TWO VERSIONS OF THE FUNDAMENTAL THEOREM OF ASSET PRICING

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ABSTRACT. Let L be a convex cone of real random variables on the probability space $(\Omega, \mathcal{A}, P_0)$. The existence of a probability P on $\mathcal{A}$ such that

$$P \sim P_0, \quad E_P|X| < \infty \text{ and } E_P(X) \leq 0 \text{ for all } X \in L$$

is investigated. Two types of results are provided, according to P is finitely additive or σ-additive. The main results concern the latter case (i.e., P is a σ-additive probability). If L is a linear space then $-X \in L$ whenever $X \in L$, so that $E_P(X) \leq 0$ turns into $E_P(X) = 0$. Hence, the results apply to various significant frameworks, including equivalent martingale measures, equivalent probability measures with given marginals, stationary Markov chains and conditional moments.

1. Introduction

Throughout, $(\Omega, \mathcal{A}, P_0)$ is a probability space and L a convex cone of real random variables on $(\Omega, \mathcal{A}, P_0)$. We focus on those probabilities P on $\mathcal{A}$ such that

$$P \sim P_0, \quad E_P|X| < \infty \text{ and } E_P(X) \leq 0 \text{ for all } X \in L. \tag{1}$$

Our main concern is the existence of one such P. Two types of results are provided. In the first, P is a finitely additive probability, while P is σ-additive in the second. The reference probability P_0 is σ-additive.

In economic applications, for instance, L could be a collection of random variables dominated by stochastic integrals of the type $\int_0^1 H\, dS$, where the semimartingale S describes the stock-price process, and H is a predictable S-integrable process ranging in some class of admissible trading strategies; see [20].

However, even if our results apply to any convex cone L, this paper has been mostly written having a linear space in mind. In fact, if L is a linear space, since $-X \in L$ whenever $X \in L$, condition (1) yields

$$E_P(X) = 0 \quad \text{for all } X \in L.$$

Therefore, the addressed problem can be motivated as follows.

Let $S = (S_t : t \in T)$ be a real process on $(\Omega, \mathcal{A}, P_0)$ indexed by $T \subset \mathbb{R}$. Suppose S is adapted to a filtration $\mathcal{G} = (\mathcal{G}_t : t \in T)$ and S_{t_0} is a constant random variable for some $t_0 \in T$. A classical problem in mathematical finance is the existence of an *equivalent martingale measure*, that is, a σ-additive probability P on $\mathcal{A}$ such that $P \sim P_0$ and S is a $\mathcal{G}$-martingale under P. But, with a suitable choice of the linear

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space L, an equivalent martingale measure is exactly a σ-additive solution P of (1). It suffices to take L as the linear space generated by the random variables

$$I_A\,(S_u - S_t) \quad \text{for all } u,\, t \in T \text{ with } u > t \text{ and } A \in \mathcal{G}_t.$$

Note also that, if L is taken to be the convex cone generated by such random variables, a σ-additive solution P of (1) is an equivalent super-martingale measure.

Equivalent martingale measures are usually requested to be σ-additive, but their economic interpretation is preserved if they are only finitely additive. Thus, to look for finitely additive equivalent martingale measures seems to be reasonable. We refer to [4]-[5] and the beginning of Section 3 for a discussion on this point.

Equivalent martingale measures (both σ-additive and finitely additive) are the obvious motivation for our problem, and this explains the title of this paper. But they are not the only motivation. Indeed, various other issues can be reduced to the existence of a probability P satisfying condition (1) for a suitable linear space L (possibly without requesting $P \sim P_0$). Examples are stationary Markov chains and equivalent probability measures with given marginals; see Examples 14 and 15. Other possible examples (not discussed in this paper) are compatibility of conditional distributions and de Finetti's coherence principle; see [1], [6] and references therein.

This paper provides two types of results and a long final section of examples. For definiteness, let us denote

$$\mathbb{S} = \big\{\text{finitely additive probabilities } P \text{ on } \mathcal{A} \text{ satisfying condition (1)}\big\},$$
$$\mathbb{T} = \big\{\sigma\text{-additive probabilities } P \text{ on } \mathcal{A} \text{ satisfying condition (1)}\big\}.$$

Clearly, $\mathbb{T} \subset \mathbb{S}$ and $\mathbb{S}$ may be empty. The members of $\mathbb{S}$ and $\mathbb{T}$ are also called *equivalent separating* probabilities; see Section 2. In particular, a (σ-additive) equivalent martingale measure is an equivalent separating measure for a suitable choice of L.

Firstly, the existence of $P \in \mathbb{S}$ is investigated. In Theorem 2, under the assumption that each $X \in L$ is bounded, $\mathbb{S} \neq \emptyset$ is given various characterizations. As an example, $\mathbb{S} \neq \emptyset$ if and only if

$$\big\{P_0(X \in \cdot) : \, X \in L,\; X \geq -1 \text{ a.s.}\big\}$$

is a tight collection of probability laws on the real line. Such condition already appears in some previous papers; see e.g. [10], [14], [15], [16]. What is new in Theorem 2 is that this tightness condition exactly amounts to $\mathbb{S} \neq \emptyset$. Furthermore, under some assumptions, Theorem 2 also applies when the elements of L are not bounded; see Corollary 5.

Secondly (and more importantly) we focus on the existence of $P \in \mathbb{T}$. Our main results are those obtained in this framework (i.e., when P is requested to be σ-additive). No assumption on the convex cone L is required.

According to Lemma 6, $\mathbb{T} \neq \emptyset$ if and only if

$$E_Q|X| < \infty \quad \text{and} \quad E_Q(X) \leq k\, E_Q(X^-)$$

for all $X \in L$, some constant $k \geq 0$ and some σ-additive probability Q such that $Q \sim P_0$. Note that, if $k = 0$, then $Q \in \mathbb{T}$. Apparently, thus, the scope of Lemma 6 is quite little (it just implies $\mathbb{T} \neq \emptyset$ even if $k > 0$). Instead, sometimes, Lemma 6 and its consequences play a role in proving $\mathbb{T} \neq \emptyset$. The main purpose of the examples in Section 5 is just to validate this claim.

