Approximation of functions by polynomials

Why?

Sometimes the analytic expression of a function is complicated, and it might be difficult to integrate it or to differentiate it.

Sometimes the analytic expression is not available, the function is known only at some points and we need to reconstruct the underlying function.

A way to overcome problems of this kind is to approximate functions with polynomials (easy to deal with).

Lagrange interpolation

Theorem 1

Let $g : [a, b] \to \mathbb{R}$ be any function and let $n \geq 0$. Given $n + 1$ distinct points $x_0, x_1, \cdots, x_n$ in $[a, b]$, there exists a unique polynomial $\Pi_n(x)$ of degree $\leq n$ such that

$$\Pi_n(x_i) = g(x_i) \qquad i = 0, 1, \cdots, n. \qquad (*)$$

Π_n is called "Lagrange interpolant of g with respect to the points $x_0, x_1, \cdots, x_n$".
The points $x_0, x_1, \cdots, x_n$ are called "interpolation nodes".

Proof

To prove the existence, we shall write explicitly the expression of Π_n. Let $L_i(x),\ i = 0, 1, \cdots, n$ be $n+1$ polynomials, of degree exactly n, defined by:

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)} \quad i = 0, 1, \cdots, n$$

For example, for $n = 3$ and nodes x_0, x_1, x_2, x_3:

$$L_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}, \qquad L_1(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}, \qquad L_3(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

These are characteristic Lagrange polynomials, that fulfil

$$L_i \in \mathbb{P}_n : \quad L_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad i = 0, 1, \cdots, n$$

continue...

Existence and uniqueness: conclusion

The polynomial $\Pi_n(x) = \sum_{i=0}^{n} g(x_i) L_i(x)$ satisfies $\Pi_n(x_j) = g(x_j)$, proving existence for arbitrary nodal values. Writing $\Pi_n(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$ gives the square Vandermonde system

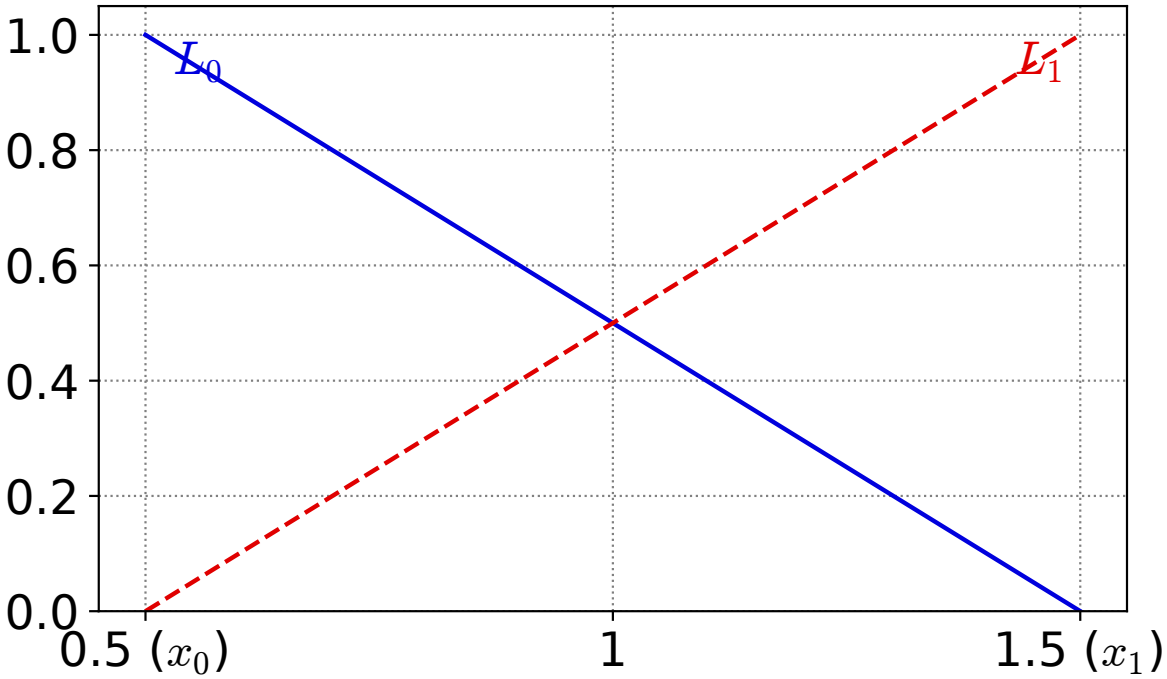
$$\begin{pmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_n) \end{pmatrix}.$$

Because it is solvable for every right-hand side, this square matrix is invertible, proving uniqueness. Equivalently, the difference of two interpolants has degree at most n and $n+1$ distinct roots, so it is identically zero.

