The Vlasov-Poisson equation with infinite mass

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The Vlasov equation is a differential equation describing the time evolution of the distribution function of a plasma consisting of charged particles with long-range (for example, Coulomb) interaction. The equation was first suggested for description of plasma by Anatoly Vlasov in 1938.

Instead of collision-based kinetic description for interaction of charged particles in plasma, Vlasov utilizes a self-consistent collective field created by the charged plasma particles. Such a description uses distribution functions $f_i(x, v, t)$ for plasma ions (i denotes the i-th species). The distribution function $f_i(x, v, t)$ for species i describes the number of particles of the species i having approximately the velocity v near the position x at time t.

The Vlasov equation is the mean field approximation ($N \to \infty$) of the time evolution of N particles of mass $\frac{1}{N}$ governed by the Newton law with a mutual interaction $\frac{1}{N^2}F(x, y)$.

Actually this limit has been proved for bounded interactions by Braun and Hepp (1977), Dobrushin(1979), Neunzert(1981), Spohn (1981), and for interactions with a singularity less than Coulomb-like (with a cut-off near the singularity) by Hauray and Jabin (2015), and Lazarovici (2016) with a combined mean field and point-particle limit, while for Coulomb interactions it is an open problem.

We do not investigate this limit, starting from a system of N particles, whereas we study the properties of the macroscopic equation.

The following system of equations was proposed for description of charged components of plasma (Vlasov-Maxwell system) in a low density regime (typical example: the solar wind):

$$\frac{d}{dt} f_i(x, v, t) =$$

$$\partial_t f_i(x, v, t) + v \cdot \nabla_x f_i(x, v, t) + q_i \Big[E(x, t) + v \wedge B(x, t) \Big] \cdot \nabla_v f_i(x, v, t) = 0$$

The Vlasov-Poisson system is an approximation of the former in the the nonrelativistic null-magnetic field limit:

$$E(x,t) = \sum_{i=1}^{n} q_i \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3}\, \rho_i(y,t)\, dy$$

$$\rho_i(x,t) = \int_{\mathbb{R}^3} f_i(x,v,t)\, dv, \quad \text{(spatial density)}$$

very reasonable for *cold* plasmas, in which particles velocities typically do not exceed $10^{-3}c$.

The Vlasov-Poisson equation has been largely studied first in two dimensions, or with some symmetry, then in three dimensions mainly assuming $f(x, v, 0) \in L^\infty \cap L^1$ and for smooth external field.

In the present talk I discuss some results on the Vlasov-Poisson equation with unbounded initial spatial density and/or singular external field, obtained in collaboration with S. Caprino (Rome 2) and C. Marchioro (Roma1).

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As well known the Vlasov-Poisson equation can be studied into two different frameworks: a Lagrangian one, following the characteristics, or an Eulerian one, by the study of the moments of the distribution.

In our papers we choose the first one and we study the characteristics of the equation, i.e. the trajectories of the "fluid particles". Let us focus on one single species i (we shall come later on more species):

$$\begin{cases} \dot{X}(t) = V(t) \\ \dot{V}(t) = E(X(t), t) + F_{ext}(X(t), V(t)) \\ (X(0), V(0)) = (x, v) \end{cases}$$

where $(X(t), V(t)) = (X(x, v, t), V(x, v, t))$ denote position and velocity of a particle starting at time $t = 0$ from (x, v).

Since f is time-invariant along this motion,

$$f(X(t), V(t), t) = f(x, v, 0)$$

it results

$$\|f(t)\|_{L^\infty} = \|f(0)\|_{L^\infty}.$$

We also remark that the measure of the phase space is conserved along the motion (Liouville Theorem).

We investigate te problem of infinite mass

(i) assuming that ρ_0 belongs to L^∞ but not to L^1, in the case of bounded velocities

(ii) removing the compactness of the velocity support for f_0.

We have considered plasmas magnetically confined in a subset of $\mathbb{R}^3$, or evolving in the whole $\mathbb{R}^3$. The reason for allowing the mass

to be infinite is not an effort for a sole mathematical generalization, but it reflects the aim to show the weak dependence of the result on the intensity of the mass of the system. Thus, in some sense, the properties of the solution do not depend on the size of the system.

