

EQUIVALENT OR ABSOLUTELY CONTINUOUS PROBABILITY MEASURES WITH GIVEN MARGINALS

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ABSTRACT. Let $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ be measurable spaces. Suppose we are given a probability α on \mathcal{A} , a probability β on \mathcal{B} and a probability μ on the product σ -field $\mathcal{A} \otimes \mathcal{B}$. Is there a probability ν on $\mathcal{A} \otimes \mathcal{B}$, with marginals α and β , such that $\nu \ll \mu$ or $\nu \sim \mu$? Such a ν , provided it exists, may be useful with regard to equivalent martingale measures and mass transportation. Various conditions for the existence of ν are provided, distinguishing $\nu \ll \mu$ from $\nu \sim \mu$.

1. INTRODUCTION

1.1. **The problem.** Let $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ be measurable spaces. Fix a probability measure (p.m.) α on \mathcal{A} , a p.m. β on \mathcal{B} and a p.m. μ on the product σ -field $\mathcal{A} \otimes \mathcal{B}$. Let \mathcal{P} denote the collection of all p.m.'s on $\mathcal{A} \otimes \mathcal{B}$ and

$$\Gamma(\alpha, \beta) = \{\nu \in \mathcal{P} : \text{the marginals of } \nu \text{ are } \alpha \text{ and } \beta\}.$$

This paper is concerned with the following questions:

- (a) Is there $\nu \in \Gamma(\alpha, \beta)$ such that $\nu \ll \mu$?
- (b) Is there $\nu \in \Gamma(\alpha, \beta)$ such that $\nu \sim \mu$?

Problems (a)-(b) are motivated in Section 2. Here, we introduce some further notation and summarize the content of this paper.

1.2. **Notation and preliminary facts.** Let (Ω, \mathcal{F}, P) be a probability space. We write

$$E_P(X) = \int X dP$$

whenever X is a real P -integrable random variable. Given another p.m. Q on \mathcal{F} , we write $P \ll Q$ to mean that $P(A) = 0$ whenever $A \in \mathcal{F}$ and $Q(A) = 0$. Similarly, $P \sim Q$ stands for $P \ll Q$ and $Q \ll P$. The notations $P \ll Q$ and $P \sim Q$ have the same meaning even if \mathcal{F} is a field (and not necessarily a σ -field) and P, Q are finitely additive probabilities (and not necessarily p.m.'s). Further, P is *perfect* if, for each measurable function $f : \Omega \rightarrow \mathbb{R}$, there is a real Borel set B such that $B \subset f(\Omega)$ and $P(f \in B) = 1$. If Ω is separable metric and \mathcal{F} the Borel σ -field, then P is perfect if and only if it is tight. Thus, for P to be perfect, it suffices that Ω is a universally measurable subset (in particular, a Borel subset) of a Polish space and \mathcal{F} the Borel σ -field. We refer to [6] for more information on perfect p.m.'s.

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In the sequel, the reference p.m. $\mu \in \mathcal{P}$ is *fixed*. Moreover, for each $\nu \in \mathcal{P}$, the marginals ν_1 and ν_2 of ν are meant as

$$\nu_1(A) = \nu(A \times \mathcal{Y}) \quad \text{and} \quad \nu_2(B) = \nu(\mathcal{X} \times B) \quad \text{where } A \in \mathcal{A} \text{ and } B \in \mathcal{B}.$$

Finally, m denotes the Lebesgue measure on the Borel σ -field of $[0, 1]$ and we adopt the following convention. If \mathcal{X} or \mathcal{Y} are topological spaces, then \mathcal{A} or \mathcal{B} are always taken to be the Borel σ -fields.

1.3. Outline. This paper consists of four sections. Section 2 provides some motivations to problems (a)-(b) while Section 5 includes concluding remarks. The core of the paper are Sections 3 and 4 which are concerned with problems (a) and (b), respectively. The main results are various conditions for problems (a)-(b) to admit a solution. Among other things, finitely additive solutions are considered as well.

2. MOTIVATIONS

In principle, problems (a)-(b) are noteworthy in all those fields where $\Gamma(\alpha, \beta)$ plays a role. In all such fields, in fact, there may be reasons for investigating proper subsets of $\Gamma(\alpha, \beta)$, such as

$$\Lambda_0 = \{\nu \in \Gamma(\alpha, \beta) : \nu \ll \mu\} \quad \text{or} \quad \Lambda_1 = \{\nu \in \Gamma(\alpha, \beta) : \nu \sim \mu\}.$$

And an obvious, preliminary question is whether $\Lambda_0 \neq \emptyset$ or $\Lambda_1 \neq \emptyset$.

We also note that, in addition to their possible applied interest, problems (a)-(b) are quite natural from the foundational point of view. Nevertheless, to our knowledge, they have been neglected so far. Apart from a recent paper [2, Example 15] we are not aware of any explicit reference.

In particular, problems (a)-(b) are not covered by the well known results by Strassen [8]. More precisely, such results do not apply to problem (b), for Λ_1 fails to be closed in any reasonable topology on \mathcal{P} . Instead, Strassen's ideas can be adapted to problem (a), since Λ_0 is sequentially closed if \mathcal{P} is given the topology of setwise convergence. Some Strassen-type solutions to problem (a) are actually provided by Theorems 5 and 6.

We next present a few examples. To fix ideas, we focus on some specific issues, but the ensuing remarks essentially extend to all areas where $\Gamma(\alpha, \beta)$ is involved.

Example 1. (Mass transportation). Let C be a non-negative measurable function on $\mathcal{X} \times \mathcal{Y}$. Here, $C(x, y)$ is regarded as the cost per unit mass for transporting a material from $x \in \mathcal{X}$ to $y \in \mathcal{Y}$. Such units are distributed according to α , before transportation, and according to β after transportation. Therefore, each member of $\Gamma(\alpha, \beta)$ is called a transport plan. Given $\Lambda \subset \Gamma(\alpha, \beta)$, say that ν is an *optimal transport plan for Λ* if $\nu \in \Lambda$ and $E_\nu(C) = \min_{\lambda \in \Lambda} E_\lambda(C)$. In this framework, it could be reasonable to choose Λ such that $\Lambda \subset \Lambda_0$ or $\Lambda \subset \Lambda_1$, provided of course $\Lambda_0 \neq \emptyset$ or $\Lambda_1 \neq \emptyset$. As to $\Lambda \subset \Lambda_0$, sometimes, it makes sense to focus only on those transport plans which have a density with respect to some reference measure μ . This happens for instance in [5], with $\mathcal{X} = \mathcal{Y} = \mathbb{R}^p$ and μ equivalent to Lebesgue measure, in order to take capacity constraints into account. A further (concrete) reason for taking $\Lambda \subset \Lambda_0$ is the following. It may be that some $H \in \mathcal{A} \otimes \mathcal{B}$ is "forbidden", in the sense that $(x, y) \in H$ does not make sense for the problem at hand. Situations of this type are usually modeled by letting $C = \infty$ on H . An alternative option could be obtained by letting $\Lambda \subset \Lambda_0$ and taking μ such that $\mu(H) = 0$. Finally, quite analogous considerations hold for $\Lambda \subset \Lambda_1$.

