Finitely additive FTAP under an atomic reference measure

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Abstract. Let L be a linear space of real bounded random variables on the probability space $(\Omega, \mathcal{A}, P_0)$. A finitely additive probability P on $\mathcal{A}$ such that

$$P \sim P_0 \quad \text{and} \quad E_P(X) = 0 \text{ for each } X \in L$$

is called EMFA (equivalent martingale finitely additive probability). In this note, EMFA's are investigated in case P_0 is atomic. Existence of EMFA's is characterized and various examples are given. Given $y \in \mathbb{R}$ and a bounded random variable Y, it is also shown that $X_n + y \xrightarrow{a.s.} Y$, for some sequence $(X_n) \subset L$, provided EMFA's exist and $E_P(Y) = y$ for each EMFA P.

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1 Introduction

In the sequel, $(\Omega, \mathcal{A}, P_0)$ is a probability space and L a linear space of real bounded random variables. We let $\mathbb{P}$ denote the set of finitely additive probabilities on $\mathcal{A}$ and $\mathbb{P}_0 = \{P \in \mathbb{P} : P \text{ is } \sigma\text{-additive}\}$. In particular, $P_0 \in \mathbb{P}_0$. We also say that $P \in \mathbb{P}$ is an *equivalent martingale finitely additive probability* (EMFA) if

$$P \sim P_0 \quad \text{and} \quad E_P(X) = 0 \text{ for each } X \in L.$$

Here, $P \sim P_0$ means that P and P_0 have the same null sets. Further, the term "martingale" (attached to P) is motivated as follows.

Let $\mathcal{F} = (\mathcal{F}_t : t \in T)$ be a filtration and $S = (S_t : t \in T)$ a real $\mathcal{F}$-adapted process on $(\Omega, \mathcal{A}, P_0)$, where $T \subset \mathbb{R}$ is any index set. Suppose S_t a bounded random variable for each $t \in T$ and define

$$L(\mathcal{F}, S) = \text{Span}\,\{I_A\,(S_t - S_s) : s, t \in T,\ s < t,\ A \in \mathcal{F}_s\}.$$

If $P \in \mathbb{P}_0$, then S is a P-martingale (with respect to $\mathcal{F}$) if and only if $E_P(X) = 0$ for all $X \in L(\mathcal{F}, S)$. If $P \in \mathbb{P}$ but $P \notin \mathbb{P}_0$, it looks natural to *define* S a P-martingale in case $E_P(X) = 0$ for all $X \in L(\mathcal{F}, S)$.

Thus, the process S is a martingale under the probability P if and only if $E_P(X) = 0$ for each X in a suitable linear space $L(\mathcal{F}, S)$. Basing on this fact, given *any* linear space L of bounded random variables, P is called a martingale probability whenever $E_P(X) = 0$ for all $X \in L$.

Existence of EMFA's is investigated in [5]. The main results are recalled in Subsection 2.2. Here, we try to motivate EMFA's and we describe the content of this note.

Quoting from [5], we list some reasons for dealing with EMFA's. As usual, a σ-additive EMFA is called *equivalent martingale measure* (EMM).

(i) Dating from de Finetti, the finitely additive theory of probability is well founded and developed, even if not prevailing. Finitely additive probabilities can be always extended to the power set and have a solid motivation in terms of coherence. Also, there are problems which can not be solved in the usual countably additive setting, while admit a finitely additive solution. Examples are in conditional probability, convergence in distribution of non measurable random elements, Bayesian statistics, stochastic integration and the first digit problem. See e.g. [4] and references therein. Moreover, in the finitely additive approach, one can clearly use σ-additive probabilities. Merely, one is not obliged to do so.

(ii) Martingale probabilities play a role in various financial frameworks. Their economic motivations, however, do not depend on whether they are σ-additive or not. See e.g. Chapter 1 of [8]. In option pricing, for instance, EMFA's give arbitrage-free prices just as EMM's. Note also that many underlying ideas, in arbitrage price theory, were anticipated by de Finetti and Ramsey.

(iii) It may be that EMM's fail to exist and yet EMFA's are available. See Examples 1 and 5. In addition, existence of EMFA's can be given simple characterizations; see Theorems 2, 3 and 4.