Typically, Lemma 6 helps when $P \in \mathbb{T}$ is requested some additional property, such as to have a bounded density with respect to P_0. This is made precise by Corollary 7 and Theorems 8-9. Theorem 8 extends to any convex cone L a previous result by [6, Theorem 5]. We next summarize these results.

There is $P \in \mathbb{T}$ such that $r\,P_0 \leq P \leq s\,P_0$, for some constants $0 < r \leq s$, if and only if the condition of Lemma 6 holds with $Q = P_0$. Similarly, if $E_{P_0}|X| < \infty$ for all $X \in L$, there is $P \in \mathbb{T}$ with bounded density with respect to P_0 if and only if

$$E_{P_0}\big(X\,I_{A_n}\big) \leq k_n\,E_{P_0}(X^-) \quad \text{for all } n \geq 1 \text{ and } X \in L,$$

where $k_n \geq 0$ is a constant, $A_n \in \mathcal{A}$ and $\lim_n P_0(A_n) = 1$. Finally, under some conditions, the sequence (A_n) is essentially unique and well known.

The main advantage of Corollary 7 and Theorems 8-9, as opposite to Lemma 6, is that they do not require the choice of Q.

2. Notation

In the sequel, as in Section 1, L is a convex cone of real random variables on the fixed probability space $(\Omega, \mathcal{A}, P_0)$. Thus,

$$\sum_{j=1}^{n} \lambda_j\,X_j \in L \quad \text{for all } n \geq 1,\ \lambda_1, \ldots, \lambda_n \geq 0 \text{ and } X_1, \ldots, X_n \in L.$$

We let $\mathbb{P}$ denote the set of finitely additive probabilities on $\mathcal{A}$ and $\mathbb{P}_0$ the subset of those $P \in \mathbb{P}$ which are σ-additive. Recall that $P_0 \in \mathbb{P}_0$.

Sometimes, L is identified with a subset of L_p for some $0 \leq p \leq \infty$, where

$$L_p = L_p(\Omega, \mathcal{A}, P_0).$$

In particular, L can be regarded as a subset of L_∞ if each $X \in L$ is bounded.

For every real random variable X, we let

$$\text{ess sup}(X) = \inf\{x \in \mathbb{R} : P_0(X > x) = 0\} \quad \text{where } \inf \emptyset = \infty.$$

Given P, $T \in \mathbb{P}$, we write $P \ll T$ to mean that $P(A) = 0$ whenever $A \in \mathcal{A}$ and $T(A) = 0$. Also, $P \sim T$ stands for $P \ll T$ and $T \ll P$.

Let $P \in \mathbb{P}$ and X a real random variable. We write

$$E_P(X) = \int X dP$$

whenever X is P-integrable. Every bounded random variable is P-integrable. If X is unbounded but $X \geq 0$, then X is P-integrable if and only if $\inf_n P(X > n) = 0$ and $\sup_n \int X\,I_{\{X \leq n\}}\,dP < \infty$. In this case,

$$\int X\,dP = \sup_n \int X\,I_{\{X \leq n\}}\,dP.$$

An arbitrary real random variable X is P-integrable if and only if X^+ and X^- are both P-integrable, and in this case $\int X dP = \int X^+ dP - \int X^- dP$.

In the sequel, a finitely additive solution P of (1) is said to be an *equivalent separating finitely additive probability* (ESFA). We let $\mathbb{S}$ denote the (possibly empty) set of ESFA's. Thus, $P \in \mathbb{S}$ if and only if

$$P \in \mathbb{P}, \quad P \sim P_0, \quad X \text{ is } P\text{-integrable} \quad \text{and} \quad E_P(X) \leq 0 \ \text{ for each } X \in L.$$

Similarly, a σ-additive solution P of (1) is an *equivalent separating measure* (ESM). That is, P is an ESM if and only if $P \in \mathbb{P}_0 \cap \mathbb{S}$. Recall that, if L is a linear space and P is an ESFA or an ESM, then $E_P(X) = 0$ for all $X \in L$.

Finally, it is convenient to recall the classical no-arbitrage condition

(NA) $$L \cap L_0^+ = \{0\} \quad \text{or equivalently} \quad (L - L_0^+) \cap L_0^+ = \{0\}.$$

3. Equivalent separating finitely additive probabilities

In [4]-[5], ESFA's are defended via the following arguments.

- The finitely additive probability theory is well founded and developed, even if not prevailing. Among its supporters, we mention B. de Finetti, L.J. Savage and L.E. Dubins.
- It may be that ESFA's are available while ESM's fail to exist.
- In option pricing, when L is a linear space, ESFA's give arbitrage-free prices just as ESM's. More generally, the economic motivations of martingale probabilities, as discussed in [11, Chapter 1], do not depend on whether they are σ-additive or not.
- Each ESFA P can be written as $P = \delta\, P_1 + (1-\delta)\, Q$, where $\delta \in [0,1)$, $P_1 \in \mathbb{P}$, $Q \in \mathbb{P}_0$ and $Q \sim P_0$. Thus, when ESM's fail to exist, one might be content with an ESFA P with δ small enough. Extreme situations of this type are exhibited in [5, Example 9] and [6, Example 11]. In such examples, ESM's do not exist, and yet, for each $\epsilon > 0$, there is an ESFA P_ϵ with $\delta < \epsilon$.

ESFA's suffer from some drawbacks as well. They are almost never unique and do not admit densities with respect to P_0. In a finitely additive setting, conditional expectations are not uniquely determined by the assessment of an ESFA P, and this makes problematic to conclude that "the stock-price process is a martingale under P". Further, it is unclear how to prescribe the dynamics of prices, needed for numerical purposes.