Example of characteristic polynomials

Ex. 1: 2 points $x_0 \neq x_1$ $(n+1=2)$

$$L_0(x) = \frac{(x - x_1)}{(x_0 - x_1)}, \quad L_1(x) = \frac{(x - x_0)}{(x_1 - x_0)}$$



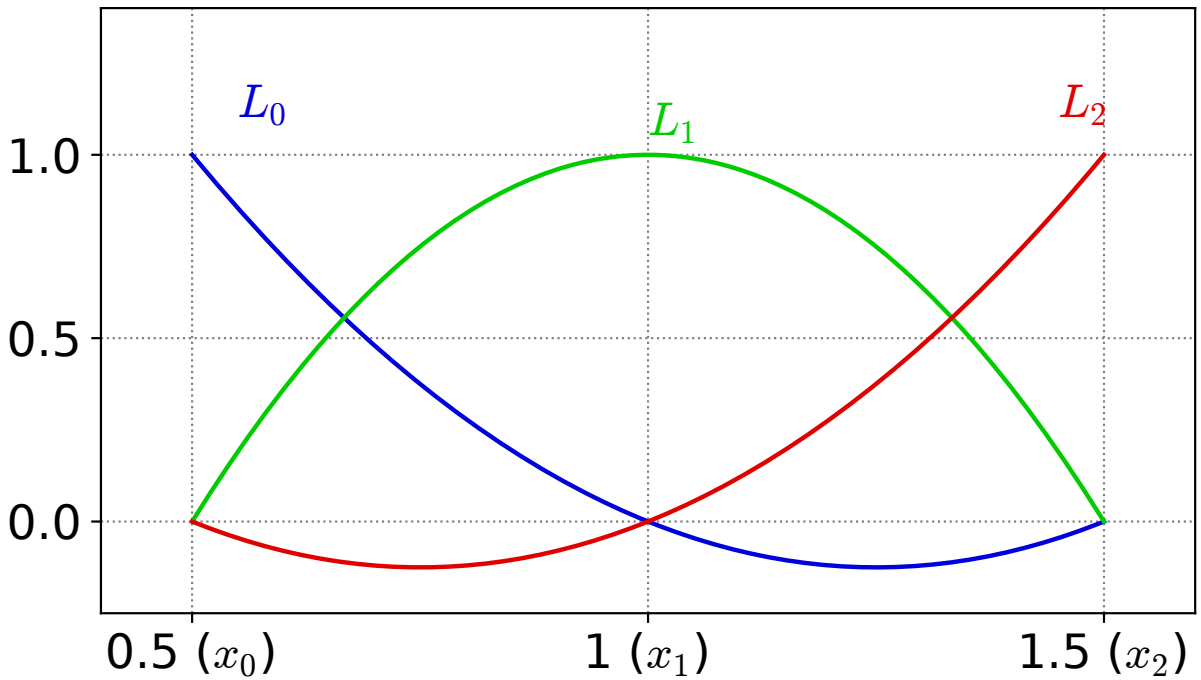
degree $n = 1$.

Example of characteristic polynomials

Ex. 2: 3 pairwise distinct points x_0, x_1, x_2 $(n+1=3)$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}, \quad L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$



degree $n = 2$.

Cases of interest

Case 1: g approximated by a constant (a polynomial of degree 0)
1 point: $x_0 = \frac{a+b}{2}$ (midpoint of the interval)