Removing the L^1 assumption on ρ_0 generates at first a well-posedness problem, since in this case the electric field is in general infinite at time $t = 0$. Hence one needs to modify the set up in order to give sense to the equations.

One possible way to get a well-posed problem is to give the density ρ_0 some properties, in such a way that the electric field is finite. Actually, what we do is to assume a slow decay (not integrable) for the spatial density.

$$\begin{cases} \dot{X}(t) = V(t) \\ \dot{V}(t) = E(X(t),t) + V(t) \wedge B(X(t)) \\ (X(0), V(0)) = (x, v) \end{cases}$$

The plasma is initially arranged in a cylinder D with x_1 as symmetry axis. B is an external magnetic field, directed along x_1 and divergent on the border of D.

Theorem

Let $f_0 \in L^\infty$ have spatial support in a cylinder $D_0 \subset D$ and compact support in the velocities. Moreover assume that

$$\int_{|i-x_1|\leq 1} \rho_0(x)\, dx \leq \frac{C}{|i|^\alpha} \qquad i \in \mathbb{Z} \setminus \{0\} \tag{1}$$

with $\alpha > 0$ arbitrary. Then, there exists a unique solution to the characteristics equations, global in time, which remains confined in D, and satisfying (1) also at time t.

While in the whole $\mathbb{R}^3$ the result is

Theorem

Let $f_0 \in L^\infty$ have compact support in the velocities and satisfy

$$\int_{|i-x|\leq 1} \rho_0(x)\, dx \leq \frac{C}{|i|^\alpha} \qquad \forall i \in \mathbb{Z}^3 \setminus \{(0,0,0)\} \tag{2}$$

with $\alpha > 2$. Then, there exists a unique solution to the characteristics equations, global in time, satisfying (2) also at time t.

The central problem is to prove the boundedness of the velocity of the plasma particles. Unfortunately to prove this we cannot rely on the energy, since it is infinite.
The local energy is a sort of energy of a bounded region interacting with the rest of the plasma.
Given a ball in $\mathbb{R}^3$ with center μ and radius R, it is defined as

$$\mathcal{E}(\mu, R, t) = \frac{1}{2} \int dx \, \varphi^{\mu,R}(x) \int dv \, |v|^2 f(x, v, t) +$$

$$\frac{1}{2} \int dx \, \varphi^{\mu,R}(x) \, \rho(x, t) \int dy \, \frac{\rho(y, t)}{|x - y|}$$

where φ is the (smooth) cutoff function

$$\varphi^{\mu,R}(x) = \varphi\left(\frac{|x - \mu|}{R}\right)$$

$$\varphi(r) = 1 \text{ if } r \in [0, 1], \quad \varphi(r) = 0 \text{ if } r \in [2, +\infty], \quad -2 \leq \varphi'(r) \leq 0.$$

Our assumptions on the initial data imply that E and $\mathcal{E}$ are finite at time zero and moreover

$$\sup_{\mu \in \mathbb{R}^3} \mathcal{E}(\mu, R, 0) \leq C R^{\beta} \qquad (*)$$

with some β less than the "dimension" of the domain (i.e., $\beta \leq 1$ in the cylinder, $\beta \leq 3$ in $\mathbb{R}^3$).

We look at a test particle and observe that the interaction with far away particles is negligible, whereas the largest contribution to the growth of $E(x, t)$ comes from particles in a ball around x, with radius $R(t)$, which is the maximal displacement of a plasma particle in the time interval $[0, t]$, covered with the maximal velocity $\mathcal{V}(t)$.

We can control the mass of these particles by means of the local energy, proving that the estimate $(*)$ still holds at time t. We prove that

$$|E(x,t)| \leq C\mathcal{V}(t)^{\gamma} \qquad \text{with} \quad \gamma < 1$$

which implies the boundedness of the particle velocities:

$$\mathcal{V}(t) \leq \mathcal{V}(0) + C \int_0^t \mathcal{V}(s)^{\gamma}\, ds$$

Removing the compactness of the velocity support of the density, we can prove (in the cylinder):

Theorem

Let f_0 satisfy the following assumptions:

$$0 \le f_0(x, v) \le Ce^{-\lambda v^2},$$

ρ_0 is supported in $D_0 \subset D$ and

$$\int_{i \le x_1 \le i+1} \rho_0(x)\, dx \le \frac{C}{|i|^\alpha} \qquad \alpha > \frac{5}{9}, \quad i \in \mathbb{Z} \setminus \{0\}$$

then there exists a unique global in time solution, gaussian in the velocities, confined in the cylinder and satisfying

$$\int_{i \le x_1 \le i+1} \rho(x, t)\, dx \le C \frac{\log(|i| + 1)}{|i|^\alpha} \qquad \alpha > \frac{5}{9}, \quad i \in \mathbb{Z} \setminus \{0\}.$$

A similar result holds for the whole $\mathbb{R}^3$.

