ON THE CONVERGENCE TO EQUILIBRIUM FOR DEGENERATE TRANSPORT PROBLEMS

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Abstract. We give a counterexample which shows that the asymptotic rate of convergence to the equilibrium state for the transport equation, with a degenerate cross section and in the periodic setting, cannot be better than $t^{-1/2}$ in the general case. We suggest moreover that the geometrical properties of the cross section are the key feature of the problem and impose, through the distribution of the forward exit time, the speed of convergence to the stationary state.

1. Introduction

The long-time behaviour of kinetic transport equations – on periodic domains or on bounded domains with specular reflection on the boundary – is well known when the cross sections are bounded from below by a strictly positive constant.

In this case, the exponential decay in time of the solutions to the unique equilibrium state of the system can be obtained, with explicit rates, by the method of hypocoercivity as in [7, 12].

This result has, however, no obvious extension in the case of cross sections vanishing in a portion of the domain. Such a transport problem is said to be degenerate, and the characterization of the long-time asymptotics in the general case is still an open problem.

Indeed, in the regions where the cross section is zero, the problem is reduced to the free transport equation, which has no equilibrium state in a periodic setting or when the problem is defined on a bounded domain with specular reflection.

A partial answer to this question has been obtained by Desvillettes and Salvarani in [3] in a special situation, namely when the cross section vanishes at a finite number of points.

The key point of the proof in [3] is the use of the Desvillettes-Villani lemma (Theorem 6.2 in [4]), based on a pair of differential inequalities that allows them to prove a polynomial (in time) speed of convergence towards equilibrium for the solution of the transport problem.

We note that there exist other phenomena that can lead to the convergence to equilibrium of the solutions of free transport equations: we cite, for example, the interaction with the boundary of the domain in the case of diffuse reflection [1, 10] or the presence of a dissipating obstacle (see [9] and the references therein).

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In this note, we consider a situation when the cross section vanishes on a set of non zero measure, and give a counterexample which shows that the L^2 distance to equilibrium cannot decay faster than $t^{-1/2}$ (see Theorem 3.3 below).

2. The problem

The problem considered here is the long-time asymptotics of the non-homogeneous (in space) transport equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \sigma(x)\,(\bar{f} - f), \tag{2.1}$$

where $f := f(t, x, v)$ represents the density of particles which at time $t \in \mathbb{R}^+$ and point $x \in \mathbb{T}^d$ ($d \in \mathbb{N}$, $d \geq 2$) move at speed 1 in the direction $v \in S^{d-1}$.

Here S^{d-1} denotes the unit sphere in $\mathbb{R}^d$ and

$$\bar{f}(t, x) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(t, x, v)\, dv,$$

where dv is the Euclidean surface element on S^{d-1} and $|S^{d-1}|$ is the total $(d-1)$-dimensional measure of S^{d-1}.

The equation is set in a periodic box, that is $x \in \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, and is supplemented with the initial condition

$$f(0, x, v) = f^{\text{in}}(x, v). \tag{2.2}$$

We assume that $f^{\text{in}} \in L^\infty(\mathbb{T}^d \times S^{d-1})$ and that $f^{\text{in}} \geq 0$ for a.e. $(x, v) \in \mathbb{T}^d \times S^{d-1}$.

The nonnegative function $\sigma(x)$ designates the absorption/scattering cross section. We assume that

(1) $\sigma \in L^\infty(\mathbb{T}^d)$ and $\sigma(x) \geq 0$ for a.e. $x \in \mathbb{T}^d$;
(2) $\|\sigma\|_{L^1(\mathbb{T}^d)} > 0$.

Since the problem (2.1)-(2.2) is a Lipschitz continuous perturbation of the free transport equation, there exists a unique mild solution f of the problem (see, for example, [8]).

It is also straightforward to prove that constants are steady solutions of Equation (2.1), and that

$$f_\infty = \frac{1}{|S^{d-1}|} \int_{\mathbb{T}^d \times S^{d-1}} f^{\text{in}}(x, v)\, dx dv$$

is the unique constant solution with the same total mass (i.e. particle number) as the initial data.

Problem: Under the assumptions above, does one have

$$\|f(t, \cdot, \cdot) - f_\infty\|_{L^2(\mathbb{T}^d \times S^{d-1})} = O(e^{-\gamma t}) \qquad \text{as } t \to +\infty$$

for some $\gamma > 0$?

