Exercises NMES

Exercise 1

Apply the Gaussian elimination method, without pivoting, to solve the linear system $Ax = b$, where

$$\begin{matrix} (r_1) \\ (r_2) \\ (r_3) \end{matrix} \begin{bmatrix} 2 & 4 & 10 \\ 2 & 6 & 20 \\ 1 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 12 \\ 18 \end{bmatrix}$$

showing the intermediate computations.

Solution: First we eliminate the first column of A under the diagonal term $a_{1,1}$. Compute $l_{2,1} = \dfrac{a_{2,1}}{a_{1,1}} = 1$ and $l_{3,1} = \dfrac{a_{3,1}}{a_{1,1}} = \dfrac{1}{2}$. Then perform

$$\begin{matrix} (r_1) \\ (r_2) = (r_2) - l_{2,1} \cdot (r_1) \\ (r_3) = (r_3) - l_{3,1} \cdot (r_1) \end{matrix} \begin{bmatrix} 2 & 4 & 10 \\ 0 & 2 & 10 \\ 0 & 2 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 18 \\ 21 \end{bmatrix}$$

In the second step of the Gaussian elimination we eliminate the second column of A under the diagonal term $a_{2,2}$. Compute $l_{3,2} = \dfrac{a_{3,2}}{a_{2,2}} = 1$. Then perform

$$\begin{matrix} (r_1) \\ (r_2) \\ (r_3) = (r_3) - l_{3,2} \cdot (r_2) \end{matrix} \begin{bmatrix} 2 & 4 & 10 \\ 0 & 2 & 10 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 18 \\ 3 \end{bmatrix}$$

Finally compute the solution by *back-substitution* method:

$$3x_3 = 3; \rightarrow x_3 = 1;$$
$$2x_2 + 10x_3 = 18; \rightarrow 2x_2 = 18 - 10; \rightarrow x_2 = 4;$$
$$2x_1 + 4x_2 + 10x_3 = -6; \rightarrow 2X_1 = -6 - 10 - 16; \rightarrow x_1 = -16;$$

Finally the solution of the linear system is $x = \begin{bmatrix} -16 \\ 4 \\ 1 \end{bmatrix}$.

Exercise 2

Write the LU factorization, without pivoting, of:

$$\begin{matrix} (r_1) \\ (r_2) \\ (r_3) \end{matrix} \begin{bmatrix} 2 & 4 & 4 \\ 1 & 5 & 7 \\ 3 & 12 & 18 \end{bmatrix}$$

showing the intermediate computations.

Solution: The steps are the same as for the Gaussian elimination method. Let $L = I_n$ and compute the entries in the first column of L while eliminating the elements in the first column of A below the diagonal: $l_{2,1} = \frac{a_{2,1}}{a_{1,1}} = \frac{1}{2}$ and $l_{3,1} = \frac{a_{3,1}}{a_{1,1}} = \frac{3}{2}$. Now replace the row (r_2) and (r_3) with

$$(r_2) - l_{2,1} \cdot (r_1) = (1, 5, 7) - \frac{1}{2} \cdot (2, 4, 4) = (1, 5, 7) - (1, 2, 2) = (0, 3, 5)$$
$$(r_3) - l_{3,1} \cdot (r_1) = (3, 12, 18) - \frac{3}{2} \cdot (2, 4, 4) = (3, 12, 18) - (3, 6, 6) = (0, 6, 12)$$

respectively. Now the matrix A becomes

$$\begin{matrix} (r_1) := (r_1) \\ (r_2) := (r_2) - l_{2,1} \cdot (r_1) \\ (r_3) := (r_3) - l_{3,1} \cdot (r_1) \end{matrix} \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 5 \\ 0 & 6 & 12 \end{bmatrix}$$

and L is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & 0 & 1 \end{bmatrix}.$$

Now repeat the computations above to eliminate the portion of the second row of A below the diagonal. So $l_{3,2} = \frac{a_{3,2}}{a_{2,2}} = 2$; replace now the third row (r_3) with $(r_3) - l_{3,2}(r_2)$.

$$(r_3) - l_{3,2} \cdot (r_2) = (0, 6, 12) - 2 \cdot (0, 3, 5) = (0, 6, 12) - (0, 6, 10) = (0, 0, 2).$$

So the matrix A has now become

$$\begin{matrix} (r_1) := (r_1) \\ (r_2) := (r_2) \\ (r_3) := (r_3) - l_{3,2} \cdot (r_2) \end{matrix} \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

and L is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & 2 & 1 \end{bmatrix}.$$