Anyhow, this section deals with ESFA's. Two distinct situations (the members of L are, or are not, bounded) are handled separately.

3.1. The bounded case.

In this Subsection, L is a convex cone of real *bounded* random variables. Hence, the elements of L are P-integrable for any $P \in \mathbb{P}$.

We aim to prove a sort of *portmanteau* theorem, that is, a result which collects various characterizations for the existence of ESFA's. To this end, the following technical lemma is needed.

Lemma 1. *Let C be a convex class of real bounded random variables, $\phi : C \to \mathbb{R}$ a linear map, and $\mathcal{E} \subset \mathcal{A}$ a collection of nonempty events such that $A \cap B \in \mathcal{E}$ whenever $A, B \in \mathcal{E}$. There is $P \in \mathbb{P}$ satisfying*

$$\phi(X) \le E_P(X) \quad \text{and} \quad P(A) = 1 \quad \text{for all } X \in C \text{ and } A \in \mathcal{E}$$

if and only if

$$\sup_A X \ge \phi(X) \quad \text{for all } X \in C \text{ and } A \in \mathcal{E}.$$

Proof. This is basically [4, Lemma 2] and so we just give a sketch of the proof. The "only if" part is trivial. Suppose $\sup_A X \ge \phi(X)$ for all $A \in \mathcal{E}$ and $X \in C$. Fix $A \in \mathcal{E}$ and define $C_A = \{X|A - \phi(X) : X \in C\}$, where $X|A$ denotes the restriction of X on A. Then, C_A is a convex class of bounded functions on A and $\sup_A Z \geq 0$ for each $Z \in C_A$. By [12, Lemma 1], there is a finitely additive probability T on the power set of A such that $E_T(Z) \geq 0$ for each $Z \in C_A$. Define

$$P_A(B) = T(A \cap B) \quad \text{for } B \in \mathcal{A}.$$

Then, $P_A \in \mathbb{P}$, $P_A(A) = 1$ and $E_{P_A}(X) = E_T(X|A) \geq \phi(X)$ for each $X \in C$. Next, let $\mathcal{Z}$ be the set of all functions from $\mathcal{A}$ into $[0,1]$, equipped with the product topology, and let

$$F_A = \big\{P \in \mathbb{P} : P(A) = 1 \text{ and } E_P(X) \geq \phi(X) \text{ for all } X \in C\big\} \quad \text{for } A \in \mathcal{E}.$$

Then, $\mathcal{Z}$ is compact and $\{F_A : A \in \mathcal{E}\}$ is a collection of closed sets satisfying the finite intersection property. Hence, $\bigcap_{A\in\mathcal{E}} F_A \neq \emptyset$, and this concludes the proof. □

We next state the portmanteau theorem for ESFA's. Conditions (a)-(b) are already known while conditions (c)-(d) are new. See [8, Theorem 2], [19, Theorem 2.1] for (a) and [4, Theorem 3], [20, Corollary 1] for (b); see also [21]. Recall that $\mathbb{S}$ denotes the (possibly empty) set of ESFA's and define

$$\mathcal{Q} = \{Q \in \mathbb{P}_0 : Q \sim P_0\}.$$

Theorem 2. *Let L be a convex cone of real bounded random variables. Each of the following conditions is equivalent to $\mathbb{S} \neq \emptyset$.*

(a) $\overline{L - L_\infty^+} \cap L_\infty^+ = \{0\}$*, with the closure in the norm-topology of* L_∞*;*

(b) *There are $Q \in \mathcal{Q}$ and a constant $k \geq 0$ such that*

$$E_Q(X) \leq k \textit{ ess sup}(-X) \quad \textit{for each } X \in L;$$

(c) *There are events $A_n \in \mathcal{A}$ and constants $k_n \geq 0$, $n \geq 1$, such that*

$$\lim_n P_0(A_n) = 1 \quad \textit{and}$$
$$E_{P_0}\big(X\, I_{A_n}\big) \leq k_n \textit{ ess sup}(-X) \quad \textit{for all } n \geq 1 \textit{ and } X \in L;$$

(d) $\big\{P_0(X \in \cdot) : X \in L,\ X \geq -1 \text{ a.s.}\big\}$ *is a tight collection of probability laws.*

Moreover, under condition (b), an ESFA is

$$P = \frac{Q + k\,P_1}{1+k} \quad \textit{for a suitable } P_1 \in \mathbb{P}.$$

Proof. First note that each of conditions (b)-(c)-(d) implies (NA), which in turn implies

$$\text{ess sup}(X^-) = \text{ess sup}(-X) > 0 \quad \text{whenever } X \in L \text{ and } P_0(X \neq 0) > 0.$$

(b) ⇒ (c). Suppose (b) holds. Define $k_n = n\,(k+1)$ and $A_n = \{n\,f \geq 1\}$, where f is a density of Q with respect to P_0. Since $f > 0$ a.s., then $P_0(A_n) \to 1$. Further, condition (b) yields

$$\begin{aligned} E_{P_0}\big(X\, I_{A_n}\big) &\leq E_{P_0}\big(X^+\, I_{A_n}\big) = E_Q\big\{X^+\,(1/f)\, I_{A_n}\big\} \leq n\, E_Q(X^+) \\ &= n\,\big\{E_Q(X) + E_Q(X^-)\big\} \leq n\,\big\{k \,\text{ess sup}(-X) + \text{ess sup}(X^-)\big\} \\ &= k_n \,\text{ess sup}(-X) \quad \text{for all } n \geq 1 \text{ and } X \in L. \end{aligned}$$

