculation that $x_m \in \tilde{E}_N \subset E_N$ implies inequality (3.6) for $k = N$. $\square$

We next study conditions for segments. For points $x, y \in \mathbb{R}^n$ we use the notation $[x, y] := \{\theta x + (1-\theta) y : \theta \in [0, 1]\}$ and refer to $[x, y]$ as the segment to the pair (x, y). Loosely speaking, we want to show that, for most segments $\Gamma \subset \Omega$, the sequence of gradients $\nabla u_k|_\Gamma$ is bounded in $L^2(\Gamma)$.

Let us start with a general comment on the construction. With u_k as above, the $L^2(\Omega)$-function ∇u_k is specified almost everywhere, hence the values of the function on segments Γ are specified almost everywhere on the segment, at least for almost every segment. In this sense, we can consider integrals of the gradient over segments.

Later on, we want to relate the gradient to traces. For $n = 2$, given a segment Γ, we consider the $H^{1/2}(\Gamma)$-functions $u_k|_\Gamma$ and their distributional (tangential) gradients $\nabla_\tau u_k|_\Gamma$. For smooth functions, these coincide with the projection of ∇u_k to the tangential space of the segment Γ. With smooth test-functions and an integration over families of parallel segments one can verify that the two constructions yield the same function $\nabla_\tau u_k|_\Gamma$ for almost all segments Γ.

Definition 3.3 (Typical segments). *For $\Omega \subset \mathbb{R}^n$ open, given a sequence $\delta_k \to 0$ and a bounded sequence $(u_k)_k \in H^1(\Omega)$, we say that a segment $\Gamma = [x, y]$ is a* typical segment *if the following holds: There exists a subsequence $k_j \to \infty$ and a constant $M_\Gamma > 0$ such that*

$$\|u_{k_j}|_\Gamma\|^2_{L^2(\Gamma)} + \|\nabla_\tau u_{k_j}|_\Gamma\|^2_{L^2(\Gamma)} \le M_\Gamma. \tag{3.10}$$

We furthermore demand that the end-points x and y are points of typical average and that the subsequence $(k_j)_j$ is a good subsequence for x and for y.

A subsequence $(k_j)_j$ with the above properties is called a good subsequence *for the segment Γ.*

Lemma 3.4 (Many typical segments). *Let $\Omega \subset \mathbb{R}^n$ be a convex domain, $\delta_k \to 0$, and let $(u_k)_k \subset H^1(\Omega)$ be a bounded sequence. Then, for almost every $x \in \Omega$, there is a good set $\mathscr{G}_x \subset \Omega$ of full measure $|\mathscr{G}_x| = |\Omega|$, such that for all $y \in \mathscr{G}_x$ the segment $[x, y]$ is a typical segment according to Definition 3.3.*

Proof. Let us first observe that almost every $x \in \Omega$ is a point of typical average by Lemma 3.2. We fix such a point x and the subsequence δ_{k_j} and apply the Lemma again. We find that almost every $y \in \Omega$ is a point of typical average. This provides, in particular, a good subsequence for both x and y.

We additionally have to verify that almost every segment (chosen in the described way) satisfies (3.10). We abbreviate the integrands as $f^k(x) := |u_k|^2(x) + |\nabla u_k|^2(x)$, a sequence of non-negative functions that are defined almost everywhere. The family f^k satisfies $\int_\Omega f^k \le C_0$. With the diameter $\mathrm{diam}(\Omega)$ of Ω we calculate for segments

$$\begin{aligned}
\int_\Omega \int_\Omega \int_{[x,y]} f^k(z)\, d\mathcal{H}^1(z)\, dy\, dx &\le \mathrm{diam}(\Omega) \int_\Omega \int_\Omega \int_0^1 f^k(\theta x + (1-\theta) y)\, d\theta\, dy\, dx \\
&= \mathrm{diam}(\Omega) \int_0^{1/2} \int_\Omega \left\{ \int_\Omega f^k(\theta x + (1-\theta) y)\, dy \right\} dx\, d\theta \\
&\quad + \mathrm{diam}(\Omega) \int_{1/2}^1 \int_\Omega \left\{ \int_\Omega f^k(\theta x + (1-\theta) y)\, dx \right\} dy\, d\theta \\
&\le \mathrm{diam}(\Omega) \int_0^{1/2} \int_\Omega 2^n C_0\, dx\, d\theta + \mathrm{diam}(\Omega) \int_{1/2}^1 \int_\Omega 2^n C_0\, dx\, d\theta = \mathrm{diam}(\Omega) |\Omega| 2^n C_0,
\end{aligned}$$

where, in the last inequality, we used the change of variables $y \mapsto \theta x + (1-\theta) y$ for the first integral, and the change of variables $x \mapsto \theta x + (1-\theta) y$ for the second integral.

