ON THE EXISTENCE OF CONTINUOUS PROCESSES WITH GIVEN ONE-DIMENSIONAL DISTRIBUTIONS

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Abstract. Let $\mathcal{P}$ be the collection of Borel probability measures on $\mathbb{R}$, equipped with the weak topology, and let $\mu : [0,1] \to \mathcal{P}$ be a continuous map. Say that μ is presentable if $X_t \sim \mu_t$, $t \in [0,1]$, for some real process X with continuous paths. It may be that μ fails to be presentable. Hence, firstly, conditions for presentability are given. For instance, μ is presentable if μ_t is supported by an interval (possibly, by a singleton) for all but countably many t. Secondly, assuming μ presentable, we investigate whether the quantile process Q induced by μ has continuous paths. The latter is defined, on the probability space $((0,1), \mathcal{B}(0,1), \text{Lebesgue measure})$, by

$$Q_t(\alpha) = \inf\{x \in \mathbb{R} : \mu_t(-\infty, x] \geq \alpha\} \qquad \text{for all } t \in [0,1] \text{ and } \alpha \in (0,1).$$

A few open problems are discussed as well.

1. Introduction

In what follows, a *process* is always meant as a real valued stochastic process indexed by $[0,1]$. A process is *continuous (cadlag)* if almost all its paths are continuous (cadlag). Also, if X and Y are processes, we write $X \sim Y$ to mean that

$$(X_{t_1}, \ldots, X_{t_k}) \sim (Y_{t_1}, \ldots, Y_{t_k}) \qquad \text{for all } k \in \mathbb{N} \text{ and } t_1, \ldots, t_k \in [0,1].$$

Let $\mathcal{P}$ be the collection of Borel probability measures on $\mathbb{R}$, equipped with the weak topology (namely, the weakest topology on $\mathcal{P}$ which makes continuous the maps $\nu \in \mathcal{P} \mapsto \int f\,d\nu$ for all bounded continuous $f : \mathbb{R} \to \mathbb{R}$). We fix a continuous function $\mu : [0,1] \to \mathcal{P}$ and we focus on the problem:

(*) Is there a continuous process X such that $X_t \sim \mu_t$ for each $t \in [0,1]$?

Question (*) arises as a natural generalization of various representation results for classes of absolutely continuous curves in the space of probability measures endowed with the Kantorovich-Rubinstein-Wasserstein metric, see e.g. [1, Chap. 8] where applications to the continuity equation and diffusion PDE's are considered. In addition, problem (*) is intriguing from a foundational point of view. A positive answer to (*), for instance, could be regarded as a strong version of Skorohod representation theorem; see Section 3. In turn, this type of versions of the Skorohod's result are useful when dealing with certain SDE's; see [4] and [8].

By a result of Blackwell and Dubins, there always exists a process X such that, for each fixed t, $X_t \sim \mu_t$ and almost all X-paths are continuous at t; see [2]; see also

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[5] for a detailed proof. Despite this fact, however, the answer to (*) is generally no. A simple example is

$$\mu_t = (1-t)\delta_0 + t\delta_1$$

where δ_x denotes the point mass at x.

Say that μ is *presentable* if question (*) has a positive answer, namely, $X_t \sim \mu_t$ for some continuous process X and all $t \in [0,1]$. To investigate presentability of μ, there is an obvious process to work with. Let $\mathcal{B}$ be the Borel σ-field on $(0,1)$, λ the Lebesgue measure, and

$$F_t(x) = \mu_t(-\infty, x] \qquad \text{for all } t \in [0,1] \text{ and } x \in \mathbb{R}.$$

Define a process Q on the probability space $((0,1), \mathcal{B}, \lambda)$ as

$$Q_t(\alpha) = \inf\big\{x \in \mathbb{R} : F_t(x) \geq \alpha\big\} \qquad \text{for all } t \in [0,1] \text{ and } \alpha \in (0,1).$$

Such a Q may be called the "quantile process" induced by μ and its finite dimensional distributions are

$$\lambda\Big(Q_{t_1} \leq x_1, \ldots, Q_{t_k} \leq x_k\Big) = \min_{1 \leq i \leq k} F_{t_i}(x_i)$$

where $k \in \mathbb{N}$, $t_1, \ldots, t_k \in [0,1]$ and $x_1, \ldots, x_k \in \mathbb{R}$. Since $Q_t \sim \mu_t$ for all t, a sufficient condition for μ to be presentable is continuity of Q.

