A UNIFYING VIEW ON SOME PROBLEMS IN PROBABILITY AND STATISTICS

PATRIZIA BERTI, LUCA PRATELLI, AND PIETRO RIGO

Abstract. Let L be a linear space of real random variables on the measurable space $(\Omega, \mathcal{A})$. Conditions for the existence of a probability P on $\mathcal{A}$ such that

$$E_P|X| < \infty \quad \text{and} \quad E_P(X) = 0 \quad \text{for all } X \in L$$

are provided. Such a P may be finitely additive or σ-additive, depending on the problem at hand, and may also be requested to satisfy $P \sim P_0$ or $P \ll P_0$ where P_0 is a reference measure. As a motivation, we note that a plenty of significant issues reduce to the existence of a probability P as above. Among them, we mention de Finetti's coherence principle, equivalent martingale measures, equivalent measures with given marginals, stationary and reversible Markov chains, and compatibility of conditional distributions.

1. Introduction

A number of problems, ranging from probability to statistics and finance, reduce to the following question. Let $(\Omega, \mathcal{A})$ be a measurable space and L a linear space of real random variables on $(\Omega, \mathcal{A})$. Is there a probability P on $\mathcal{A}$ such that

$$E_P|X| < \infty \quad \text{and} \quad E_P(X) = 0 \quad \text{for all } X \in L \text{ ?} \tag{1}$$

Such a P may be finitely additive or σ-additive, depending on the problem at hand, and may also be requested some additional requirement. For instance, in addition to (1), the probability P could be asked to satisfy

$$P \sim P_0 \quad \text{or} \quad P \ll P_0$$

where P_0 is a reference probability measure on $\mathcal{A}$.

Here are some examples. In the sequel, we let

$$\mathbb{P} = \{\text{finitely additive probabilities on } \mathcal{A}\},$$
$$\mathbb{H} = \{P \in \mathbb{P} : P \text{ satisfies equation (1)}\},$$
$$\mathbb{P}_0 = \{P \in \mathbb{P} : P \text{ is } \sigma\text{-additive}\} \quad \text{and} \quad \mathbb{H}_0 = \mathbb{H} \cap \mathbb{P}_0.$$

Note that $\mathbb{H}$ or $\mathbb{H}_0$ may be empty and $\mathbb{H}_0$ is the collection of σ-additive solutions of equation (1).

2010 *Mathematics Subject Classification.* 60A05, 60A10, 28C05.

Key words and phrases. Compatibility of conditional distributions, de Finetti's coherence principle, Disintegrability, Equivalent martingale measure, Equivalent probability measure with given marginals, Finitely additive probability, Fundamental theorem of asset pricing, Integral representation of functionals, Stationary and reversible Markov chains.

Example 1. (de Finetti's coherence principle). Let D be any collection of real bounded functions on a set Ω. A map $\phi : D \to \mathbb{R}$ is *coherent* if

$$\sup_{\omega\in\Omega} \sum_{j=1}^{n} \lambda_j \left\{ X_j(\omega) - \phi(X_j) \right\} \geq 0$$

whenever $n \geq 1$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and $X_1, \ldots, X_n \in D$. Heuristically, suppose ϕ describes your previsions on the members of D. If ϕ is coherent, it is impossible to make you a sure looser, whatever $\omega \in \Omega$ turns out to be true, by some finite combinations of bets (on $X_1, \ldots, X_n$ with stakes $\lambda_1, \ldots, \lambda_n$). Anyhow, define

$$L = \text{span}\big(X - \phi(X) : X \in D\big) \quad \text{and} \quad \mathcal{A} = \sigma(L).$$

(The notations span$(\cdot)$ and $\sigma(\cdot)$ stand for "linear space generated by" and "σ-field generated by", respectively). Then, ϕ is coherent if and only if $E_P(X) = 0$ for all $X \in L$ and some $P \in \mathbb{P}$, namely if and only if $\mathbb{H} \neq \emptyset$. This is a well known fact which follows from Hahn-Banach theorem; see e.g. [3, Section 2] and [11, Lemma 1].

In a sense, this paper stems from Example 1. Our main goal is in fact to address and unify various problems, apparently unrelated, basing on condition (1). But (1) has been suggested by Example 1 and, more generally, by de Finetti's ideas.

Another basic example is the following.

Example 2. (Equivalent martingale measures). Let $S = (S_t : t \in I)$ be a real stochastic process, indexed by $I \subset \mathbb{R}$, on the probability space $(\Omega, \mathcal{A}, P_0)$. Suppose S is adapted to a filtration $\mathcal{G} = (\mathcal{G}_t : t \in I)$ and S_{t_0} is a constant random variable for some $t_0 \in I$. Say that P is an *equivalent martingale measure* (EMM) if

$$P \in \mathbb{P}_0, \quad P \sim P_0 \quad \text{and} \quad S \text{ is a } \mathcal{G}\text{-martingale under } P.$$

Existence of EMM's is a fundamental problem in mathematical finance; see e.g. [9]. Now, with a suitable choice of L, an EMM is exactly a probability $P \in \mathbb{H}_0$ such that $P \sim P_0$. It suffices to let

$$L = \text{span}\big(I_A\,(S_u - S_t) : u,\, t \in I,\, u > t,\, A \in \mathcal{G}_t\big).$$

EMM's are usually requested to be σ-additive, but their economic interpretation is preserved if they are only finitely additive. Thus, to look for finitely additive EMM's seems to make sense; see [3], [4], [7], [12], [13].

Example 3. (Equivalent probability measures with given marginals). Let

$$\Omega = \Omega_1 \times \Omega_2 \quad \text{and} \quad \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$$

where $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are measurable spaces. Fix $P_0 \in \mathbb{P}_0$ and a probability Q_i on $\mathcal{A}_i$ for $i = 1, 2$. Is there $P \in \mathbb{P}$ such that

$$P \sim P_0 \quad \text{and} \quad P\big(\cdot \times \Omega_2\big) = Q_1(\cdot),\; P\big(\Omega_1 \times \cdot\big) = Q_2(\cdot)\ ? \tag{2}$$

If Q_1 and Q_2 are σ-additive, is there $P \in \mathbb{P}_0$ satisfying (2) ? These questions look natural (to us) and some possible answers are provided in [5, Example 12]; see also [14]. The point to be stressed here, however, is that such a P is again a solution of equation (1) (for a suitable L) such that $P \sim P_0$. Define in fact L to be the class of all random variables X on $\Omega = \Omega_1 \times \Omega_2$ of the type

$$X(\omega_1, \omega_2) = \big\{f(\omega_1) - E_{Q_1}(f)\big\} + \big\{g(\omega_2) - E_{Q_2}(g)\big\}$$

where f and g are simple functions on $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$, respectively. Then, L is a linear space and condition (2) amounts to condition (1) and $P \sim P_0$.

Example 4. (Stationary and reversible Markov chains). A kernel on $\Omega \times \mathcal{A}$ is a map α on $\Omega \times \mathcal{A}$ such that: (i) $\alpha(\omega, \cdot) \in \mathbb{P}_0$ for each $\omega \in \Omega$; (ii) $\omega \mapsto \alpha(\omega, A)$ is a measurable function for each $A \in \mathcal{A}$. Fix a kernel α and a Markov chain (X_n) with state space $(\Omega, \mathcal{A})$ and transition probability α. For $P \in \mathbb{P}_0$, say that P makes (X_n) stationary if

$$P(A) = \int \alpha(\omega, A)\, P(d\omega) \quad \text{for all } A \in \mathcal{A}.$$

This means that (X_n) is a stationary sequence as far as X_0 has distribution P. Similarly, say that P makes (X_n) reversible if

$$\int_A \alpha(\omega, B)\, P(d\omega) = \int_B \alpha(\omega, A)\, P(d\omega) \quad \text{for all } A,\, B \in \mathcal{A}.$$

If X_0 has distribution P, with P as above, then (X_n) is stationary and (X_0, X_1) is distributed as (X_1, X_0).

