A CONDITIONAL 0-1 LAW FOR THE SYMMETRIC σ-FIELD

PATRIZIA BERTI AND PIETRO RIGO

Abstract. Let $(\Omega, \mathcal{B}, P)$ be a probability space, $\mathcal{A} \subset \mathcal{B}$ a sub-σ-field, and μ a regular conditional distribution for P given $\mathcal{A}$. For various, classically interesting, choices of $\mathcal{A}$ (including tail and symmetric) the following 0-1 law is proved: There is a set $A_0 \in \mathcal{A}$ such that $P(A_0) = 1$ and $\mu(\omega)(A) \in \{0, 1\}$ for all $A \in \mathcal{A}$ and $\omega \in A_0$. Provided $\mathcal{B}$ is countably generated (and certain regular conditional distributions exist), the result applies whatever P is.

1. Introduction

Let $(\Omega, \mathcal{B}, P)$ be a probability space, $\mathcal{A} \subset \mathcal{B}$ a sub-σ-field, and $\mathbb{P}$ the set of all probability measures on $\mathcal{B}$. A *regular conditional distribution* (r.c.d.) for P given $\mathcal{A}$ is a mapping $\mu : \Omega \to \mathbb{P}$ such that $\mu(\cdot)(B)$ is a version of $E(I_B \mid \mathcal{A})$ for all $B \in \mathcal{B}$. Throughout, *P is assumed to admit a r.c.d. given $\mathcal{A}$, denoted by μ, and $\mathcal{B}$ is countably generated* (that is, $\mathcal{B}$ is generated by one of its countable subclasses).

We aim at showing that, for certain sub-σ-fields $\mathcal{A}$ (including tail and symmetric), μ obeys the following 0-1 law: There is a set $A_0 \in \mathcal{A}$ with $P(A_0) = 1$ and

$$\mu(\omega)(A) \in \{0, 1\} \quad \text{for all } A \in \mathcal{A} \text{ and } \omega \in A_0. \tag{1}$$

2. Motivations

In the sequel, A_0 denotes a set of $\mathcal{A}$ satisfying $P(A_0) = 1$.

For both foundational and technical reasons, it would be desirable that

$$\mu(\omega)(A) = I_A(\omega) \quad \text{for all } A \in \mathcal{A} \text{ and } \omega \in A_0 \tag{2}$$

for some A_0. Despite its heuristic content, however, condition (2) need not be true. In fact, by results of Blackwell and Dubins (see [3] and references therein), condition (2) holds if and only if the trace σ-field $\mathcal{A} \cap A_0 = \{A \cap A_0 : A \in \mathcal{A}\}$ is countably generated for some A_0. Unless $\mathcal{A}$ is countably generated, thus, (2) does not hold for a number of probability measures P.

When (2) fails, a natural question is whether some of its consequences are still in force. The 0-1 law in (1) is just a (intriguing) consequence of condition (2).

To give (1) some interpretation, let us fix $\omega_0 \in \Omega$. If $\mu(\omega_0)$ is 0-1 on $\mathcal{A}$, then $\mu(\omega_0)(A \cap B) = \mu(\omega_0)(A)\mu(\omega_0)(B)$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Conversely, the latter relation (with $B = A$) yields $\mu(\omega_0)(A) = \mu(\omega_0)(A)^2$ for $A \in \mathcal{A}$, so that

$$\mu(\omega_0) \text{ is 0-1 on } \mathcal{A} \quad \Leftrightarrow \quad \mathcal{B} \text{ is independent of } \mathcal{A} \text{ under } \mu(\omega_0).$$

2000 *Mathematics Subject Classification.* 60A05, 60A10, 60F20.

Key words and phrases. 0-1 law; invariant, tail and symmetric σ-fields; regular conditional distribution.

Now, roughly speaking, the probability measure $\mu(\omega)$ should embody the information conveyed by $\mathcal{A}$ for each ω in some set A_0. Accordingly, $\mathcal{B}$ should be independent of $\mathcal{A}$, under $\mu(\omega)$, for all ω in such A_0. This is precisely condition (1).

An equivalent (heuristic) argument is the following. If $\mu(\omega_0)$ already includes the information in $\mathcal{A}$, then μ should be a r.c.d. for $\mu(\omega_0)$ given $\mathcal{A}$ as well. In fact, letting $M = \{Q \in \mathbb{P} : \mu \text{ is a r.c.d. for } Q \text{ given } \mathcal{A}\}$, condition (1) holds if and only if $\mu(\omega) \in M$ for each ω in some A_0; see Theorem 12 of [2].

