FINITELY ADDITIVE MIXTURES OF PROBABILITY MEASURES

Patrizia Berti and Pietro Rigo
Universities of Modena Reggio-Emilia and Pavia

Reasoning under partial knowledge
(in honor of Giulianella Coletti)
Perugia, december 14-15, 2018

Notation

- D a linear space of real bounded functions on a set Ω
- P a prevision (i.e., a coherent functional) on D
- $\mathcal{Q}$ a collection of previsions on D
- $\Sigma(\mathcal{Q})$ the σ-field over $\mathcal{Q}$ generated by the maps $Q \mapsto Q(f)$ for all $f \in D$
- "f.a.p." stands for "finitely additive probability"

Starting point

Under mild conditions, P is a finitely additive mixture of the elements of $\mathcal{Q}$. In fact, by de Finetti's coherence principle,

- there is a f.a.p. Π on $\Sigma(\mathcal{Q})$ such that

  $\boxed{P(f) = \int Q(f)\Pi(dQ)}$ for each $f \in D$

  if and only if

- $\boxed{P(f) \geq \inf\{Q(f) : Q \in \mathcal{Q}\}}$ for each $f \in D$

Even if obvious, the previous remark provides some research hints:

(i) Prove results on finitely additive mixtures of f.a.p.'s, e.g. finitely additive mixtures of extreme points

(ii) Give alternative proofs to the classical, σ-additive results. Namely, suppose P and each $Q \in \mathcal{Q}$ are σ-additive, and you aim to show that P is a σ-additive mixture of $\mathcal{Q}$. Then, a strategy is: first prove P is a finitely additive mixture of $\mathcal{Q}$, and then show that the mixing measure Π is σ-additive

(iii) Find handy expressions for

$\inf \{Q(f) : Q \in \mathcal{Q}\}$

For instance, when such inf reduces to a Choquet integral ?

Common extensions

Let P_i be a prevision on D_i, where i ranges over some index set I. A common extension is a prevision P on $l^\infty(\Omega)$ such that

$P = P_i$ on D_i for each $i \in I$

Basing on the initial remark of this talk, we characterize the common extensions P satisfying some additional properties, such as

$P(f) \leq P(g)$ for each $(f, g) \in C$

where C is a given set of pairs of bounded functions

Example

Let μ_1, μ_2, μ_3 be f.a.p.'s on the fields $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$

- There is a f.a.p. μ such that

  $\mu = \mu_1$ on $\mathcal{F}_1$, $\mu = \mu_2$ on $\mathcal{F}_2$ and $\mu \ll \mu_3$ on $\mathcal{F}_3$

  if and only if

- $\mu_1(A) \leq \mu_2(B)$ whenever $A \in \mathcal{F}_1$, $B \in \mathcal{F}_2$ and

  $A \cap C \subset B \cap C$

  for some $C \in \mathcal{F}_3$ with $\mu_3(C) = 1$

Finitely additive mixtures of extreme points

Let $\mathcal{R}$ be a collection of f.a.p.'s on a field $\mathcal{F}$ and

$\mathcal{Q} = \{\text{extreme points of } \mathcal{R}\}$

If $\mathcal{R}$ is convex and closed under pointwise convergence, then $P \in \mathcal{R}$ if and only if

$P(\cdot) = \int Q(\cdot)\, \Pi(dQ)$ for some f.a.p. Π on $\Sigma(\mathcal{Q})$

Example: Take $\mathcal{R}$ the set of f.a.p.'s Q such that

$\boxed{Q \circ \phi^{-1} = Q}$ for all $\phi \in \Phi$

where Φ is any set of measurable functions from Ω into itself. Then, any invariant f.a.p. is a finitely additive mixture of extreme invariant f.a.p.'s, without any assumptions on Φ or $(\Omega, \mathcal{F})$

Countably additive mixtures

From now on, P and each $Q \in \mathcal{Q}$ are σ-additive probability measures on a σ-field $\mathcal{A}$

Theorem: Suppose $P(\cdot) = \int Q(\cdot)\,\Pi(dQ)$ for some f.a.p. Π on $\Sigma(\mathcal{Q})$. Then, Π is (unique and) σ-additive provided, for each $A \in \mathcal{A}$, there is an $\mathcal{A}$-measurable map $h_A : \Omega \to [0,1]$ such that

$Q\{\omega : h_A(\omega) = Q(A)\} = 1$ for each $Q \in \mathcal{Q}$

Such a theorem applies, in particular, if

$\mathcal{G} = \{A \in \mathcal{A} : Q(A) \in \{0,1\}$ for each $Q \in \mathcal{Q}\}$

is sufficient for $\mathcal{Q}$

Basing on the Theorem, it can be shown that various finitely additive mixtures are actually σ-additive mixtures

As an example, one gets the following version of de Finetti's theorem:

Let $(S, \mathcal{E})$ be a measurable space and P a σ-additive f.a.p. on $(S^\infty, \mathcal{E}^\infty)$. Let $\mathcal{Q}$ be the class of i.i.d. probability measures on $\mathcal{E}^\infty$. Then $P(\cdot) = \int Q(\cdot)\, \Pi(dQ)$, for some unique σ-additive f.a.p. Π on $\Sigma(\mathcal{Q})$, if and only if

$P(f) \geq \inf \{Q(f) : Q \in \mathcal{Q}\}$ for each simple function f

Note: Contrary to the usual versions of de Finetti's theorem, $(S, \mathcal{E})$ is arbitrary