We recall the following result obtained by Ukai, Point and Ghidouche [11]:

Theorem 2.1. (Ukai, Point, Ghidouche). *Under the assumptions above, if $\sigma(x) \geq \sigma_m > 0$ for a.e. $x \in \mathbb{T}^d$, there exist C, $\gamma > 0$ such that*

$$\|f(t, \cdot, \cdot) - f_\infty\|_{L^2(\mathbb{T}^d \times S^{d-1})} \leq C e^{-\gamma t} \|f^{\text{in}}\|_{L^2(\mathbb{T}^d \times S^{d-1})}.$$

In the next section, we answer the question above in the negative and show that the Ukai-Point-Ghidouche theorem cannot be extended to the case of degenerate cross sections.

3. A counterexample

We consider here a case of degenerate transport where convergence to equilibrium cannot be faster than algebraic. This possibility excludes the exponential convergence under the assumptions above and without additional requirements.

Following [5], for all $r \in (0, 1/2)$ we consider the periodic open set

$$Z_r = \{x \in \mathbb{R}^d \ : \ \mathrm{dist}(x, \mathbb{Z}^d) > r\}$$

together with the associated fundamental domain $Y_r = Z_r/\mathbb{Z}^d$.

A crucial tool in studying Equation (2.1) is the *forward exit time* for a particle starting from $x \in Z_r$ in the direction $v \in S^{d-1}$, defined as

$$\tau_r(x, v) = \inf\{t > 0 \ : \ x + tv \in \partial Z_r\}.$$

The forward exit time can be defined on the quotient space $Y_r \times S^{d-1}$ by periodicity since

$$\tau_r(x + k, v) = \tau_r(x, v) \text{ for all } (x, v) \in Z_r \times S^{d-1} \text{ and } k \in \mathbb{Z}^d.$$

On the measurable space $Y_r \times S^{d-1}$, equipped with its Borel σ-algebra, we define μ_r as the probability measure proportional to the Lebesgue measure on $Y_r \times S^{d-1}$, that is

$$d\mu_r(y, v) = \frac{dydv}{|Y_r|\,|S^{d-1}|}.$$

We finally define the distribution of τ_r under μ_r by

$$\Phi_r(t) := \mu_r\left(\{(x, v) \in Y_r \times S^{d-1} \ : \ \tau_r(y, v) > t\}\right).$$

The distribution of forward exit time satisfies, in the periodic case, both a lower and an upper bound. This property, proved by Bourgain, Golse and Wennberg [2, 6] is recalled in the following theorem:

Theorem 3.1. (Bourgain, Golse, Wennberg). *Let $d \geq 2$. Then there exist two positive constants C_1 and C_2 such that, for all $r \in (0, 1/2)$ and each $t > 1/r^{d-1}$*

$$\frac{C_1}{r^{d-1}}\, t^{-1} \leq \Phi_r(t) \leq \frac{C_2}{r^{d-1}}\, t^{-1}.$$

Our counterexample uses only the lower bound in Theorem 3.1. We recall that this lower bound is based on the fact that some particles never meet the scattering region, i.e. $\{x \in \mathbb{T}^d \ : \ \sigma(x) > 0\}$, because of the presence of *infinite channels* [2].

Choose

$$\sigma(x) = \mathbb{1}_{\mathbb{T}^d \setminus Y_r},$$

and

$$f^{\mathrm{in}}(x, v) = f^{\mathrm{in}}(x) = \mathbb{1}_{Y_r}$$

in (2.1)-(2.2).

It is easy to prove that the only steady solution of Equation (2.1) with the same mass as the initial condition f^{in} is the constant function $f_\infty = |Y_r|$. This property is a consequence of the following result:

Proposition 3.2. *Let $f \in L^2(\mathbb{T}^d \times S^{d-1})$ satisfy*

$$v \cdot \nabla_x f - \sigma(x)\,(\bar{f} - f) = 0. \tag{3.1}$$

Then

$$f(x,v) = \frac{1}{|S^{d-1}|} \int_{\mathbb{T}^d \times S^{d-1}} f(x,v)\,dxdv$$

for a.e. $(x,v) \in \mathbb{T}^d \times S^{d-1}$.