(iv) Each EMFA P can be written as $P = \alpha P_1 + (1-\alpha) Q$, where $\alpha \in [0, 1)$, $P_1 \in \mathbb{P}$ is purely finitely additive and $Q \in \mathbb{P}_0$ is equivalent to P_0; see Theorem 2. Even if one does not like finitely additive probabilities, when EMM's do not exist one may be content with an EMFA P whose α is small enough. In other terms, a fraction α of the total mass must be sacrificed for having equivalent martingale probabilities, but the approximation may look acceptable for small α. An extreme situation of this type is exhibited in Example 5. In such example, EMM's do not exist and yet, for each fixed $\epsilon > 0$, there is an EMFA P with $\alpha \leq \epsilon$.

In connection with points (iii)-(iv) above, and to make the notion of EMFA more transparent, we report a simple example from [5].

Example 1. (Example 7 of [5]). Let $\Omega = \{1, 2, \ldots\}$, $\mathcal{A}$ the power set of Ω, and $P_0\{\omega\} = 2^{-\omega}$ for all $\omega \in \Omega$. For each $n \geq 0$, define $A_n = \{n+1, n+2, \ldots\}$.

Define also $L = L(\mathcal{F}, S)$, where

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_n = \sigma\big(\{1\}, \dots, \{n\}\big), \quad S_0 = 1, \quad \text{and}$$
$$S_n(\omega) = \frac{1}{2^n}\, I_{A_n}(\omega) + \frac{\omega^2 + 2\omega + 2}{2^\omega}\,(1 - I_{A_n}(\omega)) \quad \text{for all } \omega \in \Omega.$$

The process S has been introduced in [1]. Loosely speaking, ω could be regarded as a (finite) stopping time and $S_n(\omega)$ as a price at time n. Such a price falls by 50% at each time $n < \omega$. Instead, for $n \geq \omega$, the price is constant with respect to n and depends on ω only.

If $P \in \mathbb{P}$ is a martingale probability, then

$$1 = E_P(S_0) = E_P(S_n) = \frac{P(A_n)}{2^n} + \sum_{j=1}^{n} \frac{j^2 + 2j + 2}{2^j}\, P\{j\}.$$

Letting $n = 1$ in the above equation yields $P\{1\} = 1/4$. By induction, one obtains $2\,P\{n\} = 1/n(n+1)$ for all $n \geq 1$. Since $\sum_{n=1}^\infty P\{n\} = 1/2$, then $P \notin \mathbb{P}_0$. Thus, EMM's do not exist. Instead, EMFA's are available. Define in fact

$$P = \frac{P_1 + Q}{2}$$

where P_1 and Q are probabilities on $\mathcal{A}$ such that $P_1\{n\} = 0$ and $Q\{n\} = 1/n(n+1)$ for all $n \geq 1$. (Note that $Q \in \mathbb{P}_0$ while P_1 is purely finitely additive). Clearly, $P \sim P_0$. Given $X \in L(\mathcal{F}, S)$, since $S_{n+1} = S_n$ on A_n^c, one obtains

$$X = \sum_{j=0}^{k} b_j\, I_{A_j}\,(S_{j+1} - S_j) \quad \text{for some } k \geq 0 \text{ and } b_0, \dots, b_k \in \mathbb{R}.$$

Since $A_j = \{j+1\} \cup A_{j+1}$ and $S_{j+1} - S_j = -1/2^{j+1}$ on A_{j+1}, it follows that

$$E_{P_1}(X) = \sum_{j=0}^{k} \frac{b_j}{2^{j+1}}\, \big\{\big((j+1)^2 + 2(j+1)\big)\, P_1\{j+1\} - P_1(A_{j+1})\big\} = -\sum_{j=0}^{k} \frac{b_j}{2^{j+1}}$$

and

$$E_Q(X) = \sum_{j=0}^{k} \frac{b_j}{2^{j+1}}\, \big\{\big((j+1)^2 + 2(j+1)\big)\, Q\{j+1\} - Q(A_{j+1})\big\}$$
$$= \sum_{j=0}^{k} \frac{b_j}{2^{j+1}}\, \Big\{\frac{(j+1)^2 + 2(j+1)}{(j+1)(j+2)} - \frac{1}{(j+2)}\Big\} = \sum_{j=0}^{k} \frac{b_j}{2^{j+1}}.$$

Therefore $E_P(X) = 0$, that is, P is an EMFA.

