HOMOGENIZATION OF THE LINEAR BOLTZMANN EQUATION WITH A HIGHLY OSCILLATING SCATTERING TERM IN EXTENDED PHASE SPACE

ÉTIENNE BERNARD AND FRANCESCO SALVARANI

Abstract. In this article, we rigorously prove the homogenization limit of the linear Boltzmann equation when the scattering term is highly oscillating with respect to the velocity variable. We prove that the limit equation keeps, in a suitably extended phase space, the same structure as the non-homogenized one. This situation does not coincide with what happens in standard phase space, where the appearance of memory terms is expected.

1. Introduction

One of the main difficulties in the mathematical study of models describing composite materials is the strongly oscillating character of some physical quantities, such as the neutron capture cross-section of Uranium-235 as a function of the energy of the incoming neutron in the range between 10^2 and 10^3 eV [6].

Such behavior makes numerical calculations very challenging and motivated the development of homogenization techniques, which consists in approximating the partial differential equation with strongly oscillating coefficients by another one with more regular coefficients [13].

The mathematical literature on the homogenization of the linear Boltzmann equation is quite developed. However, most of the articles consider the case where strong oscillations are on a single variable, in general the spatial one. We quote, for example, the simultaneous diffusion and homogenization asymptotics for the linear Boltzmann equation [3]. Nevertheless, there exist some studies which consider the homogenization problem with respect to other variables, such as the energy variable (an application of the two-scale convergence to this problem has been studied in [11]) .

Another difficulty of the homogenization theory is that the homogenized equation can be an integro-differential equation, with an integration in time, thus involving memory terms. That implies that the semi-group property of the original evolution equation can be destroyed by the homogenization limit [18, 17]. In [5], François Golse and the authors have opened a path toward both challenges by using the extended phase space trick, first applied to kinetic problems in [4, 8]. The usefulness of this approach is not limited to purely theoretical aspects, but also allows for the derivation of performant numerical methods [14].

The idea consists in considering an extended phase space involving additional variables, which allows to keep the semigroup property after passing to the homogenized limit. For instance, in [4, 8] the authors added the additional variable s to the phase space, in which the linear Boltzmann equation is defined, and then constructed a function $F_\varepsilon(t, s, x, v)$ satisfying

$$\int_0^\infty F_\varepsilon(t, s, x, v)\mathrm{d}s = f_\varepsilon(t, x, v).$$

Once proved that F_ε satisfies an evolution equation in the extended phase space, satisfying the semigroup property, it is possible to deduce that F_ε converges to a limit F, which solves the homogenized problem, and that the homogenized problem itself satisfies the semigroup property. Moreover, it is shown that

$$f_\varepsilon \to f := \int_0^\infty F\mathrm{d}s.$$

The method can be summarized by the diagram below:

$$\begin{array}{ccc} F_\varepsilon & \xrightarrow{\text{Homogenization}} & F \\ \text{Lifting of initial data} \uparrow & & \downarrow \text{Integration in } s \\ f_\varepsilon & \longrightarrow & f \end{array}$$

In [5], the method has been successfully applied to the equation of radiative transfer where the scattering processes are neglected. However, in neutron transport theory, the scattering processes cannot be neglected and thus we can not apply the method of extended phase space as directly as in [5]. Indeed, the scattering term is oscillating in the variables involved in the transport process. In the present paper, we show how to address this difficulty by using a suitable averaging velocity lemma.

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2. The model

We assume that the particles are set in the d-dimensional torus $\mathbb{T}^d$ with velocity in a bounded set $\mathcal{V} \subset \mathbb{R}^d$. We denote $f \equiv f(t,x,v)$ the density at time $t \in \mathbb{R}_+$ of particles with velocity $v \in \mathcal{V}$ and located at $x \in \mathbb{T}^d$. A classical model, describing the collective behavior of particles interacting with the background and not interacting between themselves, is the linear Boltzmann equation:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \sigma(f - \mathcal{K}f) &= 0, \\ f|_{t=0} &= f^{\mathrm{in}}(x,v). \end{cases}$$

The scattering operator $\mathcal{K} \in \mathcal{L}(L^1(\mathcal{V}); L^1(\mathcal{V}))$ denotes

$$\mathcal{K}\phi(v) := \int_{\mathcal{V}} \kappa(v,w)\phi(w)\mathrm{d}w, \quad \forall \phi \in L^1(\mathcal{V}).$$

