ASYMPTOTIC BEHAVIOUR FOR LOGARITHMIC DIFFUSION

F. SALVARANI

ABSTRACT. In this paper we prove, via the entropy dissipation method, that the solutions of the d-dimensional logarithmic diffusion equation, with non-homogeneous Dirichlet boundary data, decay exponentially in time towards its own steady state. The result is valid not only in L^1-norm (as customary when applying entropy dissipation methods), but also in any L^p-norm with $p \in [1, +\infty)$.

Keywords: asymptotic behaviour, entropy dissipation, logarithmic diffusion equation.

Mathematics Subject Classification: 35B40, 35K55.

1. INTRODUCTION

In this paper, we suggest a very compact strategy to study the long-time asymptotics of the following initial-boundary value problem

$$u_t(t,x) = \Delta \log u(t,x) \qquad (t,x) \in (0,+\infty) \times \Omega \tag{1}$$

$$u(t,x)|_{\partial\Omega} = \varphi(x) \tag{2}$$

$$u(0,x) = u_0(x) \tag{3}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded and connected open set, with boundary $\partial\Omega$ of class C^1, and $d \in \mathbb{N}$.

The initial data u_0 and the boundary conditions φ are non-negative real functions which satisfy the prescriptions of the following definition:

Definition 1.1. *The initial and boundary conditions of Problem* (1)-(3) *are said to be admissible if*

(1) $\varphi = \varphi(x)$ *is a continuous, strictly positive function of class* $W^{1,\infty}(\partial\Omega)$*;*
(2) $u_0(x) \in H^1(\Omega) \cap L^\infty(\Omega)$ *and is non-negative a.e..*

Equation (1) has many concrete applications: in particular, it arises in the dynamics of thin liquid films (see, for example, [5] and [26]).

Moreover, this equation is popular in the mathematical literature because of its connection with Ricci flow for surfaces [18].

Logarithmic diffusion equations are challenging from a mathematical point of view, and many authors have studied different kinds of problems for Equation (1), mainly in the context of the Cauchy problem or the initial-boundary value problems with homogeneous boundary conditions.

As a result of this collective research, many features of Equation (1) are now well understood. In particular, we point out the papers [2, 3, 4, 10, 11, 15, 16, 20, 23, 25] that consider many questions about existence, uniqueness, regularity, Harnack inequalities and finite blow-down.

For further results, we refer to the books of Vázquez [22, 24], that give the state-of-the-art, up to the time of their writing, on nonlinear diffusion equations of porous media type.

Our result, whose precise statement is given in Theorem 4.1, provides an explicit exponential rate of convergence towards the stationary solution of Problem (1)-(2) by using the *entropy dissipation method*. This is a powerful strategy which has been successfully employed to obtain explicit decay rates towards equilibrium of weak solutions to Cauchy problems for dissipative or hypocoercive equations and systems [1, 6, 7, 8, 9, 12, 13, 14].

Basically, the key point of the method is the choice of a Lyapunov functional for the problem, sometimes called *entropy*. Once proved that this (convex) functional is monotone decreasing in time (this property justifies the name given to the functional, on the analogy of the physical entropy), if some norm of the difference between the solution and the stationary state is controlled by the entropy, the method allows to deduce that the solutions decay in time towards equilibrium with explicit rate.

The entropy dissipation method has been mainly applied to study Cauchy problems or initial-boundary value problems with homogeneous (or periodic) boundary conditions. The use of such a procedure in the framework of initial-boundary value problems with non-homogeneous or non-periodic boundary conditions is less common.

Indeed, the presence of non-vanishing boundary conditions makes difficult to obtain convex functionals of the solution which are decreasing in time, since the source terms do not disappear easily when performing integrations by parts. Moreover, all the estimates based on mass conservation are no more valid.

We point out that a method suitable to handle non-homogeneous Dirichlet boundary conditions for the nonlinear diffusion $u_t = (u^m)_{xx}$,

$m > 0$, has been developed in [21]. The strategy of that paper – which does not cover the case of logarithmic diffusion and is limited to the 1-dimensional case – consists in considering convex functionals based not only on the solution of the equation, but involving also the stationary solution of the same problem. Then, by means of Poincaré-type inequalities, the decay of the entropy allows to obtain exponential decay for the L^1-estimate of the difference between the solution of the problem and its stationary state.