We hope to devote further work to more specific applications in the near future. As an interesting hint provided by one of the referees, the results in this paper could be applicable to find Monge solutions in those cases where the underlying optimal transport plan is not unique.

Example 2. (Equivalent martingale measures). Let $X = \{X_t : 0 \leq t \leq 1\}$ and $Y = \{Y_t : 0 \leq t \leq 1\}$ be real cadlag processes on the probability space (Ω, \mathcal{F}, P) . A p.m. Q on \mathcal{F} is said to be an equivalent martingale measure (e.m.m.) if $Q \sim P$ and both X and Y are Q -martingales.

Let D be the set of real cadlag functions on $[0, 1]$, equipped with the Skorohod topology, and let \mathcal{S} be the Borel σ -field on D . We make two simplifying assumptions. Firstly, X and Y are taken to be canonical processes. Namely, we let

$$\Omega = D \times D, \quad \mathcal{F} = \mathcal{S} \otimes \mathcal{S}, \quad X_t(\omega) = \omega_1(t), \quad Y_t(\omega) = \omega_2(t),$$

where $t \in [0, 1]$ and $\omega = (\omega_1, \omega_2) \in D \times D$. Secondly, and more importantly, X and Y are required to be Q -martingales with respect to their canonical filtrations only.

Under these assumptions, existence of an e.m.m. fits nicely into the framework of this paper. It suffices to let $\mathcal{X} = \mathcal{Y} = D$, $\mu = P$, and to choose α and β such that

$$\alpha \sim \mu_1, \quad \beta \sim \mu_2, \quad X \text{ is an } \alpha\text{-martingale}, \quad Y \text{ is a } \beta\text{-martingale}$$

where μ_1 and μ_2 are the marginals of μ . In fact, if such α and β do not exist, no e.m.m. is available. Otherwise, if α and β exist, an e.m.m. is exactly a solution to problem (b). And the condition $\mu_1 \times \mu_2 \ll \mu$ guarantees the existence of an e.m.m. by Theorem 11 below.

The situation is more complicated, even though more realistic, when X and Y are asked to be martingales with respect to a common filtration $\{\mathcal{F}_t : 0 \leq t \leq 1\}$ on $\Omega = D \times D$. In this case, existence of an e.m.m. can not be easily seen as a particular case of problem (b). In fact, to decide whether X and Y are martingales with respect to $\{\mathcal{F}_t : 0 \leq t \leq 1\}$, one needs some further information beyond α and β ; see also Section 5.

Example 3. (Contingency tables). For definiteness, a contingency table is identified with a non-negative $p \times q$ matrix $T = (t_{i,j})$ such that $\sum_{i,j} t_{i,j} = 1$. If S and T are contingency tables, write $S \ll T$ if $t_{i,j} = 0 \Rightarrow s_{i,j} = 0$, and $S \sim T$ if $t_{i,j} = 0 \Leftrightarrow s_{i,j} = 0$. Let $\alpha = (\alpha_1, \dots, \alpha_p)$ and $\beta = (\beta_1, \dots, \beta_q)$ be non-negative vectors such that $\sum_i \alpha_i = \sum_j \beta_j = 1$. Suppose we are given α , β and a contingency table T . Then, the following natural questions arise. Is there a contingency table S such that $S \ll T$ and

$$\sum_j s_{i,j} = \alpha_i, \quad \sum_i s_{i,j} = \beta_j \quad \text{for all } i, j?$$

Similarly, is there a contingency table S satisfying the above condition as well as $S \sim T$?

3. ABSOLUTELY CONTINUOUS LAWS WITH GIVEN MARGINALS

We begin with a definition. Let $\nu \in \mathcal{P}$. Say that ν is dominated by μ on rectangles, written $\nu \ll_R \mu$, if

$$\mu(A \times B) = 0 \quad \implies \quad \nu(A \times B) = 0 \quad \text{whenever } A \in \mathcal{A} \text{ and } B \in \mathcal{B}.$$

Then, $\nu \ll \mu$ implies $\nu \ll_R \mu$, but not conversely. As an example, take

$$\mathcal{X} = \mathcal{Y} = [0, 1], \quad \mu = m \times m \quad \text{and} \quad \nu(\cdot) = m\{x : (x, x) \in \cdot\}.$$

Since ν is supported by the diagonal, ν fails to be absolutely continuous with respect to μ . However, $\nu \ll_R \mu$ for

$$\nu(A \times B) = m(A \cap B) \leq m(A) \wedge m(B) \quad \text{whenever } A \in \mathcal{A} \text{ and } B \in \mathcal{B}.$$

Next result gives conditions for $\nu \ll \mu$ and $\nu \ll_R \mu$ to be equivalent. (Such equivalence is also briefly discussed in Section 5). Let μ_1 and μ_2 denote the marginals of μ .

Lemma 4. *Suppose \mathcal{X} and \mathcal{Y} are separable metric spaces. Let $\nu \in \mathcal{P}$. Then,*

$$\nu \ll \mu \quad \iff \quad \nu \ll_R \mu$$

provided (at least) one of the following conditions holds:

- (i) μ_1 or μ_2 is discrete;
- (ii) μ is supported by the graph of a measurable function (from \mathcal{X} into \mathcal{Y} or from \mathcal{Y} into \mathcal{X});
- (iii) There is a σ -finite measure γ on $\mathcal{A} \otimes \mathcal{B}$ such that

$$\nu \ll \gamma, \quad \mu \ll \gamma \quad \text{and} \quad \nu(\partial\{f = 0\}) = 0,$$

where f is a density of μ with respect to γ and $\partial\{f = 0\}$ is the boundary of the set $\{f = 0\}$.

Proof. Let ν_1 and ν_2 denote the marginals of ν . If $\nu \ll_R \mu$, as assumed throughout this proof, then $\nu_1 \ll \mu_1$ and $\nu_2 \ll \mu_2$. Furthermore, ν is dominated by μ on the open sets, that is, $\nu(C) = 0$ whenever $\mu(C) = 0$ and $C \subset \mathcal{X} \times \mathcal{Y}$ is open. In fact, each open subset of $\mathcal{X} \times \mathcal{Y}$ is a countable union of (open) rectangles.