We henceforth assume that

$$\kappa \in C(\mathcal{V} \times \mathcal{V}), \quad \kappa \geq 0, \quad \kappa(v,w) = \kappa(w,v), \quad \text{and} \quad \mathcal{K}\mathbb{1}_{\mathcal{V}} = \mathbb{1}_{\mathcal{V}}.$$

The scattering coefficient $\sigma \equiv \sigma(v)$ is independent of the spatial position and expresses the probability of interaction with the background, while the operator $\mathcal{K}$ codes the way the particles interact with the medium (see [16] pp. 225-226 for the probabilistic interpretation of the linear Boltzmann equation). We assume henceforth that the highly oscillatory behavior is encoded by the parameter $\varepsilon > 0$ and supported by the scattering coefficient σ_ε. Consider then:

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \sigma_\varepsilon(v)(f_\varepsilon - \mathcal{K}f_\varepsilon) &= 0 \\ f_\varepsilon|_{t=0} &= f^{\mathrm{in}}(x,v). \end{cases} \tag{2.1}$$

We assume that $f^{\mathrm{in}} \in L^\infty(\mathbb{T}^d \times \mathcal{V})$ and that there exist two constants $0 < c \leq C < +\infty$ such that

$$\forall \varepsilon > 0,\ \forall v \in \mathcal{V}, \quad C \geq \sigma_\varepsilon(v) \geq c. \tag{2.2}$$

For any ε, the existence and uniqueness of a nonnegative mild solution f_ε of the Cauchy problem (2.1) in $L^1(\mathbb{R}_+ \times \mathbb{T}^d \times \mathcal{V}) \cap L^\infty(\mathbb{R}_+ \times \mathbb{T}^d \times \mathcal{V})$ is classical [1]. Moreover, it is easy to see that f_ε satisfies the Maximum principle:

$$\|f_\varepsilon\|_{L^\infty(\mathbb{T}^d \times \mathcal{V})}(t) \leq \left\| f^{\mathrm{in}} \right\|_{L^\infty(\mathbb{T}^d \times \mathcal{V})} \forall t > 0.$$

As σ_ε is uniformly bounded in $L^\infty(\mathcal{V})$, the previous property implies that, up to a subsequence, there exists a measure $\mu_v \in \mathcal{M}(\mathcal{V})$ such that

$$\sigma_\varepsilon \Rightarrow \mu_v \text{ in the sense of Young measures} \tag{H1}$$

meaning that

$$\forall g \in C_0(\mathbb{R}_+), \quad g(\sigma_\varepsilon) \overset{*}{\rightharpoonup} \int_0^{+\infty} g(s)\mu_v(\mathrm{d}s) \text{ in } L^\infty(\mathbb{T}^d \times \mathcal{V}) \text{ weak-}*.$$

Denoting with $\tilde{\mu}_v$ the Laplace transform of μ_v, the property above implies

$$\forall n \in \mathbb{N}, \quad \sigma_\varepsilon^n e^{-\sigma_\varepsilon s} \overset{*}{\rightharpoonup} (-1)^n \frac{\mathrm{d}^n \tilde{\mu}_v}{\mathrm{d}s^n}(s)$$

in $L^\infty([0,T] \times \mathbb{T}^d \times \mathcal{V})$ weak-$*$ as $\varepsilon \to 0^+$ (for a more complete presentation on Young measures, we refer to [15]). Let $F \equiv F(t,s,x,v)$ be the mild solution of

$$\begin{cases} (\partial_t + v \cdot \nabla_x - \partial_s)F = \dfrac{\mathrm{d}^2 \tilde{\mu}_v}{\mathrm{d}s^2}(s)\mathcal{K} \displaystyle\int_0^{+\infty} F\mathrm{d}s \\ F(0,s,x,v) = -\dfrac{\mathrm{d}\tilde{\mu}_v}{\mathrm{d}s}(s) f^{\mathrm{in}}(x,v). \end{cases} \tag{2.3}$$

In Proposition 3.2, we will prove that F exists and is unique because the Cauchy problem (2.3) is a bounded perturbation of the free transport equation in the extended phase space.

The main result of this note is the following:

Theorem 2.1. *Under the assumptions above and up to a subsequence,*

$$f_\varepsilon \overset{*}{\rightharpoonup} \int_0^{+\infty} F\mathrm{d}s$$

in $L^\infty([0,T] \times \mathbb{T}^d \times \mathcal{V})$ *weak-*$*$ *as* $\varepsilon \to 0^+$.

