SOME REMARKS ON SKOROHOD REPRESENTATION THEOREM

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Notation and state of the art

Throughout,

(S, d) is a metric space

$\mathcal{B}$ the Borel σ-field on S

$(\mu_n : n \geq 0)$ a sequence of probability measures on $\mathcal{B}$

Skorohod representation thm

If

$\mu_n \to \mu_0$ weakly and μ_0 is separable,

there are a probability space $(\Omega, \mathcal{A}, P)$ and random variables $X_n : \Omega \to S$ such that

$X_n \sim \mu_n$ for each $n \geq 0$ and $X_n \to X_0$ a.s.

Separability of the limit μ_0

It is consistent with ZFC that non-separable probabilities on $\mathcal{B}$ do not exist. However, the existence of such probabilities cannot be currently excluded

Also, **non-separable probabilities are quite usual on sub-σ-fields** $\mathcal{G} \subset \mathcal{B}$. For instance, take

$S = \{$real cadlag functions on $[0, 1]\}$, $d =$ uniform distance,

$\mathcal{G} =$ Borel σ-field under Skorohod topology, X real cadlag process,

and define

$\mu(A) = \text{Prob}(X \in A)$ for $A \in \mathcal{G}$

Then, μ is not separable unless all jump times of X have a discrete distribution

Say that $(\mu_n : n \geq 0)$ admits a **Skorohod representation** if

$$\boxed{X_n \sim \mu_n \text{ for each } n \geq 0 \text{ and } X_n \to X_0 \text{ in probability}}$$

for some random variables X_n (defined on the same probability space)

If non-separable probabilities on $\mathcal{B}$ exist, then:

- **Separability of μ_0 cannot be dropped**, even if a.s. convergence is weakened into convergence in probability. Indeed, it may be that $\mu_n \to \mu_0$ weakly, and yet (μ_n) does not have a Skorohod representation

- **Convergence a.s. is too much**. Indeed, it may be that (μ_n) admits a Skorohod representation, but no random variables Y_n satisfy $Y_n \sim \mu_n$ for each $n \geq 0$ and $Y_n \to Y_0$ a.s.

A conjecture

If (μ_n) has a Skorohod representation, then

$$\boxed{\lim_n \sup_f \, |\mu_n(f) - \mu_0(f)| = 0}$$

where sup is over those $f : S \to [-1, 1]$ which are 1-Lipschitz. Also, when μ_0 is separable, the above condition amounts to $\mu_n \to \mu_0$ weakly. Thus, a **conjecture** is:

(μ_n) has a Skorohod representation

if and only if

$\lim_n \, D(\mu_n, \mu_0) = 0$

where D is some distance (or discrepancy index) between probability measures. Two intriguing choices of D are

$$D(\mu,\nu) = L(\mu,\nu) = \sup_f \, |\mu(f) - \nu(f)|$$

$$D(\mu,\nu) = W(\mu,\nu) = \inf \, E\{1 \wedge d(X,Y)\}$$

where inf is over those pairs (X,Y) satisfying $X \sim \mu$ and $Y \sim \nu$.

To make W well defined, we assume

$$\boxed{\sigma(d) \subset \mathcal{B} \otimes \mathcal{B}}$$

Note also that

$$L \leq 2\,W$$

We dont know whether the conjecture is true with $D = L$ or $D = W$, but we mention two attempts to prove it

First attempt: D=W

In a sense, W is the natural choice of D. However, W could not be a distance (we dont know whether the triangle inequality holds)

If (μ_n) has a Skorohod representation, then

$\lim_n W(\mu_n, \mu_0) = 0$

Conversely, under the above condition, there is a sequence $(\gamma_n : n \geq 1)$ of laws on $\mathcal{B} \otimes \mathcal{B}$ such that

γ_n has marginals μ_0 and μ_n

$\lim_n \gamma_n\{(x, y) : d(x, y) > \epsilon\} = 0$ for all $\epsilon > 0$

Thus, it would be enough a sequence $(X_n : n \geq 0)$ of random variables (defined on the same probability space) such that

$\boxed{(X_0, X_n) \sim \gamma_n \text{ for each } n \geq 1}$

Unfortunately, such sequence $(X_n : n \geq 0)$ fails to exist for an arbitrary choice of $(\gamma_n : n \geq 1)$. However, things can be adjusted in a finitely additive setting. (This is not so unusual, incidentally). In fact,

Thm: If $\lim_n W(\mu_n, \mu_0) = 0$, there are a finitely additive probability space $(\Omega, \mathcal{A}, P)$ and random variables $X_n : \Omega \to S$ such that

$X_n \to X_0$ in probability, $X_0 \sim \mu_0$ and

$E_P\{f(X_n)\} = \mu_n(f)$ for each $n \geq 1$ and each bounded continuous f

Second attempt: Skorohod thm under a stronger distance

Suppose now that (S, d) is nice, say S Polish under d, so that there are no problems with Skorohod thm under d. However, we want

$$X_n \sim \mu_n \text{ for each } n \geq 0 \text{ and } \boxed{\rho(X_n, X_0) \to 0 \text{ in probability}}$$

where ρ is another distance on S, typically stronger than d

The **motivating example** is:

$S = \{\text{real cadlag functions on } [0,1]\}$,

$d =$ Skorohod distance, $\rho =$ uniform distance

In real problems, one has cadlag processes Y_n, whose distributions are assessed on the Skorohod Borel sets. Indeed, such distributions may even fail to exist on the uniform Borel sets. Yet, one could look for some processes X_n satisfying

$$X_n \sim Y_n \text{ for each } n \geq 0 \text{ and } \boxed{\sup_t |X_n(t) - X_0(t)| \to 0 \text{ in probability}}$$

As a **further example**, for $x, y \in S$, define

$$\rho(x,y) = \sup_{f \in F} |f(x) - f(y)|$$

where F is a collection of real Borel functions on S. Then, ρ is a distance under mild conditions on F, and we could aim to random variables X_n such that

$$X_n \sim \mu_n \text{ for each } n \geq 0 \text{ and}$$

$$\boxed{\sup_{f \in F} |f(X_n) - f(X_0)| \to 0 \text{ in probability}}$$

Anyhow, the following result is available

Thm: Suppose $\rho : S \times S \to R$ is lower-semi-continuous with respect to d. There are random variables X_n such that

$X_n \sim \mu_n$ for each $n \geq 0$ and $\rho(X_n, X_0) \to 0$ in probability

if and only if

$\lim_n \sup_f |\mu_n(f) - \mu_0(f)| = 0$

where sup is over those $f : S \to [-1, 1]$ which are $\mathcal{B}$-measurable and 1-Lipschitz with respect to ρ

Remark: It is (essentially) enough that ρ is Borel measurable with respect to d. Also, instead of (S, d) Polish, it is sufficient (S, d) separable and μ_n perfect for each $n > 0$