(c) $\Rightarrow$ (d). Suppose (c) holds and define $D = \{X \in L : X \geq -1 \text{ a.s.}\}$. Up to replacing k_n with $k_n + 1$, it can be assumed $k_n \geq 1$ for all n. Define

$$u = \Big(\sum_{n=1}^{\infty} \frac{P_0(A_n)}{k_n\, 2^n}\Big)^{-1} \quad \text{and} \quad Q(\cdot) = u \sum_{n=1}^{\infty} \frac{P_0(\cdot \cap A_n)}{k_n\, 2^n}.$$

Then, $Q \in \mathcal{Q}$. Given $X \in D$, since ess sup$(-X) \leq 1$ and $X + 1 \geq 0$ a.s., one obtains

$$E_Q|X| \leq 1 + E_Q(X+1) = 2 + u \sum_{n=1}^{\infty} \frac{E_{P_0}\big(X\, I_{A_n}\big)}{k_n\, 2^n}$$
$$\leq 2 + u \sum_{n=1}^{\infty} \frac{\text{ess sup}(-X)}{2^n} \leq 2 + u.$$

Thus, $\big\{Q(X \in \cdot) : X \in D\big\}$ is tight. Since $Q \sim P_0$, then $\big\{P_0(X \in \cdot) : X \in D\big\}$ is tight as well.

(d) $\Rightarrow$ (b). Suppose (d) holds. By a result of Yan [24], there is $Q \in \mathcal{Q}$ such that $k := \sup_{X \in D} E_Q(X) < \infty$, where D is defined as above. Fix $X \in L$ with $P_0(X \neq 0) > 0$ and let $Y = X/\text{ess sup}(-X)$. Since $Y \in D$, one obtains

$$E_Q(X) = E_Q(Y)\, \text{ess sup}(-X) \leq k\, \text{ess sup}(-X).$$

Thus, (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d). This concludes the proof of the first part of the theorem, since it is already known that (b) $\Leftrightarrow$ (a) $\Leftrightarrow$ $\mathbb{S} \neq \emptyset$.

Finally, suppose (b) holds for some $Q \in \mathcal{Q}$ and $k \geq 0$. It remains to show that $P = (1+k)^{-1}(Q + k\, P_1) \in \mathbb{S}$ for some $P_1 \in \mathbb{P}$. If $k = 0$, then $Q \in \mathbb{S}$ and $P = Q$. Thus, suppose $k > 0$ and define

$$C = \{-X : X \in L\}, \quad \phi(Z) = -(1/k)\, E_Q(Z) \text{ for } Z \in C, \quad \mathcal{E} = \{A \in \mathcal{A} : P_0(A) = 1\}.$$

Given $A \in \mathcal{E}$ and $Z \in C$, since $-Z \in L$ condition (b) yields

$$\phi(Z) = (1/k)\, E_Q(-Z) \leq \text{ess sup}(Z) \leq \sup_A Z.$$

By Lemma 1, there is $P_1 \in \mathbb{P}$ such that $P_1 \ll P_0$ and $E_{P_1}(X) \leq -(1/k)\, E_Q(X)$ for all $X \in L$. Since $Q \sim P_0$ and $P_1 \ll P_0$, then $P = (1+k)^{-1}(Q + k\, P_1) \sim P_0$. Further,

$$(1+k)\, E_P(X) = E_Q(X) + k\, E_{P_1}(X) \leq 0 \quad \text{for all } X \in L.$$

$\square$

Since $L \subset L_\infty$, condition (NA) can be written as $(L - L_\infty^+) \cap L_\infty^+ = \{0\}$. Thus, condition (a) can be seen as a no-arbitrage condition. One more remark is in order. Let $\sigma(L_\infty, L_1)$ denote the topology on L_∞ generated by the maps $Z \mapsto E_{P_0}\big(Y\, Z\big)$ for all $Y \in L_1$. In the early eighties, Kreps and Yan proved that the existence of an ESM amounts to

(a*) $\qquad \overline{L - L_\infty^+} \cap L_\infty^+ = \{0\}$ with the closure in $\sigma(L_\infty, L_1)$;

see [18], [23] and [24]. But the geometric meaning of $\sigma(L_\infty, L_1)$ is not so transparent as that of the norm-topology. Hence, a question is what happens if the closure is taken in the norm-topology, that is, if (a*) is replaced by (a). The answer, due to [8, Theorem 2] and [19, Theorem 2.1], is reported in Theorem 2.

Note also that, since $L \subset L_\infty$, condition (a) agrees with the *no free lunch with vanishing risk* condition of Delbaen and Schachermayer

$$\overline{(L - L_0^+) \cap L_\infty} \cap L_\infty^+ = \{0\} \quad \text{with the closure in the norm-topology;}$$

see [10] and [14]. However, Theorem 2 applies to a different framework. In fact, in [10] and [14], L is of the form $L = \{Y_T : Y \in \mathcal{Y}\}$ where $\mathcal{Y}$ is a suitable class of real processes indexed by $[0,T]$. Instead, in Theorem 2, L is any convex cone of bounded random variables. Furthermore, the equivalence between $\mathbb{S} \neq \emptyset$ and the no free lunch with vanishing risk condition is no longer true when L includes unbounded random variables; see Example 13.

Let us turn to (b). Once $Q \in \mathcal{Q}$ has been selected, condition (b) provides a simple criterion for $\mathbb{S} \neq \emptyset$. However, choosing Q is not an easy task. The obvious choice is perhaps $Q = P_0$.

Corollary 3. *Let L be a convex cone of real bounded random variables. Condition (b) holds with $Q = P_0$, that is*

$$E_{P_0}(X) \leq k \text{ ess sup}(-X) \quad \text{for all } X \in L \text{ and some constant } k \geq 0,$$

if and only if there is $P \in \mathbb{S}$ such that $P \geq r\, P_0$ for some constant $r > 0$.