Typically, α is given, and a fundamental question is whether there is $P \in \mathbb{P}_0$ which makes (X_n) stationary or reversible. Such a P could be also asked to satisfy $P \sim P_0$ or $P \ll P_0$ for some reference probability P_0. Anyhow, P makes (X_n) stationary or reversible if and only if $P \in \mathbb{H}_0$ for a suitable choice of L. In fact, let L be the linear space of all random variables

$$X(\omega) = f(\omega) - \int f(x)\, \alpha(\omega, dx)$$

where f ranges over the simple functions on $(\Omega, \mathcal{A})$. Then, P satisfies equation (1) if and only if P makes (X_n) stationary. Likewise, define L to be the linear space generated by

$$X(\omega) = I_A(\omega)\, \alpha(\omega, B) - I_B(\omega)\, \alpha(\omega, A)$$

for all $A,\, B \in \mathcal{A}$. Then, P meets equation (1) if and only if P makes (X_n) reversible.

Example 5. (Compatibility of conditional distributions). Let $S = (S_i : i \in I)$ be a collection of real random variables. In some statistical frameworks, S is requested to have *given* conditional distributions. This means that, instead of assessing the joint distribution of S directly, one selects a class $\mathcal{H}$ of subsets of I and (tentatively) assigns the conditional distribution of

$$\big(S_i : i \in H\big) \quad \text{given} \quad \big(S_i : i \in I \setminus H\big) \quad \text{for each } H \in \mathcal{H}.$$

But a joint distribution for S with the given conditional distributions can fail to exist. Accordingly, such conditional distributions are said to be *compatible* if they are actually induced by some joint distribution for S. We refer to [6] and references therein for further details and examples. Here, we focus on the simplest case: $S = (Y, Z)$ with Y and Z real random variables.

Let $\mathcal{Y} \subset \mathbb{R}$ and $\mathcal{Z} \subset \mathbb{R}$ be Borel sets, to be regarded as the collections of "admissible" values for Y and Z, respectively. Denote by $\mathcal{B}_\mathcal{Y}$ and $\mathcal{B}_\mathcal{Z}$ the Borel σ-fields on $\mathcal{Y}$ and $\mathcal{Z}$. A conditional distribution for Y given Z is a kernel α on $\mathcal{Z} \times \mathcal{B}_\mathcal{Y}$, that is: (i) $\alpha(z, \cdot)$ is a probability measure on $\mathcal{B}_\mathcal{Y}$ for each $z \in \mathcal{Z}$; (ii) $z \mapsto \alpha(z, A)$ is $\mathcal{B}_\mathcal{Z}$-measurable for each $A \in \mathcal{B}_\mathcal{Y}$. Here, $\alpha(z, \cdot)$ should be viewed as the conditional distribution of Y given that $Z = z$. Let α be a conditional distribution for Y given

Z and β a conditional distribution for Z given Y. Then, α and β are compatible if there is a probability measure P on $\mathcal{B}_{\mathcal{Y}} \otimes \mathcal{B}_{\mathcal{Z}}$ such that

$$\int_A \beta(y,B)\, P_{\mathcal{Y}}(dy) = P(A\times B) = \int_B \alpha(z,A)\, P_{\mathcal{Z}}(dz) \quad \text{for } A\in\mathcal{B}_{\mathcal{Y}} \text{ and } B\in\mathcal{B}_{\mathcal{Z}}$$

where $P_{\mathcal{Y}}$ and $P_{\mathcal{Z}}$ are the marginals of P on $\mathcal{Y}$ and $\mathcal{Z}$.

With a suitable choice of L, compatibility of α, β amounts to condition (1). Define in fact $(\Omega,\mathcal{A}) = (\mathcal{Y}\times\mathcal{Z},\, \mathcal{B}_{\mathcal{Y}}\otimes\mathcal{B}_{\mathcal{Z}})$ and take L to be the collection of all random variables of the form

$$X(y,z) = \int f(u,z)\,\alpha(z,du) - \int f(y,u)\,\beta(y,du)$$

where $f:\mathcal{Y}\times\mathcal{Z}\to\mathbb{R}$ is a bounded continuous function. Then, L is a linear space and α, β are compatible if and only if there is $P\in\mathbb{H}_0$.

So far, all probability measures in this example have been tacitly assumed to be σ-additive. But compatibility issues arise even if such probabilities are finitely additive. In this case, compatibility of α, β essentially coincides with the notion of coherence introduced by Heath and Sudderth in [11]. See also [1] and references therein.

Example 6. (Disintegrability). Let Π be a partition of Ω and $\mathcal{F}$ a σ-field of subsets of Ω. For the sake of simplicity, assume $\Pi\subset\mathcal{F}$. A Π-strategy is a map α on $\mathcal{F}\times\Pi$ such that

$$\alpha(\cdot\mid H) \text{ is a finitely additive probability on } \mathcal{F} \text{ with } \alpha(H\mid H)=1$$

for each $H\in\Pi$. Define $\mathcal{A}(\alpha)$ to be the σ-field generated by the maps

$$\omega\mapsto\alpha\big(A\mid H(\omega)\big) \quad \text{for all } A\in\mathcal{F},$$

where $H(\omega)$ denotes the only element of Π including $\omega\in\Omega$. Let μ be a finitely additive probability on $\mathcal{F}$. A *disintegration* (for μ) is a pair (α,β) such that α is a Π-strategy, β is a finitely additive probability on $\mathcal{A}(\alpha)$, and

$$\mu(A) = \int \alpha\big(A\mid H(\omega)\big)\,\beta(d\omega) \quad \text{for each } A\in\mathcal{F}.$$

If, in addition, β and each $\alpha(\cdot\mid H)$ are σ-additive probabilities, then (α,β) is said to be a σ-additive disintegration. See e.g. [1], [2], [10] and references therein.

Suppose now that we are given μ and α, and we ask whether (α,β) is a disintegration for some β. Such a question clearly amounts to existence of $P\in\mathbb{H}$. It suffices to let $\mathcal{A}=\mathcal{A}(\alpha)$ and to take L as the linear space of all random variables of the type

$$X(\omega) = \int f(x)\,\alpha\big(dx\mid H(\omega)\big) - \int f\,d\mu$$

where f is a simple function with respect to $(\Omega,\mathcal{F})$. Note also that, if μ and each $\alpha(\cdot\mid H)$ are σ-additive, it looks natural to ask wether (α,β) is a σ-additive disintegration for some β. This amounts to existence of $P\in\mathbb{H}_0$.

The existence of $P\in\mathbb{H}$ such that $P\sim P_0$ or $P\ll P_0$ is investigated in [3]-[5]. Moreover, conditions for the existence of $P\in\mathbb{H}_0$ such that $P\sim P_0$ are given in [5]. To our knowledge, however, the case where P is requested to belong to $\mathbb{H}_0$, but not to satisfy $P\sim P_0$ or $P\ll P_0$, has been neglected so far. Similarly, we do not know of nontrivial conditions for $\mathbb{H}\neq\emptyset$ when L includes unbounded random variables.