Example 1. (Tail σ-field) Suppose $\mathcal{A} = \bigcap_n \sigma(X_n, X_{n+1}, \ldots)$ is the tail σ-field of a sequence (X_n) of real random variables on $(\Omega, \mathcal{B}, P)$. A probability measure $Q \in \mathbb{P}$ is 0-1 on $\mathcal{A}$ if and only if $\mathcal{B}$ is asymptotically independent of $(X_n, X_{n+1}, \ldots)$ under Q, in the sense that

$$\sup_{H \in \sigma(X_n, X_{n+1}, \ldots)} |Q(B \cap H) - Q(B)Q(H)| \to 0 \quad \text{for each } B \in \mathcal{B}.$$

Hence, condition (1) becomes: $\mathcal{B}$ is asymptotically independent of $(X_n, X_{n+1}, \ldots)$, under $\mu(\omega)$, for each ω in some A_0. This looks quite reasonable (to us). Indeed, in [2], condition (1) is shown to be true (whatever P is) if $\mathcal{A}$ is a tail σ-field.

A nice property of (1) is that it is preserved under an absolutely continuous change of probability measure.

Example 2. (Absolute continuity) Suppose $Q \in \mathbb{P}$ satisfies $Q \ll P$ and $\mu(\omega)$ is 0-1 on $\mathcal{A}$ for each ω in some A_0 (with $A_0 \in \mathcal{A}$ and $P(A_0) = 1$). Let f be a density of Q with respect to P and $A_1 = \{\omega : 0 < \int f(x)\mu(\omega)(dx) < \infty\}$. Then, $A_1 \in \mathcal{A}$, $Q(A_1) = 1$, and

$$\nu(\omega)(B) = \frac{\int_B f(x)\mu(\omega)(dx)}{\int f(x)\mu(\omega)(dx)}, \quad B \in \mathcal{B}, \omega \in A_1,$$

is a r.c.d. for Q given $\mathcal{A}$. If $\omega \in A_0 \cap A_1$, then $\nu(\omega) \ll \mu(\omega)$ and $\mu(\omega)$ is 0-1 on $\mathcal{A}$. Thus, $\nu(\omega) = \mu(\omega)$ on $\mathcal{A}$ for all $\omega \in A_0 \cap A_1$. This fact has two consequences. First, since $Q(A_0 \cap A_1) = 1$, condition (1) holds under Q as well. Second, if P and Q are equivalent (i.e. $Q \ll P$ and $P \ll Q$) then $\nu = \mu$ on $\mathcal{A}$ a.s.. This seems in line with intuition.

Since (1) holds in various real situations, one could suspect that it is always true, at least under mild conditions. This is not so. Define in fact

$$\mathcal{N} = \{B \in \mathcal{B} : P(B) = 0\}.$$

Then, for (1) to fail, it is enough that: (i) $\mathcal{A} \supset \mathcal{N}$; (ii) $P\{x : \mu(x) = \mu(\omega)\} = 0$ for each ω in some A_0; (iii) $P\{x : \mu(x) \text{ is not 0-1 on } \mathcal{B}\} > 0$; see Proposition 11 of [2]. Conditions (ii)-(iii) hold in most interesting problems. Thus, (1) typically fails whenever $\mathcal{A} \supset \mathcal{N}$. The next two examples illustrate this fact.

Example 3. (A failure of condition (1)**)** Let $(\mathcal{F}_t : t \geq 0)$ be a filtration on $(\Omega, \mathcal{B}, P)$. As in stochastic calculus, suppose $(\mathcal{F}_t)$ is right continuous and $\mathcal{F}_0 \supset \mathcal{N}$ (the so called "usual conditions"). Suppose also that $\mathcal{B}$ is countably generated and $X = \{X_t : t \geq 0\}$ is a real homogeneous Markov process, relative to $(\mathcal{F}_t)$, with transition kernel

$$K_t(a, H) = \text{Prob}\big(X_t \in H \mid X_0 = a\big), \quad t > 0, a \in \mathbb{R}, H \text{ a real Borel set}.$$

Letting $\mathcal{A} = \mathcal{F}_t$ for some $t > 0$, one obtains

$$\mu(\omega)\big(X_{2t} \in \cdot\big) = K_t\big(X_t(\omega), \cdot\big) \quad \text{for each } \omega \text{ in some set } A_0.$$

Thus, (1) fails under various conditions on K_t. For instance, (1) fails whenever $K_t(a, \cdot) \neq K_t(b, \cdot)$ for all $a \neq b$ and $K_t(a, \{b\}) = 0$ for all a, b. In fact, $\mu(\omega)$ is not 0-1 on $\mathcal{B}$ for each $\omega \in A_0$. Moreover, $P(X_t = b) = \int K_t(X_0, \{b\})dP = 0$ for all b. Hence,

$$P\{x : \mu(x) = \mu(\omega)\} \leq P\big(X_t = X_t(\omega)\big) = 0 \quad \text{for all } \omega \in A_0.$$

Therefore, conditions (i)-(ii)-(iii) hold.