Proof. Let $P \in \mathbb{S}$ be such that $P \geq r\, P_0$. Fix $X \in L$. Since $E_P(X) \leq 0$, then $E_P(X^+) \leq E_P(X^-)$ and ess $\sup(X^-) =$ ess $\sup(-X)$. Hence,

$$\begin{aligned} E_{P_0}(X) &\leq E_{P_0}(X^+) \leq (1/r)\, E_P(X^+) \leq (1/r)\, E_P(X^-) \\ &\leq (1/r) \text{ ess sup}(X^-) = (1/r) \text{ ess sup}(-X). \end{aligned}$$

Conversely, if condition (b) holds with $Q = P_0$, Theorem 2 implies that $P = (1+k)^{-1}(P_0 + kP_1) \in \mathbb{S}$ for suitable $P_1 \in \mathbb{P}$. Thus, $P \geq (1+k)^{-1} P_0$. □

Condition (c) is in the spirit of Corollary 3 (to avoid the choice of Q). It is a sort of localized version of (b), where Q is replaced by a suitable sequence (A_n) of events. See also [5, Theorem 5].

As shown in Section 4, if suitably strengthened, both conditions (b) and (c) become equivalent to existence of ESM's (possibly, with bounded density with respect to P_0).

We finally turn to (d). Some forms of condition (d) have been already involved in connection with the fundamental theorem of asset pricing; see e.g. [10], [14], [15], [16]. What is new in Theorem 2 is only that condition (d) amounts to existence of ESFA's. According to us, condition (d) has some merits. It depends on P_0 only and has a quite transparent meaning (mainly, for those familiar with weak convergence of probability measures). Moreover, it can be naturally regarded as a no-arbitrage condition. Indeed, basing on [7, Lemma 2.3], it is not hard to see that (d) can be rewritten as:

For each $Z \in L_0^+$, $P_0(Z > 0) > 0$, there is a constant $a > 0$ such that

$$P_0\big(X + 1 < a\, Z\big) > 0 \quad \text{whenever } X \in L \text{ and } X \geq -1 \text{ a.s.}$$

Such condition is a market viability condition, called *no-arbitrage of the first kind*, investigated by Kardaras in [15]-[16]. In a sense, Theorem 2-(d) can be seen as a generalization of [15, Theorem 1] (which is stated in a more economic framework).

3.2. **The unbounded case.** In dealing with ESFA's, it is crucial that $L \subset L_\infty$. In fact, all arguments (known to us) for existence of ESFA's are based on de Finetti's coherence principle, but the latter works nicely for bounded random variables only. More precisely, the existing notions of coherence for unbounded random variables do not grant a (finitely additive) integral representation; see [2] and [3]. On the other hand, $L \subset L_\infty$ is certainly a restrictive assumption. In this Subsection, we try to relax such assumption.

Our strategy for proving $\mathbb{S} \neq \emptyset$ is to exploit condition (d) of Theorem 2. To this end, we need a dominance condition on L, such as

$$\text{for each } X \in L, \text{ there is } \lambda > 0 \text{ such that } |X| \leq \lambda\, Y \text{ a.s.} \tag{2}$$

where Y is some real random variable. We can (and will) assume $Y \geq 1$.

Condition (2) is less strong than it appears. For instance, it is always true when L is countably generated. In fact, if L is the convex cone generated by a sequence $(X_n : n \geq 1)$ of real random variables, it suffices to let $Y_n = \sum_{i=1}^n |X_i|$ in the following lemma.

Lemma 4. *If $Y_1, Y_2, \ldots$ are non negative real random variables satisfying*

$$\textit{for each } X \in L, \textit{ there are } \lambda > 0 \textit{ and } n \geq 1 \textit{ such that } |X| \leq \lambda\, Y_n \textit{ a.s.},$$

then condition (2) holds for some real random variable Y.

Proof. For each $n \geq 1$, take $a_n > 0$ such that $P_0(Y_n > a_n) < 2^{-n}$ and define $A = \bigcup_{n=1}^\infty \{Y_j \leq a_j \text{ for each } j \geq n\}$. Then,

$$P_0(A) = 1 \quad \text{and} \quad Y := 1 + \sum_{n=1}^\infty \frac{Y_n}{2^n a_n} < \infty \text{ on } A.$$

Also, condition (2) holds trivially, since $2^n a_n\, Y > Y_n$ on A for each $n \geq 1$. □

Next result applies to those convex cones L satisfying condition (2). It provides a sufficient (sometimes necessary as well) criterion for $\mathbb{S} \neq \emptyset$.

Corollary 5. *Suppose condition (2) holds for some convex cone L and some random variable Y with values in $[1, \infty)$. Then, $\mathbb{S} \neq \emptyset$ provided*

$$\textit{for each } \epsilon > 0, \textit{ there is } c > 0 \textit{ such that} \tag{3}$$

$$P_0\big(|X| > c\, Y\big) < \epsilon \quad \textit{whenever } X \in L \textit{ and } X \geq -Y \textit{ a.s.}$$

Conversely, condition (3) holds if $\mathbb{S} \neq \emptyset$ and Y is P-integrable for some $P \in \mathbb{S}$.