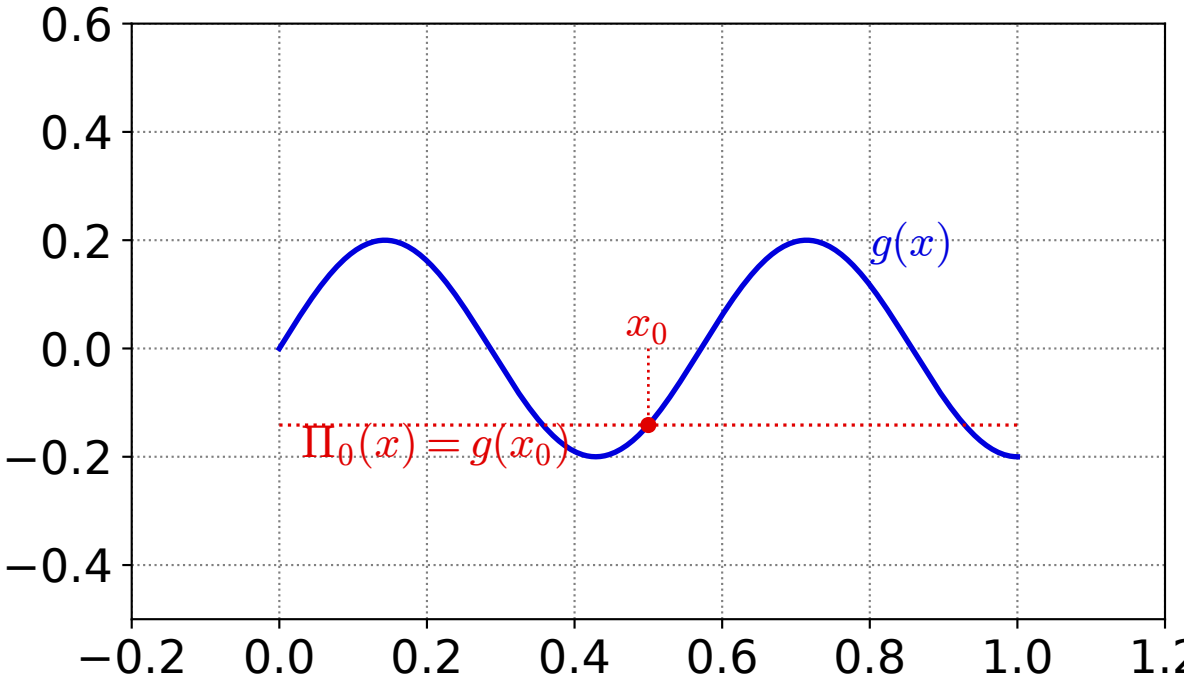
$$g(x) \simeq \Pi_0(x) = g(x_0)$$

Case 2: g approximated by a polynomial of degree 1(straight line)
2 points: $x_0 = a$, $x_1 = b$

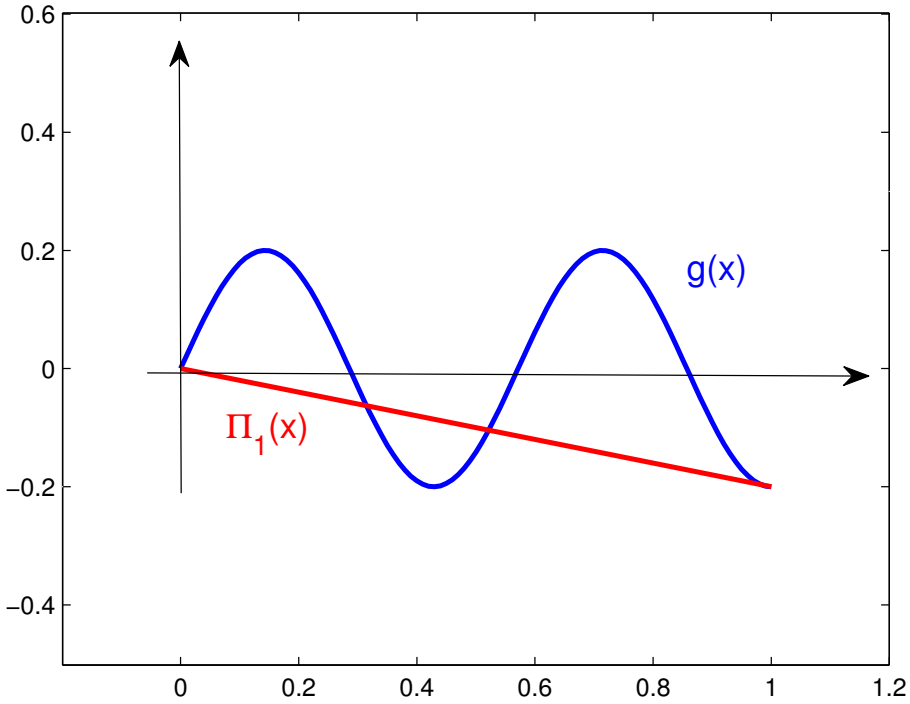
$$\begin{aligned} g(x) \simeq \Pi_1(x) &= g(x_0)L_0(x) + g(x_1)L_1(x) \\ &= g(x_0)\frac{(x - x_1)}{(x_0 - x_1)} + g(x_1)\frac{(x - x_0)}{(x_1 - x_0)} \end{aligned}$$

