EQUIVALENT OR ABSOLUTELY CONTINUOUS PROBABILITY MEASURES WITH GIVEN MARGINALS

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OUTLINE

1. Notation (just a couple of slides)

2. The problem

3. Motivations

4. Results

5. Discussion and possible developments

1. Notation

- $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ are measurable spaces and

  $(\mathcal{X} \times \mathcal{Y},\, \mathcal{A} \otimes \mathcal{B})$

  is their product

  Essentially everything in this talk holds true for **arbitrary** $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$. However, I strongly suggest to keep

  $\boxed{(\mathcal{X}, \mathcal{A}) = (\mathcal{Y}, \mathcal{B}) = (\mathcal{R}, \text{Borel } \sigma\text{-field})}$

  in your minds. **There is no real loss in doing this**

- X and Y are the coordinate projections on $\mathcal{X} \times \mathcal{Y}$:

  $X(x, y) = x$ and $Y(x, y) = y$

  for $(x, y) \in \mathcal{X} \times \mathcal{Y}$. A probability law on $\mathcal{A} \otimes \mathcal{B}$ can be thought of as a possible distribution for (X, Y)

- For any probability ν on $\mathcal{A} \otimes \mathcal{B}$, let ν_1 and ν_2 be the **marginals** of ν, namely

$$\boxed{\nu_1(A) = \nu(A \times \mathcal{Y}) \quad \text{and} \quad \nu_2(B) = \nu(\mathcal{X} \times B)}$$

  for $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

  Thus, ν_1 is the distribution of X under ν and ν_2 the distribution of Y under ν

- In the sequel, μ and ν are probabilities on $\mathcal{A} \otimes \mathcal{B}$, α is a probability on $\mathcal{A}$ and β a probability on $\mathcal{B}$

2. The problem

We moved from the following questions. **Given** a probability μ on $\mathcal{A} \otimes \mathcal{B}$, a probability α on $\mathcal{A}$ and a probability β on $\mathcal{B}$,

(a) Is there a probability ν on $\mathcal{A} \otimes \mathcal{B}$ such that

$$\boxed{\nu \sim \mu, \quad \nu_1 = \alpha, \quad \nu_2 = \beta} \ ?$$

(b) Is there a probability ν on $\mathcal{A} \otimes \mathcal{B}$ such that

$$\boxed{\nu \ll \mu, \quad \nu_1 = \alpha, \quad \nu_2 = \beta} \ ?$$

Obvious necessary conditions are

$\boxed{\alpha \sim \mu_1 \quad \text{and} \quad \beta \sim \mu_2}$ in case of (a) and

$\boxed{\alpha \ll \mu_1 \quad \text{and} \quad \beta \ll \mu_2}$ in case of (b)

Say that α and β are **admissible** if they meet such conditions

For $\mathcal{X} = \mathcal{Y} = \mathcal{R}$, our first (and wrong) intuition was to take ν with distribution function

$\boxed{F_\nu = C_\mu(F_\alpha, F_\beta)}$ where C_μ is a copula for μ.

But it does not work, even if α and β are admissible and we focus on question (b). Take for instance

$\mu(\text{diagonal}) = 1$ and $\alpha \neq \beta$

Thus,

- Copulas are not conclusive for our problem

- Such problem becomes more intriguing. It now takes the form:

  Give conditions (on μ, α, β) for the existence of ν such that

  $\boxed{\nu_1 = \alpha, \quad \nu_2 = \beta}$ and $\boxed{\nu \sim \mu \quad \text{or} \quad \nu \ll \mu}$

3. Motivations

WHERE THE PROBLEM COMES FROM

Given the real (bounded) random variables $X_1, \dots, X_k$ on the probability space $(\Omega, \mathcal{F}, P)$, is there a probability $Q \sim P$ such that

$$E_Q(X_1) = \ldots = E_Q(X_k) = 0 \quad ?$$

The question is closely related to no-arbitrage. In fact, Q exists if and only if the linear space generated by $X_1, \dots, X_k$ satisfies the classical no-arbitrage condition:

$$\boxed{P(X > 0) > 0 \iff P(X < 0) > 0}$$

for every linear combination X of $X_1, \dots, X_k$

This answer is (well known and) very nice and intuitive. But we did not know of any elementary proof (incidentally, we are still ignorant)