Proof. First note that Theorem 2 is still valid if each member of the convex cone is essentially bounded (even if not bounded). Let $L^* = \{X/Y : X \in L\}$. Then, L^* is a convex cone of essentially bounded random variables and condition (3) is equivalent to tightness of $\big\{P_0(Z \in \cdot) : Z \in L^*, Z \geq -1 \text{ a.s.}\big\}$. Suppose (3) holds. By Theorem 2-(d), L^* admits an ESFA, i.e., there is $T \in \mathbb{P}$ such that $T \sim P_0$ and $E_T(Z) \leq 0$ for all $Z \in L^*$. As noted at the beginning of this Section, such a T can be written as $T = \delta\, P_1 + (1 - \delta)\, Q$, where $\delta \in [0, 1)$, $P_1 \in \mathbb{P}$ and $Q \in \mathcal{Q}$. Since $Y \geq 1$,

$$0 < (1 - \delta)\, E_Q(1/Y) \leq E_T(1/Y) \leq 1.$$

Accordingly, one can define

$$P(A) = \frac{E_T\big(I_A/Y\big)}{E_T(1/Y)} \quad \text{for all } A \in \mathcal{A}.$$

Then, $P \in \mathbb{P}$, $P \sim P_0$, each $X \in L$ is P-integrable, and

$$E_P(X) = \frac{E_T\big(X/Y\big)}{E_T(1/Y)} \leq 0 \quad \text{for all } X \in L.$$

Thus, $P \in \mathbb{S}$. Next, suppose $\mathbb{S} \neq \emptyset$ and Y is P-integrable for some $P \in \mathbb{S}$. Define

$$T(A) = \frac{E_P\big(I_A\, Y\big)}{E_P(Y)} \quad \text{for all } A \in \mathcal{A}.$$

Again, one obtains $T \in \mathbb{P}$, $T \sim P_0$ and $E_T(Z) \leq 0$ for all $Z \in L^*$. Therefore, condition (3) follows from Theorem 2-(d). □

By Corollary 5, $\mathbb{S} \neq \emptyset$ amounts to condition (3) when L is finite dimensional. In fact, if L is the convex cone generated by the random variables $X_1, \ldots, X_d$, condition (2) holds with $Y = 1 + \sum_{i=1}^{d} |X_i|$ and such Y is certainly P-integrable if $P \in \mathbb{S}$. The case of L finite dimensional, however, is better addressed in forthcoming Example 10.

4. Equivalent separating measures

If suitably strengthened, some of the conditions of Theorem 2 become equivalent to existence of ESM's. One example is condition (a) (just replace it by (a*)). Other examples, as we prove in this section, are conditions (b) and (c).

Unlike Theorem 2, L is not requested to consist of bounded random variables.

4.1. Main result.

Recall the notation $\mathcal{Q} = \{Q \in \mathbb{P}_0 : Q \sim P_0\}$.

Lemma 6. *Let L be a convex cone of real random variables. There is an ESM if and only if*

$$\text{(b*)} \qquad E_Q|X| < \infty \quad \textit{and} \quad E_Q(X) \leq k\, E_Q(X^-), \quad X \in L,$$

for some $Q \in \mathcal{Q}$ and some constant $k \geq 0$. In particular, under condition (b*)*, there is an ESM P satisfying*

$$\frac{Q}{k+1} \leq P \leq (k+1)\, Q.$$

Proof. If there is an ESM, say P, condition (b*) trivially holds with $Q = P$ and any $k \geq 0$. Conversely, suppose (b*) holds for some $k \geq 0$ and $Q \in \mathcal{Q}$. Define $t = k + 1$ and

$$\mathcal{K} = \big\{P \in \mathbb{P}_0 : (1/t)\, Q \leq P \leq t\, Q\big\}.$$

If $P \in \mathcal{K}$, then $P \in \mathbb{P}_0$, $P \sim Q \sim P_0$ and $E_P|X| \leq t\, E_Q|X| < \infty$ for all $X \in L$. Thus, it suffices to see that $E_P(X) \leq 0$ for some $P \in \mathcal{K}$ and all $X \in L$.

We first prove that, for each $X \in L$, there is $P \in \mathcal{K}$ such that $E_P(X) \leq 0$. Fix $X \in L$ and define $P(A) = E_Q\big\{f\, I_A\big\}$ for all $A \in \mathcal{A}$, where

$$f = \frac{I_{\{X \geq 0\}} + t\, I_{\{X<0\}}}{Q(X \geq 0) + t\, Q(X < 0)}.$$

Since $E_Q(f) = 1$ and $(1/t) \leq f \leq t$, then $P \in \mathcal{K}$. Further, condition (b*) implies

$$E_P(X) = E_Q\big\{f\, X\big\} = \frac{E_Q(X^+) - t\, E_Q(X^-)}{Q(X \geq 0) + t\, Q(X < 0)} = \frac{E_Q(X) - k\, E_Q(X^-)}{Q(X \geq 0) + t\, Q(X < 0)} \leq 0.$$

Next, let $\mathcal{Z}$ be the set of all functions from $\mathcal{A}$ into $[0,1]$, equipped with the product topology. Then,

(4) $\mathcal{K}$ is compact and $\{P \in \mathcal{K} : E_P(X) \leq 0\}$ is closed for each $X \in L$.

To prove (4), we fix a net (P_α) of elements of $\mathcal{Z}$ converging to $P \in \mathcal{Z}$, that is, $P_\alpha(A) \to P(A)$ for each $A \in \mathcal{A}$. If $P_\alpha \in \mathcal{K}$ for each α, one obtains $P \in \mathbb{P}$ and $(1/t)\, Q \leq P \leq t\, Q$. Since $Q \in \mathbb{P}_0$ and $P \leq t\, Q$, then $P \in \mathbb{P}_0$, i.e., $P \in \mathcal{K}$. Hence, $\mathcal{K}$ is closed, and since $\mathcal{Z}$ is compact, $\mathcal{K}$ is actually compact. If $X \in L$, $P_\alpha \in \mathcal{K}$ and $E_{P_\alpha}(X) \leq 0$ for each α, then $P \in \mathcal{K}$ (for $\mathcal{K}$ is closed). Thus, $E_P|X| < \infty$. Define the set $A_c = \{|X| \leq c\}$ for $c > 0$. Since P_α and P are in $\mathcal{K}$, it follows that

$$|E_{P_\alpha}(X) - E_P(X)| \leq$$
$$\leq |E_{P_\alpha}\{X - X\, I_{A_c}\}| + |E_{P_\alpha}\{X\, I_{A_c}\} - E_P\{X\, I_{A_c}\}| + |E_P\{X\, I_{A_c} - X\}|$$
$$\leq E_{P_\alpha}\{|X|\, I_{\{|X|>c\}}\} + |E_{P_\alpha}\{X\, I_{A_c}\} - E_P\{X\, I_{A_c}\}| + E_P\{|X|\, I_{\{|X|>c\}}\}$$
$$\leq 2\, t\, E_Q\{|X|\, I_{\{|X|>c\}}\} + |E_{P_\alpha}\{X\, I_{A_c}\} - E_P\{X\, I_{A_c}\}|.$$