Remark: in both cases, changing the nodes can change the interpolant

Example 1



Example 2



General interpolation error

In both cases, the approximation induces an error that we want to estimate. How big is it? Is the approximation satisfactory? The following Theorem gives a precise expression of the error.

Theorem 2

Let $g \in C^{n+1}([a,b])$, and let $x_0, x_1, \cdots, x_n$ be $n+1$ distinct points in $[a,b]$. For any generic point $x \in [a,b]$ there exists a $\xi \in [a,b]$ (depending on x) such that the interpolation error is given by

$$e_n(x) := g(x) - \Pi_n(x) = \frac{g^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j) \tag{1}$$

Proof of the interpolation remainder

If $x = x_i$, both sides of the remainder formula vanish. Otherwise fix x and set

$$\omega(t) = \prod_{j=0}^{n}(t - x_j), \qquad G(t) = e_n(t) - \omega(t)\frac{e_n(x)}{\omega(x)}.$$

Since $g \in C^{n+1}([a,b])$, $G \in C^{n+1}([a,b])$. It vanishes at the $n+2$ distinct points $x_0, \ldots, x_n, x$. Repeated application of Rolle's theorem gives at least $n+2-j$ distinct zeros of $G^{(j)}$, for $j = 1, \ldots, n+1$. Thus $G^{(n+1)}(\xi) = 0$ for some $\xi \in (a,b)$.

$$\omega^{(n+1)} = (n+1)!, \qquad e_n^{(n+1)} = g^{(n+1)},$$

so

$$0 = g^{(n+1)}(\xi) - (n+1)!\frac{e_n(x)}{\omega(x)}.$$

Rearranging proves (1).

From (1) we can deduce for instance the following bound:

$$\max_{[a,b]} |e_n(x)| \leq \frac{(b-a)^{n+1}}{(n+1)!} \max_{x\in[a,b]} |g^{(n+1)}(x)| \qquad (*)$$

If we know the position of the nodes we can have sharper estimates for the term $\prod_{j=0}^{n}(x-x_j)$ that appears in (1). For instance, for the Case 1 we would obtain

Case 1 (constant approximation): Relation (1) in this case gives

$$e_0(x) = g'(\xi)(x-x_0) \quad x_0 = \frac{a+b}{2}$$

No matter where x is situated within $[a, b]$, its distance from the midpoint will be smaller than or equal to the length of half the interval (that is, $(b-a)/2$). Hence

$$\Longrightarrow \quad \max_{x\in[a,b]} |e_0(x)| \leq \frac{b-a}{2} \max_{x\in[a,b]} |g'(x)| \qquad (2)$$

($(b-a)/2$ instead of $b-a$)

Case 2 (linear approximation): For $x_0 = a$ and $x_1 = b$ we can observe that the function $x \to |(x-a)(x-b)|$, in the interval $[a, b]$ has its maximum for $x = (a+b)/2$ and such a maximum is $((b-a)/2)^2$. Hence

$$\max_{x \in [a,b]} |e_1(x)| \leq \frac{(b-a)^2}{2! \cdot 4} \max_{x \in [a,b]} |g''(x)| \tag{3}$$