This paper provides conditions for $\mathbb{H} \neq \emptyset$ and $\mathbb{H}_0 \neq \emptyset$ which apply to an arbitrary linear space L. Special attention is paid to the second case, namely to the existence of σ-additive solutions of equation (1). A new necessary and sufficient condition, for the existence of $P \in \mathbb{H}_0$ such that $P \sim P_0$ and P has a bounded density with respect to P_0, is provided as well. Finally, various counterexamples are given.

2. Notation

In the sequel, as in Section 1, L is a linear space of real random variables on the measurable space $(\Omega, \mathcal{A})$. We let $\mathbb{P}$ denote the set of finitely additive probabilities on $\mathcal{A}$ and $\mathbb{P}_0 = \{P \in \mathbb{P} : P \text{ is } \sigma\text{-additive}\}$. Also, $\mathbb{H}$ is the set of laws $P \in \mathbb{P}$ which satisfy equation (1) and

$$\mathbb{H}_0 = \mathbb{H} \cap \mathbb{P}_0 = \{P \in \mathbb{P}_0 : E_P|X| < \infty \text{ and } E_P(X) = 0 \text{ for all } X \in L\}.$$

Given P, $Q \in \mathbb{P}$, we write $P \ll Q$ to mean that $P(A) = 0$ whenever $A \in \mathcal{A}$ and $Q(A) = 0$. Also, $P \sim Q$ stands for $P \ll Q$ and $Q \ll P$.

When a reference measure $P_0 \in \mathbb{P}_0$ is given, we let

$$\text{ess sup}(X) = \inf\{x \in \mathbb{R} : P_0(X > x) = 0\} \quad \text{with } \inf \emptyset = \infty$$

for every real random variable X. We also need the notation

$$l^\infty = l^\infty(\Omega) = \{\text{real bounded functions on } \Omega\}.$$

Let $P \in \mathbb{P}$ and X a real random variable. If $X \in l^\infty$, then X is P-integrable and $\int X\,dP$ is defined to be $\int X\,dP = \lim_n \int X_n\,dP$, where (X_n) is a sequence of simple functions converging to X uniformly. If $X \geq 0$, then X is P-integrable if and only if $\inf_n P(X > n) = 0$ and $\sup_n \int X\, I_{\{X \leq n\}}\,dP < \infty$. In this case,

$$\int X\,dP = \sup_n \int X\, I_{\{X \leq n\}}\,dP.$$

If X is arbitrary real, X is P-integrable if and only if X^+ and X^- are both P-integrable, and in this case $\int X dP = \int X^+ dP - \int X^- dP$.

In this paper, we write $E_P|X| < \infty$ to mean that X is P-integrable, and we let

$$E_P(X) = \int X dP$$

whenever $E_P|X| < \infty$.

3. σ-additive solutions of equation (1)

In this section, we give conditions for $\mathbb{H} \neq \emptyset$ and $\mathbb{H}_0 \neq \emptyset$, with special attention to the second case. We point out that the solutions P of equation (1) are not requested to satisfy any other requirement (such as $P \sim P_0$ or $P \ll P_0$).

We begin with a general result, in the spirit of [5, Theorem 6].

Theorem 7. *There is $P \in \mathbb{H}$ if and only if there is $Q \in \mathbb{P}$ such that*

$$E_Q|X| < \infty \quad \text{and} \quad |\,E_Q(X)\,| \leq c\,E_Q|X| \tag{3}$$

for all $X \in L$ and some constant $c \in [0, 1)$. Furthermore, if $Q \in \mathbb{P}_0$ then $\mathbb{H}_0 \neq \emptyset$ (that is, equation (1) holds for some $P \in \mathbb{P}_0$).

Proof. If $P \in \mathbb{H}$, condition (3) trivially holds with $Q = P$ and $c = 0$. Conversely, suppose (3) holds for some $c \in [0,1)$ and $Q \in \mathbb{P}$. Define

$$t = (1+c)/(1-c) \quad \text{and} \quad \mathcal{K} = \big\{P \in \mathbb{P} : (1/t)\,Q \leq P \leq t\,Q\big\}.$$

If $P \in \mathcal{K}$, then $E_P|X| \leq t\,E_Q|X| < \infty$ for all $X \in L$, and $P \in \mathbb{P}_0$ provided $Q \in \mathbb{P}_0$. Thus, it suffices to see that $E_P(X) = 0$ for some $P \in \mathcal{K}$ and all $X \in L$.

We first prove that, for each $X \in L$, there is $P \in \mathcal{K}$ such that $E_P(X) = 0$. Fix $X \in L$ and note that condition (3) implies

$$E_Q(X^+) \leq t\,E_Q(X^-) \quad \text{and} \quad E_Q(X^-) \leq t\,E_Q(X^+).$$

If $E_Q|X| = 0$, just take $P = Q \in \mathcal{K}$. If $E_Q|X| > 0$, define

$$f = \frac{E_Q(X^-)\,I_{\{X \geq 0\}} + E_Q(X^+)\,I_{\{X<0\}}}{E_Q(X^-)\,Q(X \geq 0) + E_Q(X^+)\,Q(X<0)}$$

and $P(A) = E_Q\big\{f\,I_A\big\}$ for all $A \in \mathcal{A}$. Since $E_Q(f) = 1$ and $(1/t) \leq f \leq t$, then $P \in \mathcal{K}$. In addition,

$$E_P(X) = E_Q(f\,X) = \frac{E_Q(X^-)\,E_Q(X^+) - E_Q(X^+)\,E_Q(X^-)}{E_Q(X^-)\,Q(X \geq 0) + E_Q(X^+)\,Q(X<0)} = 0.$$

Next, let $\mathcal{Z}$ be the set of all functions from $\mathcal{A}$ into $[0,1]$, equipped with the product topology. Then,

(4) $\qquad \mathcal{K}$ is compact and $\{P \in \mathcal{K} : E_P(X) = 0\}$ is closed for each $X \in L$.

To prove (4), fix a net $(P_\alpha) \subset \mathcal{Z}$ converging to $P \in \mathcal{Z}$, that is, $P(A) = \lim_\alpha P_\alpha(A)$ for all $A \in \mathcal{A}$. If $P_\alpha \in \mathcal{K}$ for each α, then $P \in \mathbb{P}$ and $(1/t)\,Q \leq P \leq t\,Q$. Hence $P \in \mathcal{K}$, that is, $\mathcal{K}$ is closed. Since $\mathcal{Z}$ is compact, $\mathcal{K}$ is actually compact. If $P_\alpha \in \mathcal{K}$ and $E_{P_\alpha}(X) = 0$, for some $X \in L$ and each α, then $P \in \mathcal{K}$ (for $\mathcal{K}$ is closed). Hence, $E_P|X| < \infty$. Define the set $A_b = \{|X| \leq b\}$ for $b > 0$. Since $E_{P_\alpha}(X) = 0$ and both P_α and P are in $\mathcal{K}$, it follows that

$$\begin{aligned}
|E_P(X)| &= |E_P(X) - E_{P_\alpha}(X)| \leq \\
&\leq |E_P\big\{X - X\,I_{A_b}\big\}| + |E_P\big\{X\,I_{A_b}\big\} - E_{P_\alpha}\big\{X\,I_{A_b}\big\}| + |E_{P_\alpha}\big\{X\,I_{A_b} - X\big\}| \\
&\leq E_P\big\{|X|\,I_{\{|X|>b\}}\big\} + |E_P\big\{X\,I_{A_b}\big\} - E_{P_\alpha}\big\{X\,I_{A_b}\big\}| + E_{P_\alpha}\big\{|X|\,I_{\{|X|>b\}}\big\} \\
&\leq 2\,t\,E_Q\big\{|X|\,I_{\{|X|>b\}}\big\} + |E_P\big\{X\,I_{A_b}\big\} - E_{P_\alpha}\big\{X\,I_{A_b}\big\}|.
\end{aligned}$$