This note is devoted to problem (*). Our first result is that Q is continuous if and only if $\lambda(J) = 0$, where

$$J = \big\{\alpha \in (0,1) : F_t(x) = F_t(y) = \alpha \text{ for some } t \in [0,1] \text{ and some } x < y\big\}.$$

Among other things, this fact provides an useful sufficient condition for presentability. Indeed, μ is presentable whenever μ_t is supported by an interval (possibly, by a singleton) for all but countably many t.

Next, we focus on the implication

$$\mu \text{ presentable} \quad \Rightarrow \quad Q \sim X \text{ for some continuous process } X. \tag{1}$$

We do not know whether (1) *is generally true.* However, to motivate our concern about (1), we recall a well known fact. Let $D_n = \{j/2^n : j = 0, 1, \ldots, 2^n\}$. Given any process Y, there is a continuous process X such that $Y \sim X$ if and only if

(a) $Y_s \longrightarrow Y_t$ in probability, as $s \to t$, for each $t \in [0,1]$;

(b) For each $\epsilon > 0$,

$$\inf_{\delta > 0} \sup_n \text{Prob}\Big(|Y_s - Y_t| > \epsilon \text{ for some } s,\, t \in D_n \text{ with } |s-t| < \delta\Big) = 0.$$

Only the finite dimensional distributions of Y are involved in conditions (a)-(b). Hence, the existence of a continuous version of Y is actually a property of its finite dimensional distributions. In turn, μ is presentable if and only if the collection $\{\mu_t : t \in [0,1]\}$ can be extended to a suitable consistent set of finite dimensional distributions.

Now, condition (a) is automatically true if $Y = Q$; see point (i) of Theorem 1. Thus, μ is presentable if Q satisfies condition (b). If implication (1) holds true, one obtains the converse, i.e., presentability of μ amounts to condition (b) with $Y = Q$. In other terms, under (1), to decide whether μ is presentable reduces to proving condition (b) with $Y = Q$. Note also that, since the finite dimensional distributions of Q are very popular (see e.g. [7] and references therein), it is quite natural to investigate whether Q admits a continuous version whenever μ is presentable.

In this note, we prove some weaker versions of (1), namely, we show that Q is continuous under conditions stronger than presentability of μ. For instance, Q is continuous if $X_t \sim \mu_t$, $t \in [0,1]$, for some process X such that the collection of all its paths is an equicontinuous subset of $C([0,1],\mathbb{R})$. Similarly, Q is continuous if μ is presentable and all the μ_t have the same support.

Finally, a few open problems (not only the one concerning implication (1)) are discussed.

2. Results

In the sequel, for each $\alpha \in (0,1)$, we write $Q(\alpha)$ to denote the map $t \mapsto Q_t(\alpha)$. In addition,

$$C = C\big([0,1],\,\mathbb{R}\big)$$

is the set of real continuous functions on $[0,1]$.

For fixed $t \in [0,1]$, the map $\alpha \mapsto Q_t(\alpha)$ is increasing, left-continuous, and its set of discontinuity points is

$$J_t = \big\{\alpha \in (0,1) : Q_t(\alpha+) \neq Q_t(\alpha)\big\} = \big\{\alpha \in (0,1) : F_t(x) = F_t(y) = \alpha \text{ for some } x < y\big\}.$$

Define

$$J = \cup_t J_t \quad \text{and} \quad M = \big\{(\alpha,t) \in (0,1)\times[0,1] : Q_t(\alpha+) \neq Q_t(\alpha)\big\}.$$

Since M is a Borel set and J is the projection of M on $(0,1)$, then J is a Souslin set (or equivalently an analytic set). In particular, J is Lebesgue measurable.

If ν is a Borel probability measure on a topological space S, the support of ν is the intersection of all closed subsets of S with ν-probability 1.

We are now able to state our first result.