- (i) Suppose μ_1 discrete, i.e., $\mu_1(A) = 1$ for some countable $A \subset \mathcal{X}$. Since $\nu_1 \ll \mu_1$, then $\nu_1(A) = 1$. Fix $C \in \mathcal{A} \otimes \mathcal{B}$ and let $C_x = \{y \in \mathcal{Y} : (x, y) \in C\}$ denote the section of C with respect to $x \in \mathcal{X}$. Since A is countable,

$$\mu(C) = \sum_{x \in A} \mu(\{x\} \times C_x) \quad \text{and} \quad \nu(C) = \sum_{x \in A} \nu(\{x\} \times C_x).$$

If $\mu(C) = 0$, then $\nu(\{x\} \times C_x) = 0$ for all $x \in A$ because of $\nu \ll_R \mu$. Thus, $\nu(C) = 0$. The proof is exactly the same if μ_2 is discrete.

- (ii) Let $\mu(G) = 1$, where $G = \{(x, g(x)) : x \in \mathcal{X}\}$ and $g : \mathcal{X} \rightarrow \mathcal{Y}$. We first suppose g continuous. Then, G is closed, so that $\nu(G) = 1$ as well. Hence, both μ and ν can be written as

$$\mu(C) = \mu_1\{x : (x, g(x)) \in C\} \quad \text{and} \quad \nu(C) = \nu_1\{x : (x, g(x)) \in C\}$$

for all $C \in \mathcal{A} \otimes \mathcal{B}$. Thus, $\nu \ll \mu$ follows from $\nu_1 \ll \mu_1$. Next, suppose g measurable. By Lusin's theorem, given $\epsilon \in (0, 1)$, there is a closed set $F \subset \mathcal{X}$ such that $\mu_1(F^c) < \epsilon$ and g is continuous on F . Since $\nu_1 \ll \mu_1$, it can be assumed $\nu_1(F) > 0$. Define

$$\nu_F(\cdot) = \nu(\cdot \mid F \times \mathcal{Y}) \quad \text{and} \quad \mu_F(\cdot) = \mu(\cdot \mid F \times \mathcal{Y}).$$

By what already proved, since $\nu_F \ll_R \mu_F$ and g is continuous on F , one obtains $\nu_F \ll \mu_F$. Hence, $\nu \ll \mu$ follows from $\nu_1 \ll \mu_1$ and the arbitrariness

of ϵ . The proof is exactly the same if $G = \{(h(y), y) : y \in \mathcal{Y}\}$ with $h : \mathcal{Y} \rightarrow \mathcal{X}$ measurable.

(iii) Fix $C \in \mathcal{A} \otimes \mathcal{B}$ with $\mu(C) = 0$. Since

$$\int_{C \cap \{f > 0\}} f d\gamma = 0,$$

one obtains $\gamma(C \cap \{f > 0\}) = 0$. Since $\mu\{f = 0\} = 0$, one also obtains $\nu(\{f = 0\}^0) = 0$, where H^0 denotes the interior of H . Hence,

$$\nu(C) = \nu(C \cap \{f = 0\}) \leq \nu\{f = 0\} \leq \nu(\partial\{f = 0\}) = 0.$$

□

In connection with (iii) of Lemma 4 it is worth noting that, since $\nu \ll \gamma$, the condition $\nu(\partial\{f = 0\}) = 0$ is automatically true whenever

$$\gamma(\partial\{f = 0\}) = 0.$$

We next turn to problem (a). Let α be a p.m. on \mathcal{A} and β a p.m. on \mathcal{B} . For all functions $f : \mathcal{X} \rightarrow \mathbb{R}$ and $g : \mathcal{Y} \rightarrow \mathbb{R}$, denote

$$f \oplus g(x, y) = f(x) + g(y) \quad \text{for } (x, y) \in \mathcal{X} \times \mathcal{Y}.$$

Moreover, suppose \mathcal{X} and \mathcal{Y} are Polish spaces and \mathcal{P} is given the topology of weak convergence of p.m.'s.

By a classical result of Strassen [8], if $\Lambda \subset \mathcal{P}$ is convex and closed,

$$\Gamma(\alpha, \beta) \cap \Lambda \neq \emptyset$$

if and only if

$$(1) \quad E_\alpha(f) + E_\beta(g) \geq \inf_{\lambda \in \Lambda} E_\lambda(f \oplus g)$$

for all bounded continuous $f : \mathcal{X} \rightarrow \mathbb{R}$ and $g : \mathcal{Y} \rightarrow \mathbb{R}$.

Basing on this fact, it is tempting to let $\Lambda = \{\nu \in \mathcal{P} : \nu \ll \mu\}$ in condition (1). But such a Λ is not closed, and in fact Strassen's result does not apply to problem (a). As a trivial example, take $\Lambda = \{\nu \in \mathcal{P} : \nu \ll \mu\}$ and

$$\mathcal{X} = \mathcal{Y} = [0, 1], \quad \mu = m \times m, \quad \alpha = m, \quad \beta = \delta_0.$$

Since $\beta = \delta_0$ is not absolutely continuous with respect to $\mu_2 = m$, problem (a) admits no solutions. Nevertheless, if β_n is uniform on $(0, 1/n)$, then $\alpha \times \beta_n \in \Lambda$ and $\beta_n \rightarrow \beta$ weakly. Therefore,

$$E_\alpha(f) + E_\beta(g) = \lim_n \{E_\alpha(f) + E_{\beta_n}(g)\} = \lim_n E_{\alpha \times \beta_n}(f \oplus g) \geq \inf_{\lambda \in \Lambda} E_\lambda(f \oplus g)$$

for all bounded continuous f and g .

Even though Strassen's result does not work as it stands, the underlying ideas can be adapted to problem (a). In fact, $\Lambda = \{\nu \in \mathcal{P} : \nu \ll \mu\}$ is sequentially closed if \mathcal{P} is given the topology of setwise convergence, that is, the topology on \mathcal{P} generated by the maps $\lambda \mapsto \lambda(H)$ for all $H \in \mathcal{A} \otimes \mathcal{B}$. Similarly, $\Lambda = \{\nu \in \mathcal{P} : \nu \ll_R \mu\}$ is sequentially closed in such topology. This suggests to require condition (1), with $\Lambda = \{\nu \in \mathcal{P} : \nu \ll \mu\}$ or $\Lambda = \{\nu \in \mathcal{P} : \nu \ll_R \mu\}$, replacing continuous functions with *measurable* functions.