Our result provides an exponential rate which, though explicit, is not optimal: it depends indeed on the constant of a Poincaré inequality on domains and on the maximum and minimum of the initial and boundary data. This feature is not surprising, since the entropy production method usually is not enough, by itself, to prove the optimality of the decay rate.

The main novelty of our proof with respect to previous results is the use of the entropy production method also for a d-dimensional problem, with $d \geq 1$, without knowing the explicit form of the stationary state and in the presence of non-homogeneous boundary data. Moreover, we show that the method can be extended to prove exponential convergence for any L^p-norm, with $p \in [1, +\infty)$.

The structure of this article is the following: some basic results for Problem (1)-(3) are recalled in Section 2.

Section 3 introduces and studies an entropy functional of the problem: this is the fundamental tool used for proving exponential decay in time.

Finally, in Section 4, we obtain an explicit rate of convergence of the solution towards the steady state.

2. Preliminary results on the problem

Many interesting features of the non-homogeneous Dirichlet problem (1)-(3), with strictly positive boundary data, are well known.

We only recall here that the precise characterization of a weak solution to Problem (1)-(3) is given by the following definition:

Definition 2.1. *A function $u = u(t, x)$ defined on $[0, T] \times \bar{\Omega}$ is an admissible weak solution of the Dirichlet problem* (1)-(3) *with admissible initial and boundary conditions if and only if:*

(1) *u is real, non-negative and of class $L^2((0, T) \times \Omega)$;*
(2) *$u|_{\partial\Omega} = \varphi$ for all $t \in (0, T]$, where φ satisfies the properties prescribed by Definition* 1.1*;*
(3) *$\nabla \log u \in L^2(((0, T) \times \Omega)^n)$;*

(4) *u satisfies the identity*

$$\int_0^T \int_\Omega \left[u(t,x)\frac{\partial \phi}{\partial t} - \nabla \log(u(t,x)) \cdot \nabla \phi \right] dxdt + \int_\Omega u_0(x)\phi(0,x)dxdt = 0,$$

for all $\phi \in C([0,T] \times \bar{\Omega}) \cap H^1((0,T) \times \Omega)$ which vanish in $\partial\Omega$ and for $t = T$, provided that u_0 satisfies the properties required by Definition 1.1.

We note that, under the assumptions of the data prescribed by Definition 1.1, any degeneracy of the solution due to the singular character of the logarithmic diffusion is avoided: the problem is therefore standard parabolic.

The following theorem is classical:

Theorem 2.1. *Let us consider the initial-boundary value problem* (1)-(3) *with admissible (in the sense of Definition* 1.1*) non-negative initial data $u(0,x) = u_0(x)$ and boundary conditions $u(t,x)|_{\partial\Omega} = \varphi(x)$. Then Problem* (1)-(3) *admits a non-negative weak solution $u = u(t,x)$ of class $C([0,T]; H^1(\Omega))$.*

Moreover, let

$$N = \max\{\sup_{x\in\Omega} u_0(x)\,,\ \sup_{x\in\partial\Omega} \varphi(x)\}.$$

Then, $u \le N$ a.e. in $(0,T) \times \Omega$.

Indeed, for admissible initial data bounded from below by a strictly positive constant, the existence and the uniqueness of the solution follow from the standard theory of non-degenerate quasilinear parabolic equations; see for example [19].

If the initial condition vanishes somewhere in Ω a lifting argument can be used in order to prove the theorem.

We recall moreover that a non-standard approach to logarithmic diffusion, which permits to obtain the existence of solutions to the Dirichlet problem (1)-(3) as by-product of an asymptotic limit procedure is described in [17], but only in one space dimension.

In what follows, it will be useful to known that the stationary state of Problem (1)-(2) enjoys all the nice properties of the solution of the non-homogeneous Dirichlet problem for the Laplace equation. We have indeed the following result:

Lemma 2.1. *Problem* (1)-(2) *admits one and only one stationary solution $\bar{u} \in H^1(\Omega)$. Moreover, $\bar{u}$ satisfies the maximum principle, i.e. $m \le \bar{u} \le M$ a.e. in Ω, where*

$$m = \inf_{x\in\partial\Omega} \varphi(x) \quad \text{and} \quad M = \sup_{x\in\partial\Omega} \varphi(x)$$

respectively.

Proof. We only note that $\bar{u} = e^v$, where v solves, for $x \in \Omega$, the Laplace equation:

$$\Delta v = 0$$
$$v|_{\partial\Omega} = \log \varphi.$$

The thesis is hence immediate. □

3. The entropy functional

In this section we define the main tool of the proof: a functional which, on the analogy of the monotone functionals of kinetic theory, is customary called *entropy*.