Incidentally, Example 3 also suggests the following remark (unrelated to condition (1)). According to a usual naive interpretation, $\mathcal{F}_t$ describes the information at time t, in the sense that each event $A \in \mathcal{F}_t$ is known to be true or false at time t. This interpretation does not make sense in Example 3 as far as $\{X = x\} \in \mathcal{B}$ for each possible path x of the process X. In fact, $P(X = x) \leq P(X_t = x(t)) = 0$ for all x, so that $\{X = x\} \in \mathcal{F}_0$ for every path x. Under such interpretation, thus, the X-path would be already known at time $t = 0$.

Example 4. (One more failure of condition (1)**; see [2])** Let $\Omega = \mathbb{R}^2$, $\mathcal{B}$ the Borel σ-field, and $P = Q \times Q$ where Q is the $N(0, 1)$ law on the real Borel sets. Define $\mathcal{A} = \sigma\big(\mathcal{G} \cup \mathcal{N}\big)$ where $\mathcal{G}$ is the σ-field on Ω generated by $(x, y) \mapsto x$. A r.c.d. for P given $\mathcal{G}$ is $\mu((x, y)) = \delta_x \times Q$. Since $\mathcal{A} = \sigma\big(\mathcal{G} \cup \mathcal{N}\big)$, μ is also a r.c.d. for P given $\mathcal{A}$. Moreover, for all (x, y), one has $\{x\} \times [0, \infty) \in \mathcal{A}$ and

$$\mu((x, y))\Big(\{x\} \times [0, \infty)\Big) = \frac{1}{2}.$$

Though implicit in ideas of Dynkin [6] and Diaconis and Freedman [5], condition (1) has been almost neglected so far. Possible related references are [1], [2], [3] and [10], but only [2] is explicitly devoted to (1).

This note carries on the investigation started in [2]. It is proved that condition (1) holds (whatever P is) for certain sub-σ-fields $\mathcal{A}$, including the symmetric one.

3. Results

Let F be a class of measurable functions $f : \Omega \to \Omega$, where measurability means $f^{-1}\big(\mathcal{B}\big) \subset \mathcal{B}$. In case F is a group under composition, with the identity map on Ω as group-identity, we briefly say that F *is a group*. Whether or not F is a group, the F-*invariant* σ-field is

$$\mathcal{A}_F = \{B \in \mathcal{B} : f^{-1}B = B \text{ for all } f \in F\}$$

and a probability measure Q on $\mathcal{B}$ is F-invariant if $Q \circ f^{-1} = Q$ for all $f \in F$. Let $\mathbb{P}_F$ denote the set of F-invariant probability measures.

One more definition is to be recalled. Let $\mathbb{Q} \subset \mathbb{P}$ be a collection of probability measures and $\mathcal{G} \subset \mathcal{B}$ a sub-σ-field. Then, $\mathcal{G}$ is *sufficient for* $\mathbb{Q}$ in case, for each $B \in \mathcal{B}$, there is a $\mathcal{G}$-measurable function $h : \Omega \to \mathbb{R}$ which is a version of $E_Q\big(I_B \mid \mathcal{G}\big)$ for all $Q \in \mathbb{Q}$. When $\mathbb{Q} = \mathbb{P}_F$, sufficiency is a key ingredient for integral representation of invariant measures; see [5], [6], [7] and [9]. Conditions under which $\mathcal{A}_F$ is sufficient for $\mathbb{P}_F$ are given in Theorem 3 of [7]. In particular, $\mathcal{A}_F$ is sufficient for $\mathbb{P}_F$ if F is a countable group or if F includes only one function.

Arguing as Maitra in [9], we now prove that condition (1) holds whenever $P \in \mathbb{P}_F$, F is countable and $\mathcal{A}$ sufficient for $\mathbb{P}_F$. Basing on this fact we subsequently show that, if F is a finite group and $\mathcal{A} = \mathcal{A}_F$, then (1) holds whatever P is.

Lemma 5. *Suppose $\mathcal{B}$ is countably generated, F is countable, and $P \ll P_0$ for some $P_0 \in \mathbb{P}_F$ which admits a r.c.d. given $\mathcal{A}$. Then, condition* (1) *holds provided $\mathcal{A}$ is sufficient for $\mathbb{P}_F$. In particular, when $\mathcal{A} = \mathcal{A}_F$, condition* (1) *holds if F is a group or if F includes only one function.*