This calculation provides that the family of maps

$$F^k : \Omega \times \Omega \to \mathbb{R}, (x,y) \mapsto \int_{[x,y]} f^k(z)\, d\mathcal{H}^1(z)$$

is bounded by some constant $C_1 > 0$ in $L^1(\Omega\times\Omega)$. Let $E \subset \Omega\times\Omega$ be the (exceptional) set of pairs (x,y) such that there is no subsequence $(k_j)_j$ and no constant M_Γ with $F^{k_j}((x,y)) \le M_\Gamma$. Let $M > 0$ be arbitrary. We consider the sets $E_N := \{(x,y) \in \Omega \times \Omega : F^k((x,y)) \ge M\ \forall k \ge N\}$. These sets satisfy $E \subset \bigcup_N E_N$, $E_{N+1} \supset E_N$, and $|E_N| \le C_1/M$, hence also $|E| \le C_1/M$. Since M was arbitrary, this shows that E has measure 0. □

For triangles $T \subset \mathbb{R}^2$ with three typical segments as sides, we can now show the main tool for the compensated compactness result.

Proposition 3.5 (Strongly convergent extensions in $\mathbb{R}^2$)**.** *Let $\Omega \subset \mathbb{R}^2$ be a convex domain, $\delta_k \to 0$ be fixed, and let $(u_k)_k \subset H^1(\Omega)$ be a bounded sequence. Let T be a triangle, given by a triple (x_1, x_2, x_3), such that all segments $[x_l, x_m]$, $l \ne m$, are typical segments for u_k, and let $(k_j)_j$ be a good subsequence for the three segments. Then there exists a family of functions $v_{k_j} \in H^1(T)$ and a limit function $v \in H^1(T)$ such that*

$$v_{k_j} = u_{k_j} \text{ on } \partial T, \tag{3.11}$$

$$v_{k_j} \to v \text{ strongly in } H^1(T). \tag{3.12}$$

Proof. In the proof, in order to avoid the subscript of k_j, we assume that the whole sequence k is a good subsequence for u_k. Let T be a triangle as described, our aim is to construct the extensions v_k on the basis of the fact that (3.2), (3.3), and (3.10) are satisfied for the nodes and the sides.

Without loss of generality, we can assume in the sequel that $c^k_{x_l} = c_{x_l} = 0$ for all k and $l = 1, 2, 3$, where $c^k_{x_l}$ and c_{x_l} are the averages around nodes x_l as in (3.3). Indeed, in the general case, we replace u_k by $\tilde u_k = u_k - \alpha^k$, where α^k is the affine function satisfying

$$c^k_{x_l} = \fint_{B_{\delta_k}(x_l)} \alpha^k(z)\, d\mathcal{L}^2(z). \tag{3.13}$$

Since the sequences $c^k_{x_l}$ converge in $\mathbb{R}$, the functions α^k converge strongly in $H^1(\Omega)$. If $\tilde v_k$ is the strongly converging sequence for $\tilde u_k$ as in the thesis of Proposition 3.5, we can set $v_k := \tilde v_k + \alpha^k$.

Let $\phi_k \in C^\infty(\mathbb{R}^2, [0,1])$ be a sequence of cut-off functions with

$$\operatorname{supp}\phi_k \subset \bigcup_{l=1}^{3} B_{\delta_k}(x_l), \quad \phi_k(\xi) \equiv 1 \text{ on } \bigcup_{l=1}^{3} B_{\delta_k/2}(x_l), \quad \|\nabla\phi_k\| \le \frac{3}{\delta_k}. \tag{3.14}$$

We set $\psi_k := 1 - \phi_k$ and write $u_k = u_k\phi_k + u_k\psi_k$. The idea of the proof is to show that $u_k\psi_k$ admits a strongly convergent extension with the help of a compact extension operator $E : H^1_0([x_i, x_l]) \to H^1(T)$. Concerning an extension of $(u_k\phi_k)|_{\partial T}$, we will show that the family $u_k\phi_k$ itself vanishes strongly in $H^1(T)$.

Claim 1. We treat one of the sides, $\Gamma = [x_i, x_l]$. Our aim is to show that there exists $C > 0$ such that

$$\|(u_k\psi_k)|_\Gamma\|_{H^1(\Gamma)} \le C. \tag{3.15}$$

For $\delta > 0$, a set $B \subset \mathbb{R}^n$, let $B_\delta := \delta B = \{x \in \mathbb{R}^n : x/\delta \in B\}$. By a simple rescaling argument applied to the classical trace and Poincaré inequalities, for all bounded open sets $B \subset \mathbb{R}^n$ with Lipschitz boundary, there exists a constant $K = K(B)$ such that

$$\delta \int_{\partial B_\delta} |u|^2 + \int_{B_\delta} |u|^2 \le \delta^2 K \int_{B_\delta} |\nabla u|^2, \tag{3.16}$$

for all $\delta > 0$ and for all functions $u \in H^1(B_\delta)$ such that $\int_{B_\delta} u = 0$. The same estimate holds when the boundary integral over ∂B_δ is replaced by an integral over another $(n-1)$-dimensional submanifold δS, $S \subset B$.