Exercise 3

Compute the linear regression $r(x) = c_0 + c_1 x$ for the set of points

$$(-3, 0),\ (-2, 0),\ (-1, 0),\ (1, 1),\ (2, 2),\ (3, 4).$$

Solution: To compute the solution recall the least square linear problem

$$\begin{bmatrix} m & \sum_{i=1}^m x_i \\ \sum_{i=1}^m x_i & \sum_{i=1}^m (x_i)^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m y_i \\ \sum_{i=1}^m y_i x_i \end{bmatrix}.$$

which in this case is

$$\begin{bmatrix} 6 & 0 \\ 0 & 28 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix},$$

gives the solution which is

$$c_0 = \frac{7}{6}, \qquad c_1 = \frac{17}{28}.$$

Exercise 4

Compute the quadratic least-square approximation $r(x) = c_0 + c_1 x + c_2 x^2$ for the set of points

$$(-2, \frac{5}{2}),\ (-1, 0),\ (0, -1),\ (1, 0),\ (2, \frac{5}{2}).$$

Solution: Here we want to minimize the quadratic function

$$F(c_0, c_1, c_2) = \sum_{i=1}^{m} \left(F(x_i) - c_0 - c_1 \cdot x_i - c_2 \cdot x_i^2) \right)^2.$$

The problem is equivalent to solve

$$(LS2) \begin{bmatrix} m & \sum_{i=1}^m x_i & \sum_{i=1}^m (x_i)^2 \\ \sum_{i=1}^m x_i & \sum_{i=1}^m (x_i)^2 & \sum_{i=1}^m (x_i)^3 \\ \sum_{i=1}^m (x_i)^2 & \sum_{i=1}^m (x_i)^3 & \sum_{i=1}^m (x_i)^4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m y_i \\ \sum_{i=1}^m x_i y_i \\ \sum_{i=1}^m (x_i)^2 y_i \end{bmatrix}.$$

For the given set of points, it holds

$$(LS2) \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 20 \end{bmatrix}.$$

We get immediately $c_1 = 0$. And then

$$\begin{cases} 5c_0 + 10c_2 = 4 \\ 10c_0 + 34c_2 = 20 \end{cases} \rightarrow \begin{cases} 5c_0 + 10c_2 = 4 \\ 14c_2 = 12 \end{cases} \rightarrow \begin{cases} c_0 + = \frac{4}{5} - \frac{60}{35} = -\frac{32}{35} \\ c_2 = \frac{6}{7} \end{cases}$$

Therefore the solution is $r(x) = -\frac{32}{35} + \frac{6}{7}x^2$.

Exercise 5

Starting from $x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, compute 2 iterations of the Jacobi method applied to the system $Ax = b$, where

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solution: Recall that for iterative methods, we can split $A = M - N$, so that the general step of an iterative method is

$$x^{(k+1)} = x^{(k)} + M^{-1} r^{(k)}$$

In the case of Jacobi method, $M = \texttt{diag}(A)$

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Before starting, compute the residual $r^{(0)}$

$$r^{(0)} = b - Ax^{(0)} = b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

First iteration
Firstly, we have to compute $M^{-1} r^{(0)}$, which amounts to solving the linear system

$$Mu = r^{(0)}.$$

Since M is diagonal, it is easy to solve the linear system, obtaining the solution u

$$u = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix},$$

so $x^{(1)} = x^{(0)} + u = (1/2,\ 0,\ 1/2)^T$. Then compute the residual $r^{(1)} = b - Ax^{(1)}$

$$r^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{3}{2} \\ 0 \\ \frac{3}{2} \end{bmatrix} = - \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

Second iteration

As before, compute the solution of the linear system $Mu = r^{(1)}$, that yields

$$x^{(2)} = x^{(1)} + u = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{4} \end{bmatrix}.$$

Exercise 6

Starting from $x^{(0)} = (0, 0, 0)^T$, compute 2 iterations of the Gauss-Seidel method applied to the system $Ax = b$, where

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution: Recall that for iterative methods, we can split $A = M - N$, so that the general step of an iterative method is

$$x^{(k+1)} = x^{(k)} + M^{-1} r^{(k)}$$

In the case of Gauss-Seidel method, $M =$ `tril(A)`

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

Before starting, compute the residual $r^{(0)}$

$$r^{(0)} = b - Ax^{(0)} = b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