We give a sketch of the proof. Introducing

$$F_\varepsilon(t,s,x,v) := \sigma_\varepsilon(v)e^{-s\sigma_\varepsilon(v)} f_\varepsilon(t,x,v), \quad (t,s,x,v) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{T}^d \times \mathcal{V},$$

an easy computation shows that the function F_ε is solution of a kinetic equation in an extended phase space:

$$\begin{cases} (\partial_t + v \cdot \nabla_x - \partial_s)F_\varepsilon = \sigma_\varepsilon^2 e^{-\sigma_\varepsilon s}\mathcal{K} \displaystyle\int_0^{+\infty} F_\varepsilon \mathrm{d}s \\ F_\varepsilon(0,s,x,v) = \sigma_\varepsilon(v)e^{-s\sigma_\varepsilon(v)} f^{\mathrm{in}}(x,v). \end{cases} \tag{2.4}$$

In what follows, we denote with $(f_\varepsilon)_{\varepsilon>0}$ the family of ε-dependent solutions of (2.1) and with $(F_\varepsilon)_{\varepsilon>0}$ the family of ε-dependent solutions of (2.4). On the first hand, we know by the Maximum principle and by the Banach-Alaoglu Theorem that, up to a subsequence, the family $(f_\varepsilon)_{\varepsilon>0}$ converges to a function f in $L^\infty([0,T]\times\mathbb{T}^d\times\mathcal{V})$ weak-$*$ as $\varepsilon\to 0^+$. On the other hand, assume that we can show that up to a subsequence, the family $(F_\varepsilon)_{\varepsilon>0}$ converges to F (i.e., the unique mild solution of (2.3), in $L^\infty([0,T]\times\mathbb{R}_+\times\mathbb{T}^d\times\mathcal{V})$) weak-$*$ as $\varepsilon\to 0^+$, the uniqueness of limit permits to identify f with $\displaystyle\int_0^{+\infty} F(s)\mathrm{d}s$.

We have evoked above that the scattering coefficient σ is independent of the spatial position and expresses the probability of interaction with the background, while the operator $\mathcal{K}$ codes the way the particles interact with the medium. More precisely, the linear Boltzmann involves a Piecewise-deterministic Markov process (PDMP). It is a family of stochastic processes involving a deterministic motion punctuated by random jumps, the jump times following a Poisson-like fashion (here with rate σ) (on the PDMP theory and its applications, we refer to the seminal article [9] and to the more recent articles [12, 2] and references therein). In this perspective, the extra variable s in the expression of F_ε encodes the time since the last jump and in the same way that f_ε is the expectation of a PDMP, the function F_ε can be seen as the conditional expectation of this process knowing of the last jump in velocity. For this reason, the trick of the extended phase space could be useful for the homogenization problems of partial differential equations arising from PDMP.

3. The homogenization limit

The argument of the proof of Theorem 2.1 is split into several steps.

3.1. Two Cauchy problems.

First, we begin with a lemma, crucial for the homogenized equation, about the behavior of $v\mapsto\sigma_\varepsilon^n(v)e^{-\sigma_\varepsilon(v)s}$.

Lemma 3.1. *Let $(\sigma_\varepsilon)_{\varepsilon>0}$ be a family of scattering coefficients converging to μ_v in the sense of Young measures as $\varepsilon\to 0^+$. Denoting $\tilde\mu$ the Laplace Transform of a measure μ*

$$\tilde\mu(s) := \int_{\mathbb{R}_+} e^{-st}\mu(\mathrm{d}t),$$

then for any $n\in\mathbb{N}$

$$\sigma_\varepsilon^n e^{-\sigma_\varepsilon s} \overset{*}{\rightharpoonup} (-1)^n\frac{\mathrm{d}^n\tilde\mu_v}{\mathrm{d}s^n}(s) \tag{3.1}$$

in $L^\infty(\mathcal{V})$ weak-$*$ as $\varepsilon\to 0^+$. Moreover, $\tilde\mu_v(s)$ and its derivatives belong to $L^1_s(\mathbb{R}_+;L^\infty(\mathcal{V}))$.