Theorem 5. *Let α be a p.m. on \mathcal{A} and β a p.m. on \mathcal{B} . Suppose at least one between α and β is perfect and define*

$$\Lambda = \{\nu \in \mathcal{P} : \nu \ll_R \mu\}.$$

Then, $\Gamma(\alpha, \beta) \cap \Lambda \neq \emptyset$ if and only if condition (1) holds for all bounded measurable functions $f : \mathcal{X} \rightarrow \mathbb{R}$ and $g : \mathcal{Y} \rightarrow \mathbb{R}$.

Proof. If $\nu \in \Gamma(\alpha, \beta) \cap \Lambda$ and f and g are bounded measurable, then

$$E_\alpha(f) + E_\beta(g) = E_\nu(f \oplus g) \geq \inf_{\lambda \in \Lambda} E_\lambda(f \oplus g)$$

where the equality is because $\nu \in \Gamma(\alpha, \beta)$ and the inequality for $\nu \in \Lambda$. Conversely, suppose condition (1) holds for all bounded measurable f and g . Define

$$X_{f,g}(\lambda) = E_\alpha(f) + E_\beta(g) - E_\lambda(f \oplus g)$$

for all $\lambda \in \Lambda$ and all bounded measurable f and g , and let

$$S = \{X_{f,g} : f \text{ and } g \text{ bounded and measurable}\}.$$

Then, S is a linear space of real bounded functions on the set Λ . By condition (1),

$$\sup_{\lambda \in \Lambda} X(\lambda) \geq 0 \quad \text{for all } X \in S.$$

Hence, by de Finetti's coherence principle, there is a finitely additive probability P on the power set of Λ such that

$$\int_{\Lambda} X(\lambda) P(d\lambda) = 0 \quad \text{for all } X \in S;$$

see e.g. [1] and [2].

Let \mathcal{R} be the field generated by $A \times B$, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and

$$\nu_0(C) = \int_{\Lambda} \lambda(C) P(d\lambda) \quad \text{for all } C \in \mathcal{R}.$$

Such ν_0 is a finitely additive probability on \mathcal{R} . In view of [7, Theorem 2], since one between α and β is perfect, ν_0 is actually σ -additive on \mathcal{R} .

Take ν to be the (only) σ -additive extension of ν_0 to $\sigma(\mathcal{R}) = \mathcal{A} \otimes \mathcal{B}$. Given $A \in \mathcal{A}$, let

$$X(\lambda) = X_{I_A, 0}(\lambda) = \alpha(A) - \lambda(A \times \mathcal{Y}).$$

Then,

$$\nu(A \times \mathcal{Y}) = \nu_0(A \times \mathcal{Y}) = \alpha(A) - \int_{\Lambda} X(\lambda) P(d\lambda) = \alpha(A).$$

Similarly, $\nu(\mathcal{X} \times B) = \beta(B)$ for all $B \in \mathcal{B}$. Hence, $\nu \in \Gamma(\alpha, \beta)$. Further, if $\mu(A \times B) = 0$ for some $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $\lambda(A \times B) = 0$ for each $\lambda \in \Lambda$, which in turn implies $\nu(A \times B) = \nu_0(A \times B) = 0$. Thus, $\nu \in \Gamma(\alpha, \beta) \cap \Lambda$. \square

Theorem 5 provides only a partial solution to problem (a), for one only obtains $\nu \ll_R \mu$ (and not $\nu \ll \mu$) for some $\nu \in \Gamma(\alpha, \beta)$. Under the conditions of Lemma 4, however, $\nu \ll_R \mu$ amounts to $\nu \ll \mu$ and Theorem 5 yields a full solution. If μ has at least one discrete marginal, for instance, there exists $\nu \in \Gamma(\alpha, \beta)$ such that $\nu \ll \mu$ if and only if condition (1) holds, with $\Lambda = \{\lambda \in \mathcal{P} : \lambda \ll_R \mu\}$, for all bounded measurable f and g .

The argument which leads to Theorem 5 allows to obtain some other results. Next Theorems 6 and 7 are examples of this claim.

Say that $\Lambda \subset \mathcal{P}$ is *uniformly dominated by μ* if, for each $\epsilon > 0$, there is $\delta > 0$ such that $\sup_{\lambda \in \Lambda} \lambda(C) \leq \epsilon$ whenever $C \in \mathcal{A} \otimes \mathcal{B}$ and $\mu(C) < \delta$. This notion of absolute continuity is well known, mainly with regard to Vitali-Hahn-Saks theorem and related topics. A straightforward example of Λ uniformly dominated by μ is

$$\Lambda = \{\lambda \in \mathcal{P} : \lambda \leq r \mu\} \quad \text{for some constant } r.$$

Theorem 6. *Let α be a p.m. on \mathcal{A} and β a p.m. on \mathcal{B} . Suppose Λ is uniformly dominated by μ and condition (1) holds for all bounded measurable $f : \mathcal{X} \rightarrow \mathbb{R}$ and $g : \mathcal{Y} \rightarrow \mathbb{R}$. Then, there is $\nu \in \Gamma(\alpha, \beta)$ such that $\nu \ll \mu$.*

Proof. Define S as in the proof of Theorem 5. Arguing as in such a proof, condition (1) implies $\int_{\Lambda} X(\lambda) P(d\lambda) = 0$ for all $X \in S$ and some finitely additive probability P on the power set of Λ . Let $\nu(C) = \int_{\Lambda} \lambda(C) P(d\lambda)$ for all $C \in \mathcal{A} \otimes \mathcal{B}$. Then, ν is a finitely additive probability on $\mathcal{A} \otimes \mathcal{B}$ with marginals α and β . Since Λ is uniformly dominated by μ and $\nu(\cdot) \leq \sup_{\lambda \in \Lambda} \lambda(\cdot)$, one obtains $\lim_n \nu(C_n) = 0$ for every sequence $C_n \in \mathcal{A} \otimes \mathcal{B}$ such that $\lim_n \mu(C_n) = 0$. Thus, ν is σ -additive (for μ is σ -additive) so that $\nu \in \Gamma(\alpha, \beta)$ and $\nu \ll \mu$. \square

An open problem is whether condition (1) generally implies $\Gamma(\alpha, \beta) \cap \Lambda \neq \emptyset$ when Λ is taken to be $\Lambda = \{\lambda \in \mathcal{P} : \lambda \ll \mu\}$. This is actually the case under the conditions of Lemma 4. Furthermore, concerning *finitely additive* solutions to problem (a), the following result is available.