We now consider the left hand side in (3.15). Regarding the L^2-norm we note that $\|(u_k\psi_k)|_\Gamma\|_{L^2(\Gamma)} \le \|u_k|_\Gamma\|_{L^2(\Gamma)} \le C$, holds by (3.10). Regarding the gradient, we compute

$$\nabla_\tau(u_k\psi_k) = \psi_k\nabla_\tau u_k + u_k\nabla_\tau\psi_k, \tag{3.17}$$

and note that

$$\|\psi_k\nabla_\tau u_k\|_{L^2(\Gamma)} \le \|\nabla_\tau u_k\|_{L^2(\Gamma)} \le C, \tag{3.18}$$

again by (3.10). For the other term we find, using (3.14),

$$\|u_k\nabla_\tau\psi_k\|^2_{L^2(\Gamma)} \le \sum_{l=1}^{3} \|u_k\nabla_\tau\psi_k\|^2_{L^2(B_{\delta_k}(x_l)\cap\Gamma)} \le \frac{9}{\delta_k^2}\sum_{l=1}^{3}\int_{B_{\delta_k}(x_l)\cap\Gamma} |u_k|^2 d\mathcal{H}^1 .$$

With (3.16), exploiting $c^k_{x_l} = 0$, we can calculate

$$\frac{1}{\delta_k^2}\int_{B_{\delta_k}(x_l)\cap\Gamma} |u_k|^2 \le \frac{K}{\delta_k}\int_{B_{\delta_k}(x_l)} |\nabla u_k|^2 = \delta_k K\,|B_1(0)| \fint_{B_{\delta_k}(x_l)} |\nabla u_k|^2 \le CK\delta_k,$$

where we used (3.2) in the last inequality, exploiting that x_l is a point of typical average. This concludes the proof of (3.15).

Claim 2. We now construct a strongly convergent extension of $u_k\psi_k$. Using affine coordinate transformations, it is sufficient to show the following: Let Γ be the horizontal segment $\Gamma = [(0,0),(\pi,0)] \equiv [0,\pi] \subset \mathbb{R}^2$, let $\ell > 0$ be given and let R be the rectangle $(0,\pi)\times(0,\ell)$. Let $w_k \in H^1(\Gamma)$ be a bounded sequence with $w_k \equiv 0$ in $\delta_k/2$-neighborhoods of the end-points of Γ. Then there exist extensions $w_k : R \to \mathbb{R}$ with $w_k \equiv 0$ on $\partial R \setminus \Gamma$ and a limit function w such that

$$w_k \to w \qquad \text{strongly in } H^1(R). \tag{3.19}$$

We sketch a proof for this extension result with a Fourier expansion argument. In order to take Fourier series, we extend the domain with $\tilde\Gamma = (0, 2\pi)$ to $\tilde R = \tilde\Gamma \times (0,\ell)$ and take the odd extension of $w_k|_\Gamma$ to $\tilde\Gamma$, which is bounded in $H^1(\tilde\Gamma)$. Once we have constructed a 2π-periodic, odd extension $\tilde w_k : \tilde R \to \mathbb{R}$, the restriction to $w_k = \tilde w_k|_R$ is the desired function which vanishes on lateral boundaries.

Performing all calculations on the original domains we write

$$w_k|_\Gamma(s) = \sum_{m\in\mathbb{Z}} a_m^k e^{ims},$$

which satisfies, using an appropriate equivalent norm,

$$\|(u_k\psi_k)|_\Gamma\|^2_{H^1(\Gamma)} = \sum_{m\in\mathbb{Z}} |a_m^k|^2\,|m|^2 \le C. \tag{3.20}$$

The harmonic extension $(w_k)|_\Gamma$ to $R = \Gamma\times(0,\ell)$ is then

$$w_k(s,t) := \sum_{m\in\mathbb{Z}} a_m^k e^{ims} e^{-mt}.$$

This sequence is bounded in $H^1(\Gamma\times(0,\ell))$, as can be shown by a direct calculation. We choose a subsequence $k\to\infty$ such that all coefficients a_m^k converge. The corresponding formal limit function is w,

$$w(s,t) := \sum_{m\in\mathbb{Z}} a_m e^{ims} e^{-mt}, \qquad \text{where } a_m = \lim_{k\to\infty} a_m^k. \tag{3.21}$$

We claim that the strong convergence $w_k \to w$ in $H^1(\Gamma\times(0,\ell))$ holds. We compute for an arbitrary $N\in\mathbb{N}$

$$\begin{aligned}\int_0^\pi\int_0^\ell |\nabla w_k(s,t)-\nabla w(s,t)|^2\,ds\,dt &\le C\int_0^\pi\int_0^\ell \sum_{m\in\mathbb{Z}} |a_m^k - a_m|^2|m|^2 e^{-2mt} ds\,dt\\ &\le C\sum_{m\in\mathbb{Z}} |a_m^k - a_m|^2|m|^2\frac{1}{|m|} \le C\sum_{|m|\le N} |a_m^k - a_m|^2|m| + \frac{C}{N}\left(\|w_k\|^2_{H^1} + \|w\|^2_{H^1}\right)\\ &\le C\sum_{|m|\le N} |a_m^k - a_m|^2|m| + \frac{C}{N}.\end{aligned}$$

Passing to the limit as $k\to\infty$, owing to (3.21), we find

$$\lim_{k\to\infty} \|\nabla w_k - \nabla w\|^2_{L^2(\Gamma\times(0,\ell))} \le \frac{C}{N}.$$

Since $N\in\mathbb{N}$ was arbitrary, this concludes the proof of (3.19). Multiplication of all w_k and of w with a cut-off function provides additionally vanishing boundary values at the upper boundary $(0,\pi)\times\{\ell\}$.