Theorem 1. *Suppose $\mu : [0,1] \to \mathcal{P}$ is continuous. Then:*

(i) *$Q(\alpha)$ is continuous at t for each $\alpha \in (0,1) \setminus J_t$;*

(ii) *Q is continuous if and only if $\lambda(J) = 0$;*

(iii) *Q is continuous provided the support of μ_t is connected for all but countably many t.*

Proof. (i) Fix $t \in [0,1]$, $\alpha \in (0,1) \setminus J_t$ and a continuity point x of F_t. Then, $F_t(x) = \lim_{s\to t} F_s(x)$ since μ is continuous at t. If $x < Q_t(\alpha)$, then $F_t(x) < \alpha$, so that $F_s(x) < \alpha$ whenever s is close to t. It follows that $Q_s(\alpha) > x$ for each s close to t, so that $\liminf_{s\to t} Q_s(\alpha) \geq x$. Suppose now that $x > Q_t(\alpha)$. Then, $\alpha \notin J_t$ implies $F_t(x) > \alpha$, and one obtains $\limsup_{s\to t} Q_s(\alpha) \leq x$ by the previous argument. On noting that the continuity points of F_t are dense in $\mathbb{R}$, one finally obtains

$$\limsup_{s\to t} Q_s(\alpha) \leq Q_t(\alpha) \leq \liminf_{s\to t} Q_s(\alpha).$$

(ii) If $\lambda(J) = 0$, there is $A \in \mathcal{B}$ with $A \cap J = \emptyset$ and $\lambda(A) = 1$. Thus, $Q(\alpha) \in C$ for each $\alpha \in A$, because of (i) and $\alpha \notin J$. Conversely, suppose that Q is continuous and define

$$Q_t^*(\alpha) = Q_t(\alpha+) \quad \text{for } t \in [0,1] \text{ and } \alpha \in (0,1).$$

It suffices to show that Q^* is continuous as well. In that case, in fact, there is a set $A \in \mathcal{B}$ such that $\lambda(A) = 1$, $Q(\alpha) \in C$ and $Q^*(\alpha) \in C$ for each $\alpha \in A$, where $Q^*(\alpha)$ denotes the map $t \mapsto Q_t^*(\alpha)$. Given $\alpha \in A \cap J$, take $t \in [0,1]$ with $\alpha \in A \cap J_t$ and a sequence $t_n \in \mathbb{Q} \cap [0,1]$ with $t_n \to t$. Then,

$$\lim_n \{Q_{t_n}(\alpha+) - Q_{t_n}(\alpha)\} = \lim_n \{Q_{t_n}^*(\alpha) - Q_{t_n}(\alpha)\} = Q_t^*(\alpha) - Q_t(\alpha) > 0,$$

and this implies $\alpha \in \bigcup_{u \in \mathbb{Q} \cap [0,1]} J_u$. Hence, $A \cap J$ is countable, so that

$$\lambda(J) = \lambda(A \cap J) + \lambda(A^c \cap J) = \lambda(A \cap J) = 0.$$

It remains to prove that Q^* is continuous. Since Q is continuous,

$$\lim_{\delta \to 0} \sup_{|u-v| \le \delta} \Big| Q_u(\alpha) - Q_v(\alpha) \Big| = 0 \quad \text{for } \lambda\text{-almost all } \alpha \in (0,1).$$

By Egorov's theorem, given $\epsilon > 0$, there is $B \in \mathcal{B}$ such that $\lambda(B) > 1 - \epsilon$ and

$$\lim_{\delta \to 0} \sup_{\alpha \in B} \sup_{|u-v| \le \delta} \Big| Q_u(\alpha) - Q_v(\alpha) \Big| = 0.$$

Such a B can be taken to be closed, and in that case $(0,1) \setminus B = \cup_n I_n$ where the I_n are pairwise disjoint open intervals, say $I_n = (a_n, b_n)$. Letting

$$B_\epsilon = B \setminus \{a_1, b_1, a_2, b_2, \ldots\},$$

it follows that

$$(\alpha, \beta) \cap B \neq \emptyset \quad \text{whenever } \alpha \in B_\epsilon \text{ and } \beta > \alpha.$$