Proof. As $\eta\in\mathbb{R}_+$, we have that $\eta\mapsto\eta^n e^{-s\eta}\in C_0(\mathbb{R}_+)$ for all $s\in\mathbb{R}_+^*$. Hence, (3.1) is a direct consequence of assumption $(H1)$. Notice that μ_v is a family of probability measures on $\mathbb{R}_+$ and consequently, by Bernstein theorem, for every $v\in\mathcal{V}$, $\tilde\mu_v$ is a completely monotone function, meaning that it is continuous on $[0,\infty)$, infinitely differentiable on $(0,\infty)$ and for every $n\in\mathbb{N}$,

$$(-1)^n\frac{\mathrm{d}^n\tilde\mu_v}{\mathrm{d}s^n}\geq 0.$$

Besides, notice that for each $g\in L^1(\mathcal{V})$,

$$\int_\mathcal{V} g(v)\sigma_\varepsilon^n(v)e^{-\sigma_\varepsilon(v)s}\mathrm{d}v\to\int_\mathcal{V} g(v)(-1)^n\frac{\mathrm{d}^n\tilde\mu_v}{\mathrm{d}s^n}(s)\mathrm{d}v$$

which implies that, by (2.2), for every $v\in\mathcal{V}$

$$(-1)^n\frac{\mathrm{d}^n\tilde\mu_v}{\mathrm{d}s^n}(s)\leq C^n e^{-cs}.$$

Consequently, $(s,v)\mapsto(-1)^n\dfrac{\mathrm{d}^n\tilde\mu_v}{\mathrm{d}s^n}(s)\in L^1(\mathbb{R}_+;L^\infty(\mathcal{V}))$, which concludes the proof. □

We introduce now some new notations. First, we define the positive semigroup:

$$S_t : g\in L^1_{s,x,v}\mapsto S_t g(s,x,v) = g(s+t,x-vt,v)\in L^1_{s,x,v}$$

and the operator

$$\mathcal{F} : g\mapsto\frac{\mathrm{d}^2\tilde\mu_v}{\mathrm{d}s^2}(s)\mathcal{K}\int_0^\infty g\,\mathrm{d}s$$

Notice that $\mathcal{F}$ is bounded in $L^1_{s,x,v}$ by Lemma 3.1. Then we can now state the following result:

Proposition 3.2. *Under the assumptions above, for any $T>0$, the Cauchy problem* (2.3) *has a unique mild solution $F\in L^\infty_t([0,T];L^1_{s,x,v})$ satisfying:*

$$F(t,s,x,v) = -\frac{\mathrm{d}\tilde\mu_v}{\mathrm{d}s}(s+t)f^{in}(x-vt,v)+\int_0^t S_{t-\tau}\mathcal{F}F(\tau,s,x,v)\mathrm{d}\tau.$$

Moreover, we have

$$F\in L^\infty_{t,s,x,v}([0,T]\times\mathbb{R}_+\times\mathbb{T}^d\times\mathcal{V})\cap L^1_s(\mathbb{R}_+;L^\infty_{t,x,v}([0,T]\times\mathbb{T}^d\times\mathcal{V})).$$

Proof. First, we note that $v\cdot\nabla_x-\partial_s$ is an advection operator, which can be written as $(v,-1)\cdot\nabla_{(x,s)}$. Hence, by the method of characteristics, the free transport problem in extended phase space

$$\begin{cases}(\partial_t+v\cdot\nabla_x-\partial_s)F=0\\ F(0,s,x,v)=F^{\rm in}(s,x,v)\end{cases}$$

generates a positive semigroup $(S_t)_{t\in\mathbb{R}_+}$ on $L^1(\mathbb{R}_+\times\mathbb{T}^d\times\mathcal{V})$ defined by

$$(S_tF^{\rm in})(t,s,x,v):=F^{\rm in}(s+t,x-vt,v),\ \text{ for any } F^{\rm in}\in L^1(\mathbb{R}_+\times\mathbb{T}^d\times\mathcal{V}).$$

Besides, the operator $\mathcal{F}:F\mapsto\dfrac{\mathrm{d}^2\tilde\mu_v}{\mathrm{d}s^2}(s)\mathcal{K}\displaystyle\int_0^\infty F\mathrm{d}s$ is bounded in $L^1(\mathbb{R}_+\times\mathbb{T}^d\times\mathcal{V})$ since $\dfrac{\mathrm{d}^2\tilde\mu_v}{\mathrm{d}s^2}\in L^1(\mathbb{R}_+;L^\infty(\mathcal{V}))$ by Lemma 3.1. Therefore, by the bounded perturbation theorem (see Theorem III.1.3 p. 158 in [10]), the Cauchy problem

$$\begin{cases}(\partial_t+v\cdot\nabla_x-\partial_s)F=\mathcal{F}F\\ F(0,s,x,v)=F^{\rm in}(s,x,v)\end{cases}$$

generates a positive semigroupe $(T_t)_{t\ge0}$ that, by the variation of parameters formula, satisfies (see Corollary III.1.7 p. 161 in [10])