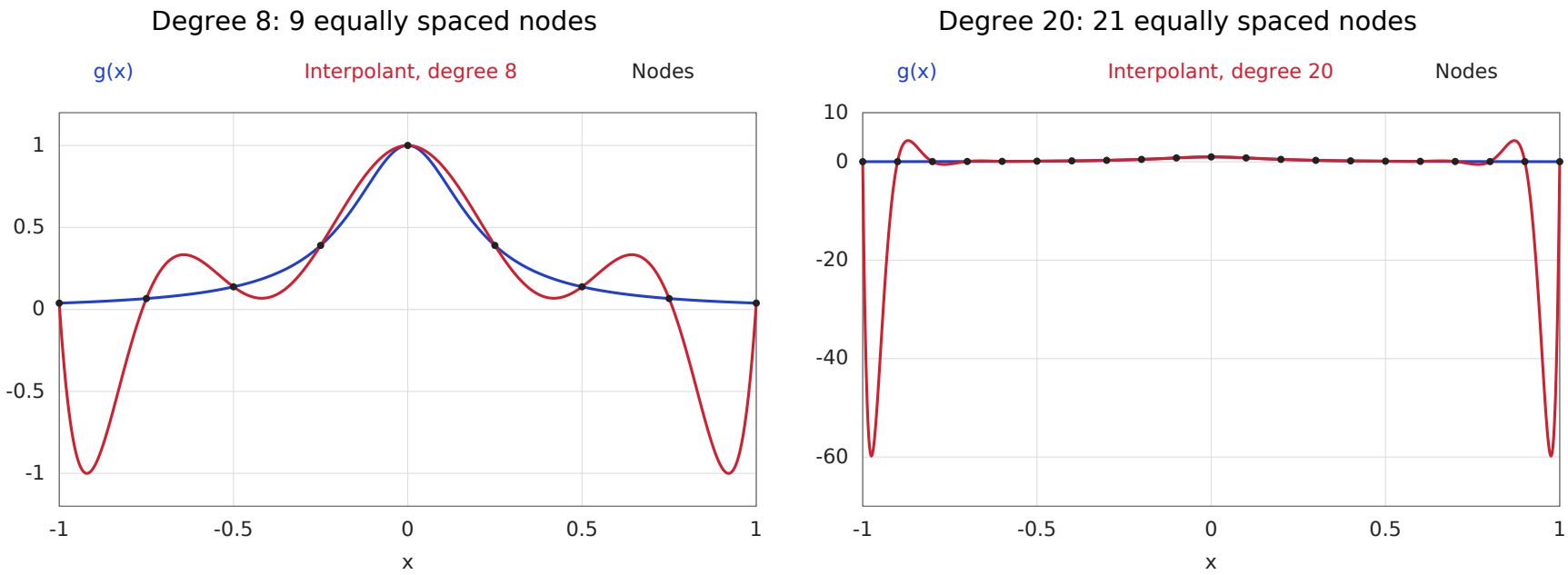
(and we have $(b-a)^2/8$ instead of $(b-a)^2/2$).

Runge function

Increasing the degree does not always reduce the error at equally spaced nodes. The classical example is the Runge function:

$$g(x) = \frac{1}{1 + 25x^2}, \qquad x \in [-1, 1]$$

Equally spaced nodes give the following interpolants:



Degree 8 (left) and degree 20 (right).

Composite Lagrange interpolation

Use of Lagrange interpolation (piecewise) to have a good approximation of a function.

Given $f : [a, b] \to \mathbb{R}$ (smooth enough), subdivide $[a, b]$ in N subintervals, for simplicity of notation all equal. We have then a uniform subdivision of $[a, b]$ into intervals of length $h = (b - a)/N$. In each subinterval we approximate f with a Lagrange interpolant polynomial.

To fix ideas, let us consider again Cases 1 and 2.