Theorem 7. *Let α be a p.m. on \mathcal{A} and β a p.m. on \mathcal{B} . Suppose condition (1) holds, with $\Lambda = \{\lambda \in \mathcal{P} : \lambda \ll_R \mu\}$, for all bounded measurable $f : \mathcal{X} \rightarrow \mathbb{R}$ and $g : \mathcal{Y} \rightarrow \mathbb{R}$. Then, there is a finitely additive probability ν on $\mathcal{A} \otimes \mathcal{B}$, with marginals α and β , such that $\nu \ll \mu$.*

Proof. We first prove a claim.

Claim: Let P_i be a finitely additive probability on the field \mathcal{F}_i , $i = 1, 2$, and let $\mathcal{F}_1 \subset \mathcal{F}_2$. Then, P_1 can be extended to a finitely additive probability P on \mathcal{F}_2 such that $P \ll P_2$ if and only if $P_1 \ll (P_2|_{\mathcal{F}_1})$, where $P_2|_{\mathcal{F}_1}$ is the restriction of P_2 on \mathcal{F}_1 .

In fact, the "only if" part is trivial. Conversely, suppose $P_1 \ll (P_2|_{\mathcal{F}_1})$ and define $\mathcal{D} = \{B \in \mathcal{F}_2 : P_2(B) \in \{0, 1\}\}$. Fix $A \in \mathcal{F}_1$ and $B \in \mathcal{D}$ with $A \subset B$. If $P_2(B) = 1$, then $P_1(A) \leq P_2(B)$. If $P_2(B) = 0$, then $A \subset B$ implies $P_2(A) = 0$. Since $A \in \mathcal{F}_1$ and $P_1 \ll (P_2|_{\mathcal{F}_1})$, one obtains $P_1(A) = 0$, and again $P_1(A) \leq P_2(B)$. By [3, Theorem 3.6.1], there is a finitely additive probability P on \mathcal{F}_2 such that $P = P_1$ on \mathcal{F}_1 and $P = P_2$ on \mathcal{D} . Such a P does the job.

We next prove Theorem 7. Define ν_0 as in the proof of Theorem 5. Such ν_0 is a finitely additive probability, defined on the field \mathcal{R} generated by rectangles, with marginals α and β . It is straightforward to verify that $\nu_0 \ll (\mu|_{\mathcal{R}})$. Thus, it suffices to apply the previous claim with

$$\mathcal{F}_1 = \mathcal{R}, \quad \mathcal{F}_2 = \mathcal{A} \otimes \mathcal{B}, \quad P_1 = \nu_0, \quad P_2 = \mu.$$

\square

Incidentally, unlike Theorem 5, Theorems 6 and 7 do not request α or β to be perfect. It may be that perfectness can be dropped from Theorem 5 as well, but we have not a proof of this fact.

So far, μ , α and β are all fixed. We now take a different point of view, we fix μ only while α and β are allowed to vary subject to the condition $\alpha \ll \mu_1$ and $\beta \ll \mu_2$ (recall that μ_1 and μ_2 are the marginals of μ). Such condition can not be bypassed, being necessary for problem (a) to admit a solution.

Suppose μ dominates the product of its marginals, namely

$$\mu_1 \times \mu_2 \ll \mu.$$

Then, for all $\alpha \ll \mu_1$ and $\beta \ll \mu_2$, one trivially obtains $\alpha \times \beta \in \Gamma(\alpha, \beta)$ and $\alpha \times \beta \ll \mu_1 \times \mu_2 \ll \mu$. Hence, $\alpha \times \beta$ is a solution to problem (a). In other terms, if $\mu_1 \times \mu_2 \ll \mu$, then

- (2) for all p.m.'s α on \mathcal{A} and β on \mathcal{B} , satisfying $\alpha \ll \mu_1$
and $\beta \ll \mu_2$, there is $\nu \in \Gamma(\alpha, \beta)$ such that $\nu \ll \mu$.

As a last result on problem (a) we now show that, under the conditions of Lemma 4, the converse of the above implication is true as well.

Theorem 8. *Let $\gamma = \gamma_1 \times \gamma_2$, where γ_1 and γ_2 are σ -finite measures on \mathcal{A} and \mathcal{B} respectively, and let \mathcal{X} and \mathcal{Y} be separable metric spaces. Suppose μ_1 is discrete, or μ_2 is discrete, or $\mu \ll \gamma$ and $\gamma(\partial\{f = 0\}) = 0$ where f is a density of μ with respect to γ . Then,*

$$\mu_1 \times \mu_2 \ll \mu \iff \text{condition (2)} \iff \mu_1 \times \mu_2 \ll_R \mu.$$

Proof. It has been proved in the text that $\mu_1 \times \mu_2 \ll \mu$ implies condition (2). Next, suppose μ fails to dominate $\mu_1 \times \mu_2$ on rectangles, that is, $\mu(A \times B) = 0$, $\mu_1(A) > 0$ and $\mu_2(B) > 0$ for some $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Let $\alpha(\cdot) = \mu_1(\cdot | A)$ and $\beta(\cdot) = \mu_2(\cdot | B)$. Then, $\alpha \ll \mu_1$ and $\beta \ll \mu_2$. However, if $\nu \in \Gamma(\alpha, \beta)$ and $\nu \ll \mu$, one obtains the absurd conclusion

$$\nu(A \times B^c) = \nu(A \times \mathcal{Y}) = \alpha(A) = 1 \quad \text{and} \quad \nu(A^c \times B) = \nu(\mathcal{X} \times B) = \beta(B) = 1.$$

Hence, condition (2) yields $\mu_1 \times \mu_2 \ll_R \mu$. Finally, suppose $\mu_1 \times \mu_2 \ll_R \mu$. If μ_1 or μ_2 is discrete, Lemma 4 implies $\mu_1 \times \mu_2 \ll \mu$. Hence, suppose $\mu \ll \gamma$ and $\gamma(\partial\{f = 0\}) = 0$. Since $\mu \ll \gamma$, then $\mu_1 \ll \gamma_1$ and $\mu_2 \ll \gamma_2$. Thus, $\mu_1 \times \mu_2 \ll \gamma_1 \times \gamma_2 = \gamma$ and again Lemma 4 yields $\mu_1 \times \mu_2 \ll \mu$. \square

Note that, under the assumptions of Theorem 8, condition (2) amounts to

$$\mu(A \times B) = 0 \iff \mu_1(A) \wedge \mu_2(B) = 0 \quad \text{whenever } A \in \mathcal{A} \text{ and } B \in \mathcal{B}.$$

The above condition, in fact, is clearly equivalent to $\mu_1 \times \mu_2 \ll_R \mu$.