Fix $\alpha \in B_\epsilon$ and take a sequence $\alpha_n \in B \cap (\alpha, 1)$ with $\alpha_n \to \alpha$. For all s, $t \in [0,1]$,

$$|Q_s^*(\alpha) - Q_t^*(\alpha)| = \lim_n |Q_s(\alpha_n) - Q_t(\alpha_n)| \le \sup_{\beta \in B} \sup_{|u-v| \le |s-t|} \Big| Q_u(\beta) - Q_v(\beta) \Big|.$$

Therefore, $Q^*(\alpha) \in C$ for each $\alpha \in B_\epsilon$, where $\lambda(B_\epsilon) = \lambda(B) > 1 - \epsilon$. To conclude the proof, it suffices to let $H = \cup_n B_{1/n}$ and to note that $\lambda(H) = 1$ and $Q^*(\alpha) \in C$ for every $\alpha \in H$.

(iii) Just note that J_t is countable for fixed t and $J_t = \emptyset$ if the support of μ_t is connected.

□

We next focus on the special case where all μ_t have the same support, say

$$\text{support}\Big(\mu_t\Big) = F \qquad \text{for all } t \in [0,1] \text{ and some closed set } F \subset \mathbb{R}. \tag{2}$$

If $F = \mathbb{R}$, Theorem 1 implies that Q is continuous. Otherwise, the following result is available.

Theorem 2. *Assume condition* (2) *with $F \neq \mathbb{R}$ and write $F^c = \cup_n I_n$, where the I_n are pairwise disjoint open intervals. Letting $a_n = \inf I_n$, the following statements are equivalent:*

(i) *Q is continuous;*

(ii) *μ is presentable;*

(iii) $F_t(a_n) = F_0(a_n)$ *for all* $t \in [0,1]$ *and* $n \in \mathbb{N}$ *with* $a_n > -\infty$.

Proof. (i) ⇒ (ii). Obvious.

(ii) ⇒ (iii). Let X be a continuous process on the probability space $(\Omega, \mathcal{A}, P)$ such that $X_t \sim \mu_t$ for all t. Up to modifying X on a null set, it can be assumed that *all* the X-paths are continuous. Fix $t \in [0,1]$ and $n \in \mathbb{N}$ with $a_n > -\infty$. Define $b_n = \sup I_n$ and

$$u_n = P\big(X_s \leq a_n \text{ for each } s \in [0,1]\big), \quad v_n = P\big(X_s \geq b_n \text{ for each } s \in [0,1]\big).$$

Since X is continuous and I_n is open,

$$u_n + v_n = P\big(X_s \notin I_n \text{ for each } s \in [0,1]\big) = P\big(X_s \notin I_n \text{ for each } s \in \mathbb{Q} \cap [0,1]\big) = 1.$$

Hence, if $F_t(a_n) > u_n$, one obtains the contradiction

$$P\big(X_t \notin I_n\big) = F_t(a_n) + P\big(X_t \geq b_n\big) \geq F_t(a_n) + v_n > u_n + v_n = 1.$$

Therefore, $F_t(a_n) \leq u_n$. Since $F_t(a_n) \geq u_n$ (by definition of u_n) one finally obtains $F_t(a_n) = u_n$.

(iii) ⇒ (i). If $\alpha \in J$, then $\alpha = F_t(a_n)$ for some t and n with $a_n > -\infty$. Hence, by (iii), J is included in the countable set $\big\{F_0(a_n) : n \in \mathbb{N},\ a_n > -\infty\big\}$. Therefore, $\lambda(J) = 0$ and (i) follows from Theorem 1.

□

We now turn to implication (1), namely, we investigate whether presentability of μ implies $Q \sim X$ for some continuous process X. As claimed in Section 1, we do not know whether (1) is generally true, but we have some partial results. One is Theorem 2 above. In fact, in the special case where all the μ_t have the same support, implication (1) is actually true. Another (partial) result is the following.

Theorem 3. *Q is continuous provided there is a process X, defined on some probability space $(\Omega, \mathcal{A}, P)$, such that $X_t \sim \mu_t$ for all $t \in [0,1]$ and*

$$\big\{X(\omega) : \omega \in \Omega\big\} \text{ is an equicontinuous subset of } C.$$

(Here, $X(\omega)$ denotes the map $t \mapsto X_t(\omega)$).