Claim 3. We finally claim that the extensions $u_k\phi_k$ of $(u_k\phi_k)|_{\partial T}$ converges strongly to 0 in $H^1(T)$. Indeed, we can compute

$$\nabla(u_k\phi_k) = \phi_k\nabla u_k + u_k\nabla\phi_k, \tag{3.22}$$

and use (3.2) to find

$$\int_{B_{\delta_k}(x_l)} |\nabla u_k|^2|\phi_k|^2\,d\mathcal{L}^2 \le \int_{B_{\delta_k}(x_l)} |\nabla u_k|^2\,d\mathcal{L}^2 \le CM_l\delta_k^2.$$

For the term $u_k\nabla\phi_k$ we use (3.14), the Poincaré inequality (3.16), and (3.2),

$$\int_{B_{\delta_k}(x_l)} |u_k|^2|\nabla\phi_k|^2 \le \frac{9}{\delta_k^2}\int_{B_{\delta_k}(x_l)} |u_k|^2 \le 9K\int_{B_{\delta_k}(x_l)} |\nabla u_k|^2 \le CM_l\delta_k^2.$$

This yields the thesis of Claim 3 and concludes the proof of the proposition. □

We wish to emphasize that the extension of $w_k|_\Gamma$ with a Fourier series exploits that w_k vanishes in the nodes. It was in order to cut out the corners in the above proof that we introduced the notion of a point of typical average.

As a preparation for the three-dimensional case we make a remark on another possible extension. We will use such an improved extension in the next subsection in order to extend the two-dimensional extension further into the third dimension.

Lemma 3.6. *The extensions v_{k_j} of Proposition 3.5 can be chosen such that all segments $\Gamma = [x_i, x_l]$, $i \neq l$, are also typical segments for v_{k_j}, and such that v_{k_j} satisfies additionally, for some number $M_\Gamma > 0$*

$$\fint_{B_{\delta_{k_j}}(\Gamma)} |\nabla v_{k_j}(z)|^2 \, d\mathcal{L}^2(z) \leq M_\Gamma \quad \forall j \in \mathbb{N}. \tag{3.23}$$

Proof. We analyze the construction of the last proof. One part of the extended function v_k is $u_k \phi_k$. For these contributions, the boundedness (3.23) was actually shown in Claim 3.

The extension of $w_k|_{(0,\pi)}$ to functions w_k on $R = (0,\pi) \times (0,\ell)$ was performed with Fourier series. The construction can be altered by using the original function $w_k|_{(0,\pi)}$ in a δ_k-strip and then the extension of the above proof, i.e.

$$\tilde{w}_k(s,t) = \begin{cases} w_k(s,0) & \text{if } t < \delta_k \\ w_k(s, t - \delta_k) & \text{else.} \end{cases}$$

With this choice, in $B_{\delta_k}(\Gamma)$, the values $|\nabla \tilde{w}_k(x)|$ are bounded by multiples of corresponding point-values of $|\nabla_\tau w_k|_\Gamma|$ and $|w_k|_\Gamma|$. These are bounded by (3.20).

One easily verifies that the segment Γ is a typical segment also for v_k. □

Definition 3.7 (Adapted grid for $n = 2$)**.** *Let $Q \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $(u_k)_k$ a bounded sequence in $H^1(Q)$, $h > 0$ fixed and $\delta_k \searrow 0$. We say that a family $\mathscr{T}_h = \{T_i\}_{i \in \Lambda_h}$ of triangles is an* adapted grid *for $(u_k)_k$ if the boundaries of all triangles are typical segments according to Definition 3.3. We furthermore require that one subsequence $(k_j)_j$ is a good subsequence for all segments.*

The above observations on points of typical average, on typical segments, and on strongly convergent extensions allow to prove Theorem 1.3 in the two-dimensional case. We remark that the proof for the three-dimensional case will be almost identical.

Proof of Theorem 1.3 for $n = 2$. Existence of adapted grids. Denoting the sequences with a subscript k, we are given the family u_k and want to construct an adapted grid for u_k. We fix the sequence $\delta_k = 1/k$. The grid can be chosen by subsequently adding grid-points and by subsequently passing to subsequences. Every node x is chosen as a point of typical average and such that almost every segment with x as an end-point is a typical segment. Since almost every x has both properties by Lemmas 3.2 and 3.4, we can construct a grid to prescribed $h > 0$ in this way.

Compensated compactness. Our aim is to verify the convergence (1.13). It suffices to show this convergence for each single triangle. Let therefore T be one triangle of the grid and note that, by assumption, there holds

$$q_k \rightharpoonup q \qquad \text{weakly in } L^2(T), \tag{3.24}$$

$$f_k := \nabla \cdot q_k \to f \qquad \text{strongly in } H^{-1}(T). \tag{3.25}$$

Since T is a triangle with typical sides as described in Proposition 3.5, we can use the strongly $H^1(T)$-convergent extension v_k of the boundary values of u_k of Proposition 3.5, $v_k \to v$ in $H^1(T)$. The boundary values are always expressed through the trace theorem, hence, by definition of identical traces, we have

$$\int_T q_k \cdot \nabla u_k + \int_T \nabla \cdot q_k u_k = \int_T q_k \cdot \nabla v_k + \int_T \nabla \cdot q_k v_k \,.$$

We can therefore calculate

$$\int_T q_k \cdot \nabla u_k = \int_T q_k \cdot \nabla v_k - \langle f_k, u_k - v_k \rangle_{H^{-1},H_0^1} \to \int_T q \cdot \nabla v - \langle f, u - v \rangle_{H^{-1},H_0^1} \,.$$

We use here the weak L^2-convergence of q_k and the strong L^2-convergence of ∇v_k. In the term containing f, we use the weak H_0^1-convergence $u_k - v_k \to u - v$ and the strong H^{-1}-convergence $f_k \to f$.