Since $X\,I_{A_b}$ is bounded, $E_P\big\{X\,I_{A_b}\big\} = \lim_\alpha E_{P_\alpha}\big(X\,I_{A_b}\big)$. Thus,

$$|E_P(X)| \leq 2\,t\,E_Q\big\{|X|\,I_{\{|X|>b\}}\big\} \quad \text{for every } b > 0.$$

Since $E_Q|X| < \infty$, then $\lim_{b\to\infty} Q(|X| > b) = 0$, which in turn implies

$$|E_P(X)| \leq 2\,t \lim_{b\to\infty} E_Q\big\{|X|\,I_{\{|X|>b\}}\big\} = 0.$$

Hence, $\{P \in \mathcal{K} : E_P(X) = 0\}$ is closed.

Because of (4), to conclude the proof it suffices to see that

$$\big\{P \in \mathcal{K} :\, E_P(X_1) = \ldots = E_P(X_n) = 0\big\} \neq \emptyset$$

for all $n \geq 1$ and $X_1, \ldots, X_n \in L$.

Given $n \geq 1$ and $X_1, \ldots, X_n \in L$, define

$$C = \big\{\big(E_P(X_1), \ldots, E_P(X_n)\big) :\, P \in \mathcal{K}\big\}.$$

Then, C is a convex closed subset of $\mathbb{R}^n$. Thus, C is the intersection of all half-planes $\{f \geq u\}$ including it, where $u \in \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a linear functional. Fix f and u such that $C \subset \{f \geq u\}$. Since $f(X_1, \ldots, X_n) \in L$, then $E_P\big\{f(X_1, \ldots, X_n)\big\} = 0$ for some $P \in \mathcal{K}$. Since $\big(E_P(X_1), \ldots, E_P(X_n)\big) \in C \subset \{f \geq u\}$, one obtains

$$f(0, \ldots, 0) = 0 = E_P\big\{f(X_1, \ldots, X_n)\big\} = f\big(E_P(X_1), \ldots, E_P(X_n)\big) \geq u.$$

This proves $(0, \ldots, 0) \in C$ and concludes the proof.

□

Roughly speaking Theorem 7 states that, for $P \in \mathbb{H}$ to exist, it is enough to check a weaker condition (that is, condition (3)) for some probability Q possibly different from P. In addition, P can be taken to be σ-additive whenever Q is σ-additive. These facts can be useful in real problems. Note also that, if $L \subset l^\infty$, then $E_Q|X| < \infty$ can be dropped from condition (3).

Theorem 7 unifies various different situations ($L \subset l^\infty$ or not, $P \in \mathbb{H}$ or $P \in \mathbb{H}_0$). However, in the particular case where $L \subset l^\infty$ and P is not asked to be σ-additive, a better result is available. In fact, if $L \subset l^\infty$, there is $P \in \mathbb{H}$ if and only if

$$\sup_{\omega \in \Omega} X(\omega) \geq 0 \quad \text{for all } X \in L.$$

This fact is well known (it is basically Example 1) and will be exploited in the next result.

We need the following notation. Let I be an index set and $V = (X_i : i \in I)$ a collection of functions $X_i : \Omega \to \mathbb{R}$. Then, V can be regarded as a map $V : \Omega \to \mathbb{R}^I$, where $\mathbb{R}^I$ is the set of functions from I into $\mathbb{R}$ equipped with the product topology. It suffices to let

$$V(\omega)(i) = X_i(\omega) \quad \text{for } \omega \in \Omega \text{ and } i \in I.$$

For $A \subset \Omega$, we let $V(A) = \{V(\omega) : \omega \in A\}$ denote the range of V on A.

Theorem 8. *Let L be the linear space generated by $V = (X_i : i \in I)$. Suppose*

$$V(A) \text{ compact} \quad \text{and} \quad \sup_{\omega \in A} X(\omega) \geq 0$$

for all $X \in L$ and some (nonempty) $A \subset \Omega$. Then, there is a σ-additive probability P on $\sigma(L)$ satisfying equation (1) and $P^*(A) = 1$, where P^* is the P-outer measure.

Proof. Suppose first $A = \Omega$. Let $K = V(\Omega)$ and $f_i(x) = x(i)$ for all $i \in I$ and $x \in K$. Also, let $\mathcal{C}$ be the set of real continuous functions on K and $\mathcal{B}_0 = \sigma(\mathcal{C})$ the Baire σ-field on K. Since $\mathcal{B}_0$ agrees with the σ-field generated by the maps $\{f_i : i \in I\}$, then $\sigma(L) = V^{-1}(\mathcal{B}_0)$. Since K is compact, $X_i(\Omega) = f_i(K)$ is compact for each $i \in I$, so that $L \subset l^\infty$. Since $L \subset l^\infty$ and $\sup_\Omega X \geq 0$ for all $X \in L$, there is $T \in \mathbb{H}$. Define

$$\phi(f) = E_T\big\{f(V)\big\} \quad \text{for all } f \in \mathcal{C}.$$

By the Riesz theorem, since K is compact and Hausdorff, there is a σ-additive probability Q on $\mathcal{B}_0$ such that $E_Q(f) = \phi(f) = E_T\big\{f(V)\big\}$ for all $f \in \mathcal{C}$.

Next, since $\sigma(L) = V^{-1}(\mathcal{B}_0)$, each $U \in \sigma(L)$ can be written as $U = \{V \in B\}$ for some $B \in \mathcal{B}_0$. Since $B \subset K = V(\Omega)$, such a B is unique. Accordingly, one can define

$$P(U) = P(V \in B) = Q(B).$$

Then, P is a σ-additive probability on $\sigma(L)$ and

$$E_P(X_i) = E_P\big\{f_i(V)\big\} = E_Q(f_i) = E_T\big\{f_i(V)\big\} = E_T(X_i) = 0$$

for all $i \in I$. This concludes the proof if $A = \Omega$. If $A \neq \Omega$, just apply the above argument with A in the place of Ω. Then, there is a σ-additive probability P_A on the trace σ-field $\sigma(L) \cap A$ such that $E_{P_A}(X|A) = 0$ for all $X \in L$. (Here, $X|A$ denotes the restriction of X on A). Let

$$P(B) = P_A(A \cap B) \quad \text{for } B \in \sigma(L).$$

Then, for $X \in L$ and $B \in \sigma(L)$ with $B \supset A$, one obtains

$$E_P(X) = E_{P_A}(X|A) = 0 \quad \text{and} \quad P(B) = P_A(A) = 1.$$

This concludes the proof. □

Next example provides some possible applications of Theorem 8.