$$T_t=S_t+\int_0^t S_{t-\tau}\mathcal{F}T_\tau\mathrm{d}\tau.$$

So that specializing in $F^{\rm in}(s,x,v)=-\dfrac{\mathrm{d}\tilde\mu_v}{\mathrm{d}s}(s)f^{\rm in}(x,v)$, there exists a unique mild solution F of the Cauchy problem (2.3) in $L^\infty_{\rm loc}(\mathbb{R}_+;L^1(\mathbb{R}_+\times\mathbb{T}^d\times\mathcal{V}))$, that satisfies the formula:

$$F(t,s,x,v)=-\frac{\mathrm{d}\tilde\mu_v}{\mathrm{d}s}(s+t)f^{\rm in}(x-vt,v)+\int_0^t S_{t-\tau}\mathcal{F}F(\tau,s,x,v)\mathrm{d}\tau.$$

It remains to show that F belongs to $L^\infty_{t,s,x,v}([0,T]\times\mathbb{R}_+\times\mathbb{T}^d\times\mathcal{V})\cap L^1_s(\mathbb{R}_+;L^\infty_{t,x,v}([0,T]\times\mathbb{T}^d\times\mathcal{V}))$. It is known that, as $\mathcal{F}$ is bounded, we have (see Corollary III.1.11 p. 163 in [10])

$$T_t=\sum_{n\ge0}S_n(t)$$

with $S_0(t)=S_t$ and, for any $n\ge1$,

$$S_n(t):=\int_0^t S_{t-\tau}\mathcal{F}S_{n-1}(\tau)\mathrm{d}\tau.$$

Now, let $g\in L^1_{t,s}(\mathbb{R}_+\times\mathbb{R}_+;L^\infty_{x,v}(\mathbb{T}^d\times\mathcal{V}))$ be nonnegative. We have

$$\mathcal{F}g(\tau,s,x,v)=\frac{\mathrm{d}^2\tilde\mu_v}{\mathrm{d}s^2}(s)\int_0^\infty\int_{\mathcal{V}}\kappa(v,w)g(\tau,s,x,w)\mathrm{d}w\mathrm{d}s=\frac{\mathrm{d}^2\tilde\mu_v}{\mathrm{d}s^2}(s)\|\mathcal{K}g\|_{L^1_s}$$

and

$$\begin{aligned}S_{t-\tau}\mathcal{F}g(\tau,s,x,v)&=\frac{\mathrm{d}^2\tilde\mu_v}{\mathrm{d}s^2}(s+t-\tau)\int_0^\infty\int_{\mathcal{V}}\kappa(v,w)g(\tau,s,x-v(t-\tau),w)\mathrm{d}w\\&\le C^2e^{-c(s+t-\tau)}\int_0^\infty\|g\|_{L^\infty_{x,v}}(\tau,s)\mathrm{d}s\le C^2e^{-cs}\int_0^\infty\|g\|_{L^\infty_{x,v}}(\tau,s)\mathrm{d}s.\end{aligned}$$

Notice that

$$\int_0^\infty S_{t-\tau}\mathcal{F}g(\tau,s,x,v)\mathrm{d}s\le\frac{C^2}{c}\int_0^\infty\|g\|_{L^\infty_{x,v}}(\tau,s)\mathrm{d}s.$$

Thus

$$\left\|\int_0^t S_{t-\tau}\mathcal{F}g(\tau,s,x,v)\mathrm{d}\tau\right\|_{L^1_s(L^\infty_{x,v})}\le\int_0^t\frac{C^2}{c}\|g\|_{L^1_s(L^\infty_{x,v})}\mathrm{d}\tau.$$

That being said, we observe that, for any $G(s,x,v)$,

$$\begin{aligned}S_n(t)G&=\int_0^t S_{t-t_n}\mathcal{F}S_{n-1}(t_n)G\mathrm{d}t_n\\&\le C^2\int_0^t S_{t-t_n}e^{-cs}\|S_{n-1}G\|_{L^1_sL^\infty_{x,v}}(t_n)\mathrm{d}t_n\\&\le C^2e^{-cs}\int_0^t\left\|\int_0^t S_{t-t_{n-1}}\mathcal{F}S_{n-2}(t_n-1)\mathrm{d}t_{n-1}\right\|_{L^1_sL^\infty_{x,v}}\mathrm{d}t_n\\&\le C^2e^{-cs}\int_0^t\int_0^t\frac{C^2}{c}\|S_{n-2}G\|_{L^1_sL^\infty_{x,v}}\mathrm{d}t_{n-1}\mathrm{d}t_n\\&\le C^2e^{-cs}\int_0^t\cdots\int_0^t\left(\frac{C^2}{c}\right)^{n-1}\|S_0G\|_{L^1_sL^\infty_{x,v}}\mathrm{d}t_1\cdots\mathrm{d}t_n.\end{aligned}$$