Proof. By Example 2, it is enough to prove that P_0 meets (1). Thus, it can be assumed $P \in \mathbb{P}_F$. Let $\mathcal{M} = \{B \in \mathcal{B} : Q(B) = 0 \text{ for all } Q \in \mathbb{P}_F\}$. Since $\mathcal{A}$ is sufficient for $\mathbb{P}_F$ and $\mathcal{B}$ is countably generated, by Theorem 1 of [4], there is a countably generated σ-field $\mathcal{D}$ such that $\mathcal{D} \subset \mathcal{A} \subset \sigma\big(\mathcal{D} \cup \mathcal{M}\big)$. Since $\mathcal{D}$ is countably generated and $\mathcal{D} \subset \mathcal{A}$, there is $A_1 \in \mathcal{A}$ such that $P(A_1) = 1$ and $\mu(\omega)(D) = I_D(\omega)$ for all $D \in \mathcal{D}$ and $\omega \in A_1$. Since $P \in \mathbb{P}_F$, given $B \in \mathcal{B}$ and $f \in F$, one obtains $\mu(\omega)(f^{-1}B) = \mu(\omega)(B)$ for almost all ω. Since F is countable and $\mathcal{B}$ countably generated, it follows that $\mu(\omega) \in \mathbb{P}_F$ for each ω in some set $A_2 \in \mathcal{A}$ with $P(A_2) = 1$. Fix $\omega \in A_1 \cap A_2$. Then, $\mu(\omega)$ is 0-1 on $\mathcal{D} \cup \mathcal{M}$, so that $\mu(\omega)$ is 0-1 on $\sigma\big(\mathcal{D} \cup \mathcal{M}\big)$ as well. Since $\mathcal{A} \subset \sigma\big(\mathcal{D} \cup \mathcal{M}\big)$, for getting condition (1) it suffices to let $A_0 = A_1 \cap A_2$. □

Theorem 6. *If $\mathcal{B}$ is countably generated, F is a finite group and $\mathcal{A} = \mathcal{A}_F$, then condition* (1) *holds.*

Proof. Define

$$Q = \frac{\sum_{f \in F} P \circ f^{-1}}{card(F)}, \quad \nu(\omega) = \frac{\sum_{f \in F} \mu(\omega) \circ f^{-1}}{card(F)} \quad \text{for all } \omega \in \Omega,$$

and note that $Q = P$ and $\nu(\omega) = \mu(\omega)$ on $\mathcal{A} = \mathcal{A}_F$. Further, $\nu(\omega) \in \mathbb{P}$ for all $\omega \in \Omega$, $\omega \mapsto \nu(\omega)(B)$ is $\mathcal{A}$-measurable for all $B \in \mathcal{B}$, and for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$ one obtains:

$$\int_A \nu(\omega)(B) Q(d\omega) = \int_A \nu(\omega)(B) P(d\omega)$$

$$= \frac{1}{card(F)} \sum_{f \in F} \int_A \mu(\omega)(f^{-1}B) P(d\omega) = \frac{1}{card(F)} \sum_{f \in F} P\big(A \cap f^{-1}B\big)$$

$$= \frac{1}{card(F)} \sum_{f \in F} P\big(f^{-1}A \cap f^{-1}B\big) = Q(A \cap B).$$

Hence, ν is a r.c.d. for Q given $\mathcal{A}$. Since F is a finite group, then $Q \in \mathbb{P}_F$ and $\mathcal{A} = \mathcal{A}_F$ is sufficient for $\mathbb{P}_F$. Accordingly, by applying Lemma 5 to Q and ν, there is $A_0 \in \mathcal{A}$ such that $Q(A_0) = 1$ and $\nu(\omega)$ is 0-1 on $\mathcal{A}$ for each $\omega \in A_0$. Since $Q = P$ and $\nu = \mu$ on $\mathcal{A}$, this concludes the proof. □

Among other things, given a single measurable function $f : \Omega \to \Omega$, Lemma 5 and Theorem 6 apply to $\mathcal{A} = \mathcal{A}_{\{f\}} = \{B \in \mathcal{B} : f^{-1}B = B\}$. Precisely, Lemma 5 grants condition (1) in case $P \ll P_0$ for some f-invariant P_0 (admitting a r.c.d. given $\mathcal{A}$). By Theorem 6, instead, condition (1) holds whatever P is in case f is bijective with $f = f^{-1}$.

Towards our main example, concerning the symmetric σ-field (cf. Example 11), we mention one more consequence of Theorem 6.

Example 7. (Permutations of order n) Fix a measurable space $(\mathcal{X},\mathcal{U})$, with $\mathcal{U}$ countably generated, and define $(\Omega,\mathcal{B})=(\mathcal{X}^\infty,\mathcal{U}^\infty)$. Denote points of $\Omega=\mathcal{X}^\infty$ by $\omega=(\omega_1,\omega_2,\ldots)$. A *permutation of order* n is a map $f:\Omega\to\Omega$ of the form

$$f(\omega)=(\omega_{\pi_1},\ldots,\omega_{\pi_n},\omega_{n+1},\ldots),\quad \omega\in\Omega,$$

for some permutation $(\pi_1,\ldots,\pi_n)$ of $(1,\ldots,n)$. The set F_n of permutations of order n is a group with $n!$ elements, and $\mathcal{A}_{F_n}$ includes those $B\in\mathcal{B}$ invariant under permutations of the first n coordinates. By Theorem 6, every r.c.d. μ (for some law $P\in\mathbb{P}$) given $\mathcal{A}=\mathcal{A}_{F_n}$ meets condition (1).