Example 9. Suppose that $\sup_A X \geq 0$ for all $X \in L$ and some $A \subset \Omega$. To fix ideas, suppose also that $\mathcal{A} = \sigma(L)$. Then, $\mathbb{H}_0 \neq \emptyset$ if L is finite dimensional and each $X \in L$ takes only a finite number of values on A. Similarly, $\mathbb{H}_0 \neq \emptyset$ if A is a compact topological space and each $X \in L$ is continuous on A. In fact, the map $V : A \to \mathbb{R}^I$ is continuous if each member of L is continuous on A. As a particular case, $\mathbb{H}_0 \neq \emptyset$ whenever $V = (X_1, \ldots, X_n)$ and

$$\big\{h(X_1) = a_1, \ldots, h(X_n) = a_n\big\} \neq \emptyset \quad \text{for all } a_1, \ldots, a_n \in \{0,1\} \text{ where } h = I_{[0,\infty)}.$$

In fact, take A such that $A \cap \{h(X_1) = a_1, \ldots, h(X_n) = a_n\}$ consists of exactly one point for all $a_1, \ldots, a_n \in \{0,1\}$. Since A is finite, it is compact and each $X \in L$ is continuous on A. Thus, it suffices to check $\sup_A X \geq 0$ for all $X \in L$. Let $X \in L$, say $X = \sum_{j=1}^n \lambda_j X_j$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Then, $X(\omega) \geq 0$ if ω is the point of A satisfying $\lambda_j X_j(\omega) \geq 0$ for all j.

Apart from Example 9 (and similar situations) compactness of $V(A)$ is certainly a strong assumption. A more realistic condition would be

$$(X_{i_1}, \ldots, X_{i_n})(A) \text{ is a compact subset of } \mathbb{R}^n \tag{5}$$

for all $n \geq 1$ and $i_1, \ldots, i_n \in I$. While condition (5) does not work for $\mathbb{H}_0 \neq \emptyset$ (see Example 11) it suffices for the existence of $P \in \mathbb{H}$ which is σ-additive on $\sigma(X_1, \ldots, X_n)$ for all finite $\{X_1, \ldots, X_n\} \subset L$. We close this section by proving the latter fact.

Theorem 10. *Let L be the linear space generated by $V = (X_i : i \in I)$. Suppose condition* (5) *holds and $\sup_A X \geq 0$ for all $X \in L$ and some (nonempty) $A \subset \Omega$. Then, there is $P \in \mathbb{H}$ such that $P^*(A) = 1$ and*

$$P \text{ is } \sigma\text{-additive on } \sigma(X_1, \ldots, X_n) \text{ for all } n \geq 1 \text{ and } X_1, \ldots, X_n \in L.$$

Proof. Let $A = \Omega$. Since each X_i has compact range, it can be assumed $|X_i| \leq 1$ for all $i \in I$. Hence, $V(\Omega) \subset [-1,1]^I$. Also, since $L \subset l^\infty$ and $\sup_\Omega X \geq 0$ for all $X \in L$, there is $T \in \mathbb{H}$. Arguing as in the proof of Theorem 8, there is also a σ-additive probability Q on the Baire σ-field of $[-1,1]^I$ such that $E_Q(f) = E_T\big\{f(V)\big\}$ for all continuous $f : [-1,1]^I \to \mathbb{R}$.

Next, let $f_i(x) = x(i)$ for $i \in I$ and $x \in [-1,1]^I$ and let $\mathcal{A}_0$ be the field of subsets of Ω of the form $\{(X_{i_1}, \ldots, X_{i_n}) \in B\}$ where $n \geq 1$, $i_1, \ldots, i_n \in I$ and $B \subset \mathbb{R}^n$ is a Borel set. Suppose $\{(X_{i_1}, \ldots, X_{i_n}) \in B\} = \{(X_{j_1}, \ldots, X_{j_m}) \in D\}$ for some $i_1,\ldots,i_n,j_1,\ldots,j_m\in I$ and some Borel sets $B\subset\mathbb{R}^n$ and $D\subset\mathbb{R}^m$. Define $Y=(X_{i_1},\ldots,X_{i_n},X_{j_1},\ldots,X_{j_m})$ and

$$K=\{(f_{i_1},\ldots,f_{i_n},f_{j_1},\ldots,f_{j_m})\in Y(\Omega)\}.$$

Because of (5), $Y(\Omega)$ is compact. Hence, K is compact, and this implies $Q(K)=1$. Further,

$$K\cap\{(f_{i_1},\ldots,f_{i_n})\in B\}=K\cap\{(f_{j_1},\ldots,f_{j_m})\in D\}.$$

It follows that

$$Q\big((f_{i_1},\ldots,f_{i_n})\in B\big)=Q\big(K\cap\{(f_{i_1},\ldots,f_{i_n})\in B\}\big)$$
$$=Q\big(K\cap\{(f_{j_1},\ldots,f_{j_m})\in D\}\big)=Q\big((f_{j_1},\ldots,f_{j_m})\in D\big).$$

Therefore, it makes sense to define P_0 on $\mathcal{A}_0$ as

$$P_0\big((X_{i_1},\ldots,X_{i_n})\in B\big)=Q\big((f_{i_1},\ldots,f_{i_n})\in B\big).$$

Such a P_0 is a finitely additive probability on $\mathcal{A}_0$ which is σ-additive on $\sigma(X_1,\ldots,X_n)$ for all $n\geq 1$ and $X_1,\ldots,X_n\in L$. Further,

$$E_{P_0}(X_i)=E_Q(f_i)=E_T\big\{f_i(V)\big\}=E_T(X_i)=0\quad\text{for all } i\in I.$$

To conclude the proof for $A=\Omega$, take any finitely additive extension P of P_0 to $\mathcal{A}$. Finally, for $A\neq\Omega$, it suffices to argue as in the proof of Theorem 8. □

4. Equivalent martingale measures with bounded density

This section includes some miscellaneous material. The main result is a necessary and sufficient condition for the existence of $P\in\mathbb{H}_0$ such that $P\sim P_0$ and P has a bounded density with respect to P_0. Various examples, connected to the previous section as well as to [3]-[5], are given as well.

As remarked after Example 9, condition (5) does not imply $\mathbb{H}_0\neq\emptyset$. This fact is pointed out in the following example.

Example 11. Let $\mathcal{A}$ be the power set of $\Omega=\{1,2,\ldots\}$ and L the linear space generated by

$$X_0=I_{\{1\}}-1/3,\quad X_n=I_{\{n\}}-3\,I_{\{n+1\}}\quad\text{for } n\geq 1.$$

Then, $\mathcal{A}=\sigma(L)$ and, for each $P\in\mathbb{P}$, one obtains $E_P(X)=0$ for all $X\in L$ if and only if $P\{n\}=3^{-n}$ for all $n\geq 1$. Since $\sum_{n=1}^\infty P\{n\}=1/2$, it follows that $\mathbb{H}_0=\emptyset$. On the other hand, $\sup_\Omega X\geq E_P(X)=0$ for all $X\in L$ if $P=(P_1+Q)/2$, where $P_1\in\mathbb{P}$, $Q\in\mathbb{P}_0$, $P_1\{n\}=0$ and $Q\{n\}=2\,3^{-n}$ for all $n\geq 1$. Finally, condition (5) trivially holds with $A=\Omega$.

If $V(A)$ is compact, $\mathcal{A}=\sigma(L)$ and $\sup_A X\geq 0$ for all $X\in L$, then Theorem 8 grants $\mathbb{H}_0\neq\emptyset$. For finite V, say $V=(X_1,\ldots,X_n)$, a question is whether compactness of $V(A)$ can be weakened into compactness of $X_i(A)$ for every i. It is not hard to see that the answer is no.