Performing the above interpretation of identical boundary values again for u and v instead of u_k and v_k provides

$$\int_T q_k \cdot \nabla u_k \to \int_T q \cdot \nabla v - \langle f, u - v \rangle_{H^{-1},H_0^1} = \int_T q \cdot \nabla u,$$

and thus

$$\lim_{k\to\infty} \int_T q_k \cdot \nabla u_k \, dx = \int_T q \cdot \nabla u \, dx, \tag{3.26}$$

which provides (1.13). □

3.2 Adapted grids in three dimensions

We are again given a sequence $u_k \in H^1(\Omega)$, now with $\Omega \subset \mathbb{R}^3$ an open domain. Our aim is to show that almost all simplices S contained in Ω are "typical" in the sense that $u_k|_{\partial S}$ has a strongly convergent extension for a subsequence $(k_j)_j$. Since objects of different dimensions appear in the sequel, we find it convenient to indicate the dimension with a superscript. We will typically use Γ^1 for segments, E^2 for planes, and S^3 for three-dimensional simplices.

In two space dimensions, we considered typical segments and points of typical average. Regarding segments we demanded boundedness of u_k on the segment, regarding points, we demanded more, namely a boundedness property in a neighborhood. Transferring these concepts to three space dimensions, we will demand that u_k is bounded on triangles T^2, and that averages of u_k are bounded in neighborhood of segments Γ^1. We therefore introduce below *segments of typical average*, which have stronger requirements than typical segments.

Definition 3.8 (Segments of typical average and typical triangles)**.** *Let $n = 3$ and $\Gamma^1 = [x, y] \subset \Omega$ be a segment, contained in a two-dimensional plane $E^2 \subset \mathbb{R}^3$. We say that Γ^1 is a* segment of typical average *for $(u_k)_k$, $\delta_k \to 0$ and E^2, if, along a subsequence $(k_j)_j$, the family $u_{k_j}|_{E^2}$ is an $H^1(E^2 \cap \Omega)$-bounded sequence and if*

1. *The segment Γ^1 is a typical segment in E^2 for $u_{k_j}|_{E^2}$ and δ_{k_j} according to Definition 3.3.*

2. *For a constant $M_0 > 0$, holds*

$$\fint_{B_{\delta_{k_j}}(\Gamma^1)\cap\Omega} |u_{k_j}(z)|^2 + |\nabla u_{k_j}(z)|^2 \, d\mathcal{L}^3(z) \le M_0, \tag{3.27}$$

$$\fint_{B_{\delta_{k_j}}(\Gamma^1)\cap E^2} |u_{k_j}(z)|^2 + |\nabla u_{k_j}(z)|^2 \, d\mathcal{L}^2(z) \le M_0. \tag{3.28}$$

We say that a triangle $T^2 \subset \mathbb{R}^3$ is a typical triangle*, if the three sides are segments of typical average for the plane E^2 containing T^2, for a single subsequence $(k_j)_j$.*

We note that, by definition of a typical triangle, for some $M_0 > 0$,

$$\|u_{k_j}|_{T^2}\|^2_{L^2(T^2)} + \|\nabla_\tau u_{k_j}|_{T^2}\|^2_{L^2(T^2)} \le M_0. \tag{3.29}$$

Lemma 3.9 (Many typical triangles)**.** *Let $\Omega \subset \mathbb{R}^3$ be a convex domain, $\delta_k \to 0$ be fixed, and $(u_k)_k$ be an $H^1(\Omega)$-bounded sequence. Then, successively chosen, for almost all $x_1 \in \Omega$, for almost all $x_2 \in \Omega$, for almost all $x_3 \in \Omega$, the triangle T^2 given by (x_1, x_2, x_3) is a typical triangle.*

Sketch of proof. For almost every plane E^2 defined by (x_1, x_2, x_3), the family $u_k|_{E^2}$ is bounded in $H^1(E^2)$. This follows from Fubini's theorem, arguing as in Lemma 3.4.

Let E^2 be such a plane. Then, by Lemma 3.4, applied with $n = 2$, almost all segments in E^2 are typical segments in E^2. This provides the property of item 1.