Our naive (and wrong) proof was:

Think of $X_1, \ldots, X_k$ as coordinate projections on $\mathcal{R}^k$, so that P is actually a Borel law on $\mathcal{R}^k$

For each $1 \leq j \leq k$, take a Borel probability Q_j on $\mathcal{R}$ such that

$$Q_j \sim P_j \quad \text{and} \quad E_{Q_j}(X_j) = 0$$

where P_j is the j-th marginal of P

Finally, take Q such that $Q \sim P$ and Q has marginals $Q_1, \ldots, Q_k$

Perfect ! Just it does not work, since the third step generally fails, even if $k = 2$. Anyhow, our tentative strategy suggested the problem

Incidentally, in this talk I am assuming $k = 2$, but everything could be adapted to arbitrary k

This is how we came across the problem. But such a problem can be regarded under various sides

Coupling: We are just looking for a coupling ν of α and β satisfying an additional property (i.e., $\nu \sim \mu$ or $\nu \ll \mu$)

Stochastic dependence: Any law on $\mathcal{A} \otimes \mathcal{B}$ fixes the stochastic dependence between the coordinate projections X and Y. Turning from μ to ν, thus, we are actually changing such dependence. The new stochastic dependence ν is required to have given marginals α and β, to preserve the null sets (if $\nu \ll \mu$) or even to have exactly the same null sets (if $\nu \sim \mu$)

But our favorite view on the problem is **optimal transportation**

Suppose we pay $C(x, y)$ for moving a unit of some good from $x \in \mathcal{X}$ into $y \in \mathcal{Y}$. Think of α and β as the initial and final distributions, respectively, of such units. Here, "initial" and "final" stand for before and after the transportation of the units from $\mathcal{X}$ to $\mathcal{Y}$. A **transport plan** is a probability λ on $\mathcal{A} \otimes \mathcal{B}$ with marginals α and β. In Kantorovich's formulation, we focus on

$$\boxed{\inf_\lambda E_\lambda(C)}$$

where inf is over all transport plans λ. In Monge's formulation, we only consider those transport plans λ_f which are supported by the graph of a measurable function $f : \mathcal{X} \to \mathcal{Y}$, namely

$$\lambda_f(\cdot) = \alpha\{x : (x, f(x)) \in \cdot\}$$

(Think of f as the function on which the transportation is based). Thus, in Monge's formulation, we investigate $\inf_f E_{\lambda_f}(C)$

In this framework, it is quite natural (and realistic) to impose some constraint on the transport plan λ

In our problem, such constraint is absolute continuity or equivalence with respect to a given law μ. As a trivial example, think of a region $\mathcal{X} \times \mathcal{Y}$ including some lakes. This is modeled by

$\mu(\text{lake}) = 0$

Since the lakes can not be crossed, every sound transport plan ν should satisfy $\nu(\text{lake}) = 0$

One more connection between optimal transportation and our problem is that, in both cases, the transport plans λ_f (supported by the graph of some function f) play a special role

4. Results

Say that ν **is dominated by** μ **on rectangles**, written

$\boxed{\nu \ll_R \mu}$, if

$$\mu(A \times B) = 0 \implies \nu(A \times B) = 0 \quad \text{for } A \in \mathcal{A} \text{ and } B \in \mathcal{B}$$

Now, $\nu \ll_R \mu$ is strictly weaker than $\nu \ll \mu$. Take in fact

μ uniform on the unit square $[0,1]^2$ and

$\nu(\text{diagonal}) = 1$ with $\nu_1 = \nu_2 = m$

where m is Lebesgue measure on $[0,1]$. Then, ν is not dominated by μ, and yet $\nu \ll_R \mu$ since

$$\nu(A \times B) = m(A \cap B) \leq m(A) \wedge m(B)$$

Under some conditions, however, $\boxed{\nu \ll_R \mu \;\Leftrightarrow\; \nu \ll \mu}$