First iteration

Firstly, we have to compute $M^{-1} r^{(0)}$, which amounts to solving the linear system

$$Mu = r^{(0)}.$$

Since M is lower triangular, it is easier to solve the linear system by forward-substitution, obtaining the solution u

$$u = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix},$$

so $x^{(1)} = x^{(0)} + u = (1, -1, 3)^T$. Then compute the residual $r^{(1)} = b - Ax^{(1)}$

$$r^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Second iteration As before, compute the solution of the linear system

$$Mu = r^{(1)}$$

using forward substitution. This yields

$$x^{(2)} = x^{(1)} + u = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}.$$

Exercise 7

Starting from $v^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, compute 2 iterations of the power method on the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

returning the eigenvector and eigenvalue approximation and showing the intermediate steps.

Solution: In the first iteration of the power method we compute:

$$\begin{aligned}
w_1 &= Av_0 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} =; \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
\|w_1\| &= \sqrt{2^2 + 1^2} = \sqrt{5}; \\
v_1 &= \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}; \\
\lambda_1 &= v_1^T A v_1 = \frac{1}{5} \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \frac{14}{5}.
\end{aligned}$$

In the second iteration of the power method we compute:

$$\begin{aligned}
w_2 &= Av_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 5 \\ 4 \end{bmatrix}; \\
\|w_2\| &= \frac{1}{\sqrt{5}} \sqrt{5^2 + 4^2} = \sqrt{\frac{41}{5}}; \\
v_2 &= \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{41}} \begin{bmatrix} 5 \\ 4 \end{bmatrix}; \\
\lambda_2 &= v_2^T A v_2 = \frac{1}{41} \begin{bmatrix} 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} 14 \\ 13 \end{bmatrix} = \frac{122}{41}.
\end{aligned}$$

Exercise 8

Apply two bisection iterations to solve the equation

$$x^3 + 3x - 2 = 0 \qquad in\,[0, 1].$$

Solution: Let $F(x) = x^3 + 3x - 2$. We are looking the solution in the interval $[0, 1]$, so first check that $F(0)$ and $F(1)$ have different sign. Indeed

$$F(0) = -2, \qquad F(1) = 2$$

.

First iteration. We start with $a = 0$ and $b = 1$. Compute the midpoint $m = \frac{a+b}{2}$, and evaluate $F(m)$.

$$m = \frac{1}{2}, \qquad F(m) = \frac{1}{8} + \frac{3}{2} - 2 = -\frac{3}{8}.$$

Therefore, since $F(m) < 0$, replace update a, $a := m$.

Second iteration. Compute the midpoint $m = \frac{a+b}{2}$ and evaluate $F(m)$.

$$m = \frac{3}{4}, \qquad F(m) = \frac{27}{64} + \frac{9}{4} - 2 = \frac{43}{64}.$$

Since, $F(m) > 0$, then for the next iteration we update b, $b := m$.

Exercise 9

With initial guess $x^{(0)} = 1$ apply one Newton iteration to find an approximate solution of the equation

$$(2x - 1)\left(3x^2 - 2x + 1\right) = 0$$

Solution: We have to solve the equation $F(x) = 0$ where

$$F(x) := (2x - 1)\left(3x^2 - 2x + 1\right)$$

Recall that the generic Newton iteration is given as follows

$$x^{(k+1)} = x^{(k)} - \frac{F(x^{(k)})}{F'(x^{(k)})},$$

so firstly compute F'

$$F'(x) = 2(3x^2 - 2x + 1) + (2x - 1)(6x - 2).$$

So, compute the first iteration of the Newton method

$$x^{(1)} = x^{(0)} - \frac{F(x^{(0)})}{F'(x^{(0)})} = 1 - \frac{2}{8} = 1 - \frac{1}{4} = \frac{3}{4}.$$

Exercise 10

With initial guess $\underline{x}^{(0)} = [1, 1]^T$ apply one Newton iteration to find an approximate solution of the system

$$\underline{F}(\underline{x}) = \begin{bmatrix} x_1^2 - 2x_1 + x_2 + 7 \\ 2x_1 - x_2 + 2 \end{bmatrix}.$$

Solution: Recall that the Newton's iteration for multivariate functions is given by

$$\underline{x}^{(k+1)} = \underline{x}^{(k)} - \left(J_F(\underline{x}^{(k)})\right)^{-1} \cdot \underline{F}(\underline{x}^{(k)}),$$

where J_F is the Jacobian matrix of $\underline{F}$. We have

$$J_F(\underline{x}) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 - 2 & 1 \\ 2 & -1 \end{bmatrix}.$$