It remains to check properties (3.27) and (3.28) of item 2 for almost every choice of (x_1, x_2, x_3). Let $0 \neq \gamma \in \mathbb{R}^3$ be an arbitrary vector such that $\Gamma_x := [x, x+\gamma]$ defines a segment in $\mathbb{R}^3$ for every $x \in \mathbb{R}^3$. With fixed γ, we now consider

$$f_k : \mathbb{R}^3 \to \mathbb{R}, \quad f_k(x) = \int_{(x+\mathbb{R}\gamma)\cap\Omega} |u_k|^2 + |\nabla u_k|^2.$$

Let $F^2 \subset \mathbb{R}^3$ be an arbitrary plane orthogonal to γ. We consider the restriction $f_k : F^2 \to \mathbb{R}$, which is a bounded family in $L^1(F^2)$. Arguing as in the proof of Lemma 3.2, we conclude that for almost all $x \in F^2$, the δ_k-averages of f_k are bounded. This implies (3.27).

The estimate (3.28) follows in the same way when we choose a line $F^1 \subset E^2$, which is orthogonal to γ. $\square$

Lemma 3.10 (Strongly convergent extensions in $\mathbb{R}^3$)**.** *Let $\Omega \subset \mathbb{R}^3$, $\delta_k \searrow 0$, and let $(u_k)_k$ be a bounded sequence in $H^1(\Omega)$. Let $S^3 \subset \Omega$ be a simplex such that the four sides T^2_m, $m = 1, 2, 3, 4$, are typical triangles. Then there exists a subsequence $(k_j)_j$*

and extensions $v_{k_j} \in H^1(S^3)$ of the boundary values $u_{k_j}|_{\partial S^3}$ such that, for a limit function $v \in H^1(S^3)$,

$$v_{k_j} = u_{k_j} \text{ on } \partial S^3, \tag{3.30}$$

$$v_{k_j} \to v \quad \text{strongly in } H^1(S^3). \tag{3.31}$$

Proof. Once more, we assume that the original sequence is a good sequence and omit in the proof the subscript of k_j.

Step 1. Modification of u_k to $\tilde{u}_k$ with vanishing values along the edges. Our first aim is to modify u_k such that we only have to treat functions that vanish on the edges Γ_i^1, $i = 1, ..., 6$. To this end we note that, since every side T_m^2, $m = 1, ..., 4$, is a typical triangle, we may use the two-dimensional result of Proposition 3.5 on each face. This provides extensions $w_k : T_m^2 \to \mathbb{R}$ with $w_k|_{\Gamma_i^1} = u_k|_{\Gamma_i^1}$ that are strongly convergent in $H^1(T_m^2)$. With a rotation of the functions w_k around Γ_i^1, using additionally linear transformations and cut-off functions, we can construct extensions

$$\tilde{w}_k : S^3 \to \mathbb{R}, \quad \tilde{w}_k|_{T_m^2} = w_k, \quad \tilde{w}_k \text{ strongly convergent in } H^1(S^3).$$

The last property follows from the strong convergence of w_k in $H^1(T_m^2)$. By Lemma 3.6, we can achieve that each edge Γ_i^1 is a segment with typical averages not only for the sequence u_k, but also for the sequence $\tilde{w}_k$ (compare Definition 3.8 and estimate (3.23), which remains valid after the extension by rotation).

We now consider the modified sequence of functions $\tilde{u}_k := u_k - \tilde{w}_k$. This sequence has vanishing values on all edges Γ_i^1. Since the sequences $\tilde{w}_k$ converges strongly in $H^1(S^3)$, it is sufficient to show for $\tilde{u}_k$ the existence of a strongly $H^1(S^3)$-convergent subsequence. It is important to note that our construction guarantees that the edges Γ_i^1 are segments of typical averages also for the sequence $\tilde{u}_k$.

Step 2. Extension of $\tilde{u}_k$. We treat one of the faces T^2, let $\Gamma^1 \subset \partial T^2$ be one edge. We use a family of smooth cut-off functions $\phi_k : \mathbb{R}^3 \to [0, 1]$ with $\text{supp}(\phi_k) \subset B_{\delta_k}(\Gamma^1)$ and $\|\nabla\phi_k\|_\infty \le C/\delta_k$, such that $\phi_k \equiv 1$ on $B_{\delta_k/2}(\Gamma^1) \subset \mathbb{R}^3$. Analogous to Proposition 3.5, we want to extend the trace $[(1 - \phi_k)\tilde{u}_k]|_{T^2}$ as a harmonic function to S^3. We calculate

$$\int_{T^2} |\nabla_\tau[(1 - \phi_k)\tilde{u}_k]|^2 \, d\mathcal{L}^2 \le C \frac{1}{\delta_k^2} \int_{B_{\delta_k}(\Gamma^1)\cap T^2} |\tilde{u}_k|^2 \, d\mathcal{L}^2 + C \int_{T^2} |\nabla_\tau \tilde{u}_k|^2 \, d\mathcal{L}^2.$$

The last integral is bounded by (3.29). For the other integral on the right hand side we use the boundedness of the gradient in $B_{\delta_k}(\Gamma^1) \cap T^2$ and Poincaré's inequality, exploiting $\tilde{u}_k \equiv 0$ on Γ^1. We find that $[(1-\phi_k)\tilde{u}_k]|_{T^2}$ is a bounded sequence in $H^1(T^2)$, which vanishes in a neighborhood of the boundary. This allows to extend the function harmonically to S^3 with vanishing values on $\partial S^3 \setminus T^2$. As calculated for Proposition 3.5, the harmonic extension has a strongly $H^1(S^3)$-convergent subsequence.