Proof. Since $\big\{X(\omega) : \omega \in \Omega\big\}$ is equicontinuous,

$$\sup_{\omega \in \Omega} \sup_{|s-t|<\delta} \Big| X_s(\omega) - X_t(\omega) \Big| \leq g(\delta)$$

for some function g on $(0, \infty)$ such that $\lim_{\delta \to 0} g(\delta) = 0$. Fix $\alpha \in (0,1)$, $t \in (0,1)$, and take $\delta > 0$ such that $(t-\delta,\, t+\delta) \subset [0,1]$. Then,

$$X_t(\omega) - g(\delta) \leq X_s(\omega) \leq X_t(\omega) + g(\delta)$$

for all $\omega \in \Omega$ and $s \in (t-\delta,\, t+\delta)$. On noting that such inequality holds for every $\omega \in \Omega$, one obtains

$$Q_t(\alpha) - g(\delta) \leq Q_s(\alpha) \leq Q_t(\alpha) + g(\delta)$$

for each $s \in (t-\delta,\, t+\delta)$, which in turn implies

$$Q_t(\alpha) - g(\delta) \leq \liminf_{s \to t} Q_s(\alpha) \leq \limsup_{s \to t} Q_s(\alpha) \leq Q_t(\alpha) + g(\delta).$$

Hence, $Q(\alpha)$ is continuous at t (for $\lim_{\delta \to 0} g(\delta) = 0$). Up to obvious modifications, the previous argument works for $t = 0$ and $t = 1$ as well. Therefore, $Q(\alpha) \in C$. □

Among other things, Theorem 3 has the following consequence. For definiteness, given any map $\nu : [0,1] \to \mathcal{P}$, say that ν is *canonically presentable* if the quantile process induced by ν is continuous.

Corollary 4. *μ is presentable if and only if admits the representation*

$$\mu_t = \sum_{n=1}^{\infty} c_n\, \mu_t^n \qquad \text{for all } t \in [0,1], \tag{3}$$

where $\mu^n : [0,1] \to \mathcal{P}$ is a canonically presentable map, $c_n \geq 0$ a constant and $\sum_{n=1}^{\infty} c_n = 1$. In particular, if (3) holds, a continuous process X such that $X_t \sim \mu_t$ for all t can be defined on the probability space $((0,1)^2, \mathcal{B}^2, \lambda^2)$ as follows

$$X_t(\alpha,\beta) = \sum_{n=1}^{\infty} 1_{(d_{n-1},d_n]}(\beta)\, Q_t^n(\alpha),$$

where $t \in [0,1]$, $(\alpha,\beta) \in (0,1)^2$, $d_0 = 0$, $d_n = \sum_{i=1}^n c_i$ and Q^n is the quantile process induced by μ^n.

Proof. Suppose (3) holds and define X according to the Corollary. Then, X is continuous and

$$\begin{aligned}\lambda^2(X_t \in A) &= \int_0^1 \lambda\big\{\alpha \in (0,1) : X_t(\alpha,\beta) \in A\big\}\, d\beta \\ &= \sum_{n=1}^{\infty} (d_n - d_{n-1})\, \lambda\big\{\alpha \in (0,1) : Q_t^n(\alpha) \in A\big\} \\ &= \sum_{n=1}^{\infty} c_n\, \mu_t^n(A) = \mu_t(A)\end{aligned}$$

for all $t \in [0,1]$ and all Borel sets $A \subset \mathbb{R}$. Conversely, suppose μ presentable, and take a continuous process Y on some probability space $(\Omega, \mathcal{A}, P)$ such that $Y_t \sim \mu_t$ for all t. Since Y is continuous,

$$\sup_{|s-t|<1/n} |Y_s - Y_t| \overset{a.s.}{\longrightarrow} 0 \qquad \text{as } n \to \infty.$$

By Egorov's theorem, there is an increasing sequence $B_1 \subset B_2 \subset \ldots$ of sets in $\mathcal{A}$ such that, for each fixed $k \in \mathbb{N}$,

$$P(B_k) > 1 - 1/k \quad \text{and} \quad \lim_n \sup_{\omega \in B_k} \sup_{|s-t|<1/n} |Y_s(\omega) - Y_t(\omega)| = 0.$$