Proof. Multiply Equation (3.1) by f and integrate with respect to $(x,v) \in \mathbb{T}^d \times S^{d-1}$. We deduce the energy estimate

$$\int_{\mathbb{T}^d \times S^{d-1}} \sigma(x)(\bar{f} - f)^2 dxdv = 0.$$

Since $\sigma(x) \geq 0$ a.e., the previous equation implies that $\sigma(x)\,(\bar{f} - f) = 0$ for a.e. $(x,v) \in \mathbb{T}^d \times S^{d-1}$.

Hence f satisfies the equation $v \cdot \nabla_x f = 0$. Therefore, by applying the Fourier transform with respect to the space variable, we obtain that $v \cdot k\hat{f}(k,v) = 0$, that is $\hat{f}(k,v) = 0$ for all k and for a.e. $v \in S^{d-1}$.

Thus, $\mathrm{supp}(\hat{f}) \subset \{0\} \times S^{d-1}$ which means that $f = f(v)$. Again by the energy estimate, $\sigma(x)f(v) = \sigma(x)\bar{f}$ and hence, in the region $\{x \in \mathbb{T}^d : \sigma(x) > 0\}$, $f(v) = \bar{f}$.

Since this region is of positive measure the announced result follows. ☐

Since the solution of the Cauchy problem (2.1)-(2.2) satisfies

$$\int_{\mathbb{T}^d \times S^{d-1}} f(t,x,v)\,dxdv = \int_{\mathbb{T}^d \times S^{d-1}} f^{\text{in}}(x,v)\,dxdv,$$

the only equilibrium solution to which f can converge in $L^2(\mathbb{T}^d \times S^{d-1})$ as $t \to +\infty$ is

$$f_\infty = \frac{1}{|S^{d-1}|} \int_{\mathbb{T}^d \times S^{d-1}} f^{\text{in}}(x,v)\,dxdv = |Y_r|.$$

Thus

$$\begin{aligned}
\int_{\mathbb{T}^d \times S^{d-1}} (f - f_\infty)^2\,dxdv &\geq \int_{Y_r \times S^{d-1}} (f - f_\infty)^2\,dxdv \\
&= \int_{Y_r \times S^{d-1}} \mathbb{1}_{\tau_r(x,-v) > t} (f - f_\infty)^2\,dxdv \\
&+ \int_{Y_r \times S^{d-1}} \mathbb{1}_{\tau_r(x,-v) \leq t} (f - f_\infty)^2\,dxdv \\
&= I + J.
\end{aligned} \tag{3.2}$$

By Duhamel's formula

$$\begin{aligned} f(t,x,v) &= f^{\mathrm{in}}(x-tv,v)\exp\left(-\int_0^t \sigma(x-sv)\,ds\right) \\ &\quad+\int_0^t \exp\left(-\int_0^s \sigma(x-\tau v)\,d\tau\right)\sigma(x-sv)\bar{f}(s,x-sv)\,ds \\ &\geq f^{\mathrm{in}}(x-tv,v)\exp\left(-\int_0^t \sigma(x-sv)\,ds\right) \end{aligned}$$

since $f \geq 0$ by the maximum principle for (2.1)-(2.2).

Thus

$$f(t,x,v)\mathbb{1}_{\tau_r(x,-v)>t} \geq f^{\mathrm{in}}(x-tv,v)\mathbb{1}_{\tau_r(x,-v)>t}$$

since

$$\tau_r(x,-v) > t \Longrightarrow \sigma(x-sv) = 0 \text{ for all } s \in [0,t],$$

and therefore

$$f(t,x,v)\mathbb{1}_{\tau_r(x,-v)>t} \geq \mathbb{1}_{\tau_r(x,-v)>t}$$

since

$$\tau_r(x,-v) > t \Longrightarrow x - tv \in Y_r \Longrightarrow f^{\mathrm{in}}(x-tv,v) = 1.$$