It remains to verify the smallness in $H^1(S^3)$ of the functions $\phi_k\tilde{u}_k$. We calculate

$$\begin{aligned}\int_{S^3} |\nabla(\phi_k\tilde{u}_k)|^2 \, d\mathcal{L}^3 &\le C \frac{1}{\delta_k^2} \int_{B_{\delta_k}(\Gamma^1)\cap S^3} |\tilde{u}_k|^2 \, d\mathcal{L}^3 + C \int_{S^3} |\phi_k|^2 |\nabla\tilde{u}_k|^2 \, d\mathcal{L}^3 \\ &\le C \fint_{B_{\delta_k}(\Gamma^1)\cap S^3} |\tilde{u}_k|^2 \, d\mathcal{L}^3 + C\delta_k^2 \fint_{B_{\delta_k}(\Gamma^1)\cap S^3} |\nabla\tilde{u}_k|^2 \, d\mathcal{L}^3 \to 0.\end{aligned}$$

The convergence to 0 of the second term is an immediate consequence of the boundedness of the integral, which follows from property (3.27) of segments with typical averages. For the first term we use once more Poincaré's inequality: the gradients are bounded on planes and in space by (3.28) and (3.27), the vanishing values $\tilde{u}_k \equiv 0$ on Γ^1 imply smallness of averages in the neighborhood. □

As in two space dimensions, Theorem 1.3 for three space dimensions is based on the fact that, for every grid-size $h > 0$, we can choose an adapted grid to h and a given sequence u_k.

Definition 3.11 (Adapted grid in three dimensions)**.** *Let $Q \subset \mathbb{R}^3$ be a bounded domain and let $(u_k)_k$ be a bounded sequence in $H^1(Q)$. We say that a subdivision $\mathscr{T}_h = \{S_i\}_{i\in\Lambda_h}$ of $Q_h \subset Q$ in simplices S_i is an* adapted grid *for $(u_k)_k$ if all sides T^2_m of the simplices are typical triangles with one single subsequence $(k_j)_j$ according to Definition 3.8.*

Proof of Theorem 1.3 for $n = 3$. Existence of adapted grids. The three-dimensional Lemmata on typical triangles imply that for a given family u_k and given $h > 0$ an adapted grid for u_k can be constructed by subsequently adding grid-points.

Compensated compactness. The statement of the compensated compactness is shown as in the two-dimensional case. □

A Ergodic homogenization cell problem

In [18], a probability space setting is introduced to treat homogenization of stochastic coefficients. The authors use dynamical systems $T_{x/\varepsilon} : \omega \to T_{x/\varepsilon}(\omega)$ on the probability space $(\Omega_{\mathcal{P}}, \mathcal{P})$ to construct stochastic coefficients $a^\varepsilon(x) = \tilde{a}(x/\varepsilon;\omega)$. Under ergodicity assumptions, they obtain the following result on solutions of cell problems.

Theorem A.1. *Under ergodicity assumptions, for some matrix $a^* \in \mathbb{R}^{n\times n}$, the following holds. For $\mathcal{P}$-almost every ω and coefficients $\tilde{a}(y) = \tilde{a}(y;\omega)$ there exists $\psi_k(.;\omega) : \mathbb{R}^n \to \mathbb{R}^n$ solving*

$$\nabla_y \cdot (\tilde{a}(y)\psi_k(y)) = 0 \quad \text{on } \mathbb{R}^n, \tag{1.1}$$

$$\operatorname{curl} \psi_k = 0 \quad \text{on } \mathbb{R}^n, \tag{1.2}$$

such that the average of ψ_k is e_k and the average of $\tilde{a} \cdot \psi_k$ is $a^* \cdot e_k$, in the following sense: For every subset $K \subset \mathbb{R}^n$ holds

$$\psi_k(./\varepsilon;\omega) \rightharpoonup e_k \quad \text{in } L^2(K), \tag{1.3}$$

$$\tilde{a}(./\varepsilon;\omega)\,\psi_k(./\varepsilon;\omega) \rightharpoonup a^* \cdot e_k \text{ in } L^2(K). \tag{1.4}$$

From Theorem A.1 one easily deduces the property of Definition 1.1. In particular, stochastic coefficients as constructed in [18] allow averaging of the constitutive relation.

Acknowledgement

The authors wish to thank the unknown referees for the careful reading of the manuscript and many useful hints that helped to improve this contribution. The support by DFG-grant SCHW 639/3-1 is gratefully acknowledged.

References

[1] G. Allaire. Homogenization of the Stokes flow in a connected porous medium. *Asymptotic Analysis*, 2:203–222, 1989.

[2] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518, 1992.

[3] G. Allaire. Dispersive limits in the homogenization of the wave equation. *Ann. Fac. Sci. Toulouse Math. (6)*, 12(4):415–431, 2003.

[4] G. Allaire and R. Brizzi. A multiscale finite element method for numerical homogenization. *Multiscale Model. Simul.*, 4(3):790–812 (electronic), 2005.

[5] I. Babuška. Homogenization and its application. Mathematical and computational problems. In *Numerical solution of partial differential equations, III (Proc. Third Sympos. (SYNSPADE), Univ. Maryland, College Park, Md., 1975)*, pages 89–116. Academic Press, New York, 1976.