Example 12. Let $\Omega=[0,1]$, $\mathcal{A}$ the Borel σ-field, and L the linear space generated by $V=(X_1,X_2)$, where

$$X_1(\omega)=\omega\quad\text{and}\quad X_2(\omega)=I_{\{0\}}(\omega).$$

Then, $\mathcal{A}=\sigma(L)$. If $P\in\mathbb{P}_0$ and $E_P(X_1)=0$, then P is the point mass at 0, so that $E_P(X_2)=1$. Thus, $\mathbb{H}_0=\emptyset$. Nevertheless, both X_1 and X_2 have compact range, and $E_P(X_1) = E_P(X_2) = 0$ if $P \in \mathbb{P}$ meets $P(0, \epsilon) = 1$ for each $\epsilon > 0$. Hence, $\sup_\Omega X \geq E_P(X) = 0$ for all $X \in L$.

In the rest of this paper, we fix a reference measure $P_0 \in \mathbb{P}_0$ and we focus on those P which satisfy equation (1) as well as $P \sim P_0$ or $P \ll P_0$.

To begin with, we recall a (nice) result which applies in the finite dimensional case. Indeed, if L is finite dimensional, there is $P \in \mathbb{H}_0$ such that $P \sim P_0$ if and only if L satisfies the classical *no-arbitrage* condition, namely

$$P_0(X > 0) > 0 \quad \Longleftrightarrow \quad P_0(X < 0) > 0 \quad \text{for each } X \in L.$$

This follows from [8, Theorem 2.4]. An elementary proof is given in [5, Example 7]. Incidentally, to our knowledge, no analogous result is available when P_0 is not given and one is only looking for some $P \in \mathbb{H}_0$ (Example 9 provides only sufficient conditions for this case).

A second remark is that the proof of Theorem 7 actually implies the following statement.

Theorem 13. *There is $P \in \mathbb{H}$ such that $P \ll P_0$ (or $P \sim P_0$) if and only if there is $Q \in \mathbb{P}$ such that*

$$(6) \qquad Q \ll P_0 \text{ (or } Q \sim P_0\text{)}, \quad E_Q|X| < \infty \quad \text{and} \quad |\,E_Q(X)\,| \leq c\,E_Q|X|$$

for all $X \in L$ and some constant $c \in [0, 1)$. Moreover, if $Q \in \mathbb{P}_0$, there is $P \in \mathbb{H}_0$ such that $P \ll P_0$ (or $P \sim P_0$).

In real problems, to apply Theorem 13, one has to select some $Q \in \mathbb{P}$ such that $Q \ll P_0$ or $Q \sim P_0$. The choice of Q is clearly a drawback. At least when looking for $P \in \mathbb{H}_0$ satisfying $P \sim P_0$, however, one option is to take $Q = P_0$. Indeed, condition (6) holds with $Q = P_0$ if and only if there is $P \in \mathbb{H}_0$ such that $r\,P_0 \leq P \leq s\,P_0$ for some constants $0 < r \leq s$; see [5, Theorem 6].

We next prove a new result concerning the case $P \sim P_0$.

In view of [5, Theorem 2], if $L \subset l^\infty$, there is $P \in \mathbb{H}$ such that $P \sim P_0$ if and only if

$$E_{P_0}\big(X\,I_{A_n}\big) \leq k_n \,\text{ess sup}(-X) \quad \text{for all } n \geq 1 \text{ and } X \in L,$$

where (k_n) is a sequence of nonnegative constants and $(A_n) \subset \mathcal{A}$ a sequence of events such that $\lim_n P_0(A_n) = 1$. An obvious strengthening is

$$(7) \qquad E_{P_0}\big(X\,I_{A_n}\big) \leq k_n\,E_{P_0}(X^-) \quad \text{for all } n \geq 1 \text{ and } X \in L$$

with (k_n) and (A_n) as above. Condition (7) is potentially useful in applications, for it involves P_0 only (while Theorem 13 requires the choice of Q). Furthermore, condition (7) looks like condition (6) applied with $Q = P_0$, which in turn suffices for the existence of $P \in \mathbb{H}_0$ such that $r\,P_0 \leq P \leq s\,P_0$. Thus, it seems natural to investigate (7). It turns out that condition (7) works and $L \subset l^\infty$ may be weakened into $L \subset L_1(P_0)$.

Theorem 14. *Suppose $E_{P_0}|X| < \infty$ for all $X \in L$. Then, condition* (7) *is equivalent to the existence of $P \in \mathbb{H}_0$ such that $P \sim P_0$ and $P \leq s\,P_0$ for some constant $s > 0$.*

Proof. Let $P \in \mathbb{H}_0$ be such that $P \sim P_0$ and $P \leq s\,P_0$. Define $k_n = ns$ and $A_n = \{n\,f \geq 1\}$, where f is a density of P with respect to P_0. Then, $\lim_n P_0(A_n) =$

$P_0(f > 0) = 1$. Given $X \in L$, since $E_P(X^+) = E_P(X^-)$, one also obtains

$$E_{P_0}\big(X\, I_{A_n}\big) \leq E_{P_0}\big(X^+ I_{A_n}\big) = E_P\big(X^+ (1/f)\, I_{A_n}\big)$$
$$\leq n\, E_P(X^+) = n\, E_P(X^-) \leq k_n\, E_{P_0}(X^-).$$

Conversely, suppose condition (7) holds for some $k_n \geq 0$ and $A_n \in \mathcal{A}$ such that $\lim_n P_0(A_n) = 1$. If there is a subsequence n_j such that $k_{n_j} \leq 1$ for all j, taking the limit as $j \to \infty$ condition (7) yields $E_{P_0}(X) \leq E_{P_0}(X^-)$ for all $X \in L$. It follows that $|\, E_{P_0}(X)\, | \leq (1/3)\, E_{P_0}|X|$ for all $X \in L$, namely condition (6) holds with $Q = P_0$. As noted above, this implies the existence of $P \in \mathbb{H}_0$ such that $r\, P_0 \leq P \leq s\, P_0$ for some $0 < r \leq s$.

Hence, it can be assumed $k_n > 1$ for large n, say $k_n > 1$ for each $n \geq m$. Define

$$k = \Big(\sum_{n=m}^{\infty} \frac{P_0(A_n)}{k_n\, 2^n}\Big)^{-1} \quad \text{and} \quad Q(\cdot) = k \sum_{n=m}^{\infty} \frac{P_0(\cdot \cap A_n)}{k_n\, 2^n}.$$

Then, $1 < k < \infty$, $Q \in \mathbb{P}_0$ and $Q \sim P_0$. For any random variable $Y \geq 0$, one obtains

$$E_Q(Y) = k \sum_{n=m}^{\infty} \frac{E_{P_0}\big(Y\, I_{A_n}\big)}{k_n\, 2^n} \leq k\, E_{P_0}(Y).$$

In particular, $Q(A) \leq k\, P_0(A)$ and $E_Q|X| \leq k\, E_{P_0}|X| < \infty$ whenever $A \in \mathcal{A}$ and $X \in L$. Further, condition (7) implies

$$E_Q(X) = k \sum_{n=m}^{\infty} \frac{E_{P_0}\big(X\, I_{A_n}\big)}{k_n\, 2^n} \leq k\, E_{P_0}(X^-) \quad \text{for all } X \in L.$$

Having proved the above facts about Q, the proof essentially proceeds as that of Theorem 7. We sketch the fundamental points.

Define

$$\mathcal{K} = \big\{P \in \mathbb{P}_0 :\, (k+1)^{-1} Q \leq P \leq Q + k\, P_0\big\}.$$

If $P \in \mathcal{K}$, then $E_P|X| \leq E_Q|X| + k\, E_{P_0}|X| \leq 2\, k\, E_{P_0}|X| < \infty$ for all $X \in L$. Also, $P \in \mathbb{P}_0$, $P \sim P_0$ and $P \leq 2\, k\, P_0$. Hence, it suffices to show that $E_P(X) = 0$ for all $X \in L$ and some $P \in \mathcal{K}$.