We give an idea of the proof in the case of the cylinder.

We introduce a cutoff dynamics, by considering initial data

$$f_0^N(x, v) = f_0(x, v)\chi(|v| \leq N).$$

Hence, by previous results, the solution does exist and it is unique, and our aim is to remove the cutoff.

We take the difference in the two dynamics:

$$\left|X^N(t) - X^{N+1}(t)\right| \leq$$
$$\int_0^t dt_1 \int_0^{t_1} dt_2 \Big\{ \left|E^N(X^N(t),t) - E^N(X^{N+1}(t),t)\right|$$
$$+ \left|E^N(X^{N+1}(t),t) - E^{N+1}(X^{N+1}(t),t)\right|$$
$$+ \left|V^N(t) \wedge B(X^N(t)) - V^{N+1}(t) \wedge B(X^{N+1}(t))\right| \Big\}$$

and we want to prove that $\left|X^N(t) - X^{N+1}(t)\right|$ goes to zero as $N \to \infty$ summably, so to prove existence and uniqueness.

We fix arbitrarily a time $T > 0$. Main result

$$\forall t \in [0, T] \qquad \int_0^t E^N(x,s)\,ds \le C\,N^{\gamma} \qquad \gamma < \frac{2}{3}$$

Such finer estimate ($\gamma < \frac{2}{3}$) is done on the *time integral* of the electric field (with respect to the pointwise estimate with $\gamma < 1$). It relies on a *bootstrap method*: starting with the *a priori* estimate

$$|E^N(x,t)| \le C\left[\mathcal{V}^N(t)\right]^{\frac{4}{3}}\left[\sup_{\mu\in\mathbb{R}^3}\mathcal{E}(\mu, R^N(t), t)\right]^{\frac{1}{3}}$$

we obtain better on the time average of the electric field over a suitable (small) time interval Δ_1

$$\frac{1}{\Delta_1}\int_t^{t+\Delta_1} |E^N(x,s)|ds$$

and so on, enlarging at each step the time interval by a constant factor g: $\Delta_n = g\Delta_{n-1}$, until we arrive at the final estimate

$$\frac{1}{\Delta_{\bar{n}}}\int_t^{t+\Delta_{\bar{n}}} |E^N(x,s)|ds \leq C\left[\mathcal{V}^N(t)\right]^\gamma$$

(there are technical reasons for which we cannot proceed indefinitely).

Consequences

$$\mathcal{V}^N(t) \leq C\,N \qquad \text{and} \qquad \|\rho^N(t)\|_{L^\infty} \leq C\,N^{3\gamma}.$$

Put $\left|X^N(t) - X^{N+1}(t)\right| = \delta^N(t)$, then

$$\left|E^N(X^N(t),t) - E^N(X^{N+1}(t),t)\right| \leq C\,N^{3\gamma}\delta^N(t)\left(|\log\delta^N(t)| + 1\right)$$

which is a quasi-lipschitz property for E^N.

The second term $\left|E^N(X^{N+1}(t),t) - E^{N+1}(X^{N+1}(t),t)\right|$ can be bounded analogously, and we omit the third term with B (which complicates unessentially the iterative scheme).