Case 1: $N \to h = (b - a)/N, \quad x_0 = a, x_1 = x_0 + h, \cdots, x_N = b$

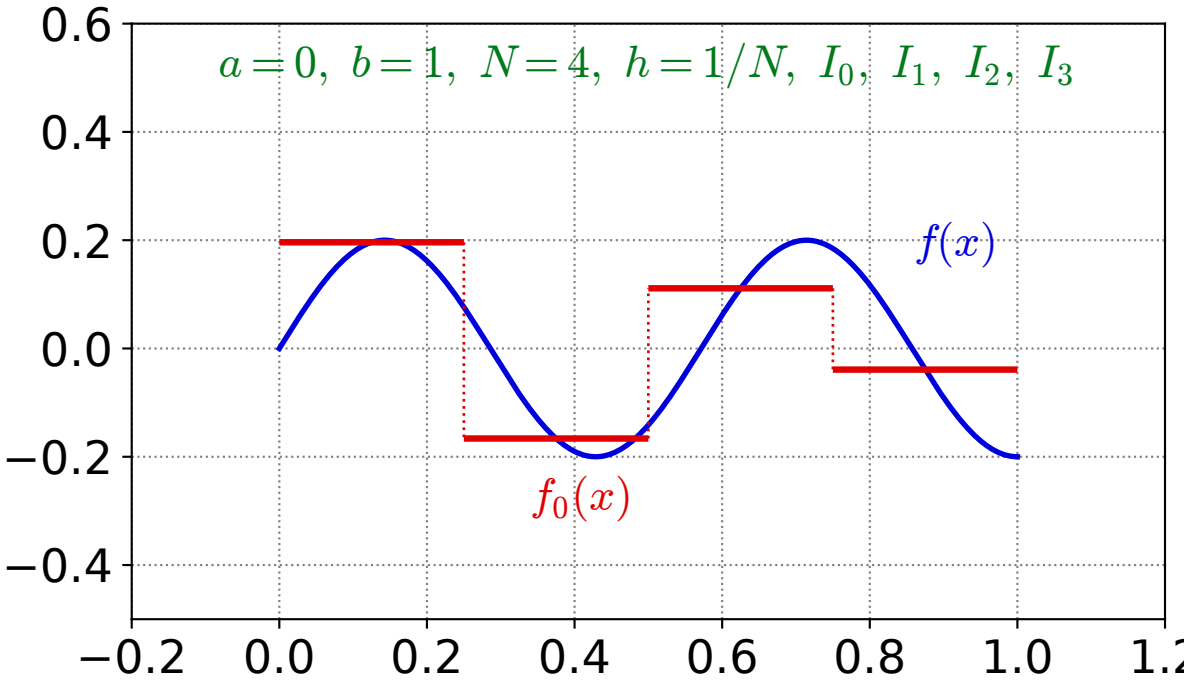
$I_0 = [x_0, x_1], \cdots, I_j = [x_j, x_{j+1}], \cdots, I_{N-1} = [x_{N-1}, x_N]$

x_j^M =midpoint of the interval I_j : $x_j^M = (x_j + x_{j+1})/2$

$f(x)_{|[a,b]} \simeq f_0(x)$ piecewise constant function given by

$$f_0(x)_{|I_j} = f(x_j^M) \quad j = 0, 1, \cdots, N - 1$$

Piecewise constant interpolation



Piecewise constant interpolation: error

Assume $f \in C^1([a,b])$. On I_j, $f_0(x) = f(x_j^M)$ and $|x - x_j^M| \le h/2$. For $x \neq x_j^M$, the mean value theorem gives

$$f(x) - f(x_j^M) = (x - x_j^M) f'(c), \qquad c \in I_j.$$

At $x = x_j^M$ the error is zero. Therefore on every panel,

$$|f(x) - f_0(x)| \le \frac{h}{2} \|f'\|_{\infty, I_j} \le \frac{h}{2} \|f'\|_{\infty, [a,b]}.$$

Taking the supremum over $[a,b]$ yields

$$\|f - f_0\|_\infty \le \frac{h}{2} \|f'\|_\infty = O(h).$$

At shared endpoints choose either neighbouring constant value; the same bound holds. Equivalently use half-open panels, with the last panel closed on the right.

Case 2: $N \to h = (b-a)/N, \quad x_0 = a, x_1 = x_0 + h, \cdots, x_N = b$

$I_0 = [x_0, x_1], \cdots, I_j = [x_j, x_{j+1}], \cdots, I_{N-1} = [x_{N-1}, x_N]$

$f(x)_{|[a,b]} \simeq f_1(x)$ piecewise linear function which, on each interval I_j, is the Lagrange interpolant of degree ≤ 1 with respect to the endpoints of I_j, i.e.,

$$f_1(x)_{|I_j} = f(x_j)\frac{(x - x_{j+1})}{(x_j - x_{j+1})} + f(x_{j+1})\frac{(x - x_j)}{(x_{j+1} - x_j)} \quad j = 0, 1, \cdots, N-1$$

Assume $f \in C^2([a,b])$. Using the bound (3) for the error $E_1(x) = f(x) - f_1(x)$ and proceeding as before we then obtain

$$\max_{[a,b]} |E_1(x)| \leq \frac{h^2}{8} \max_{[a,b]} |f''(x)| = C\, h^2 \qquad C = \frac{\max_{[a,b]} |f''(x)|}{8}$$

Hence: if f'' exists and is bounded, the interpolation error goes to zero quadratically with h (halving h divides this error bound by four), and we can choose h as small as we want!

Piecewise linear interpolation