We now turn to our main result. Let $\mathcal{A}_n\subset\mathcal{B}$ be a sub-σ-field, $n=1,2,\ldots$, and

$$\mathcal{A}_*=\sigma\big(\bigcup_{n\geq 1}\bigcap_{j\geq n}\mathcal{A}_j\big),\quad \mathcal{A}^*=\sigma\big(\bigcap_{n\geq 1}\bigcup_{j\geq n}\mathcal{A}_j\big).$$

It seems reasonable that condition (1) holds provided it holds for every $\mathcal{A}_n$ and $\mathcal{A}_n\to\mathcal{A}$ in some sense. In fact, this is true if $\mathcal{A}\subset\mathcal{A}_*$ and $\mathcal{A}_n\to\mathcal{A}$ is meant as

$$E\big(I_B\mid\mathcal{A}_n\big)\overset{P}{\to}E\big(I_B\mid\mathcal{A}\big)\quad\text{for each } B\in\mathcal{B}. \tag{3}$$

Furthermore, condition (1) holds for $\mathcal{A}\subset\mathcal{A}^*$ (and not only for $\mathcal{A}\subset\mathcal{A}_*$) if (3) is strengthened into

$$E\big(I_B\mid\mathcal{A}_n\big)\overset{a.s.}{\to}E\big(I_B\mid\mathcal{A}\big)\quad\text{for each } B\in\mathcal{B}. \tag{3*}$$

Note that, by the martingale convergence theorem, if $\mathcal{A}_n$ is a monotonic sequence then (3*) holds with $\mathcal{A}=\mathcal{A}_*=\mathcal{A}^*$.

Theorem 8. *Suppose $\mathcal{B}$ is countably generated and, for each $n\geq 1$:*

(4) *There are a r.c.d. ν_n for P given $\mathcal{A}_n$ and a set $C_n\in\mathcal{A}_n$ such that $P(C_n)=1$ and $\nu_n(\omega)$ is 0-1 on $\mathcal{A}_n$ for all $\omega\in C_n$.*

If (3) *holds and $\mathcal{A}\subset\mathcal{A}_*$, or if* (3*) *holds and $\mathcal{A}\subset\mathcal{A}^*$, then*

$$\mu(\omega)\text{ is 0-1 on }\mathcal{A}\text{ for each }\omega\in A_0,\ \text{where } A_0\in\mathcal{A}\text{ and } P(A_0)=1$$

(that is, condition (1) *holds). In particular, condition* (1) *holds whenever $\mathcal{A}_n$ is a monotonic sequence and $\mathcal{A}=\mathcal{A}_*=\mathcal{A}^*$.*

Proof. Suppose $\mathcal{A}\subset\mathcal{A}_*$ and (3) holds. Define

$$V_n^B(\omega)=\sup_{H\in\mathcal{A}_n}\Big|\mu(\omega)(B\cap H)-\mu(\omega)(B)\mu(\omega)(H)\Big|,\quad n\geq 1, B\in\mathcal{B},\omega\in\Omega,$$

and let $\mathcal{B}_0$ be a countable field such that $\mathcal{B}=\sigma(\mathcal{B}_0)$. It is enough proving that:

(5) There are a subsequence (n_j) and a set $A_0\in\mathcal{A}$ such that $P(A_0)=1$ and $\lim_j V_{n_j}^B(\omega)=0$ for all $\omega\in A_0$ and $B\in\mathcal{B}_0$.

Suppose in fact (5) holds. Fix $\omega\in A_0$, $B\in\mathcal{B}$ and $\epsilon>0$. Since $\mathcal{B}_0$ is a field which generates $\mathcal{B}$, there is $B_0\in\mathcal{B}_0$ such that $\mu(\omega)(B\Delta B_0)<\epsilon$. Hence,

$$V_n^B(\omega)\leq\sup_{H\in\mathcal{A}_n}\Big|\mu(\omega)(B\cap H)-\mu(\omega)(B_0\cap H)\Big|+$$

$$+\sup_{H\in\mathcal{A}_n}\Big|\mu(\omega)(B_0\cap H)-\mu(\omega)(B_0)\mu(\omega)(H)\Big|+\sup_{H\in\mathcal{A}_n}\Big|\mu(\omega)(B_0)\mu(\omega)(H)-\mu(\omega)(B)\mu(\omega)(H)\Big|$$

$$\leq V_n^{B_0}(\omega)+2\mu(\omega)(B\Delta B_0)<V_n^{B_0}(\omega)+2\epsilon\quad\text{for all } n.$$

Since $\omega \in A_0$ and $B_0 \in \mathcal{B}_0$, condition (5) yields

$$\limsup_j V_{n_j}^B(\omega) \le 2\epsilon + \limsup_j V_{n_j}^{B_0}(\omega) = 2\epsilon.$$