As explained in the introduction, the presence of non-homogeneous or non periodic boundary conditions does not allow the mass conservation and could introduce surface terms in the integrations by part, which are not easy to treat.

A way to overcome this problem is to define an auxiliary function $f = u/\bar{u}$. Thanks to the positive lower bound imposed to the boundary condition φ, we deduce that $\bar{u}(x) > 0$ for almost every $x \in \Omega$. Hence f is well defined and non-negative.

It is straightforward to prove that the function f satisfies the following initial-boundary value problem:

$$\bar{u}\, f_t = \Delta \log f \tag{4}$$

$$f(t,x)|_{\partial\Omega} = 1, \tag{5}$$

$$f(0,x) = f_0(x) = u_0(x)/\bar{u}(x) \tag{6}$$

in Ω. We then consider the convex function Φ_n given by

$$\Phi_n(f) = [(n+3)f(f-1)^{n+2} - 2(f-1)^{n+3}] = (f-1)^{n+2}[(n+1)f+2], \tag{7}$$

where $n \geq 0$ and even. It is easy to deduce the first and second derivative of Φ_n: they are given by

$$\Phi_n'(f) = (n+3)[(n+2)f(f-1)^{n+1} - (f-1)^{n+2}]$$

and

$$\Phi_n''(f) = (n+1)(n+2)(n+3)f(f-1)^n.$$

Note that Φ_n is convex and non-negative definite for $f \geq 0$. Moreover $\Phi_n(1) = \Phi_n'(1) = 0$.

In this section, in order to make the computations more readable, we will assume that the solutions are smooth so that the different computations are valid. This assumption is then eliminated by approximation, which is justified according to known theory (see, for example, [4, 6]).

The following lemma characterises the time evolution of the entropy of the system:

Lemma 3.1. *Let u_0 and φ be admissible initial-boundary conditions. Then the functional*

$$H_n(f) = \int_\Omega \bar{u}\Phi_n(f)\,dx \tag{8}$$

with $\Phi_n(f)$ given by (7)*, is an entropy for Problem* (4)-(6)*, that is*

$$\frac{dH_n(f)}{dt} = -I_n(f), \tag{9}$$

where the (nonnegative) entropy production $I_n(f)$ is

$$I_n(f) = (n+1)(n+2)(n+3)\int_\Omega (f-1)^n |\nabla f|^2\,dx. \tag{10}$$

Proof. We multiply both sides of Equation (4) by $\Phi_n'(f)$. Integrating over Ω with respect to x, we obtain

$$\int_\Omega \bar{u}\Phi_n'(f)\frac{\partial f}{\partial t}\,dx = \int_\Omega \Phi_n'(f)\Delta \log f\,dx.$$

We integrate now by parts the right-hand side:

$$\int_\Omega \bar{u}\Phi_n'(f)\frac{\partial f}{\partial t}\,dx = \int_{\partial\Omega} \Phi_n'(f)\nabla \log f\;\cdot \nu\,dS - \int_\Omega \Phi_n''(f)\nabla f\cdot\nabla \log f\,dx,$$

where ν is the outward pointing unit normal vector field along $\partial\Omega$.

Thanks to the boundary conditions (5) and to the property $\Phi_n'(1) = 0$ we obtain

$$\frac{d}{dt}\int_\Omega \bar{u}\Phi_n(f)dx = -\int_\Omega \Phi_n''(f)\nabla f\cdot\nabla \log f\,dx.$$

Since ${\Phi_n}''(f) = (n+1)(n+2)(n+3)f(f-1)^n$, we deduce that

$$\frac{d}{dt}\int_\Omega \bar{u}\Phi_n(f)dx = -(n+1)(n+2)(n+3)\int_\Omega (f-1)^n|\nabla f|^2\,dx.$$

Hence the lemma is fully proven. □

4. Convergence towards equilibrium

This section is devoted to the proof of the exponential convergence in time towards equilibrium for solutions of the non-homogeneous Dirichlet problem (1)-(3).