For $k = 0$, that is the first iteration, we have

$$\underline{F}(\underline{x}^{(0)}) = \begin{bmatrix} 1^2 - 2 + 1 + 7 \\ 2 - 1 + 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}, \quad \text{and} \quad J_F(\underline{x}^{(0)}) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix},$$

and we have to compute $\underline{u} = [u_1, u_2]^T$ solution of $J_F(\underline{x}^{(0)}) \cdot \underline{u} = \underline{F}(\underline{x}^{(0)})$. The solution of this linear system is

$$\begin{cases} u_2 = 7 \\ 2u_1 - u_2 = 3 \end{cases} \quad \rightarrow \begin{cases} u_2 = 7 \\ u_1 = \dfrac{3 + 7}{2} = 5 \end{cases}$$

and the first newton iteration step gives

$$\underline{x}^{(1)} = \underline{x}^{(0)} - \underline{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 7 \end{bmatrix} = - \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

Exercise 11

Given the function $f(x) = \cos(2\pi\, x)$ compute its Lagrange interpolant of degree 2 through the points $x_1 = 0$, $x_2 = 1/2$, $x_3 = 1$.

Solution 1: Recall that the Lagrange interpolant of f of degree k, over the points $x_1, x_2, \ldots, x_{k+1}$, is

$$\Pi_k(f) := \sum_{i=1}^{k+1} f(x_i) L_i(x), \quad \text{where} \quad L_i(x) := \prod_{\substack{j=1 \\ j \neq i}}^{k+1} \frac{(x - x_j)}{(x_i - x_j)}.$$

In order to compute $\Pi_2(f)$ on the points x_1, x_2, x_3, we have

$$\begin{aligned}
L_1(x) &= \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(x-\frac{1}{2})(x-1)}{(-\frac{1}{2})(-1)} = (2x-1)(x-1),\\
L_2(x) &= \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(x-0)(x-1)}{(\frac{1}{2})(-\frac{1}{2})} = -4x(x-1),\\
L_3(x) &= \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(x-0)(x-\frac{1}{2})}{(1)(\frac{1}{2})} = x(2x-1),
\end{aligned}$$

and also

$$\begin{aligned}
f(x_1) &= \cos(2\pi x_1) = \cos(0) = 1,\\
f(x_2) &= \cos(2\pi x_2) = \cos(\pi) = -1,\\
f(x_3) &= \cos(2\pi x_3) = \cos(2\pi) = 1.
\end{aligned}$$

Finally

$$\begin{aligned}
\Pi_2(f) &= f(x_1)L_1(x) + f(x_2)L_2(x) + f(x_3)L_3(x)\\
&= 1\cdot(2x-1)(x-1) + (-1)\cdot(-4x)(x-1) + 1\cdot x(2x-1)\\
&= 2x^2 - 2x - x + 1 + 4x^2 - 4x + 2x^2 - x\\
&= 8x^2 - 8x + 1.
\end{aligned}$$

Solution 2: An alternative solution consists in the following observation. $\Pi_2(f)$ is the degree two polynomial $c_1 + c_2x + c_3x^2$, where the coefficients c_1, c_2, c_3 are solution of the following linear system

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}$$

For this exercise we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

In particular $c_1 = 1$ and it remains to solve

$$\begin{cases} 2c_2 + c_3 = -8 \\ c_2 + c_3 = 0 \end{cases} \rightarrow \begin{cases} 2c_2 + c_3 = -8 \\ c_3 = -c_2 \end{cases} \rightarrow \begin{cases} c_2 = -8 \\ c_3 = 8 \end{cases}$$

Again we found that $\Pi_2(f) = 1 - 8x + 8x^2$.

Exercise 12

Given the Cauchy problem:

$$\begin{cases} y'(t) = -2ty(t) + 2t^3 \text{ for } t > 0 \\ y(0) = -1; \end{cases}$$

compute two steps by the implicit Euler method, with $\Delta t = 1$, in order to approximate $y(2)$. Report the intermediate computations.