Thus, $\lim_j V_{n_j}^B(\omega) = 0$ for all $B \in \mathcal{B}$ and $\omega \in A_0$. Denote $\mathcal{C} = \bigcup_{n\ge1}\bigcap_{j\ge n}\mathcal{A}_j$ and fix $\omega \in A_0$ and $A \in \mathcal{C}$. If $A \in \mathcal{A}_k$ for some k, then

$$|\mu(\omega)(A) - \mu(\omega)(A)^2| \le \sup_{H\in\mathcal{A}_k}\Big|\mu(\omega)(A\cap H) - \mu(\omega)(A)\mu(\omega)(H)\Big| = V_k^A(\omega).$$

Since $A \in \mathcal{C}$, there is n such that $A \in \mathcal{A}_k$ for each $k \ge n$, so that

$$|\mu(\omega)(A) - \mu(\omega)(A)^2| \le \lim_j V_{n_j}^A(\omega) = 0.$$

Therefore, $\mu(\omega)$ is 0-1 on $\mathcal{C}$, which implies that $\mu(\omega)$ is 0-1 on $\sigma(\mathcal{C}) = \mathcal{A}_*$. Since $\mathcal{A} \subset \mathcal{A}_*$, condition (1) holds.

It remains to prove condition (5). The proof is split into three steps.

(i) Fix n and take ν_n and C_n as in condition (4). Since $C_n \in \mathcal{A}_n$ and $P(C_n) = 1$, up to modifying ν_n on C_n^c, it can be assumed that $\nu_n(\omega)$ is 0-1 on $\mathcal{A}_n$ for all $\omega \in \Omega$. We now prove that, for each ω in some set $M_n \in \mathcal{A}$ with $P(M_n) = 1$, one has

$$\mu(\omega)(B\cap H) = \int_{\{\nu_n(H)=1\}} \nu_n(x)(B)\mu(\omega)(dx) \quad \text{for all } H \in \mathcal{A}_n \text{ and } B \in \mathcal{B}$$

where $\{\nu_n(H) = 1\}$ denotes the set $\{x : \nu_n(x)(H) = 1\}$. Define

$$\mu_n(\omega)(B) = \int \nu_n(x)(B)\mu(\omega)(dx), \quad \omega \in \Omega, B \in \mathcal{B}.$$

Since $\mathcal{B}$ is countably generated and μ_n is a r.c.d. for P given $\mathcal{A}$, there is $M_n \in \mathcal{A}$ such that $P(M_n) = 1$ and $\mu_n(\omega) = \mu(\omega)$ for all $\omega \in M_n$. Let $H \in \mathcal{A}_n$, $B \in \mathcal{B}$ and $\omega \in M_n$. Since $\nu_n(\cdot)(H) \in \{0, 1\}$, then $\nu_n(x)(B \cap H) = \nu_n(x)(B)I_{\{\nu_n(H)=1\}}(x)$ for all $x \in \Omega$. Thus,

$$\mu(\omega)(B\cap H) = \mu_n(\omega)(B\cap H) = \int \nu_n(x)(B\cap H)\mu(\omega)(dx)$$
$$= \int_{\{\nu_n(H)=1\}} \nu_n(x)(B)\mu(\omega)(dx).$$

(ii) We next prove that, for each $\omega \in M_n \cap T$, where $T \in \mathcal{A}$ and $P(T) = 1$, one also has

$$\mu(\omega)(B)\mu(\omega)(H) = \int_{\{\nu_n(H)=1\}} \mu(x)(B)\mu(\omega)(dx) \quad \text{for all } H \in \mathcal{A}_n \text{ and } B \in \mathcal{B}.$$

Let $\sigma(\mu)$ be the σ-field generated by $\mu(\cdot)(B)$ for all $B \in \mathcal{B}$. Then, $\sigma(\mu) \subset \mathcal{A}$ and $\sigma(\mu)$ is countably generated since $\mathcal{B}$ is countably generated. Hence, there is $T \in \mathcal{A}$ with $P(T) = 1$ and $\mu(\omega)(D) = I_D(\omega)$ for all $D \in \sigma(\mu)$ and $\omega \in T$. Given $C \in \mathcal{B}$ and a bounded $\sigma(\mu)$-measurable function $h : \Omega \to \mathbb{R}$, it follows that

$$\int_C h(x)\mu(\omega)(dx) = h(\omega)\mu(\omega)(C) \quad \text{whenever } \omega \in T.$$

Fix $\omega \in M_n \cap T$, $H \in \mathcal{A}_n$ and $B \in \mathcal{B}$. Letting $h(x) = \mu(x)(B)$ and $C = \{\nu_n(H) = 1\}$, one obtains

$$\mu(\omega)(B)\mu(\omega)(H) = \mu(\omega)(B)\mu(\omega)\big(\nu_n(H) = 1\big) \quad \text{since } \omega \in M_n$$
$$= \int_{\{\nu_n(H)=1\}} \mu(x)(B)\mu(\omega)(dx) \quad \text{since } \omega \in T.$$