[6] A. Bensoussan, J.-L. Lions, and G. C. Papanicolaou. Homogenization in deterministic and stochastic problems. In *Stochastic problems in dynamics (Sympos., Univ. Southampton, Southampton, 1976)*, pages 106–115. Pitman, London, 1977.

[7] G. Bouchitté and B. Schweizer. Homogenization of Maxwell's equations in a split ring geometry. *Multiscale Model. Simul.*, 8(3):717–750, 2010.

[8] A. Bourgeat, A. Mikelic, and S. Wright. Stochastic two-scale convergence in the mean and applications. *J. Reine Angew. Math.*, 456:19–51, 1994.

[9] P. G. Ciarlet. *The finite element method for elliptic problems*, volume 40 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Reprint of the 1978 original [North-Holland, Amsterdam; MR0520174 (58 #25001)].

[10] D. Cioranescu, A. Damlamian, and G. Griso. Periodic unfolding and homogenization. *C. R. Math. Acad. Sci. Paris*, 335(1):99–104, 2002.

[11] S. Conti and B. Schweizer. Rigidity and gamma convergence for solid-solid phase transitions with SO(2) invariance. *Comm. Pure Appl. Math.*, 59(6):830–868, 2006.

[12] S. Conti and B. Schweizer. A sharp-interface limit for a two-well problem in geometrically linear elasticity. *Arch. Ration. Mech. Anal.*, 179(3):413–452, 2006.

[13] G. Dal Maso and L. Modica. Nonlinear stochastic homogenization. *Ann. Mat. pura appl.*, 144:347–389, 1986.

[14] E. De Giorgi and S. Spagnolo. Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine. *Boll. Un. Mat. Ital. (4)*, 8:391–411, 1973.

[15] W. E and B. Engquist. The heterogeneous multiscale methods. *Commun. Math. Sci.*, 1(1):87–132, 2003.

[16] W. E, P. Ming, and P. Zhang. Analysis of the heterogeneous multiscale method for elliptic homogenization problems. *J. Amer. Math. Soc.*, 18(1):121–156 (electronic), 2005.

[17] T. Y. Hou and X.-H. Wu. A multiscale finite element method for elliptic problems in composite materials and porous media. *J. Comput. Phys.*, 134(1):169–189, 1997.

[18] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik. *Homogenization of differential operators and integral functionals.* Springer-Verlag, Berlin, 1994.

[19] S. Kozlov. Averaging of random operators. *Math. USSR Sbornik*, 37:167–179, 1980.

[20] C. Melcher and B. Schweizer. Direct approach to L^p estimates in homogenization theory. *Ann. Mat. Pura Appl. (4)*, 188(3):399–416, 2009.

[21] F. Murat and L. Tartar. H-convergence. In *Topics in the mathematical modelling of composite materials*, volume 31 of *Progr. Nonlinear Differential Equations Appl.*, pages 21–43. Birkhäuser Boston, Boston, MA, 1997.

[22] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.*, 20(3):608–623, 1989.

[23] G. Nguetseng. Homogenization structures and applications. I. *Z. Anal. Anwendungen*, 22(1):73–107, 2003.

[24] H. Owhadi and L. Zhang. Metric-based upscaling. *Comm. Pure Appl. Math.*, 60(5):675–723, 2007.

[25] W. Rudin. *Real and complex analysis.* McGraw-Hill Book Co., New York, third edition, 1987.

[26] E. Sánchez-Palencia. *Nonhomogeneous media and vibration theory*, volume 127 of *Lecture Notes in Physics.* Springer-Verlag, Berlin, 1980.

[27] B. Schweizer. Homogenization of degenerate two-phase flow equations with oil trapping. *SIAM J. Math. Anal.*, 39(6):1740–1763, 2008.

[28] B. Schweizer and M. Veneroni. On non-periodic homogenization of time-dependent equations. Submitted to: *Nonlinear Anal. B: Real World Appl.*

[29] B. Schweizer and M. Veneroni. Periodic homogenization of the Prandtl-Reuss model with hardening. *J. Multiscale Modeling*, 2(1-2):69–106, 2010.

[30] L. Tartar. Problèmes de contrôle des coefficients dans des équations aux dérivées partielles. In *Control theory, numerical methods and computer systems modelling (Internat. Sympos., IRIA LABORIA, Rocquencourt, 1974)*, pages 420–426. Lecture Notes in Econom. and Math. Systems, Vol. 107. Springer, Berlin, 1975.

[31] L. Tartar. *The general theory of homogenization. A personalized introduction*, volume 7 of *Lecture Notes of the Unione Matematica Italiana*. Springer-Verlag, Berlin, 2009.

[32] M. Veneroni. Stochastic homogenization of subdifferential inclusions via scale integration. *International Journal of Structural Changes in Solids*, 3(1):83–98, 2011.

[33] S. Wright. On the steady flow of an incompressible fluid through a randomly perforated porous medium. *J. Differ. Equations*, 146(2):261–286, 1998.

[34] V. V. Zhikov. Estimates for an averaged matrix and an averaged tensor. *Uspekhi Mat. Nauk*, 46(3(279)):49–109, 239, 1991.