This note investigates EMFA's when the reference probability measure P_0 is *atomic*. There are essentially two reasons for focusing on atomic P_0. One is that

P_0 is actually atomic in several real situations. The other reason is a version of the FTAP (fundamental theorem of asset pricing). Indeed, when P_0 is atomic, existence of EMFA's amounts to

$$\overline{L - L_\infty^+} \cap L_\infty^+ = \{0\} \quad \text{with the closure in the norm-topology;}$$

we refer to Subsection 2.2 for details.

Three results are obtained for atomic P_0. First, existence of EMFA's is given a new characterization (Theorem 4). Such a characterization looks practically more useful than the existing ones. Second, the extreme situation mentioned in point (iv) is realized (Example 5). Third, the following problem is addressed (Theorem 6 and Example 8). Suppose EMFA's exist and fix a bounded random variable Y. If

$$E_P(Y) = y \quad \text{ for some } y \in \mathbb{R} \text{ and all EMFA's } P,$$

does $Y - y$ belong to the closure of L in some topology ? Or else, if $E_P(Y) \geq 0$ for all EMFA's P, can Y be approximated by random variables of the form $X + Z$ with $X \in L$ and $Z \geq 0$? Indeed, with EMM's instead of EMFA's, these questions are classical; see [6], [9], [10], [13] and references therein. For instance, if Y is regarded as a contingent claim, $E_P(Y) = y$ for all EMFA's P means that y is the unique arbitrage-free price of Y. Similarly $Y - y \in \overline{L}$, with the closure in a suitable topology, can be seen as a weak form of completeness for the underlying market.

A last note deals with the assumption that L consists of *bounded* random variables. Even if strong, such an assumption can not be dropped. In fact, while de Finetti's coherence principle (our main tool) can be extended to unbounded random variables, the extensions are very far from granting an integral representation; see [2], [3] and references therein.

2 Known results

2.1 Notation

For each essentially bounded random variable X, we let

$$\text{essup}(X) = \inf\{a \in \mathbb{R} : P_0(X > a) = 0\} = \inf\{\sup_A X : A \in \mathcal{A},\, P_0(A) = 1\},$$
$$\|X\| = \|X\|_\infty = \text{essup}(|X|).$$

Given P, $T \in \mathbb{P}$, we write $P \ll T$ if $P(A) = 0$ whenever $A \in \mathcal{A}$ and $T(A) = 0$, and $P \sim T$ if $P \ll T$ and $T \ll P$. We also write

$$E_P(X) = \int X\, dP$$

whenever $P \in \mathbb{P}$ and X is a real bounded random variable.

A probability $P \in \mathbb{P}$ is *pure* if it does not have a non trivial σ-additive part. Precisely, if P is pure and Γ is a σ-additive measure such that $0 \leq \Gamma \leq P$, then $\Gamma = 0$. By a result of Yosida-Hewitt, any $P \in \mathbb{P}$ can be written as $P = \alpha P_1 + (1-\alpha) Q$ where $\alpha \in [0,1]$, $P_1 \in \mathbb{P}$ is pure (unless $\alpha = 0$) and $Q \in \mathbb{P}_0$.

A P_0-*atom* is a set $A \in \mathcal{A}$ with $P_0(A) > 0$ and $P_0(\cdot \mid A) \in \{0,1\}$; P_0 is *atomic* if there is a countable partition $A_1, A_2, \ldots$ of Ω such that A_n is a P_0-atom for all n.

2.2 Existence of EMFA's

We next state a couple of results from [5]. Let

$$\mathbb{M} = \{P \in \mathbb{P} : P \sim P_0 \text{ and } E_P(X) = 0 \text{ for all } X \in L\}$$

be the set of EMFA's. Note that $\mathbb{M} \cap \mathbb{P}_0$ is the set of EMM's.