Taking $G(s,x,v) := -\frac{\mathrm{d}\tilde{\mu}_v}{\mathrm{d}s}(s) f^{\mathrm{in}}(x,v)$, we have, by Lemma 3.1,

$$0 \le G(s,x,v) \le Ce^{-cs} \left\| f^{\mathrm{in}} \right\|_{L^\infty(\mathbb{T}^d\times\mathcal{V})}$$

and

$$S_t G(s,x,v) \le Ce^{-cs} e^{-ct} \left\| f^{\mathrm{in}} \right\|_{L^\infty(\mathbb{T}^d\times\mathcal{V})}$$

thus

$$\|S_t G(s,x,v)\|_{L^1_s L^\infty_{x,v}} \le \frac{C}{c} \left\| f^{\mathrm{in}} \right\|_{L^\infty(\mathbb{T}^d\times\mathcal{V})}$$

thus

$$S_n G \le Ce^{-cs} \left\| f^{\mathrm{in}} \right\|_{L^\infty(\mathbb{T}^d\times\mathcal{V})} \left(\frac{C^2}{c} \right)^n \frac{t^n}{n!}$$

which implies that

$$F(t,s,x,v) \le Ce^{-cs} \left\| f^{\mathrm{in}} \right\|_{L^\infty(\mathbb{T}^d\times\mathcal{V})} \left(1 + e^{\frac{C^2}{c} t} \right).$$

Consequently, $F \in L^\infty_{t,s,x,v}([0,T] \times \mathbb{R}_+ \times \mathbb{T}^d \times \mathcal{V}) \cap L^1_s(\mathbb{R}_+; L^\infty_{t,x,v}([0,T] \times \mathbb{T}^d \times \mathcal{V}))$, which is the desired conclusion. □

In the same way, introducing the following operator:

$$\mathcal{F}_\varepsilon : F \mapsto \sigma_\varepsilon^2 e^{-\sigma_\varepsilon s} \mathcal{K} \int_0^\infty F \mathrm{d}s,$$

we prove, as in the proof of Proposition 3.2, the following proposition:

Proposition 3.3. *Under the assumptions above, for any $T > 0$ and for any $\varepsilon > 0$ the Cauchy problem* (2.4) *has a unique mild solution $F_\varepsilon \in L^\infty([0,T]; L^1_{s,x,v})$ satisfying*

$$F_\varepsilon(t,s,x,v) = \sigma_\varepsilon(v) e^{-\sigma_\varepsilon(v)(t+s)} f^{in}(x - vt, v) + \int_0^t S_{t-\tau} \mathcal{F}_\varepsilon F_\varepsilon(\tau, x, v) \mathrm{d}\tau.$$

Moreover, we have $F \in L^\infty_{t,s,x,v}([0,T] \times \mathbb{R}_+ \times \mathbb{T}^d \times \mathcal{V}) \cap L^1_s(\mathbb{R}_+; L^\infty_{t,x,v}([0,T] \times \mathbb{T}^d \times \mathcal{V}))$ and we have $\forall \varepsilon > 0, \forall (t,s,x,v) \in [0,T] \times \mathbb{R}_+ \times \mathbb{T}^d \times \mathcal{V}$,

$$F_\varepsilon(t,s,x,v) = \sigma_\varepsilon(v) e^{-\sigma_\varepsilon(v)s} f_\varepsilon(t,s,v).$$

The existence and uniqueness of the mild solutions of Problems (2.3) and (2.4) being established, we now show the relative compactness of $(\mathcal{K} f_\varepsilon)_{\varepsilon>0}$ thanks to a velocity averaging lemma.

3.2. **A velocity averaging lemma.**

We recall first a velocity averaging lemma that is, in fact, a special case of Theorem 1.8 p.29 in [7].