4. EQUIVALENT LAWS WITH GIVEN MARGINALS

A general approach to problem (b), introduced in [1]-[2], is the following.

Recall that a *determining* class for a measurable space (Ω, \mathcal{F}) is a class H of real bounded measurable functions on Ω such that

$$P = Q \iff E_P(h) = E_Q(h) \quad \text{for each } h \in H$$

whenever P and Q are p.m.'s on \mathcal{F} . For instance, $H = \{I_A : A \in \mathcal{F}_0\}$ is a determining class if \mathcal{F}_0 is a field such that $\mathcal{F} = \sigma(\mathcal{F}_0)$. Or else, $H = \{\text{bounded continuous functions on } \Omega\}$ is a determining class if Ω is a metric space and \mathcal{F} the Borel σ -field.

Fix a determining class F for $(\mathcal{X}, \mathcal{A})$ and a determining class G for $(\mathcal{Y}, \mathcal{B})$. It is also assumed that F and G are linear spaces. Further, given α and β , define

$$L_0 = \left\{ f \oplus g - E_\alpha(f) - E_\beta(g) : f \in F \text{ and } g \in G \right\}.$$

Such L_0 is a linear space of bounded random variables on the measurable space $(\mathcal{X} \times \mathcal{Y}, \mathcal{A} \otimes \mathcal{B})$ and has the property that

$$\nu \in \Gamma(\alpha, \beta) \iff \nu \in \mathcal{P} \text{ and } E_\nu(X) = 0 \text{ for each } X \in L_0.$$

Thus, problem (b) can be stated as: Is there $\nu \in \mathcal{P}$ satisfying $\nu \sim \mu$ and $E_\nu(X) = 0$ for each $X \in L_0$? Next result gives a tool for answering this question.

Theorem 9. (Lemma 6 of [2]). *Let L be a linear space of real bounded random variables on the probability space $(\Omega, \mathcal{F}, P_0)$. There is a p.m. P on \mathcal{F} such that*

$$P \sim P_0 \quad \text{and} \quad E_P(X) = 0 \quad \text{for all } X \in L$$

if and only if there are a p.m. Q on \mathcal{F} and a constant $c < 1$ such that

$$Q \sim P_0 \quad \text{and} \quad |E_Q(X)| \leq c E_Q|X| \quad \text{for all } X \in L.$$

Letting

$$(\Omega, \mathcal{F}, P_0) = (\mathcal{X} \times \mathcal{Y}, \mathcal{A} \otimes \mathcal{B}, \mu) \quad \text{and} \quad L = L_0,$$

Theorem 9 yields the following result.

Theorem 10. *Let $\Lambda = \{\lambda \in \mathcal{P} : \lambda \sim \mu\}$ and $M = \{X \in L_0 : E_\mu|X| > 0\}$. Then, $\Gamma(\alpha, \beta) \cap \Lambda \neq \emptyset$ if and only if*

$$\inf_{\lambda \in \Lambda} \sup_{X \in M} \frac{|E_\lambda(X)|}{E_\lambda|X|} < 1.$$

On one hand, Theorem 10 formally solves problem (b). On the other hand, Theorem 10 is not very helpful in real problems, since the proposed condition is quite hard to be checked. There are some exceptions, however. In [2, Example 15], a usable condition for solving problem (b) is obtained through Theorem 10. We now (slightly) improve such condition. We also provide a new and simpler proof.

Theorem 11. *Let α and β be p.m.'s on \mathcal{A} and \mathcal{B} , respectively. If*

$$\alpha \sim \mu_1, \quad \beta \sim \mu_2 \quad \text{and} \quad \mu_1 \times \mu_2 \ll \mu,$$

there is $\nu \in \Gamma(\alpha, \beta)$ such that $\nu \sim \mu$.

Proof. For each $\lambda \in \mathcal{P}$, let λ_1 and λ_2 denote the marginals of λ . Then,

$$(3) \quad \mu = c\lambda + (1-c)\rho$$

for some $c \in (0, 1)$ and some $\lambda, \rho \in \mathcal{P}$ satisfying

$$\lambda \sim \mu, \quad \alpha \geq c\lambda_1 \quad \text{and} \quad \beta \geq c\lambda_2.$$

To prove (3), fix a density f_1 of α with respect to μ_1 , a density f_2 of β with respect to μ_2 , and define

$$\phi(x, y) = f_1(x) \wedge f_2(y) \wedge (1/2) \quad \text{for } (x, y) \in \mathcal{X} \times \mathcal{Y}.$$

Since $\alpha \sim \mu_1$ and $\beta \sim \mu_2$, then $\mu(\phi > 0) = 1$. Hence, $0 < E_\mu(\phi) \leq 1/2$. Let

$$c = E_\mu(\phi) \quad \text{and} \quad \lambda(H) = \frac{E_\mu\{I_H \phi\}}{c} \quad \text{for all } H \in \mathcal{A} \otimes \mathcal{B}.$$

Then, $\lambda \in \mathcal{P}$ and $\mu(H) - c\lambda(H) = E_\mu\{I_H(1-\phi)\} \geq (1/2)\mu(H)$ for all $H \in \mathcal{A} \otimes \mathcal{B}$. Hence, $\mu - c\lambda \geq 0$ so that

$$\rho := \frac{\mu - c\lambda}{1-c} \in \mathcal{P}.$$

Also, $\mu(\phi > 0) = 1$ implies $\lambda \sim \mu$, and

$$\alpha(A) - c\lambda_1(A) = \alpha(A) - \int_{A \times \mathcal{Y}} \phi d\mu \geq \alpha(A) - \int_A f_1 d\mu_1 = \alpha(A) - \alpha(A) = 0$$

for all $A \in \mathcal{A}$. Similarly, one obtains $\beta - c\lambda_2 \geq 0$.

For future purposes, we note that μ admits representation (3) provided $\alpha \sim \mu_1$ and $\beta \sim \mu_2$, even if $\mu_1 \times \mu_2 \ll \mu$ fails to be true.