Theorem 2 *Each $P \in \mathbb{M}$ admits the representation $P = \alpha P_1 + (1-\alpha) Q$ where $\alpha \in [0,1)$, $P_1 \in \mathbb{P}$ is pure (unless $\alpha = 0$), $Q \in \mathbb{P}_0$ and $Q \sim P_0$. Moreover, $\mathbb{M} \neq \emptyset$ if and only if*

$$E_Q(X) \leq k \, essup(-X), \quad X \in L, \tag{1}$$

for some constant $k > 0$ and $Q \in \mathbb{P}_0$ with $Q \sim P_0$. In particular, under condition (1)*, one obtains*

$$\frac{k P_1 + Q}{k+1} \in \mathbb{M} \quad \textit{for some } P_1 \in \mathbb{P}.$$

In addition to characterizing $\mathbb{M} \neq \emptyset$, Theorem 2 provides some information on the weight $1-\alpha$ of the σ-additive part Q of an EMFA. Indeed, under (1), there is $P \in \mathbb{M}$ such that $\alpha \leq k/(k+1)$. On the other hand, condition (1) is not very helpful in real problems, for it requires to have Q in advance. A characterization independent of Q would be more effective. We will come back to this point in the next section.

We next turn to separation theorems. Write $U - V = \{u - v : u \in U, v \in V\}$ whenever U, V are subsets of a linear space. Let $L_p = L_p(\Omega, \mathcal{A}, P_0)$ for all $p \in [1, \infty]$. We regard L as a subspace of L_∞ and we let $L_\infty^+ = \{X \in L_\infty : X \geq 0\}$. Since L_∞ is the dual of L_1, it can be equipped with the weak-star topology $\sigma(L_\infty, L_1)$. Thus, $\sigma(L_\infty, L_1)$ is the topology on L_∞ generated by the maps $X \mapsto E_{P_0}(X\,Y)$ for all $Y \in L_1$.

By a result of Kreps [11] (see also [12]) existence of EMM's amounts to

$$\overline{L - L_\infty^+} \cap L_\infty^+ = \{0\} \quad \text{with the closure in } \sigma(L_\infty, L_1).$$

On the other hand, it is usually argued that the norm topology on L^∞ is geometrically more transparent than $\sigma(L_\infty, L_1)$, and results involving the former are often viewed as superior. Thus, a (natural) question is what happens if the closure is taken in the norm-topology.

Theorem 3 $\mathbb{M} \neq \emptyset$ *if and only if*

$$L_\infty^+ \subset U \cup \{0\} \quad \text{and} \quad (L - L_\infty^+) \cap U = \emptyset$$
$$\text{for some norm-open convex set } U \subset L_\infty.$$

In particular, a necessary condition for $\mathbb{M} \neq \emptyset$ *is*

$$\overline{L - L_\infty^+} \cap L_\infty^+ = \{0\} \quad \text{with the closure in the norm-topology.} \tag{2}$$

If P_0 *is atomic, condition* (2) *is sufficient for* $\mathbb{M} \neq \emptyset$ *as well.*

Note that, in the particular case where L is a suitable class of stochastic integrals (in a fixed time interval and driven by a fixed semi-martingale), condition (2) agrees with the *no free lunch with vanishing risk* condition of [7]. See also [8]. The main difference with [7] is that, in this note, L is an arbitrary subspace of L_∞.

It is still open whether condition (2) implies $\mathbb{M} \neq \emptyset$ for arbitrary $P_0 \in \mathbb{P}_0$. However, (2) is equivalent to $\mathbb{M} \neq \emptyset$ when P_0 is atomic. This is a first reason for paying special attention to the latter case. A second (and more important) reason is that P_0 is actually atomic in various real situations. Accordingly, in the sequel we focus on the atomic case.

3 New results in case of atomic P_0

In this section, P_0 *is atomic*. Everything is well understood if P_0 has finitely many atoms only (such a case can be reduced to that of Ω finite). Thus, the P_0-atoms are assumed to be *infinitely many*. Let $A_1, A_2, \ldots$ be a countable partition of Ω such that A_n is a P_0-atom for each n. Also, $X|A_n$ denotes the a.s.-constant value of the random variable X on A_n.

Theorem 2 gives a general characterization of existence of EMFA's. As already noted, however, a characterization not involving Q would be more usable in real problems. In case P_0 is atomic, one such characterization is actually available.

