TWO VERSIONS OF THE FTAP

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THE PROBLEM

Given

- $(\Omega, \mathcal{A}, P)$ probability space

- L linear space of real random variables

we are looking for a probability Q on $\mathcal{A}$ such that

$$\boxed{Q \sim P \ , \ E_Q|X| < \infty \ \text{ and } \ E_Q(X) = 0 \text{ for all } X \in L}$$

Such a Q may be finitely additive or σ-additive (depending on the problem at hand)

MOTIVATIONS

A number of meaningful problems reduce to the previous one (possibly, without requesting $Q \sim P$):

- $\boxed{\text{Equivalent martingale measures}}$
- de Finetti's coherence principle
- Equivalent probability measures with given marginals
- Compatibility of conditional distributions
- Stationary and reversible Markov chains

NOTE: All results we are going to state hold even if L is a convex cone only, up to replacing $E_Q(X) = 0$ with $E_Q(X) \leq 0$

EQUIVALENT MARTINGALE MEASURES

Let $(S_t : t \in T)$ be a real process indexed by $T \subset \mathcal{R}$, where $0 \in T$. Suppose S adapted to a filtration $(\mathcal{G}_t : t \in T)$ and S_0 constant. An EMM is precisely a (σ-additive) solution Q of our problem, with L the linear space generated by

$$I_A(S_u - S_t)$$

for all $t, u \in T$ with $t < u$ and $A \in \mathcal{G}_t$.

But it also makes sense to take Q finitely additive. Let us call EMFA a finitely additive solution Q of our problem.

WHY TO LOOK FOR EMFA's ?

- The finitely additive probability theory is well founded and developed, even if not prevailing. Among its supporters, we mention B. de Finetti, L.J. Savage and L.E. Dubins

- It may be that EMFA's are available while EMM's fail to exist

- In option pricing, EMFA's give arbitrage-free prices just as EMM's. More generally, the economic motivations of martingale probabilities do not depend on whether they are σ-additive or not

- Each EMFA Q can be written as

  $$Q = \alpha Q_1 + (1-\alpha)Q_2,$$

  where $\alpha \in [0,1)$, Q_1 is purely finitely additive, Q_2 is σ-additive and $Q_2 \sim P$. Thus, when EMM's fail to exist, one might be content with an EMFA with α small enough

FIRST VERSION OF THE FTAP

If $L \subset L_\infty$, the following conditions are equivalent:

(a) There is an EMFA

(b) $\{P \circ X^{-1} : X \in L,\, X \geq -1 \text{ a.s.}\}$ is tight

(c) There are a constant $k \geq 0$ and a probability measure $T \sim P$ such that

$$E_T(X) \leq k \,\text{ess sup}(-X) \quad \text{for all } X \in L$$

(d) $\overline{L - L_\infty^+} \cap L_\infty^+ = \{0\}$ with the closure in the norm-topology of L_∞

REM 1: (a) $\Leftrightarrow$ (d) has been proved by G. Cassese and D.B. Rokhlin long before us. (It is life !).

REM 2: Since $L \subset L_\infty$, condition (d) agrees with the NFLVR condition of Delbaen and Schachermayer

REM 3: Since $L \subset L_\infty$, there is an EMM if and only if

$\overline{L - L_\infty^+} \cap L_\infty^+ = \{0\}$ with the closure in $\sigma(L_\infty, L_1)$

But the geometric meaning of $\sigma(L_\infty, L_1)$ is not so transparent. Hence, a question is what happens if the closure is taken in the norm-topology. The answer is just given by the previous theorem

REM 4: The assumption $L \subset L_\infty$ can be dropped (but at the price of a less clean characterization)

SECOND VERSION OF THE FTAP

There is an EMM if and only if there are a constant $k \geq 0$ and a probability measure $T \sim P$ such that $E_T|X| < \infty$ and

$\boxed{E_T(X) \leq k\, E_T(X^-)}$ for all $X \in L$

REM 1: The above condition can be written as

$$\boxed{\sup_X \frac{|\,E_T(X)\,|}{E_T|X|} < 1}$$

where sup is over those $X \in L$ with $P(X \neq 0) > 0$

REM 2: To apply such result in real problems, one has to select $T \sim P$. A natural choice is $T = P$. In fact, the following corollary is available:

There is an EMM Q such that

$\boxed{r\,P \le Q \le s\,P}$ for some constants $0 < r \le s$

if and only if $E_P|X| < \infty$ and

$\boxed{E_P(X) \le k\,E_P(X^-)}$, $X \in L$, for some constant $k \ge 0$.

Our last result is in the spirit of the previous one (to avoid the choice of T). It is a sort of *localized* version of the FTAP.

Suppose $E_P|X| < \infty$ for all $X \in L$. There is an EMM Q such that

$\boxed{Q \le s\,P}$ for some constant $s > 0$

if and only if

$\boxed{E_P(X|A_n) \le k_n\,E_P(X^-)}$, $X \in L$, $n \ge 1$,

for some constants $k_n \ge 0$ and some events $A_n \in \mathcal{A}$ with $P(A_n) \to 1$