If $P(B_k) = 1$ for some k, Theorem 3 implies that μ is canonically presentable. Hence, assume $P(B_k) < 1$ for all k. To avoid trivialities, assume also that $P\big(B_k \setminus B_{k-1}\big) > 0$ for all k (with $B_0 = \emptyset$). For fixed $n \in \mathbb{N}$, define

$$C_n = B_n \setminus B_{n-1}, \quad c_n = P(C_n),$$
$$Y_n(\omega) = Y(\omega) \text{ if } \omega \in C_n \text{ and } Y_n(\omega) = 0 \text{ if } \omega \notin C_n,$$
$$\mu_t^n(\cdot) = P\big(Y_t \in \cdot \mid C_n\big) \qquad \text{for all } t \in [0,1].$$

Then, equation (3) is trivially true and, since $\{Y_n(\omega) : \omega \in \Omega\}$ is equicontinuous, μ^n is canonically presentable because of Theorem 3. □

In Theorems 2 and 3, under suitable conditions, one obtains that μ is canonically presentable, i.e., Q is continuous. A condition for the weaker conclusion $Q \sim X$, for some continuous process X, follows from the Chentsov-Kolmogorov criterion.

Proposition 5. *Fix the constants $a \geq 1$ and $b > 1$ and suppose $\int |x|^a \, \mu_t(dx) < \infty$ for all $t \in [0,1]$. Then $Q \sim X$, for some continuous process X, provided*

$$\sup_{s \neq t} \frac{E\{|Y_s - Y_t|^a\}}{|s-t|^b} < \infty$$

for some process Y such that $Y_t \sim \mu_t$ for all $t \in [0,1]$.

Proof. Since $Y_t \sim Q_t \sim \mu_t$ and $\int |x|^a \, \mu_t(dx) < \infty$, it is well known that

$$E_\lambda\{|Q_s - Q_t|^a\} \leq E\{|Y_s - Y_t|^a\};$$

see e.g. [1, Theorem 6.0.2]. Hence,

$$E_\lambda\{|Q_s - Q_t|^a\} \leq c\,|s-t|^b \qquad \text{for all } s,\, t \in [0,1] \text{ and some constant } c > 0.$$

Thus, by the Chentsov-Kolmogorov criterion, there is a continuous process X on $((0,1), \mathcal{B}, \lambda)$ such that $\lambda\big(X_t \neq Q_t\big) = 0$ for all t. □

As an example, suppose Y is defined on the probability space $(\Omega, \mathcal{A}, P)$ and has Holder-continuous paths, say

$$|Y_s(\omega) - Y_t(\omega)| \leq L(\omega)\,|s-t|^\gamma$$

for all $\omega \in \Omega$ and $s,\, t \in [0,1]$, where $\gamma \in (0,1]$ is a constant and L a random variable. Suppose also that $Y_t \sim \mu_t$ for all t. If $\sup_\omega L(\omega) < \infty$, then Q is continuous because of Theorem 3. Under the weaker assumption $E(L^a) < \infty$, for some $a > 1/\gamma$, Proposition 5 yields $Q \sim X$ for some continuous process X.

3. Concluding remarks and open problems

Let $(\nu_n : n \geq 0)$ be a sequence of Borel probability measures on a metric space S. According to the Skorohod representation theorem, if $\nu_n \to \nu_0$ weakly and ν_0 is separable, there are S-valued random variables Q_n, defined on some probability space, such that $Q_n \sim \nu_n$ for all $n \geq 0$ and $Q_n \overset{a.s.}{\longrightarrow} Q_0$. Furthermore, in case $S = \mathbb{R}$, it suffices to let

$$Q_n(\alpha) = \inf\big\{x \in \mathbb{R} : \nu_n(-\infty, x] \geq \alpha\big\} \qquad \text{for all } n \geq 0 \text{ and } \alpha \in (0,1).$$

In other terms, if $S = \mathbb{R}$, one obtains a Skorohod representation taking Q_n to be the quantile map induced by ν_n. It is worth noting that this simple fact implies a Skorohod representation, for any Polish space S, by means of some general topological results; see [3] and [6]. In a sense, this is one more reason for taking implication (1) into account.

Let us turn to open problems. The most intriguing is whether implication (1) is always true. But this is not the only open problem. We next mention a few possible hints for future research.