For each $X \in L$, there is $P \in \mathcal{K}$ such that $E_P(X) = 0$. Fix in fact $X \in L$. If $E_Q(X) = 0$, just take $P = Q \in \mathcal{K}$. If $E_Q(X) > 0$, take a density h of Q with respect to P_0 and define

$$f = \frac{E_Q(X)\, I_{\{X<0\}} + E_{P_0}(X^-)\, h}{E_Q(X)\, P_0(X < 0) + E_{P_0}(X^-)} \quad \text{and} \quad P(A) = E_{P_0}\big(f\, I_A\big) \quad \text{for } A \in \mathcal{A}.$$

Since $E_Q(X) \leq k\, E_{P_0}(X^-)$, then $(k+1)^{-1} h \leq f \leq h + k$. Hence, $P \in \mathcal{K}$ and

$$E_P(X) = E_{P_0}\big(f\, X\big) = \frac{-E_Q(X)\, E_{P_0}(X^-) + E_{P_0}(X^-)\, E_Q(X)}{E_Q(X)\, P_0(X < 0) + E_{P_0}(X^-)} = 0.$$

Finally, if $E_Q(X) < 0$, just replace X with $-X$ and repeat the above argument.

From now on, the proof agrees exactly with that of Theorem 7. In fact, $\mathcal{K}$ is compact and $\{P \in \mathcal{K} : E_P(X) = 0\}$ is closed for each $X \in L$ (under the same topology as in the proof of Theorem 7). In addition, for each finite subset $\{X_1, \ldots, X_n\} \subset L$, one obtains $E_P(X_1) = \ldots = E_P(X_n) = 0$ for some $P \in \mathcal{K}$. This concludes the proof. $\square$

Let us turn to the case $P \ll P_0$.

If $L \subset l^\infty$, there is $P \in \mathbb{H}$ such that $P \ll P_0$ if and only if

$$\text{ess sup}(X) \geq 0 \quad \text{for all } X \in L;$$

see [3, Theorem 3]. This is a simple and intuitive result and one may wonder how long it depends on $L \subset l^\infty$. Say that L is dominated if there is a random variable $Y \geq 1$ satisfying

$$\text{for each } X \in L, \text{ there is } \lambda > 0 \text{ such that } |X| \leq \lambda\, Y, \;\; P_0\text{-a.s.}$$

Such condition is not so unusual. It holds, for instance, if L is countably generated; see [5, Lemma 4]. If L is dominated, a conjecture is that there is $P \in \mathbb{H}$ such that $P \ll P_0$ if and only if ess sup$(X) \geq 0$ for all $X \in L$. However, this conjecture is false.

Example 15. Let $\Omega = (0, \infty)$, $\mathcal{A}$ the Borel σ-field and P_0 the exponential law with parameter 1. Take L to be the linear space generated by

$$X_1(\omega) = 1/\omega \quad \text{and} \quad X_2(\omega) = I_{(0,\pi/2)}(\omega)\, \tan(\omega) - I_{[\pi/2,\infty)}(\omega).$$

Then, L is dominated (just take $Y = 1 + |X_1| + |X_2|$) and it is not hard to check that ess sup$(X) \geq 0$ for all $X \in L$. However, if $P \in \mathbb{P}$ and $E_P(X_1) = 0$, then $P(t, \infty) = 1$ for all $t > 0$, so that $E_P(X_2) = -1$. Therefore, $\mathbb{H} = \emptyset$.

Next, each $P \in \mathbb{H}$ with $P \sim P_0$ can be written as $P = \alpha\, P_1 + (1 - \alpha)\, Q$, where $\alpha \in [0, 1)$, $P_1 \in \mathbb{P}$ is pure, $Q \in \mathbb{P}_0$ and $Q \sim P_0$. Suppose that

$$\text{for each } \epsilon > 0, \text{ there is } P \in \mathbb{H} \text{ such that } P \sim P_0 \text{ and } \alpha(P) < \epsilon, \tag{8}$$

where $\alpha(P)$ is the weight of the pure component of P. Notwithstanding such condition, it may be that no $P \in \mathbb{H}_0$ satisfies $P \sim P_0$; see [4, Example 9] and [5, Example 10]. In both the quoted examples, however, there is $P \in \mathbb{H}_0$ such that $P \ll P_0$ (even if not $P \sim P_0$). We now give a more extreme example, where condition (8) holds and no $P \in \mathbb{H}_0$ meets $P \ll P_0$.

Example 16. Let $\Omega = (-1, 1)$. Take $\mathcal{A}$ the Borel σ-field, P_0 uniform on Ω, and L the linear space generated by

$$X_0(\omega) = -I_{(-1,0)}(\omega) + (1 + \omega)\, I_{(0,1)}(\omega), \quad X_n(\omega) = \text{sgn}(\omega)\, |\omega|^n \text{ for } n \geq 1.$$

Fix $P \in \mathbb{P}_0$ such that $P \ll P_0$ and $E_P(X_n) = 0$ for $n \geq 1$. If f is a density of P with respect to P_0, then

$$\begin{aligned} \int_0^1 f(\omega)\, \omega^n\, d\omega &= 2\, E_P\big\{X_n\, I_{(0,1)}\big\} = -2\, E_P\big\{X_n\, I_{(-1,0)}\big\} \\ &= \int_{-1}^0 f(\omega)\, |\omega|^n\, d\omega = \int_0^1 f(-\omega)\, \omega^n\, d\omega \quad \text{for all } n \geq 1. \end{aligned}$$

It follows that $f(\omega) = f(-\omega)$ for almost all ω (with respect to Lebesgue measure) which in turn implies

$$E_P(X_0) = -P(-1, 0) + P(0, 1) + \int_0^1 \omega\, P(d\omega) = (1/2) \int_0^1 \omega\, f(\omega)\, d\omega > 0.$$

Hence, there is no $P \in \mathbb{H}_0$ satisfying $P \ll P_0$. However, condition (8) holds. In fact, given $\epsilon \in (0,1)$, define

$$Q(A) = \frac{P_0\big(A \cap (-\epsilon,\epsilon)\big) + \epsilon^2 P_0\big(A \cap (-\epsilon,\epsilon)^c\big)}{\epsilon + \epsilon^2(1-\epsilon)} \quad \text{for all } A \in \mathcal{A}.$$

Then, $Q \in \mathbb{P}_0$ and $Q \sim P_0$. Since $Q(-1,0) = Q(0,1)$,

$$E_Q(X_0) = \int_0^1 \omega\, Q(d\omega) = \frac{1}{4}\,\frac{2\,\epsilon - \epsilon^3}{\epsilon\,(1-\epsilon)+1} < \epsilon.$$

Fix $X \in L$, say $X = \sum_{j=0}^n b_j X_j$ where $n \geq 1$ and $b_0, \ldots, b_n \in \mathbb{R}$, and define $Y = \sum_{j=1}^n b_j X_j$. Since Y is continuous and $Y(0) = 0$, then $\text{ess sup}(-X) \geq |b_0|$. Thus, $E_Q(Y) = 0$ yields

$$E_Q(X) = b_0\, E_Q(X_0) \leq \epsilon\,|b_0| \leq \epsilon\, \text{ess sup}(-X).$$

In view of [5, Theorem 2], letting

$$P = \frac{Q + \epsilon\, P_1}{1+\epsilon} \quad \text{for suitable } P_1 \in \mathbb{P},$$

one obtains $P \in \mathbb{H}$ and $P \sim P_0$. On noting that $\alpha(P) \leq \epsilon(1+\epsilon)^{-1} < \epsilon$, condition (8) is shown to be true.