Lemma 3.4 (Velocity Averaging)**.** *Let $p > 1$ and assume that $(f_\varepsilon)_{\varepsilon>0}$ is a bounded family in $L^p(\mathbb{R}_+ \times \mathbb{T}^d \times \mathcal{V})$ such that*

$$\sup_{\varepsilon>0} \int_0^T \iint_{\mathbb{T}^d\times\mathcal{V}} |\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon| \mathrm{d}x\mathrm{d}v\mathrm{d}t < +\infty$$

for each $T > 0$. Then for each $\psi \in C(\mathcal{V} \times \mathcal{V})$, the family $(\rho_\psi[f_\varepsilon])_{\varepsilon>0}$ defined by

$$\rho_\psi[f_\varepsilon](t,x,v) = \int_{\mathcal{V}} f_\varepsilon(t,x,w) \psi(v,w) \mathrm{d}w, \qquad \textit{for all } \varepsilon > 0$$

is relatively compact in $L^1([0,T] \times \mathbb{T}^d \times \mathcal{V})$-strong.

The previous lemma implies:

Lemma 3.5. *Let $(f_\varepsilon)_{\varepsilon>0}$ be the family of solutions of the Cauchy problem* (2.4)*. Then the family $(K f_\varepsilon)_{\varepsilon>0}$ is relatively compact in $L^1([0,T] \times \mathbb{T}^d \times \mathcal{V})$-strong.*

Proof. By the Maximum principle, we have that, for all $\varepsilon > 0$ and for any $t > 0$,

$$|f_\varepsilon(t,x,v)| \le \left\| f^{\mathrm{in}} \right\|_{L^\infty(\mathbb{T}^d\times\mathcal{V})}.$$

So that

$$\sup_{\varepsilon>0} \|f_\varepsilon\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^d\times\mathcal{V})} \le \left\| f^{\mathrm{in}} \right\|_{L^\infty(\mathbb{T}^d\times\mathcal{V})}.$$

Hence, we have, on $\mathbb{T}^d \times \mathcal{V}$, that

$$|\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon| \le C \left\| f^{\mathrm{in}} \right\|_{L^\infty(\mathbb{T}^d\times\mathcal{V})} (1 + \|\kappa\|_{L^\infty(\mathcal{V}\times\mathcal{V})}).$$

Thus, the family $(|\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon|)_{\varepsilon>0}$ is uniformly bounded in $L^1\big([0,T] \times \mathbb{T}^d \times \mathcal{V}\big)$. Consequently, the strong compactness in $L^1([0,T] \times \mathbb{T}^d \times \mathcal{V})$ of $(\mathcal{K} f_\varepsilon)_{\varepsilon>0}$ is a direct consequence of Lemma 3.4. □

3.3. Proof of Theorem 2.1.

Proof. By Proposition 3.3 and the Maximum principle, we have

$$\|F_\varepsilon\|_{L^\infty_{t,s,x,v}} \le C\left\|f^{\rm in}\right\|_{L^\infty_{x,v}}.$$

Thus, by the Banach-Alaoglu theorem, $(F_\varepsilon)_{\varepsilon>0}$ is relatively weak-$*$ compact in $L^\infty(\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{T}^d\times\mathcal{V})$. Up to a subsequence, we denote F the weak-$*$ limit point of F_ε. In the same way, up to subsequence, we denote f the weak-$*$ limit point of $(f_\varepsilon)_{\varepsilon>0}$. Observe that f is also the weak-$*$ limit point of $\displaystyle\int_0^\infty F_\varepsilon \mathrm{d}s$. Denote

$$F_{\varepsilon,1}(t,s,x,v) := \sigma_\varepsilon(v)e^{-\sigma_\varepsilon(v)(t+s)}f^{\rm in}(x-vt,v) \quad\text{and}\quad F_{\varepsilon,2}(t,s,x,v) := \int_0^t S_{t-\tau}\mathcal{F}_\varepsilon F_\varepsilon(\tau,x,v)\mathrm{d}\tau.$$

Hence, by Proposition 3.3, we have

$$F_{\varepsilon,1} \overset{*}{\rightharpoonup} \left((t,s,x,v)\mapsto -f^{\rm in}(x-tv,v)\frac{\mathrm{d}\tilde\mu_v}{\mathrm{d}s}(s+t)\right).$$

As for $F_{\varepsilon,2}$, by the Fubini-Tonelli Theorem,

$$\begin{aligned}
F_{\varepsilon,2}(t,s,x,v) &= \int_0^t S_{t-\tau}\mathcal{F}_\varepsilon F_\varepsilon(\tau,x,v)\mathrm{d}\tau\\
&= \int_0^t S_{t-\tau}(\sigma_\varepsilon(v)^2e^{-s\sigma_\varepsilon(v)}\mathcal{K}f_\varepsilon(\tau,x,v))\mathrm{d}\tau\\
&= \int_0^t \sigma_\varepsilon(v)^2e^{-(t+s-\tau)\sigma_\varepsilon(v)}\mathcal{K}f_\varepsilon(\tau,x-(t-\tau)v,v)\mathrm{d}\tau.
\end{aligned}$$