The previous results are the basis of the following theorem:

Theorem 4.1. *Let $u \in C([0,T]; H^1(\Omega))$ be the solution of Problem* (1)-(3)*, with admissible initial and boundary conditions u_0 and φ respectively. Then u decays exponentially fast towards the stationary solution $\bar{u}$ in L^p-norm, for any $p \in [1, +\infty)$. Moreover the following bounds hold:*

$$\|u - \bar{u}\|_{L^p(\Omega)} \leq X(u_0, \varphi, p, \Omega) \exp\left(-\frac{C_{d,\Omega} m(p^2-1)}{Mp^2[(p-1)N + 2m]} t\right),$$

where

$$X(u_0, \varphi, p, \Omega) = M^{(p-1)/p} \left(\frac{1}{2} \int_\Omega \bar{u} \left(\frac{u_0}{\bar{u}} - 1\right)^p \left[(p-1)\frac{u_0}{\bar{u}} + 2\right] dx\right)^{1/p},$$

when p is an even natural number and

$$\|u - \bar{u}\|_{L^p(\Omega)} \leq \operatorname{meas}(\Omega)^{(r-p)/rp} \|u - \bar{u}\|_{L^r(\Omega)}$$

otherwise, where r is any even natural number greater than p.

Proof. As a first step in investigating the entropy decay rate, we note that a lower bound to the entropy production functional $I_n(f)$ can be easily obtained by means of the Poincaré inequality. Indeed, if f belongs to $H^1(\Omega)$, there exists a strictly positive constant $C_{d,\Omega}$, which does not depend on x nor on f, such that

$$\int_\Omega (f-1)^n |\nabla f|^2 \, dx \geq \frac{C_{d,\Omega}}{(n+2)^2} \int_\Omega (f-1)^{n+2} \, dx.$$

In order to obtain a proof of the exponential convergence of the solution $u(t,x)$ to Problem (1)-(3) towards its stationary state $\bar{u}(x)$, it remains to show that the relative entropy is bounded from above an from below by the entropy production, modulo multiplication by a positive constant.

Thanks to the maximum principle, we deduce easily that

$$\int_\Omega \Phi_n(f) \bar{u} \, dx \leq \left[(n+1)\frac{N}{m} + 2\right] \int_\Omega (f-1)^{n+2} \bar{u} \, dx.$$

On the other hand, by using the elementary inequality

$$(f-1)^{n+2}[(n+1)f + 2] \geq 2(f-1)^{n+2},$$

valid for any $f \geq 0$ and $n \geq -1$, we obtain that

$$\int_\Omega \Phi_n(f)\bar{u}\,dx \geq 2\int_\Omega (f-1)^{n+2}\bar{u}\,dx.$$

As a consequence of the previous bounds for the entropy, we deduce from Equation (9) that H_n satisfies the differential inequality

$$\frac{dH_n}{dt} \leq -C_{d,\Omega}\frac{m(n+1)(n+3)}{M(n+2)[(n+1)N+2m]}H_n(t),$$

hence, for all $n \geq 0$ and even,

$$\|u-\bar{u}\|_{n+2} \leq M^{(n+1)/(n+2)}\left(\frac{1}{2}\int_\Omega \bar{u}\Phi_n(u_0)\,dx\right)^{1/(n+2)} e^{-\gamma t},$$

where

$$\gamma = \gamma(u_0,\varphi,n,\Omega) = \frac{C_{d,\Omega}(n+1)(n+3)m}{M(n+2)^2[(n+1)N+2m]}.$$

For non-even values of $p \in [1,+\infty)$, we use the standard interpolation inequality

$$\|g\|_{L^p(\Omega)} \leq \text{meas}\,(\Omega)^{(r-p)/rp}\|g\|_{L^r(\Omega)},$$

where r is any even natural number greater than p.

The previous computations imply the thesis of the theorem. □

Acknowlegdments: The author is grateful to the referee for his/her suggestions and comments concerning the paper.

References

[1] A. Arnold, P. Markowich, G. Toscani, and A. Unterreiter. On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. *Comm. Partial Differential Equations*, 26(1-2):43–100, 2001.

[2] J. G. Berryman and C. J. Holland. Asymptotic behavior of the nonlinear diffusion equation $n_t = (n^{-1}n_x)_x$. *J. Math. Phys.*, 23(6):983–987, 1982.

[3] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, and J. L. Vázquez. Asymptotics of the fast diffusion equation via entropy estimates. *Arch. Rational Mech. Anal.*, 191:347–385, 2009.

[4] M. Bonforte and J. L. Vázquez. Positivity, local smoothing, and harnack inequalities for very fast diffusion equations. *Preprint*, 2009.