We use an elementary bound, valid for any $0 < a < 1$ and $0 < \varepsilon < 1$,

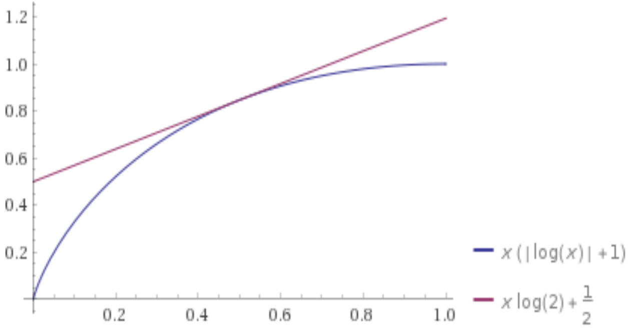
$$a(1 + |\log a|) \leq a|\log \varepsilon| + \varepsilon$$

hence

$$\delta^N(t)\left(1 + |\log \delta^N(t)|\right) \leq \delta^N(t)|\log \varepsilon| + \varepsilon$$

and choosing $\varepsilon = e^{-\lambda N^2}$, we get

$$\delta^N(t)\left(1 + |\log \delta^N(t)|\right) \leq \lambda N^2 \delta^N(t) + e^{-\lambda N^2}$$


1.2
1.0
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x (| log(x) | + 1)
x log(2) + 1/2

and finally

$$\delta^N(t) \le C \int_0^t dt_1 \int_0^{t_1} dt_2 \Big[N^{3\gamma+2} \delta^N(t_2) + e^{-\frac{\lambda}{2} N^2} \Big].$$

We iterate in time, using the same inequality for $\delta^N(t_2)$ and so on, after k steps we stop using for $\delta^N(t)$ the a priori bound $\delta^N(t) \le C\,N$, obtaining

$$\delta^N(t) \le C\, e^{-\frac{\lambda}{2} N^2} N \sum_{i=0}^{k-1} (C\,N^{3\gamma+2})^i \frac{t^{2i}}{(2i)!} + N(C\,N^{3\gamma+2})^k \frac{t^{2k}}{(2k)!}$$

$$\le C\, e^{-\frac{\lambda}{2} N^2} N\, e^{CN^{\frac{3\gamma+2}{2}}} + \text{exponentially small term}$$

provided that k has been chosen sufficiently large depending on N. Since $\gamma < \frac{2}{3}$,

$$\lim_{N\to\infty} \delta^N(t) = 0.$$

We focus now on the treatment of the external field, which plays the role of forbidding the entrance of the plasma in a certain region of the physical space.

We consider the simple example of a particle of unit mass and charge, moving in the half-space $x_1 > 0$ under the action of a bounded electric field $E(x,t)$ and a magnetic field $B(x) = (0, 0, b(x_1))$, where $b(x_1)$ is a smooth function, diverging (with its primitive) for $x_1 = 0$.
The equation of motion is

$$\dot{V}(t) = E(X(t), t) + V(t) \wedge B(X(t))$$

and its second component gives

$$\dot{V}_2(t) = E_2(X(t), t) - V_1(t) b(X_1(t)) = E_2(X(t), t) - \dot{h}(X_1(t))$$

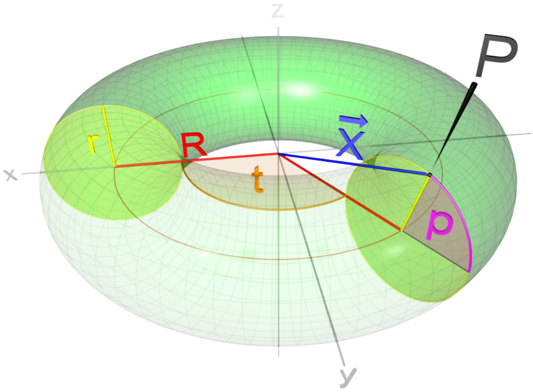
where h is a primitive of b. Integrating in time, we get

$$h(X_1(t)) - h(X_1(0)) = -[V_2(t) - V_2(0)] + \int_0^t ds\, E_2(X(s), s)$$

and the r.h.s. is always bounded during the motion, hence $X_1(t)$ cannot reach $x_1 = 0$ (where h diverges).

In case of a plasma the problem is much more difficult, in fact the plasma, especially in case of infinite mass, could push a fluid element toward the barrier $x_1 = 0$.

We have faced the problem of how to forbid the entrance of the plasma inside a torus



We adopt toroidal coordinates

$$\begin{cases} x_1 = (R + r\cos\alpha)\cos\theta \\ x_2 = (R + r\cos\alpha)\sin\theta \\ x_3 = r\sin\alpha \end{cases}$$

$0 \leq \alpha < 2\pi, \qquad 0 \leq \theta < 2\pi$, and we choose the external magnetic field of the form

$$B(x) = \nabla \wedge A(x), \qquad A(x) = \frac{a(r)}{R + r\cos\alpha}\hat{e}_\theta$$

where $a(r)$ is a smooth function singular for $r \to r_0$ (border of the torus).