Theorem 4 *Suppose that, for each* $n \geq 1$, *there is a constant* $k_n > 0$ *such that*

$$X|A_n \leq k_n \, essup(-X) \quad \text{for each } X \in L. \tag{3}$$

Letting $\beta = \inf_n k_n$, *for each* $\alpha \in \left(\frac{\beta}{1+\beta}, 1\right)$ *one obtains*

$$\alpha P_1 + (1-\alpha) Q \in \mathbb{M} \quad \text{for some } P_1 \in \mathbb{P} \text{ and } Q \in \mathbb{P}_0 \text{ with } Q \sim P_0.$$

Moreover, condition (3) *is necessary for* $\mathbb{M} \neq \emptyset$ *(so that* $\mathbb{M} \neq \emptyset$ *if and only if* (3) *holds).*

Proof. Suppose first $\mathbb{M} \neq \emptyset$. Fix $P \in \mathbb{M}$, $n \geq 1$ and $X \in L$. Since $E_P(X) = 0$,

$$\begin{aligned} P(A_n)\, X|A_n &\leq P(A_n)\, X^+|A_n \leq E_P(X^+) \\ &= E_P(X) + E_P(X^-) = E_P(X^-) \leq \text{essup}(-X). \end{aligned}$$

Therefore, condition (3) holds with $k_n = 1/P(A_n)$. Conversely, suppose (3) holds. Fix any sequence $(q_n : n \geq 1)$ satisfying $q_n > 0$ for all n, $\sum_n q_n = 1$ and $\sum_n (q_n/k_n) < \infty$. For each $A \in \mathcal{A}$, define

$$I(A) = \{n : P_0(A \cap A_n) > 0\} \quad \text{and} \quad Q(A) = \frac{\sum_{n \in I(A)} (q_n/k_n)}{\sum_n (q_n/k_n)}.$$

Then, $Q \in \mathbb{P}_0$ and $Q \sim P_0$. Also, for each $X \in L$, condition (3) yields

$$E_Q(X) = \sum_n Q(A_n)\, X|A_n \leq \text{essup}(-X) \sum_n Q(A_n)\, k_n = \frac{\text{essup}(-X)}{\sum_n (q_n/k_n)}.$$

Thus, condition (1) holds with $k = \left\{\sum_n (q_n/k_n)\right\}^{-1}$. By Theorem 2, there is $P_1 \in \mathbb{P}$ such that $(k\, P_1 + Q)/(k+1) \in \mathbb{M}$. Finally, fix $\alpha \in \left(\frac{\beta}{1+\beta}, 1\right)$. Condition (3) remains true if the k_n are replaced by arbitrary constants $k_n^* \geq k_n$. Thus, it can be assumed $\sup_n k_n = \infty$. In this case, it suffices to note that

$$k = \frac{1}{\sum_n (q_n/k_n)} = \frac{\alpha}{1-\alpha}$$

for a suitable choice of $(q_n : n \geq 1)$.

Next example has been discussed in point (iv) of Section 1.

Example 5. Let $\Omega = \{1, 2, \ldots\}$. Take $\mathcal{A}$ to be the power set and $P_0\{n\} = 2^{-n}$ for all $n \in \Omega$. Define $T = (2P_0 + P^* - \delta_1)/2$, where $P^* \in \mathbb{P}$ is any pure probability and δ_1 the point mass at 1. Since $P^*\{n\} = 0$ for all $n \in \Omega$, then $T\{1\} = 0$ and $T \in \mathbb{P}$. Let $B = \{2, 3, \ldots\}$ and define L to be the linear space generated by $\{I_A - T(A)\, I_B : A \subset B\}$. If $P \in \mathbb{P}$ is a martingale probability, then

$$P\{n, n+1, \ldots\} = T\{n, n+1, \ldots\}\, P(B) \geq \frac{P(B)}{2} \quad \text{for all } n > 1.$$

Thus, $P \notin \mathbb{P}_0$ as far as $P(B) > 0$, so that $\mathbb{M} \cap \mathbb{P}_0 = \emptyset$. On the other hand, $P_\epsilon := \epsilon\, T + (1-\epsilon)\, \delta_1 \in \mathbb{M}$ for all $\epsilon \in (0, 1)$. In fact, $P_\epsilon\{n\} > 0$ for all $n \in \Omega$ (so that $P_\epsilon \sim P_0$) and

$$E_{P_\epsilon}(X) = \epsilon\, E_T(X) + (1-\epsilon)\, X(1) = 0 \quad \text{for all } X \in L.$$

To sum up, in this example, EMM's do not exist and yet, for each $\epsilon > 0$, there is $P \in \mathbb{M}$ such that $\alpha(P) \leq \epsilon$. Here, $\alpha(P)$ denotes the weight of the pure part of P, in the sense that $P = \alpha(P)\, P_1 + (1 - \alpha(P))\, Q$ for some pure $P_1 \in \mathbb{P}$ and $Q \in \mathbb{P}_0$ with $Q \sim P_0$.