Since $X\, I_{A_c}$ is bounded, $E_P\{X\, I_{A_c}\} = \lim_\alpha E_{P_\alpha}(X\, I_{A_c})$. Thus,

$$\limsup_\alpha |E_{P_\alpha}(X) - E_P(X)| \leq 2\, t\, E_Q\{|X|\, I_{\{|X|>c\}}\} \quad \text{for every } c > 0.$$

As $c \to \infty$, one obtains $E_P(X) = \lim_\alpha E_{P_\alpha}(X) \leq 0$. Hence, $\{P \in \mathcal{K} : E_P(X) \leq 0\}$ is closed.

Because of (4), to conclude the proof it suffices to see that

$$\{P \in \mathcal{K} : E_P(X_1) \leq 0, \ldots, E_P(X_n) \leq 0\} \neq \emptyset \tag{5}$$

for all $n \geq 1$ and $X_1, \ldots, X_n \in L$. Our proof of (5) is inspired to [17, Theorem 1].

Given $n \geq 1$ and $X_1, \ldots, X_n \in L$, define

$$C = \bigcup_{P \in \mathcal{K}} \{(a_1, \ldots, a_n) \in \mathbb{R}^n : E_P(X_j) \leq a_j \text{ for } j = 1, \ldots, n\}.$$

Then, C is a convex closed subset of $\mathbb{R}^n$. To prove C closed, suppose

$$(a_1^{(m)}, \ldots, a_n^{(m)}) \to (a_1, \ldots, a_n), \text{ as } m \to \infty, \text{ where } (a_1^{(m)}, \ldots, a_n^{(m)}) \in C.$$

For each m, take $P_m \in \mathcal{K}$ such that $E_{P_m}(X_j) \leq a_j^{(m)}$ for all j. Since $\mathcal{K}$ is compact, $P_\alpha \to P$ for some $P \in \mathcal{K}$ and some subnet (P_α) of the sequence (P_m). Hence,

$$a_j = \lim_\alpha a_j^{(\alpha)} \geq \lim_\alpha E_{P_\alpha}(X_j) = E_P(X_j) \quad \text{for } j = 1, \ldots, n.$$

Thus $(a_1, \ldots, a_n) \in C$, that is, C is closed.

Since C is convex and closed, C is the intersection of all half-planes $\{f \geq u\}$ including it, where $u \in \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a linear functional. Fix f and u such that $C \subset \{f \geq u\}$. Write f as $f(a_1, \ldots, a_n) = \sum_{j=1}^n \lambda_j\, a_j$, where $\lambda_1, \ldots, \lambda_n$ are real coefficients. If $(a_1, \ldots, a_n) \in C$, then $(a_1 + b, a_2, \ldots, a_n) \in C$ for $b > 0$, so that

$$b\, \lambda_1 + f(a_1, \ldots, a_n) = f(a_1 + b, a_2, \ldots, a_n) \geq u \quad \text{for all } b > 0.$$

Hence, $\lambda_1 \geq 0$. By the same argument, $\lambda_j \geq 0$ for all j, and this implies $f(X_1, \ldots, X_n) \in L$. Take $P \in \mathcal{K}$ such that $E_P\{f(X_1, \ldots, X_n)\} \leq 0$. Since $(E_P(X_1), \ldots, E_P(X_n)) \in C \subset \{f \geq u\}$, it follows that

$$u \leq f\Big((E_P(X_1), \ldots, E_P(X_n))\Big) = E_P\{f(X_1, \ldots, X_n)\} \leq 0 = f(0, \ldots, 0).$$

This proves $(0, \ldots, 0) \in C$ and concludes the proof.

□

Basically, Lemma 6 turns the original problem into a (slightly) simpler one. In order that $E_P(X) \leq 0$ for all $X \in L$ and some $P \in \mathcal{Q}$, which is the goal, it is enough that $E_Q(X) \leq k\, E_Q(X^-)$ for all $X \in L$, some constant $k \geq 0$ and some $Q \in \mathcal{Q}$. Apparently, the gain is really small. Sometimes, however, such a gain is not trivial and allows to address the problem. Subsection 4.2 and Section 5 are mostly devoted to validate this claim.

We finally note that, if L is a linear space, condition (b*) can be written as

$$\text{(b**)} \qquad E_Q|X| < \infty \quad \text{and} \quad |\, E_Q(X)\,| \leq c\, E_Q|X|$$

for all $X \in L$, some $Q \in \mathcal{Q}$ and some constant $c < 1$. In fact, (b*) implies (b**) with $c = k/(k+2)$ while (b**) implies (b*) with $k = 2c/(1-c)$. However, (b**) is stronger than (b*) if L is not a linear space. For instance, (b*) holds and (b**) fails for the convex cone $L = \{X_b : b \leq 0\}$, where $X_b(\omega) = b$ for all $\omega \in \Omega$.

4.2. **Equivalent separating measures with bounded density.** As in case of condition (b) of Theorem 2, to apply Lemma 6 one has to choose $Q \in \mathcal{Q}$ and a (natural) choice is $Q = P_0$. This is actually the only possible choice if the density of the ESM is requested to be bounded, from above and from below, by some strictly positive constants.