(iii) Since $\mathcal{B}_0$ is countable, by (3) and a diagonalization argument, there is a subsequence (n_j) such that

$$E\big(I_B \mid \mathcal{A}_{n_j}\big) \overset{a.s.}{\to} E\big(I_B \mid \mathcal{A}\big), \text{ as } j \to \infty, \text{ for all } B \in \mathcal{B}_0.$$

Define $Z_n^B(\cdot) = \nu_n(\cdot)(B) - \mu(\cdot)(B)$. If $B \in \mathcal{B}_0$, then $Z_{n_j}^B \overset{a.s.}{\to} 0$ as $j \to \infty$, and $|Z_{n_j}^B| \leq 1$ for all j. Hence,

$$\int |Z_{n_j}^B(x)|\mu(\cdot)(dx) = E\big(|Z_{n_j}^B| \mid \mathcal{A}\big) \overset{a.s.}{\to} 0.$$

Define further

$$S = \{\omega : \lim_j \int |Z_{n_j}^B(x)|\mu(\omega)(dx) = 0 \text{ for each } B \in \mathcal{B}_0\} \quad \text{and} \quad A_0 = \bigcap_n (M_n \cap S \cap T).$$

Then, $A_0 \in \mathcal{A}$ and $P(A_0) = 1$. Given $B \in \mathcal{B}_0$ and $\omega \in A_0$, points (i)-(ii) yield:

$$V_{n_j}^B(\omega) = \sup_{H \in \mathcal{A}_{n_j}} \Big|\mu(\omega)(B \cap H) - \mu(\omega)(B)\mu(\omega)(H)\Big|$$
$$= \sup_{H \in \mathcal{A}_{n_j}} \Big|\int_{\{\nu_{n_j}(H)=1\}} \nu_{n_j}(x)(B)\mu(\omega)(dx) - \int_{\{\nu_{n_j}(H)=1\}} \mu(x)(B)\mu(\omega)(dx)\Big|$$
$$= \sup_{H \in \mathcal{A}_{n_j}} \Big|\int_{\{\nu_{n_j}(H)=1\}} Z_{n_j}^B(x)\mu(\omega)(dx)\Big|$$
$$\leq \int |Z_{n_j}^B(x)|\mu(\omega)(dx) \to 0 \quad \text{as } j \to \infty.$$

Thus (5) holds, and this concludes the proof in case $\mathcal{A} \subset \mathcal{A}_*$ and (3) holds.

Finally, suppose $\mathcal{A} \subset \mathcal{A}^*$ and (3*) holds. By using (3*) instead of (3), in point (iii) there is no need of taking a subsequence (n_j), and one obtains $\lim_n V_n^B(\omega) = 0$ for all $B \in \mathcal{B}_0$ and ω in a set $A_0 \in \mathcal{A}$ with $P(A_0) = 1$. Arguing as at the beginning of this proof, this in turn implies $\lim_n V_n^B(\omega) = 0$ for all $B \in \mathcal{B}$ and $\omega \in A_0$. Denote $\mathcal{L} = \bigcap_{n \geq 1} \bigcup_{j \geq n} \mathcal{A}_j$ and fix $\omega \in A_0$ and $A \in \mathcal{L}$. Since $A \in \mathcal{L}$, there is a subsequence (m_j) (possibly depending on A) such that $A \in \mathcal{A}_{m_j}$ for all j. Hence,

$$|\mu(\omega)(A) - \mu(\omega)(A)^2| \leq \sup_{H \in \mathcal{A}_{m_j}} \Big|\mu(\omega)(A \cap H) - \mu(\omega)(A)\mu(\omega)(H)\Big| = V_{m_j}^A(\omega) \to 0.$$

Therefore, $\mu(\omega)$ is 0-1 on $\mathcal{L}$, which implies that $\mu(\omega)$ is 0-1 on $\sigma(\mathcal{L}) = \mathcal{A}^*$. Since $\mathcal{A} \subset \mathcal{A}^*$, this concludes the proof. □

For Theorem 8 to apply, it is useful to have some condition implying existence of r.c.d.'s whatever the sub-σ-field is. One such condition is: *P admits a r.c.d. given $\mathcal{G}$, for any sub-σ-field $\mathcal{G} \subset \mathcal{B}$, provided P is perfect and $\mathcal{B}$ countably generated*; see [8]. We recall that P is *perfect* in case each $\mathcal{B}$-measurable function $f : \Omega \to \mathbb{R}$ meets $P(f \in I) = 1$ for some real Borel set $I \subset f(\Omega)$. For P to be perfect, it is enough that Ω is a universally measurable subset of a Polish space (in particular, a Borel subset) and $\mathcal{B}$ the Borel σ-field on Ω.