The rest of this note is concerned with the following problem. Suppose $\mathbb{M} \neq \emptyset$ and fix $Y \in L_\infty$. If $E_P(Y) = y$ for some $y \in \mathbb{R}$ and all $P \in \mathbb{M}$, does $Y - y$ belong to the closure of L in some reasonable topology ? Or else, if $E_P(Y) \geq 0$ for all $P \in \mathbb{M}$, can Y be approximated by random variables of the form $X + Z$ with $X \in L$ and $Z \in L_\infty^+$? Up to replacing EMFA's with EMM's, questions of this type are classical; see [6], [9], [10], [13] and references therein. Indeed, regarding Y as a contingent claim, $E_P(Y) = y$ for all $P \in \mathbb{M}$ means that y is the unique arbitrage-free price of Y. Similarly $Y - y \in \overline{L}$, with the closure in a suitable topology, can be seen as a weak form of completeness for the underlying market.

In what follows, L_∞ is equipped with the norm topology. Accordingly, for each $H \subset L_\infty$, $\overline{H}$ denotes the closure of H in the norm topology.

Theorem 6 *Suppose $\mathbb{M} \neq \emptyset$ and fix $Y \in L_\infty$. Then,*

(a) $Y \in \overline{L - L_\infty^+} \quad \Longleftrightarrow \quad E_P(Y) \leq 0$ *for each* $P \in \mathbb{M}$,

(b) $Y \in \bigcap_{P \in \mathbb{M}} \overline{L}^P \quad \Longleftrightarrow \quad E_P(Y) = 0$ *for each* $P \in \mathbb{M}$,

where $\overline{L}^P$ *denotes the closure of* L *in the* $L_1(P)$*-topology. In addition, if* $E_P(Y) = 0$ *for each* $P \in \mathbb{M}$*, then* $X_n \xrightarrow{a.s.} Y$ *for some sequence* $(X_n) \subset L$.

Proof. First note that "$\Longrightarrow$" is obvious in both (a) and (b). Suppose $Y \notin \overline{L - L_\infty^+}$. Fix $A \in \mathcal{A}$ with $P_0(A) > 0$ and define

$$U = \overline{L - L_\infty^+}, \quad V = \{\alpha I_A + (1-\alpha)Y : 0 \leq \alpha \leq 1\}.$$

Then, $U \cap V = \emptyset$. In fact, $I_A \notin U$ because of $\mathbb{M} \neq \emptyset$ and Theorem 3. If $\alpha I_A + (1-\alpha)Y \in U$ for some $\alpha < 1$, there are $(X_n) \subset L$ and $(Z_n) \subset L_\infty^+$ such that $X_n - Z_n \xrightarrow{L_\infty} \alpha I_A + (1-\alpha)Y$, which in turn implies

$$\frac{X_n - (Z_n + \alpha I_A)}{1-\alpha} \xrightarrow{L_\infty} Y.$$

But this is a contradiction, as $Y \notin U$. Next, since U and V are convex and closed with V compact, some linear (continuous) functional $\Phi : L_\infty \to \mathbb{R}$ satisfies

$$\inf_{f \in V} \Phi(f) > \sup_{f \in U} \Phi(f).$$

It is routine to verify that Φ is positive and $\Phi(1) > 0$. Hence, $\Phi(f) = \Phi(1)\, E_{P_A}(f)$ for all $f \in L_\infty$ and some $P_A \in \mathbb{P}$ with $P_A \ll P_0$. Since L is a linear space and $\sup_{f \in L} \Phi(f) \leq \sup_{f \in U} \Phi(f) < \infty$, then $\Phi = 0$ on L. To sum up, P_A satisfies $P_A \ll P_0$, $P_A(A) > 0$, $E_{P_A}(Y) > 0$, and $E_{P_A}(X) = 0$ for all $X \in L$. It follows that

$$P := \sum_n \frac{1}{2^n} P_{A_n} \in \mathbb{M} \quad \text{and} \quad E_P(Y) > 0.$$