Hence

$$\begin{aligned} I &= \int_{Y_r\times S^{d-1}} \mathbb{1}_{\tau_r(x,-v)>t}(f-f_\infty)^2\,dxdv \\ &= \int_{Y_r\times S^{d-1}} (\mathbb{1}_{\tau_r(x,-v)>t}f - \mathbb{1}_{\tau_r(x,-v)>t}f_\infty)^2\,dxdv \\ &\geq \int_{Y_r\times S^{d-1}} \mathbb{1}_{\tau_r(x,-v)>t}(1-f_\infty)^2\,dxdv \\ &= (1-|Y_r|)^2\int_{Y_r\times S^{d-1}} \mathbb{1}_{\tau_r(x,-v)>t}\,dxdv \\ &= (1-|Y_r|)^2|Y_r|\,|S^{d-1}|\Phi_r(t), \end{aligned}$$

where the inequality above follows from the fact that $f_\infty < 1$, so that

$$\mathbb{1}_{\tau_r(x,-v)>t}f_\infty \leq \mathbb{1}_{\tau_r(x,-v)>t} \leq \mathbb{1}_{\tau_r(x,-v)>t}f(t,x,v).$$

Therefore

$$I \geq (1-|Y_r|)^2|Y_r|\,|S^{d-1}|\frac{C_1}{r^{d-1}}\,t^{-1}$$

for all $t > r^{1-d}$.

Since

$$J = \int_{Y_r\times S^{d-1}} \mathbb{1}_{\tau_r(x,-v)\leq t}(f-f_\infty)^2\,dxdv \geq 0,$$

(3.2) implies

$$\int_{\mathbb{T}^d\times S^{d-1}} (f-f_\infty)^2\,dxdv \geq \frac{C_1}{r^{d-1}}(1-|Y_r|)^2|Y_r|\,|S^{d-1}|\,t^{-1}$$

or, equivalently,

$$\|f-f_\infty\|_{L^2(\mathbb{T}^d\times S^{d-1})} \geq \frac{C}{\sqrt{t}}.$$

This particular example shows that the convergence cannot be better than polynomial in the general case. Here the L^2-norm of the difference between the time-dependent solution and the corresponding stationary state decays at best like $t^{-1/2}$.

Our result can be summarized in the following theorem:

Theorem 3.3. *For all $r \in (0, 1/2)$, there exists an initial condition $f^{\text{in}} \in L^\infty(\mathbb{T}^d \times S^{d-1})$ satisfying $f^{\text{in}}(x, v) \geq 0$ for a.e. $(x, v) \in \mathbb{T}^d \times S^{d-1}$ and such that, for each cross section $\sigma \in L^\infty(\mathbb{T}^d)$ satisfying $\sigma(x) \geq 0$ for a.e. $x \in \mathbb{T}^d$ and $\sigma(x) = 0$ for a.e. $x \in Y_r$, the solution f of the Cauchy problem* (2.1)-(2.2) *satisfies*

$$\|f - f_\infty\|_{L^2(\mathbb{T}^d \times S^{d-1})} \geq \frac{C}{\sqrt{t}}$$

for each $t > r^{1-d}$, where

$$f_\infty = \frac{1}{|S^{d-1}|} \int_{\mathbb{T}^d \times S^{d-1}} f^{\text{in}}(x, v)\, dxdv$$

and C is a positive constant.

Remark 3.4. *The initial data f^{in} chosen in the proof of Theorem 3.3 is independent of the velocity variable v. Therefore, regularity in v cannot help in obtaining exponential convergence.*

Remark 3.5. *The same argument shows that one can choose $f^{\text{in}} \in C^\infty(\mathbb{T}^d)$ provided that $f^{\text{in}} = 0$ on $\mathbb{T}^d \setminus Y_r$. Thus regularity in x cannot help either in obtaining exponential decay.*

Remark 3.6. *The same result holds if the isotropic scattering model considered here is replaced with a transport equation of the form*

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \sigma(x) \left[f - \frac{1}{|S^{d-1}|} \int_{S^{d-1}} p(v, v') f(t, x, v')\, dv' \right] = 0,$$

where $p(v, v') \in L^2(S^{d-1} \times S^{d-1})$ is a scattering kernel such that

$$p(v, v') = p(v', v) \geq 0 \text{ and } \frac{1}{|S^{d-1}|} \int_{S^{d-1}} p(v, v')\, dv' = 1.$$

Remark 3.7. *The distribution of the forward exit time, induced by the geometrical properties of the scattering region, is the key ingredient in the computations.*

Hence, further hypotheses on the geometry of the scattering region are necessary in order to improve the convergence rate.

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