In order to $\nu \ll_R \mu \;\Leftrightarrow\; \nu \ll \mu$, it suffices that

- μ is supported by the graph of a continuous function

- At least one marginal of μ is discrete (i.e., μ_1 or μ_2 is discrete)

- Both μ and ν are dominated by γ and

  $\boxed{\gamma(\,\partial\{f = 0\}\,) = 0}$

  where γ is a σ-finite measure on $\mathcal{A} \otimes \mathcal{B}$ and f a density of μ with respect to γ

Given the (bounded measurable) functions

$f : \mathcal{X} \to \mathcal{R}$ and $g : \mathcal{Y} \to \mathcal{R}$

write

$\alpha(f) = \int f \, d\alpha$, $\beta(g) = \int g \, d\beta$, and

$\boxed{f \oplus g(x, y) = f(x) + g(y)}$ for $(x, y) \in \mathcal{X} \times \mathcal{Y}$

THM1 Fix μ, α and β. There is ν such that

$\nu \ll_R \mu$, $\nu_1 = \alpha$, $\nu_2 = \beta$

if and only if

$\boxed{\alpha(f) + \beta(g) \geq \inf \{\lambda(f \oplus g) : \lambda \ll_R \mu\}}$

for all bounded measurable f and g

- The condition

  $$\alpha(f) + \beta(g) \geq \inf\{\lambda(f \oplus g) : \lambda \ll_R \mu\}$$

  is basically a **coherence condition (in de Finetti's sense)**

- The necessity is obvious

  $$\alpha(f) + \beta(g) = \nu(f \oplus g) \geq \inf\{\lambda(f \oplus g) : \lambda \ll_R \mu\}$$

  where the equality is because ν has marginals α, β and the inequality for $\nu \ll_R \mu$

- Checking such condition in real problems looks quite hard. Being necessary, however, it can not be bypassed

- THM1 is a partial solution only, for it implies $\nu \ll_R \mu$ but not $\nu \ll \mu$. Under the conditions listed above, however, one obtains $\nu \ll_R \mu \;\Leftrightarrow\; \nu \ll \mu$, and THM1 becomes a full solution

Suppose now that only μ is assigned while α and β are allowed to vary among the admissible laws, namely

$\alpha \ll \mu_1$ and $\beta \ll \mu_2$

THM2 Given μ, the following statements are equivalent

(i) For all admissible α and β, there is ν such that $\nu_1 = \alpha$, $\nu_2 = \beta$ and $\nu \ll_R \mu$

(ii) For all admissible α and β, there is a **finitely additive** probability ν such that $\nu_1 = \alpha$, $\nu_2 = \beta$ and $\nu \ll \mu$

(iii) $\boxed{\mu_1 \times \mu_2 \ll_R \mu}$

Condition (iii) is unusual (and so potentially intriguing). Roughly,

$\mu_1 \times \mu_2 \ll_R \mu$

means that the support of μ can not have rectangular holes

Also, under a mild condition on μ, THM2 can be improved

THM3 Suppose

$\mu \ll \gamma$ and $\gamma(\partial\{f = 0\}) = 0$

where γ is a σ-finite product measure on $\mathcal{A} \otimes \mathcal{B}$ and f a density of μ with respect to γ. Then,

$\boxed{\mu_1 \times \mu_2 \ll \mu}$

if and only if

For all admissible α and β, there is ν such that $\nu_1 = \alpha$, $\nu_2 = \beta$ and $\nu \ll \mu$

So far, we were concerned with $\nu \ll \mu$. I next turn to $\boxed{\nu \sim \mu}$.