Next, having proved (3), define

$$\alpha^* = \frac{\alpha - c\lambda_1}{1-c} \quad \text{and} \quad \beta^* = \frac{\beta - c\lambda_2}{1-c}.$$

Then, α^* is a p.m. on \mathcal{A} and β^* a p.m. on \mathcal{B} . Further, $\alpha^* \ll \mu_1$ and $\beta^* \ll \mu_2$. Hence, a solution to problem (b) is given by

$$\nu = c\lambda + (1-c)(\alpha^* \times \beta^*).$$

In fact, $\nu \sim \mu$ follows from $\lambda \sim \mu$ and $\alpha^* \times \beta^* \ll \mu_1 \times \mu_2 \ll \mu$. Further,

$$\nu_1 = c\lambda_1 + (1-c)\alpha^* = \alpha \quad \text{and} \quad \nu_2 = c\lambda_2 + (1-c)\beta^* = \beta.$$

This concludes the proof. \square

Theorem 11 provides a sufficient condition for problem (b) to admit a solution. Note that $\alpha \sim \mu_1$ and $\beta \sim \mu_2$ are necessary for solving problem (b). Thus, the real requirement of Theorem 11 is $\mu_1 \times \mu_2 \ll \mu$. Note also that, under the conditions of Theorem 8, $\mu_1 \times \mu_2 \ll \mu$ reduces to $\mu_1 \times \mu_2 \ll_R \mu$.

Among other things, Theorem 11 allows to settle the following conjecture. Let us consider the condition

- (4) for all p.m.'s α on \mathcal{A} and β on \mathcal{B} , satisfying $\alpha \sim \mu_1$ and $\beta \sim \mu_2$, there is $\nu \in \Gamma(\alpha, \beta)$ such that $\nu \sim \mu$.

Condition (4) is trivially true if $\mu \sim \gamma_1 \times \gamma_2$, with γ_1 and γ_2 σ -finite measures on \mathcal{A} and \mathcal{B} respectively. Indeed, given $\alpha \sim \mu_1$ and $\beta \sim \mu_2$, it suffices to let $\nu = \alpha \times \beta$. A (natural) question is whether the converse is true as well, and our first conjecture was that condition (4) actually amounts to $\mu \sim \gamma_1 \times \gamma_2$ for some σ -finite γ_1 and γ_2 . Such a conjecture fails to be true, however.

Example 12. Let $\mathcal{X} = \mathcal{Y} = [0, 1]$ and

$$\mu = \frac{(m \times m) + \mu'}{2} \quad \text{where} \quad \mu'(\cdot) = m\{x \in [0, 1] : (x, x) \in \cdot\}.$$

Since $\mu_1 = \mu_2 = m$, then $\mu_1 \times \mu_2 \ll \mu$. Thus, condition (4) follows from Theorem 11. Suppose now that $\mu \sim \gamma_1 \times \gamma_2$ with γ_1 and γ_2 σ -finite. Let $D = \{(x, x) : x \in [0, 1]\}$ be the diagonal. Since $\gamma_1 \sim \mu_1 = m$, then $\gamma_1\{x\} = 0$ for all $x \in [0, 1]$, which in turn implies $\gamma_1 \times \gamma_2(D) = 0$. But this is a contradiction, for $2\mu(D) = \mu'(D) = 1$.

As for problem (a), one might be also interested in a finitely additive solution to problem (b). In this case, the conditions of Theorem 11 may be weakened.

Theorem 13. *Let α and β be p.m.'s on \mathcal{A} and \mathcal{B} , respectively. If*

$$\alpha \sim \mu_1, \quad \beta \sim \mu_2 \quad \text{and} \quad \mu_1 \times \mu_2 \ll_R \mu,$$

there is a finitely additive probability ν on $\mathcal{A} \otimes \mathcal{B}$, with marginals α and β , such that $\nu \sim \mu$.

Proof. As in the proof of Theorem 11, since $\alpha \sim \mu_1$ and $\beta \sim \mu_2$, one obtains $\mu = c\lambda + (1-c)\rho$ where

$$c \in (0, 1), \quad \lambda, \rho \in \mathcal{P}, \quad \lambda \sim \mu, \quad \alpha \geq c\lambda_1 \quad \text{and} \quad \beta \geq c\lambda_2.$$

Let $\alpha^* = (1-c)^{-1}(\alpha - c\lambda_1)$, $\beta^* = (1-c)^{-1}(\beta - c\lambda_2)$ and $\Lambda = \{\tau \in \mathcal{P} : \tau \ll_R \mu\}$. Since $\alpha^* \ll \mu_1$ and $\beta^* \ll \mu_2$, then $\alpha^* \times \beta^* \ll \mu_1 \times \mu_2 \ll_R \mu$. Hence, $\alpha^* \times \beta^* \in \Lambda$, and this implies

$$E_{\alpha^*}(f) + E_{\beta^*}(g) = E_{\alpha^* \times \beta^*}(f \oplus g) \geq \inf_{\tau \in \Lambda} E_{\tau}(f \oplus g)$$

for all bounded measurable $f : \mathcal{X} \rightarrow \mathbb{R}$ and $g : \mathcal{Y} \rightarrow \mathbb{R}$. By Theorem 7, there is a finitely additive probability ν^* on $\mathcal{A} \otimes \mathcal{B}$, with marginals α^* and β^* , such that $\nu^* \ll \mu$. Therefore, it suffices to let

$$\nu = c\lambda + (1-c)\nu^*.$$

In fact, $\nu \sim \mu$ follows from $\lambda \sim \mu$ and $\nu^* \ll \mu$, while it is straightforward to verify that ν has marginals α and β . \square

Still concerning problem (b), we close this section with a result analogous to Theorem 8.

Theorem 14. *Under the assumptions of Theorem 8,*

$$\mu_1 \times \mu_2 \ll \mu \quad \iff \quad \text{condition (4)}.$$

Proof. If $\mu_1 \times \mu_2 \ll \mu$, condition (4) follows from Theorem 11. Conversely, assume condition (4). By Theorem 8, it suffices to prove that $\mu_1 \times \mu_2 \ll_R \mu$. Toward a contradiction, suppose $\mu(A \times B) = 0$, $\mu_1(A) > 0$ and $\mu_2(B) > 0$ for some $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then,

$$\mu_1(A) = \mu(A \times \mathcal{Y}) = \mu(A \times B^c) \leq \mu(\mathcal{X} \times B^c) = \mu_2(B^c) < 1.$$

Similarly, $\mu_2(B) < 1$, and one can define

$$\alpha(\cdot) = \frac{3\mu_1(\cdot | A) + \mu_1(\cdot | A^c)}{4} \quad \text{and} \quad \beta(\cdot) = \frac{3\mu_2(\cdot | B) + \mu_2(\cdot | B^c)}{4}.$$

Then, $\alpha \sim \mu_1$ and $\beta \sim \mu_2$. By condition (4), there is $\nu \in \Gamma(\alpha, \beta)$ such that $\nu \sim \mu$. If such a ν exists, however, one obtains the absurd conclusion

$$\nu(A \times B^c) = \nu(A \times \mathcal{Y}) = \alpha(A) = 3/4 \quad \text{and} \quad \nu(A^c \times B) = \nu(\mathcal{X} \times B) = \beta(B) = 3/4.$$

This concludes the proof. \square

5. CONCLUDING REMARKS

This section collects some miscellaneous material, connected to parts of the paper. Some problems to be investigated are mentioned as well.