For this model, we have the following result, using the techniques previously sketched

Theorem

Let f_0 satisfy the following assumptions:

$$0 \leq f_0(x, v) \leq C e^{-\lambda |v|^q} g(|x|), \qquad \frac{18}{7} < q < 3$$

g is supported outside the torus and

$$g(|x|) \leq \frac{C}{|x|^\alpha}, \qquad \frac{8}{3} < \alpha \leq 3$$

then there exists a unique global in time solution, supported outside the torus and with the same properties of the initial datum f_0.

We turn now to the case in which the plasma is constituted by more species, with different charge signs in $\mathbb{R}^3$.
Such case is straightforward only in the case of an initial distribution with compact support, or anyway with finite total mass.
In case of infinite mass the above technique which makes use of the energy of a bounded region, suitably smoothened by a mollifier function, does not work any more.

Theorem

Let us fix an arbitrary positive time T. For any $i = 1 \dots n$, let $f_{i,0}$ satisfy the following hypotheses:

$$0 \leq f_{i,0}(x,v) \leq C\, e^{-\lambda|v|^2} \frac{1}{(1+|x|)^\alpha} \tag{3}$$

with $\alpha > 1$, and λ, C, positive constants. Then there exists a unique global in time solution, gaussian in the velocities, satisfying (3) also at time t.

We make use here also of a truncated partial dynamics, in the form

$$f_{i,0}^{N}(x,v) = f_{i,0}(x,v)\chi_{\{|x|\leq N^{\beta}\}}(x)\chi_{\{|v|\leq N\}}(v) \tag{4}$$

where we introduce a velocity cutoff, N, and a spatial cutoff, N^{β} (with $\beta > 0$ to be fixed suitably), and we want to investigate the limit $N \to \infty$.

What fails to be true, with respect to the previous cases of one species (or more species having the same charge signs), is the following Proposition:

$$\sup_{\mu\in\mathbb{R}^3} \mathcal{E}^N(\mu, R^N(t), t) \leq C \sup_{\mu\in\mathbb{R}^3} \mathcal{E}^N(\mu, R^N(t), 0)$$

$$R^N(t) = 1 + \int_0^t \mathcal{V}^N(s)ds$$

We overcome (the lack of) this technical tool by using the energy conservation. For coulomb interaction we use also the well known relation:

$$\mathcal{E}^N(0) = \mathcal{E}^N(t) = \frac{1}{2}\int dx \int dv\, |v|^2 f^N(x,v,t) + \frac{1}{2}\int dx\, |E^N(x,t)|^2$$

in spite of the (*a priori*) ambiguity in sign of the potential energy, when written as

$$\frac{1}{2}\int dx \int dy\, \rho^N(x,t)\, \rho^N(y,t) \frac{1}{|x-y|},$$

$$f^N(x,v,t) = \sum_{i=1}^{n} f_i^N(x,v,t) \qquad \text{and} \qquad \rho^N(x,t) = \sum_{i=1}^{n} q_i\, \rho_i^N(x,t).$$

Hence what we are able to prove is a bound of the type

$$\mathcal{E}^N(t) = \mathcal{E}^N(0) \leq CN^{3\beta}$$

by exploiting the properties of the initial data, and using energy conservation.
Here β is the exponent of the spatial cutoff introduced previously ($|x| \leq N^\beta$). The parameter β is now chosen in order that the iterative method does work, performing so the limit $N \to \infty$.

Proof of $\mathcal{E}^N(t) \leq CN^{3\beta}$.

First of all we have

$$\begin{aligned} \mathcal{E}^N(t) =& \frac{1}{2}\int dx \int dv\, |v|^2 f^N(x,v,t) + \frac{1}{2}\int dx\, |E^N(x,t)|^2 \\ =& \mathcal{E}^N(0) \leq CN^{\beta(3-\alpha)} + \frac{1}{2}\int dx\, |E^N(x,0)|^2 \end{aligned} \tag{5}$$

as we obtain by using in the kinetic energy the decreasing property of the initial density and the spatial cutoff N^β.