One is to replace "continuous" with "cadlag" in problem (*). Namely, to assume $\mu : [0,1] \to \mathcal{P}$ cadlag and investigate the problem

(**) Is there a cadlag process X such that $X_t \sim \mu_t$ for each $t \in [0,1]$?

Incidentally, we are not aware of any μ which provides a negative answer to (**). In particular, if $\mu_t = (1-t)\delta_0 + t\delta_1$, a cadlag process X satisfying $X_t \sim \mu_t$ for all t is available. It suffices to let

$$X_t = 1_{[U,1]}(t),$$

where the random variable U is uniformly distributed on $(0,1)$.

Another issue arises if $\mathbb{R}$ is replaced with an arbitrary metric space S. More precisely, let $\mathcal{P}(S)$ be the set of Borel probability measures on S, equipped with the weak topology, and let $\mu : [0,1] \to \mathcal{P}(S)$ be a continuous map. Then, problem (*) turns into

(***) Is there a continuous, S-valued process X such that $X_t \sim \mu_t$ for each $t \in [0,1]$?

If $S = \mathbb{R}$ and each μ_t is supported by an interval, μ is presentable by Theorem 1. Hence, a question is whether (***) has a positive answer under some assumption on the supports of the μ_t. Our last result is an attempt to answer when $S = \mathbb{R}^2$ and all the μ_t have the same marginal on the first coordinate.

Example 6. Suppose $S = \mathbb{R}^2$ and

$$\mu_t\big(\cdot \times \mathbb{R}\big) = \gamma(\cdot) \qquad \text{for all } t \in [0,1] \text{ and some probability measure } \gamma \in \mathcal{P}.$$

For fixed t, take a regular version

$$\big\{\pi_t(\cdot \mid x) : x \in \mathbb{R}\big\}$$

of the conditional distribution of the second coordinate given the first under μ_t. This means that

- $\pi_t(\cdot \mid x) \in \mathcal{P}$ for each $x \in \mathbb{R}$;
- $x \mapsto \pi_t(B \mid x)$ is Borel measurable for each Borel set $B \subset \mathbb{R}$;
- $\int_A \pi_t(B \mid x)\,\gamma(dx) = \mu_t\big(A \times B\big)$ for all Borel sets A, $B \subset \mathbb{R}$.

Then, there is a continuous, $\mathbb{R}^2$-valued process X such that $X_t \sim \mu_t$ for all t provided

$$t \mapsto \pi_t(\cdot \mid x) \text{ is continuous, as a map from } [0,1] \text{ into } \mathcal{P}, \text{ for each } x \in \mathbb{R}; \tag{4}$$

$$\pi_t(\cdot \mid x) \quad \text{is supported by an interval for all } t \in [0,1] \text{ and } x \in \mathbb{R}. \tag{5}$$

It suffices to let

$$X_t(\alpha,\beta) = \Big(Y(\alpha),\, Z_t(\alpha,\beta)\Big) \qquad \text{for all } t \in [0,1] \text{ and } (\alpha,\beta) \in (0,1)^2$$

where

$$Y(\alpha) = \inf\big\{x \in \mathbb{R} : \gamma(-\infty, x] \geq \alpha\big\} \quad \text{and}$$
$$Z_t(\alpha,\beta) = \inf\Big\{x \in \mathbb{R} : \pi_t\Big((-\infty, x] \mid Y(\alpha)\Big) \geq \beta\Big\}.$$

In fact, $t \mapsto Z_t(\alpha,\beta)$ is a continuous map because of (4)-(5) and Theorem 1. And, on the probability space $((0,1)^2, \mathcal{B}^2, \lambda^2)$, one obtains

$$\lambda^2\big(Y \in A,\, Z_t \in B\big) = \int_0^1 \lambda\big\{\beta \in (0,1) : Y(\alpha) \in A,\, Z_t(\alpha,\beta) \in B\big\}\, d\alpha$$
$$= \int_0^1 1_A(Y(\alpha))\, \pi_t(B \mid Y(\alpha))\, d\alpha = \int_A \pi_t(B \mid x)\, \gamma(dx) = \mu_t\big(A \times B\big)$$

whenever $t \in [0,1]$ and A, $B \subset \mathbb{R}$ are Borel sets.

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