- **Extreme points.** Various questions arise once $\Lambda_0 = \{\nu \in \Gamma(\alpha, \beta) : \nu \ll \mu\}$ or $\Lambda_1 = \{\nu \in \Gamma(\alpha, \beta) : \nu \sim \mu\}$ are shown to be non-empty. One of such questions is whether each $\nu \in \Lambda_0$ is a mixture of extreme points of Λ_0 . Precisely, let Σ be the σ -field on \mathcal{P} generated by the maps $\lambda \mapsto \lambda(H)$ for all $H \in \mathcal{A} \otimes \mathcal{B}$. Given $\nu \in \Lambda_0$, is there a p.m. π on Σ , supported by the extreme points of Λ_0 , such that

$$\nu(\cdot) = \int \lambda(\cdot) \pi(d\lambda) ?$$

A further problem, connected to mass transportation, concerns conditions for Λ_0 or Λ_1 to admit an optimal transport plan.

- **Equivalent martingale measures.** As noted in Example 2, the case where X and Y are required to be martingales with respect to a common filtration is not covered by problem (b). To capture this case, problems (a)-(b) should be generalized as follows. Let (Ω, \mathcal{F}, P) be a probability space and P_i a p.m. on the sub- σ -field $\mathcal{F}_i \subset \mathcal{F}$, where $i = 1, 2$. Is there a p.m. Q on \mathcal{F} such that $Q \ll P$, or $Q \sim P$, and $Q = P_i$ on \mathcal{F}_i for each i ? This question looks intriguing but also quite hard to be answered in general.
- **Domination on rectangles.** Let \mathcal{R} be the field generated by the measurable rectangles; $\nu \ll_R \mu$ just means that ν is dominated by μ on \mathcal{R} but not necessarily on $\sigma(\mathcal{R}) = \mathcal{A} \otimes \mathcal{B}$. Nevertheless, to our knowledge, domination on rectangles has not been explicitly investigated so far. Lemma 4 provides conditions under which $\nu \ll_R \mu$ implies $\nu \ll \mu$, but possibly some other conditions can be singled out. However, condition (ii) of Lemma 4 can not be improved by asking μ to be supported by countably many graphs. In fact, next example exhibits a situation where ν is not dominated by μ even if $\nu \ll_R \mu$ and $\mu(\cup_n G_n) = 1$ where each G_n is the graph of a measurable function.

Example 15. Let q_1, q_2, \dots be an enumeration of the rational numbers in the interval $[0, 1)$. For each $n \geq 1$, define

$$f_n(x) = x + q_n \text{ if } x \in [0, 1 - q_n) \quad \text{and} \quad f_n(x) = x + q_n - 1 \text{ if } x \in [1 - q_n, 1).$$

Take $\mathcal{X} = \mathcal{Y} = [0, 1)$, $\nu = m \times m$ and $\mu = \sum_n 2^{-n} \lambda_n$, where

$$\lambda_n(\cdot) = m\{x \in [0, 1) : (x, f_n(x)) \in \cdot\}.$$

Then, $\mu(\cup_n G_n) = 1$ and $\nu(\cup_n G_n) = 0$ where $G_n = \{(x, f_n(x)) : x \in [0, 1)\}$. Hence, μ is supported by countably many graphs and ν is not dominated by μ . However, $\nu \ll_R \mu$. To prove the latter fact, since $\mu_1 \sim \mu_2 \sim m$, we need to show that

$$\lambda_n(A \times B) = 0 \text{ for all } n \quad \implies \quad m(A) \wedge m(B) = 0$$

whenever $A, B \subset [0, 1)$ are Borel sets. Fix A and B such that $\lambda_n(A \times B) = 0$ for all n , and define

$$B^* = \cup_n f_n^{-1}B = \{x \in [0, 1) : f_n(x) \in B \text{ for some } n\}.$$

If $m(B) = 0$, we are done. Otherwise, if $m(B) > 0$, it can be shown that $m(B^*) = 1$; see e.g. Exercise 30, p. 39, of [4]. In this case, since

$$m(A \cap f_n^{-1}B) = \lambda_n(A \times B) = 0 \quad \text{for all } n,$$

one obtains

$$m(A) = m(A \cap B^*) = m[\cup_n (A \cap f_n^{-1}B)] = 0.$$

Therefore, $\nu \ll_R \mu$.

- **An open problem.** Let $\Lambda = \{\lambda \in \mathcal{P} : \lambda \ll \mu\}$. For such a Λ , as already noted, we do not know whether condition (1) (required for all bounded measurable f and g) implies $\Lambda_0 = \Gamma(\alpha, \beta) \cap \Lambda \neq \emptyset$.
- **Finitely additive probabilities.** Problems (a)-(b) are basically extension problems. Define in fact

$$\nu^*(A \times \mathcal{Y}) = \alpha(A), \quad \nu^*(\mathcal{X} \times B) = \beta(B), \quad \nu^*(H) = 0,$$

whenever $A \in \mathcal{A}$, $B \in \mathcal{B}$, $H \in \mathcal{A} \otimes \mathcal{B}$ and $\mu(H) = 0$. In problem (a), one is looking for a (countably additive) extension ν of ν^* to $\mathcal{A} \otimes \mathcal{B}$. In problem (b), ν is also required to be strictly positive whenever μ is strictly positive. Now, since problems (a)-(b) are of the extension type, allowing for finitely additive probabilities makes easier to solve them. This is confirmed by Theorems 7 and 13. Note also that, up to technical details, condition (1) is essentially a *coherence* condition in de Finetti's sense. Indeed, in the proof of Theorem 5, condition (1) is actually used as a coherence condition. And, a coherent map can be coherently extended to any larger domain.

- **A curious fact.** Say that α and β are admissible for problem (a) (or for problem (b)) whenever $\alpha \ll \mu_1$ and $\beta \ll \mu_2$ (or $\alpha \sim \mu_1$ and $\beta \sim \mu_2$). By Theorems 8 and 14, under some assumptions, one obtains

$$\text{condition (2)} \iff \text{condition (4)}.$$

Thus, problem (a) admits a solution for all admissible α and β if and only if the same happens to problem (b).

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