A NOTE ON DUALITY THEOREMS IN MASS TRANSPORTATION

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Notation

- $(\mathcal{X}, \mathcal{F}, \mu)$ and $(\mathcal{Y}, \mathcal{G}, \nu)$ probability spaces

- $c : \mathcal{X} \times \mathcal{Y} \to R$ measurable function such that

  $f_1 + g_1 \leq c \leq f_2 + g_2$

  for some $f_1, f_2 \in L_1(\mu)$ and $g_1, g_2 \in L_1(\nu)$

- $\Gamma(\mu, \nu) = \{$probabilities on $\mathcal{F} \otimes \mathcal{G}$ with marginals μ and $\nu\}$

- For $P \in \Gamma(\mu, \nu)$, we write: $P(c) = E_P(c) = \int c \, dP$

In Kantorovich approach to **mass transportation**, one focus on

$$\boxed{\alpha(c) = \inf\{P(c) : P \in \Gamma(\mu, \nu)\}}$$

As in classical linear programming, define also

$$\boxed{\beta(c) = \sup\{\mu(f) + \nu(g) : f \in L_1(\mu),\ g \in L_1(\nu),\ f + g \leq c\}}$$

Since $P(f + g) = \mu(f) + \nu(g)$ for each $P \in \Gamma(\mu, \nu)$, then $\beta(c) \leq \alpha(c)$. Thus, a (natural) question is whether

$\alpha(c) = \beta(c)$?

Similarly, if

$$\boxed{\alpha^*(c) = \sup\{P(c) : P \in \Gamma(\mu,\nu)\}}$$

$$\boxed{\beta^*(c) = \inf\{\mu(f) + \nu(g) : f \in L_1(\mu),\ g \in L_1(\nu),\ f + g \geq c\}}$$

then $\alpha^*(c) \leq \beta^*(c)$ and a question is whether $\alpha^*(c) = \beta^*(c)$?

A **duality theorem** (for both $\alpha(c)$ and $\alpha^*(c)$) is the assertion that

$$\boxed{\alpha(c) = \beta(c) \text{ and } \alpha^*(c) = \beta^*(c)}$$

Example: Wasserstein distance

Let $(\mathcal{X}, d)$ be a metric space, $\mathcal{Y} = \mathcal{X}$ and $c = d$. Suppose d measurable with respect to $\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{X})$ and

$$\int d(x, x_0)\, \mu(dx) + \int d(x, x_0)\, \nu(dx) < \infty \quad \text{for some } x_0 \in \mathcal{X}$$

Then, $\alpha(d)$ reduces to the Wasserstein distance while $\beta(d)$ to the Kantorovich-Rubinstein distance:

$$\beta(d) = \sup_f |\mu(f) - \nu(f)|$$

where sup is over the 1-Lipschitz functions $f : \mathcal{X} \to R$. Thus,

Wasserstein metric = Kantorovich-Rubinstein metric

if duality holds for $c = d$

Example: Arveson's problem

Let $H \in \mathcal{F} \otimes \mathcal{G}$. If $P(H) = 0$ for each $P \in \Gamma(\mu, \nu)$, are there $A \in \mathcal{F}$ and $B \in \mathcal{G}$ such that

$$H \subset (A \times \mathcal{Y}) \cup (\mathcal{X} \times B) \text{ and } \mu(A) = \nu(B) = 0 \text{ ?}$$

The answer is **yes** if duality holds for $c = 1_H$

In fact, $\alpha^*(H) = 0$ and it is not hard to see that

$$H \subset (A \times \mathcal{Y}) \cup (\mathcal{X} \times B) \text{ and } \beta^*(H) = \mu(A) + \nu(B)$$

for some $A \in \mathcal{F}$ and $B \in \mathcal{G}$. Thus,

$$\mu(A) + \nu(B) = \beta^*(H) = \alpha^*(H) = 0$$

State of the art

To my knowledge, the best duality theorem is due to Ramachandran and Ruschendorf (PTRF, 1995):

$\alpha(c) = \beta(c)$ and $\alpha^*(c) = \beta^*(c)$ provided (at least) one between μ and ν is **perfect**

Open problem: Prove or disprove that $\alpha(c) = \beta(c)$ and $\alpha^*(c) = \beta^*(c)$ without any assumption

In this talk, unfortunately, I do not solve the above problem. I just give a few duality theorems not requiring perfectness

Perfect and separable probability measures

A probability measure P on a measurable space $(\Omega, \mathcal{A})$ is perfect if, for each measurable $f : \Omega \to R$, there is $B \in \mathcal{B}(R)$ such that

$$\boxed{B \subset f(\Omega) \quad \text{and} \quad P(f \in B) = 1}$$

Basically, perfectness is a non topological notion of tightness. In fact, if Ω is metric, $\mathcal{A} = \mathcal{B}(\Omega)$ and $\text{card}(\Omega) \leq \text{card}(R)$ then

P perfect $\Leftrightarrow$ P tight $\Rightarrow$ P separable (but **not conversely**)

Here, P separable means $P(A) = 1$ for some separable $A \in \mathcal{B}(\Omega)$. In particular, if Ω is separable metric, P is trivially separable but may fail to be perfect

Theorem 1: Suppose $\mathcal{X}$ and $\mathcal{Y}$ are metric spaces and (at least) one one between μ and ν is separable. Then, duality holds provided

$\boxed{x \mapsto c(x,y) \text{ and } y \mapsto c(x,y)}$ are continuous

Remarks:

- Theorem 1 improves the result by Ramachandran and Ruschendorf provided c has continuous sections and the cardinalities of $\mathcal{X}$ and $\mathcal{Y}$ do not exceed the continuum

- By Theorem 1, it is consistent with the usual axioms of set theory (ZFC) that duality holds for every c with continuous sections. In fact, it is consistent with ZFC that any Borel probability on any metric space is separable

- By Theorem 1, Wasserstein metric = Kantorovich-Rubinstein metric whenever one between μ and ν is separable

Corollary: If $\mathcal{X}$ and $\mathcal{Y}$ are metric spaces, duality holds provided

- both μ and ν are separable

- c is bounded

- $x \mapsto c(x,y)$ **or** $y \mapsto c(x,y)$ is continuous

Theorem 2: Duality holds provided, for each $\epsilon > 0$, there is a countable partition $\{A_0, A_1, \ldots\} \subset \mathcal{F}$ of $\mathcal{X}$ such that $\mu(A_0) = 0$ and

$$\sup_{y \in \mathcal{Y}} |c(x,y) - c(z,y)| \leq \epsilon \text{ whenever } x, z \in A_i \text{ and } i > 0$$

Corollary: Let $H = \cup_n (A_n \times B_n)$ be a countable union of measurable rectangles. Then, $\alpha^*(H) = \beta^*(H)$. Also, $\alpha(H) = \beta(H)$ (so that duality holds for $c = 1_H$) provided

$$\mu(A_n \text{ infinitely often}) = 0 \quad \text{or} \quad \nu(B_n \text{ infinitely often}) = 0$$

Example: Let $\mathcal{X} = \mathcal{Y}$ and $\mu = \nu$, with $\mathcal{X}$ a metric space and μ separable and diffuse. Let H be a countable union of measurable rectangles. Then, duality holds for $c = 1_{H_1}$ and $c = 1_{H_2}$ where

$$H_1 = H \cap \Delta, \quad H_2 = H \cap \Delta^c \quad \text{and } \Delta = \{(x,x) : x \in \mathcal{X}\}$$