- Let μ, α and β be given. There is ν such that $\nu_1 = \alpha$, $\nu_2 = \beta$ and $\nu \sim \mu$ provided

  $\mu_1 \times \mu_2 \ll \mu$ and

  $a\,\mu_1 \leq \alpha \leq b\,\mu_1, \quad c\,\mu_2 \leq \beta \leq d\,\mu_2$

  for some constants $a, b, c, d > 0$

- **Even if μ is not equivalent to a σ-finite product measure**, it may be that, for all admissible α and β, there is ν such that $\nu_1 = \alpha$, $\nu_2 = \beta$ and $\nu \sim \mu$. This actually happens for

  $\mu = (1/2)\,\{\lambda' + \lambda''\}$

  where λ' is uniform on the unit square and λ'' is supported by the diagonal and has uniform marginals

5. Discussion and possible developments

de Finetti's coherence principle: Let $E : \mathcal{D} \to \mathcal{R}$, where $\mathcal{D}$ is **any** class of real bounded functions on a set Ω. Then, E is coherent if

$$\boxed{\sup_{\omega\in\Omega} \sum_{i=1}^{n} c_i \{X_i(\omega) - E(X_i)\} \geq 0}$$

for all $n \geq 1$, $c_1, \ldots, c_n \in \mathcal{R}$ and $X_1, \ldots, X_n \in \mathcal{D}$

Interpretation: Suppose E describes your previsions (e.g., expectations) on the elements of $\mathcal{D}$. This means that, for each $X \in \mathcal{D}$ and $c \in \mathcal{R}$, you agree:

$$\boxed{\text{to pay} \quad c\,E(X) \quad \text{for receiving} \quad c\,X(\omega) \quad \text{if } \omega \in \Omega \text{ occurs}}$$

You are coherent provided **you can not be made a sure looser by a finite combinations of bets** (on $X_1, \ldots, X_n$ with stakes $c_1, \ldots, c_n$)

Hahn-Banach theorem: E is coherent if and only if there is a finitely additive probability P on the power set of Ω such that

$\boxed{E(X) = \int X\, dP = E_P(X)}$ for all $X \in \mathcal{D}$

So far, $\mathcal{D}$ and E were arbitrary. But, if $\mathcal{D}$ is a linear space and E a linear functional, coherence reduces to

$$\sup_{\omega \in \Omega} \{X(\omega) - E(X)\} \geq 0 \quad \text{for all } X \in \mathcal{D}$$

Letting $E = 0$ (which is a linear functional) one obtains:

If $\mathcal{D}$ is a linear space of bounded functions, there is a finitely additive probability P such that

$$\boxed{E_P(X) = 0 \quad \text{for all } X \in \mathcal{D}}$$

if and only if

$$\boxed{\sup_{\omega \in \Omega} X(\omega) \geq 0 \quad \text{for all } X \in \mathcal{D}}$$

Now, **under some conditions, such a P can be taken σ-additive and equivalent to a given reference probability P_0**

This suggests a possible approach to our problem. Just take

$$\Omega = \mathcal{X} \times \mathcal{Y}, \quad P_0 = \mu,$$

and $\mathcal{D}$ the collection of those random variables X of the type

$$X(x, y) = \{f(x) - \alpha(f)\} + \{g(y) - \beta(g)\}$$

where $f : \mathcal{X} \to \mathcal{R}$ and $g : \mathcal{Y} \to \mathcal{R}$ are bounded and measurable

Then, for each probability P on $\mathcal{A} \otimes \mathcal{B}$,

$$\boxed{E_P(X) = 0 \quad \text{for all } X \in \mathcal{D} \text{ if and only if } P \text{ has marginals } \alpha \text{ and } \beta}$$

Thus, it suffices to let $\nu = P$ with P as above and satisfying $P \sim P_0$

A last remark. In addition to the problem of this talk, many other issues reduce to existence of a probability P such that

$$E_P(X) = 0 \quad \text{for all } X \in \mathcal{D}$$

with $\mathcal{D}$ a **suitable** linear space of bounded random variables. Among other things, we mention:

- Equivalent martingale measures
- Stationary and reversible Markov chains
- Compatibility of conditional distributions