By Lemma 3.5, up to a subsequence, we know that $\mathcal{K}f_\varepsilon \to \mathcal{K}f$ in $L^1([0,T]\times\mathbb{T}^d\times\mathcal{V})$-strong as $\varepsilon\to 0^+$. Thus

$$F_{\varepsilon,2} \overset{*}{\rightharpoonup} \int_0^t \frac{\mathrm{d}^2\tilde\mu_v}{\mathrm{d}s^2}(t+s-\tau)\mathcal{K}f(\tau,x-(t-\tau)v,v)\mathrm{d}\tau.$$

We recognize immediately that

$$F_\varepsilon(t,s,x,v) = F_{\varepsilon,1}(t,s,x,v) + F_{\varepsilon,2}(t,s,x,v).$$

Consequently,

$$F_\varepsilon \overset{*}{\rightharpoonup} -f^{\rm in}(x-tv,v)\frac{\mathrm{d}\tilde\mu_v}{\mathrm{d}s} + \int_0^t \frac{\mathrm{d}^2\tilde\mu_v}{\mathrm{d}s^2}(t+s-\tau)\mathcal{K}f(\tau,x-(t-\tau)v,v)\mathrm{d}\tau := \tilde F.$$

We know that $\displaystyle\int_0^\infty F_\varepsilon(t,s,x,v) \overset{*}{\rightharpoonup} f$. Besides

$$\int_0^\infty F_\varepsilon(t,s,x,v)\mathrm{d}s \overset{*}{\rightharpoonup} \int_0^\infty \tilde F(t,s,x,v)\mathrm{d}s.$$

Thus, by the uniqueness of the limit, we have

$$\int_0^\infty \tilde F(t,s,x,v)\mathrm{d}s = f(t,x,v).$$

Consequently,

$$F_\varepsilon \overset{*}{\rightharpoonup} -f^{\rm in}(x-tv,v)\frac{\mathrm{d}\tilde\mu_v}{\mathrm{d}s}(s+t) + \int_0^t S_{t-\tau}\mathcal{F}F(\tau,x,v)\mathrm{d}\tau.$$

By Proposition 3.2, we can conclude that $F_\varepsilon \overset{*}{\rightharpoonup} F$, where F is the unique mild solution to (2.3). □

4. Final remarks

As the main aim of the present paper is to explain the interest of the application of the extended phase space technique to the homogenization of kinetic equations, for the sake of simplicity, we have assumed the simplest case of $x\in\mathbb{T}^d$. However, we emphasize that the result holds for the more general (and more interesting for industrial applications) case of an open bounded domain Ω with absorbing boundary conditions. Indeed, we can extend the solutions of the Cauchy problems by 0 outside Ω such as

$$\{f\}(t,x,v) := \begin{cases} f(t,x,v) & \text{whenever } x\in\Omega\\ 0 & \text{otherwise,}\end{cases}$$

and we can show that the extensions satisfy, in the distributional sense, the same equations, with an additional term related to the presence of the absorbing boundary. This additional term is a family of Radon measures uniformly bounded in $\mathcal{M}([-R,R]^d\times\mathcal{V})$ for any $R>0$ and thus the averaging velocity lemma used above holds here. We refer to [4] for an example of this method.

We have assumed that the scattering coefficient σ does not depend upon the spatial variable x in order to be able to compute the limit in the sense of Young measures in Lemma 3.1 and Proposition 3.2. The more general

case of the scattering process involving the spatial variable x seems to necessitate new arguments.
In the same way, we have assumed $\mathcal{V}$ being bounded to use a compactness averaging lemma that are only valid in the velocity averaging on compact sets. Generalizing the approach in the case $\mathcal{V}$ unbounded asks to be able to show that $\mathcal{K}f_\varepsilon$ is strongly compact in L^1. We hope to return to these questions in a forthcoming publication.

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É. B. : CERMICS - Ecole des Ponts ParisTech 6 et 8 avenue Blaise Pascal Cité Descartes - Champs sur Marne 77455 Marne la Vallée Cedex 2 (FRANCE)
Email address: etienne.bernard@enpc.fr

F.S.: Léonard de Vinci Pôle Universitaire, Research Center, 92916 Paris La Défense, France & Dipartimento di Matematica "F. Casorati", Università degli Studi di Pavia, Via Ferrata 1, 27100 Pavia, Italy
Email address: francesco.salvarani@unipv.it