The potential energy in $\mathcal{E}^N(0)$ is bounded by

$$\int_{x\in\mathbb{R}^3} dx\,|E^N(x,0)|^2 \le \int_{x\in\mathbb{R}^3} dx \left[\int_{|y|\le N^\beta} dy\,\frac{C}{(1+|y|)^\alpha}\frac{1}{|x-y|^2}\right]^2, \tag{6}$$

and

$$\begin{aligned} &\int_{|y|\le N^\beta} dy\,\frac{C}{(1+|y|)^\alpha}\frac{1}{|x-y|^2} \le \\ &\frac{2}{|x|^2}\,\chi_{\{|x|\ge 2N^\beta\}}(x)\int_{|y|\le N^\beta} dy\,\frac{C}{(1+|y|)^\alpha} \\ &+\chi_{\{|x|\le 2N^\beta\}}(x)\int_{|y|\le N^\beta} dy\,\frac{C}{(1+|y|)^\alpha}\frac{1}{|x-y|^2}. \end{aligned} \tag{7}$$

The first term on the right hand side of (7) is bounded by

$$\frac{C}{|x|^2} N^{\beta(3-\alpha)} \chi_{\{|x|\geq 2N^\beta\}}(x),$$

while for the second one we proceed as follows,

$$\int_{|y|\leq N^\beta} dy \, \frac{C}{(1+|y|)^\alpha} \frac{1}{|x-y|^2} \leq \int_{\mathbb{R}^3} dy \, \frac{C}{(1+|y|)^\alpha} \frac{1}{|x-y|^2} \leq$$
$$\int_{|x-y|\leq 1} dy \, \frac{C}{(1+|y|)^\alpha} \frac{1}{|x-y|^2} + \int_{|x-y|>1} dy \, \frac{C}{(1+|y|)^\alpha} \frac{1}{|x-y|^2}$$

and, by Hölder inequality,

$$\int_{|x-y|\leq 1} dy\, \frac{C}{(1+|y|)^\alpha}\frac{1}{|x-y|^2} \leq$$

$$\left[\int_{|x-y|\leq 1} dy \left(\frac{C}{(1+|y|)^\alpha}\right)^p\right]^{1/p} \left[\int_{|x-y|\leq 1} dy \left(\frac{1}{|x-y|^2}\right)^q\right]^{1/q} \leq$$

const

by choosing $q < \frac{2}{3}$ and $p > 3$

whereas, using again Hölder inequality with different exponents,

$$\int_{|x-y|>1} dy \, \frac{C}{(1+|y|)^\alpha} \frac{1}{|x-y|^2} \leq$$

$$\left[\int_{|x-y|>1} dy \left(\frac{C}{(1+|y|)^\alpha}\right)^p\right]^{1/p} \left[\int_{|x-y|>1} dy \left(\frac{1}{|x-y|^2}\right)^q\right]^{1/q} \leq$$

$$\left[\int_{\mathbb{R}^3} dy \left(\frac{C}{(1+|y|)^\alpha}\right)^p\right]^{1/p} \left[\int_{|x-y|>1} dy \left(\frac{1}{|x-y|^2}\right)^q\right]^{1/q} \leq const$$

by choosing $q > \frac{3}{2}$ and $p < 3$, but in such a way that $\alpha p > 3$ (that is possible since $\alpha > 1$).

Coming back to (6) we have obtained

$$|E^N(x,0)| \le \frac{C}{|x|^2} N^{\beta(3-\alpha)} \, \chi_{\{|x|\ge 2N^\beta\}}(x) + C\, \chi_{\{|x|\le 2N^\beta\}}(x),$$

and for the corresponding integral of $|E^N(x,0)|^2$

$$\int_{x\in\mathbb{R}^3} dx\, |E^N(x,0)|^2 \le CN^{2\beta(3-\alpha)} \frac{1}{N^{\beta(1-\nu)}} \int_{\mathbb{R}^3} dx\, \frac{1}{|x|^{3+\nu}} + CN^{3\beta} \le$$
$$CN^{5\beta-2\beta\alpha+\beta\nu} + CN^{3\beta} \le CN^{3\beta},$$

taking $0 < \nu < \min\{1, 2\alpha - 2\}$. Inserting the last estimate in (5) we obtain the bound.