We finally mention a (slightly) different approach. Let L^* be the linear space generated by L and $\{I_A : A \in \mathcal{A},\, P_0(A) = 0\}$. Then, $P \in \mathbb{H}$ and $P \ll P_0$ if and only if

$$E_P|X| < \infty \quad \text{and} \quad E_P(X) = 0 \quad \text{for all } X \in L^*.$$

Roughly speaking, to find $P \in \mathbb{H}$ such that $P \ll P_0$ reduces to find P which satisfies equation (1) alone for a larger linear space (namely, L^*). Similarly, $P \in \mathbb{H}$ and $P \sim P_0$ if and only if P meets equation (1) with L^* in the place of L and $E_P(Y) \neq 0$ for each $Y \in U$, where $U = \{I_A : A \in \mathcal{A},\, P_0(A) > 0\}$. This suggests the following problem. Given a class U of real random variables, is there $P \in \mathbb{P}$ or $P \in \mathbb{P}_0$ such that

$$E_P\big(|X| + |Y|\big) < \infty, \quad E_P(X) = 0 \quad \text{and} \quad E_P(Y) \neq 0 \quad \text{for all } X \in L \text{ and } Y \in U\,?$$

Our last result is in this direction.

Proposition 17. *Let $L \subset l^\infty$ and Y a bounded random variable. Define*

$$a = \inf_{X \in L} \sup_{\omega \in \Omega} \big\{X(\omega) - Y(\omega)\big\} \quad \textit{and} \quad b = \inf_{X \in L} \sup_{\omega \in \Omega} \big\{X(\omega) + Y(\omega)\big\}.$$

Given $y \in \mathbb{R}$, there is $P \in \mathbb{H}$ such that $E_P(Y) = y$ if and only if

$$\sup_{\omega \in \Omega} X(\omega) \geq 0 \ \textit{ for all } \ X \in L \ \textit{ and } \ -a \leq y \leq b.$$

In particular, Proposition 17 implies that $E_P(Y) \neq 0$ for some $P \in \mathbb{H}$ if and only if $\sup_\Omega X \geq 0$ for all $X \in L$ and $|a| + |b| > 0$.

Proof of Proposition 17. If $P \in \mathbb{H}$ and $E_P(Y) = y$, for each $X \in L$ one obtains

$$\sup_\Omega X \geq E_P(X) = 0, \quad \sup_\Omega (X - Y) \geq E_P(X - Y) = -y, \quad \sup_\Omega (X + Y) \geq E_P(X + Y) = y.$$

Hence, $-a \leq y \leq b$. Conversely, suppose $\sup_\Omega X \geq 0$ for all $X \in L$ and $-a \leq y \leq b$. If $Y \in L$, then $a \leq 0$ and $b \leq 0$, so that $y = 0$. Thus, it suffices to take any $P \in \mathbb{H}$ (which exists for $\sup_\Omega X \geq 0$ for all $X \in L$). If $Y \notin L$, let M be the linear space generated by L and Y. Since $Y \notin L$, each $Z \in M$ can be written as $Z = X + \lambda Y$ for some unique $X \in L$ and $\lambda \in \mathbb{R}$. Define

$$\phi(Z) = \phi(X + \lambda Y) = \lambda y \quad \text{for all } Z \in M.$$

Then, ϕ is a linear functional on M. Since $M \subset l^\infty$, if $\phi(Z) \leq \sup_\Omega Z$ for all $Z \in M$, then ϕ meets de Finetti's coherence principle, so that $\phi(Z) = E_P(Z)$ for all $Z \in M$ and some $P \in \mathbb{P}$. In particular, $E_P(Y) = \phi(Y) = y$ and $E_P(X) = \phi(X) = 0$ for all $X \in L$. Thus, it remains only to see that $\phi(Z) \leq \sup_\Omega Z$ for all $Z \in M$. Let $Z = X + \lambda Y \in M$. If $\lambda = 0$, then $Z = X$ and $\phi(X) = 0 \leq \sup_\Omega X$. Since $y \geq -a$, if $\lambda < 0$ one obtains

$$\phi(Z) = \lambda y \leq |\lambda|\, a \leq |\lambda| \sup_\Omega \Big\{ \frac{X}{|\lambda|} - Y \Big\} = \sup_\Omega \big\{ X + \lambda Y \big\} = \sup_\Omega Z.$$

Since $y \leq b$, the same argument works if $\lambda > 0$. This concludes the proof.

References

[1] Berti P., Rigo P. (1996) On the existence of inferences which are consistent with a given model, *Ann. Statist.*, 24, 1235-1249.
[2] Berti P., Rigo P. (1999) Sufficient conditions for the existence of disintegrations, *J. Theoret. Probab.*, 12, 75-86.
[3] Berti P., Pratelli L., Rigo P. (2013) Finitely additive equivalent martingale measures, *J. Theoret. Probab.*, 26, 46-57.
[4] Berti P., Pratelli L., Rigo P. (2014) Price uniqueness and fundamental theorem of asset pricing with finitely additive probabilities, *Stochastics*, 86, 135-146.
[5] Berti P., Pratelli L., Rigo P. (2014) Two versions of the fundamental theorem of asset pricing, *Submitted*, available at: `http://www-dimat.unipv.it/~rigo/`
[6] Berti P., Dreassi E., Rigo P. (2014) Compatibility results for conditional distributions, *J. Multivar. Analysis*, 125, 190-203.
[7] Cassese G. (2007) Yan theorem in L_∞ with applications to asset pricing, *Acta Math. Appl. Sin. Engl. Ser.*, 23, 551-562.
[8] Dalang R., Morton A., Willinger W. (1990) Equivalent martingale measures and no-arbitrage in stochastic securities market models, *Stoch. and Stoch. Reports*, 29, 185-201.
[9] Delbaen F., Schachermayer W. (2006) *The mathematics of arbitrage*, Springer.
[10] Dubins L.E., Prikry K. (1995) On the existence of disintegrations, *Seminaire de Probab.* XXIX, Lect. Notes in Math., Springer, 1613, 248-259.
[11] Heath D., Sudderth W.D. (1978) On finitely additive priors, coherence and extended admissibility, *Ann. Statist.*, 6, 333-345.
[12] Rokhlin D.B. (2005) The Kreps-Yan theorem for L^∞, *Intern. J. Math. Math. Sci.*, 17, 2749-2756.
[13] Rokhlin D.B., Schachermayer W. (2006) A note on lower bounds of martingale measure densities, *Illinois J. Math.*, 50, 815-824.
[14] Strassen V. (1965) The existence of probability measures with given marginals, *Ann. Math. Statist.*, 36, 423-439.

Patrizia Berti, Dipartimento di Matematica Pura ed Applicata "G. Vitali", Universita' di Modena e Reggio-Emilia, via Campi 213/B, 41100 Modena, Italy
E-mail address: patrizia.berti@unimore.it

Luca Pratelli, Accademia Navale, viale Italia 72, 57100 Livorno, Italy
E-mail address: pratel@mail.dm.unipi.it

Pietro Rigo (corresponding author), Dipartimento di Matematica "F. Casorati", Universita' di Pavia, via Ferrata 1, 27100 Pavia, Italy
E-mail address: pietro.rigo@unipv.it