Solution: We denote by $y_n := y(t_n)$, where $t_n = t_0 + n\Delta t$, and here we have $t_0 = 0,\ t_1 = 1$ and $t_2 = 2$. The Implicit Euler (IE) method is given by

$$\frac{y_{n+1} - y_n}{\Delta t} = f(t_{n+1}, y_{n+1}),$$

and for this exercise, we have $f(t, y(t)) = -2ty(t)+2t^3$. Thus, the IE method applied to our Cauchy problem leads to solving, for each step, the following equation:

$$\frac{y_{n+1} - y_n}{\Delta t} = -2t_{n+1}y_{n+1} + 2t_{n+1}^3, \quad \text{(Implicit in the unknown } y_{n+1}\text{)}.$$

Therefore we manipulate it as

$$y_{n+1} + 2\Delta t \cdot t_{n+1} \cdot y_{n+1} = y_n + 2\Delta t \cdot t_{n+1}^3;$$

$$y_{n+1} = \frac{y_n + 2\Delta t \cdot t_{n+1}^3}{1 + 2\Delta t \cdot t_{n+1}}, \quad \text{(Explicitated in } y_{n+1}\text{)}.$$

Finally, the computation of the first step is

$$y_1 = \frac{y_0 + 2\Delta t \cdot t_1^3}{1 + 2\Delta t \cdot t_1} = \frac{-1+2}{1+2} = \frac{1}{3},$$

and the computation of the second step is

$$y_2 = \frac{y_1 + 2\Delta t \cdot t_2^3}{1 + 2\Delta t \cdot t_2} = \frac{1/3 + 16}{1+4} = \frac{1+48}{15} = \frac{49}{15},$$

Exercise 13

Write the pseudo-code of the composite trapezoidal quadrature rule, then use the composite trapezoidal quadrature rule to compute an approximation of

$$\int_0^{2\pi} \sin^2(t)\, dt$$

by splitting the integration interval $[0, 2\pi]$ into four uniform subintervals. Report the intermediate computations.

Solution:

The composite trapezoidal quadrature rule amounts to approximating the function to be integrated with the Lagrange piecewise-linear approximation and integrating it. The extema of the four subintervals are $t_1 = 0$, $t_2 = \frac{\pi}{2}$, $t_3 = \pi$, $t_4 = \frac{3\pi}{2}$, $t_5 = 2\pi$. The width of such subintervals is $\frac{\pi}{2}$. So

$$\int_0^{2\pi} \sin^2(t)dt \simeq \frac{\pi}{4} \sum_{i=1}^{4} \int_{t_i}^{t_{i+1}} \Pi_1(\sin^2)(t)dt = \frac{\pi}{4} \sum_{i=1}^{4} [\sin^2(t_i) + \sin^2(t_{i+1})].$$

Compute each term of the sum individually:

- $\sin^2(0) + \sin^2(\frac{\pi}{2}) = 1$,
- $\sin^2(\frac{\pi}{2}) + \sin^2(\pi) = 1$,
- $\sin^2(\pi) + \sin^2(\frac{3\pi}{2}) = 1$,
- $\sin^2(\frac{3\pi}{2}) + \sin^2(2\pi) = 1$,

combining all the terms yields

$$\int_0^{2\pi} \sin^2(t)dt \simeq \frac{\pi}{4}(1 + 1 + 1 + 1) = \pi.$$

Exercise 14

Describe the Crank-Nicolson scheme for the solution of an ODE and explain its relation with the trapezoidal quadrature rule. Then, compute one step of the Crank-Nicolson scheme for the problem

$$\begin{cases} y'(t) = 2t\,(1 - y(t)) \\ y(0) = 3 \end{cases}$$

selecting $\Delta t = 1$.

Solution: Let $t_n = t_0 + n\Delta t$ and $y_n = y(t_n)$. We are given $t_0 = 0$ and $\Delta t = 1$, and have to compute one step of Crank-Nicolson, i.e. compute y_1, for $t_1 = 1$.

For the Crank-Nicolson scheme, we approximate the derivative using a finite difference and the ODE field with the average of the fields at two consecutive timesteps

$$\frac{y_n - y_{n-1}}{\Delta t} = \frac{1}{2}(f(t_n, y_n) + f(t_{n-1}, y_{n-1})).$$

In our case, $f(t, y) = 2t(1 - y)$ and $\Delta = 1$, therefore

$$y_n - y_{n-1} = \frac{1}{2}(2t_n(1 - y_n) + 2t_{n-1}(1 - y_{n-1})).$$

In order to compute y_1, substitute t_0, t_1, y_0 in the equation above and solve for y_1.

$$
\begin{aligned}
y_1 - y_0 &= \frac{1}{2}(2t_1(1 - y_1) + 2t_0(1 - y_0)) \\
y_1 - 3 &= \frac{1}{2}(2 \cdot 1 \cdot (1 - y_1) + 2 \cdot 0 \cdot (1 - 3)),
\end{aligned}
$$

to obtain $y_1 = 2$.