[5] J. P. Burelbach, S. G. Bankoff, and S. H. Davis. Nonlinear stability of evaporating/condensating liquid films. *J. Fluid Mech.*, 195:463–494, 1988.

[6] J. A. Carrillo, A. Jüngel, P. A. Markowich, G. Toscani, and A. Unterreiter. Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities. *Monatsh. Math.*, 133(1):1–82, 2001.

[7] J. A. Carrillo and G. Toscani. Asymptotic L^1-decay of solutions of the porous medium equation to self-similarity. *Indiana Univ. Math. J.*, 49(1):113–142, 2000.

[8] J. A. Carrillo and G. Toscani. Long-time asymptotics for strong solutions of the thin film equation. *Comm. Math. Phys.*, 225(3):551–571, 2002.
[9] J. A. Carrillo and J. L. Vázquez. Fine asymptotics for fast diffusion equations. *Comm. Partial Differential Equations*, 28(5-6):1023–1056, 2003.
[10] P. Daskalopoulos and M. del Pino. On the Cauchy problem for $u_t = \Delta \log u$ in higher dimensions. *Math. Ann.*, 313(2):189–206, 1999.
[11] S. H. Davis, E. DiBenedetto, and D. J. Diller. Some a priori estimates for a singular evolution equation arising in thin-film dynamics. *SIAM J. Math. Anal.*, 27(3):638–660, 1996.
[12] M. Del Pino and J. Dolbeault. Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *J. Math. Pures Appl.*, 81(9):847–875, 2002.
[13] J. Denzler and R. J. McCann. Fast diffusion to self-similarity: complete spectrum, long-time asymptotics, and numerology. *Arch. Ration. Mech. Anal.*, 175(3):301–342, 2005.
[14] L. Desvillettes. Hypocoercivity: the example of linear transport. In *Recent trends in partial differential equations*, volume 409 of *Contemp. Math.*, pages 33–53. Amer. Math. Soc., Providence, RI, 2006.
[15] E. DiBenedetto and D. J. Diller. About a singular parabolic equation arising in thin film dynamics and in the Ricci flow for complete $\mathbf{R}^2$. In *Partial differential equations and applications*, volume 177 of *Lecture Notes in Pure and Appl. Math.*, pages 103–119. Dekker, New York, 1996.
[16] J. R. Esteban, A. Rodríguez, and J. L. Vázquez. A nonlinear heat equation with singular diffusivity. *Comm. Partial Differential Equations*, 13(8):985–1039, 1988.
[17] F. Golse and F. Salvarani. The nonlinear diffusion limit for generalized Carleman models: the initial-boundary value problem. *Nonlinearity*, 20(4):927–942, 2007.
[18] R. S. Hamilton. The Ricci flow on surfaces. In *Mathematics and general relativity (Santa Cruz, CA, 1986)*, volume 71 of *Contemp. Math.*, pages 237–262. Amer. Math. Soc., Providence, RI, 1988.
[19] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva. *Linear and quasilinear equations of parabolic type.* Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967.
[20] A. Rodriguez, J. L. Vazquez, and J. R. Esteban. The maximal solution of the logarithmic fast diffusion equation in two space dimensions. *Adv. Differential Equations*, 2(6):867–894, 1997.
[21] F. Salvarani and G. Toscani. Large-time asymptotics for nonlinear diffusions: the initial-boundary value problem. *J. Math. Phys.*, 46(2):023502, 11, 2005.
[22] J. L. Vázquez. *Smoothing and decay estimates for nonlinear diffusion equations*, volume 33 of *Oxford Lecture Series in Mathematics and its Applications.* Oxford University Press, Oxford, 2006. Equations of porous medium type.
[23] J. L. Vázquez. Finite-time blow-down in the evolution of point masses by planar logarithmic diffusion. *Discrete Contin. Dyn. Syst.*, 19(1):1–35, 2007.
[24] J. L. Vázquez. *The porous medium equation.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2007. Mathematical theory.

[25] J. L. Vázquez. Measure-valued solutions and the phenomenon of blow-down in logarithmic diffusion. *J. Math. Anal. Appl.*, 352:515–547, 2009.

[26] M. B. Williams and S. H. Davis. Non linear theory of film rupture. *J. of Colloidal and Interface Sc.*, 90:220–228, 1982.

Dipartimento di Matematica, Università degli Studi di Pavia. Via Ferrata, 1 - 27100 Pavia, Italy - e-mail: francesco.salvarani@unipv.it