Thus, Theorem 8 applies whenever P is perfect, $\mathcal{A}_n$ is a decreasing sequence of countably generated σ-fields, and $\mathcal{A} = \bigcap_n \mathcal{A}_n$. This particular case, where $\mathcal{A}$ is a tail σ-field, has been already proved in Theorem 15 of [2]. But Theorem 8 covers various other real situations.

Example 9. (Increasing unions of tail σ-fields) Let $(X_{i,j} : i, j = 1, 2, \ldots)$ be an array of real random variables on $(\Omega, \mathcal{B}, P)$ and

$$Z_m^{(n)} = (X_{m,1}, \ldots, X_{m,n}), \quad \mathcal{A}_n = \bigcap_m \sigma(Z_m^{(n)}, Z_{m+1}^{(n)}, \ldots).$$

Then, $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots$ and, for each n, $\mathcal{A}_n$ is the tail σ-field of the sequence $(Z_m^{(n)} : m \geq 1)$. Thus, $\mathcal{A}_n$ meets condition (4) as far as $\mathcal{B}$ is countably generated and P perfect. In that case, by Theorem 8, condition (1) holds for $\mathcal{A} = \sigma(\cup_n \mathcal{A}_n)$.

Example 10. (Increasing unions of finite groups) Let $F_1 \subset F_2 \subset \ldots$ be an increasing sequence of finite groups of measurable functions of Ω into itself, and

$$\mathcal{A} = \mathcal{A}_{\bigcup_n F_n} = \bigcap_n \mathcal{A}_{F_n}.$$

Suppose $\mathcal{B}$ is countably generated and P perfect. Then, Theorems 6 and 8 imply that $\mu(\omega)$ is 0-1 on $\mathcal{A}$ for each $\omega \in A_0$, where $A_0 \in \mathcal{A}$ and $P(A_0) = 1$.

Finally, as a last and most important example, we mention the *symmetric* σ-field.

Example 11. (Symmetric σ-field) As in Example 7, let $(\Omega, \mathcal{B}) = (\mathcal{X}^\infty, \mathcal{U}^\infty)$ where $(\mathcal{X}, \mathcal{U})$ is a measurable space and $\mathcal{U}$ is countably generated. Denoting F_n the group of permutations of order n, the symmetric σ-field is

$$\mathcal{A} = \{B \in \mathcal{B} : f^{-1}B = B \text{ for all } f \in \bigcup_n F_n\} = \bigcap_n \mathcal{A}_{F_n}.$$

This is just a case of Example 10. If P is perfect, thus, there is $A_0 \in \mathcal{A}$ such that $P(A_0) = 1$ and $\mu(\omega)$ is 0-1 on $\mathcal{A}$ for each $\omega \in A_0$.

References

[1] Berti, P., Rigo, P. (1999) Sufficient conditions for the existence of disintegrations, *J. Theoret. Probab.* **12**, 75-86.

[2] Berti, P., Rigo, P. (2007) 0-1 laws for regular conditional distributions, *Ann. Probab.* **35**, 649-662.

[3] Blackwell, D., Dubins, L.E. (1975) On existence and non-existence of proper, regular, conditional distributions, *Ann. Probab.* **3**, 741-752.

[4] Burkholder, D.L. (1961) Sufficiency in the undominated case, *Ann. Math. Statist.* **32**, 1191-1200.

[5] Diaconis, P., Freedman, D. (1981) Partial exchangeability and sufficiency, *Proc. of the Indian Statistical Institute Golden Jubilee International Conference on Statistics: Applications and New Directions*, Calcutta 16 December - 19 December, 205-236.

[6] Dynkin, E.B. (1978) Sufficient statistics and extreme points, *Ann. Probab.* **6**, 705-730.

[7] Farrell, R.H. (1962) Representation of invariant measures, *Illinois J. of Math.* **6**, 447-467.

[8] Jirina, M. (1954) Conditional probabilities on σ-algebras with countable basis, *Czechoslovak Math. J.* **4 (79)**, 372-380. English translation in: *Selected Transl. Math. Stat. and Prob.* **2**, Amer. Math. Soc. (1962) 79-86.

[9] Maitra, A. (1977) Integral representations of invariant measures, *Trans. Amer. Math. Soc.* **229**, 209-225.

[10] Seidenfeld, T., Schervish, M.J., Kadane, J. (2001) Improper regular conditional distributions, *Ann. Probab.* **29**, 1612-1624. Corrections in: *Ann. Probab.* **34** (2006) 423-426.

Patrizia Berti, Dipartimento di Matematica Pura ed Applicata "G. Vitali", Universita' di Modena e Reggio-Emilia, via Campi 213/B, 41100 Modena, Italy

E-mail address: `berti.patrizia@unimore.it`

Pietro Rigo (corresponding author), Dipartimento di Economia Politica e Metodi Quantitativi, Universita' di Pavia, via S. Felice 5, 27100 Pavia, Italy

E-mail address: `prigo@eco.unipv.it`