This concludes the proof of (a). Suppose now that $E_P(Y)=0$ for all $P\in\mathbb{M}$. By (a), there are sequences $(X_n)\subset L$ and $(Z_n)\subset L_\infty^+$ such that $X_n-Z_n\overset{L_\infty}{\longrightarrow}Y$. For each $P\in\mathbb{M}$, since $Z_n\in L_\infty^+$ and $E_P(X_n)=E_P(Y)=0$, one obtains

$$\begin{aligned}E_P|X_n-Y| &\le E_P|X_n-Z_n-Y|+E_P(Z_n)\\ &= E_P|X_n-Z_n-Y|-E_P(X_n-Z_n-Y)\le 2\,\|X_n-Z_n-Y\|\longrightarrow 0.\end{aligned}$$

This proves (b). Finally, take $P\in\mathbb{M}$, say $P=\alpha\,P_1+(1-\alpha)\,Q$ where $\alpha\in[0,1)$, $P_1\in\mathbb{P}$, $Q\in\mathbb{P}_0$ and $Q\sim P_0$. Arguing as above,

$$E_Q(Z_n)\le\frac{E_P(Z_n)}{1-\alpha}\le\frac{\|X_n-Z_n-Y\|}{1-\alpha}\longrightarrow 0.$$

Thus, $Z_{n_j}\overset{a.s.}{\longrightarrow}0$ and $X_{n_j}=Z_{n_j}+(X_{n_j}-Z_{n_j})\overset{a.s.}{\longrightarrow}Y$ for some subsequence (n_j).

As regards part (b) of Theorem 6, a question is whether $E_P(Y)=0$ for all $P\in\mathbb{M}$ implies $Y\in\overline{L}$. We now prove that the answer is no. The following lemma is useful.

Lemma 7 *Let $P\in\mathbb{P}$. If $P\ll P_0$ and $P(A_n)=P_0(A_n)$ for all n, then $P=P_0$.*

Proof. Fix $A\in\mathcal{A}$ and $n\ge 1$. If $P_0(A\cap A_n)=0$, then $P(A\cap A_n)=0=P_0(A\cap A_n)$. If $P_0(A\cap A_n)>0$, then $P_0(A^c\cap A_n)=0$, and thus

$$P(A\cap A_n)=P(A_n)=P_0(A_n)=P_0(A\cap A_n).$$

It follows that $P(A)\ge\sum_{i=1}^n P(A\cap A_i)=\sum_{i=1}^n P_0(A\cap A_i)$. As $n\to\infty$, one obtains $P(A)\ge P_0(A)$. Finally, taking complements yields $P=P_0$.

Example 8. Let L be the linear space generated by $\{I_{A_n}-P_0(A_n):n\ge 1\}$ and

$$Y=\frac{I_A}{P_0(A)}-\frac{I_{A^c}}{P_0(A^c)}\quad\text{where } A=\cup_{n=1}^\infty A_{2n}.$$

Each $P\in\mathbb{M}$ meets $P\ll P_0$ and $P(A_n)=P_0(A_n)$ for all n. Thus, Lemma 7 yields $\mathbb{M}=\{P_0\}$. Further, $E_{P_0}(Y)=0$. However, $Y\notin\overline{L}$. Fix in fact $X\in L$. Since $X=x$ a.s. on the set $\left(\cup_{i=1}^n A_i\right)^c$, for some $n\ge 1$ and $x\in\mathbb{R}$, one obtains

$$\|Y-X\|=\sup_i|(Y-X)|A_i|\ge\sup_{i>n}|(Y-x)|A_i|\ge\frac{1}{P_0(A)}\wedge\frac{1}{P_0(A^c)}.$$

In Example 8, $\mathbb{M}$ is quite small (its cardinality is 1). It looks reasonable to conjecture that, with $\mathbb{M}$ large enough, $E_P(Y)=0$ for all $P\in\mathbb{M}$ could imply $Y\in\overline{L}$. One more question is whether Theorem 6 holds in case P_0 is not atomic. Incidentally, this question is related to the open problem mentioned after Theorem 3: is condition (2) equivalent to $\mathbb{M}\neq\emptyset$ when P_0 is not atomic ?

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