Corollary 7. *Let L be a convex cone of real random variables. There is an ESM P such that*

$$r\, P_0 \leq P \leq s\, P_0,$$

for some constants $0 < r \leq s$, if and only if

$$E_{P_0}|X| < \infty \quad \textit{and} \quad E_{P_0}(X) \leq k\, E_{P_0}(X^-)$$

for all $X \in L$ and some constant $k \geq 0$.

Proof. The "if" part follows from Lemma 6. Conversely, let P be an ESM such that $r\, P_0 \leq P \leq s\, P_0$. Given $X \in L$, one obtains $E_{P_0}|X| \leq (1/r)\, E_P|X| < \infty$ and

$$E_{P_0}(X) \leq E_{P_0}(X^+) \leq (1/r)\, E_P(X^+) \leq (1/r)\, E_P(X^-) \leq (s/r)\, E_{P_0}(X^-).$$

□

Suppose now that the density of the ESM is only asked to be bounded (from above). This situation can be characterized through an obvious strengthening of condition (c). Thus, from a practical point of view, the choice of $Q \in \mathcal{Q}$ is replaced by that of a suitable sequence (A_n) of events. Sometimes, however, the choice of (A_n) is essentially unique.

Suppose $L \subset L_1$ and

$$\text{(c*)} \qquad E_{P_0}\big(X\, I_{A_n}\big) \leq k_n\, E_{P_0}(X^-) \quad \text{for all } n \geq 1 \text{ and } X \in L,$$

where $k_n \geq 0$ is a constant, $A_n \in \mathcal{A}$ and $\lim_n P_0(A_n) = 1$. If L is a linear space, as shown in [6, Theorem 5], condition (c*) amounts to existence of an ESM with bounded density. Here, we prove that (c*) works for a convex cone as well.

Theorem 8. *Suppose $E_{P_0}|X| < \infty$ for all $X \in L$, where L is a convex cone of real random variables. There is an ESM P such that $P \leq s\, P_0$, for some constant $s > 0$, if and only if condition (c*) holds.*

Proof. Let P be an ESM such that $P \leq s\,P_0$. Define $k_n = ns$ and $A_n = \{n\,f \geq 1\}$, where f is a density of P with respect to P_0. Then, $\lim_n P_0(A_n) = P_0(f > 0) = 1$. For $X \in L$, one also obtains

$$E_{P_0}\big(X\,I_{A_n}\big) \leq E_{P_0}\big(X^+\,I_{A_n}\big) = E_P\big\{X^+\,(1/f)\,I_{A_n}\big\} \leq n\,E_P(X^+) \leq n\,E_P(X^-) \leq k_n\,E_{P_0}(X^-).$$

Conversely, suppose condition (c*) holds for some k_n and A_n. It can be assumed $k_n \geq 1$ for all n (otherwise, just replace k_n with $k_n + 1$). Define

$$u = \Big(\sum_{n=1}^\infty \frac{P_0(A_n)}{k_n\,2^n}\Big)^{-1} \quad \text{and} \quad Q(\cdot) = u\sum_{n=1}^\infty \frac{P_0(\cdot \cap A_n)}{k_n\,2^n}.$$

Then, $Q \in \mathcal{Q}$. For any random variable $Y \geq 0$,

$$E_Q(Y) = u\sum_{n=1}^\infty \frac{E_{P_0}\big(Y\,I_{A_n}\big)}{k_n\,2^n} \leq u\,E_{P_0}(Y).$$

Thus, $Q(A) \leq u\,P_0(A)$ and $E_Q|X| \leq u\,E_{P_0}|X| < \infty$ whenever $A \in \mathcal{A}$ and $X \in L$. Similarly, condition (c*) implies $E_Q(X) \leq u\,E_{P_0}(X^-)$ for all $X \in L$.

Define

$$\mathcal{K} = \big\{P \in \mathbb{P}_0 : \,(u+1)^{-1}Q \leq P \leq Q + u\,P_0\big\}.$$

If $P \in \mathcal{K}$, then $E_P|X| \leq E_Q|X| + u\,E_{P_0}|X| \leq 2\,u\,E_{P_0}|X| < \infty$ for all $X \in L$. Also, $P \in \mathcal{Q}$ and $P \leq 2\,u\,P_0$. Hence, it suffices to show that $E_P(X) \leq 0$ for all $X \in L$ and some $P \in \mathcal{K}$.

For each $X \in L$, there is $P \in \mathcal{K}$ such that $E_P(X) \leq 0$. Fix in fact $X \in L$. If $E_Q(X) \leq 0$, just take $P = Q \in \mathcal{K}$. If $E_Q(X) > 0$, take a density h of Q with respect to P_0 and define

$$f = \frac{E_Q(X)\,I_{\{X<0\}} + E_{P_0}(X^-)\,h}{E_Q(X)\,P_0(X<0) + E_{P_0}(X^-)} \quad \text{and} \quad P(A) = E_{P_0}\big(f\,I_A\big) \quad \text{for } A \in \mathcal{A}.$$

Since $E_Q(X) \leq u\,E_{P_0}(X^-)$, then $(u+1)^{-1}h \leq f \leq h + u$. Hence, $P \in \mathcal{K}$ and

$$E_P(X) = E_{P_0}\big(f\,X\big) = \frac{-E_Q(X)\,E_{P_0}(X^-) + E_{P_0}(X^-)\,E_Q(X)}{E_Q(X)\,P_0(X<0) + E_{P_0}(X^-)} = 0.$$

From now on, the proof agrees exactly with that of Lemma 6. In fact, $\mathcal{K}$ is compact and $\{P \in \mathcal{K} : E_P(X) \leq 0\}$ is closed for each $X \in L$ (under the same topology as in the proof of Lemma 6). In addition, for each finite subset $\{X_1, \ldots, X_n\} \subset L$, one obtains $E_P(X_1) \leq 0, \ldots, E_P(X_n) \leq 0$ for some $P \in \mathcal{K}$. This concludes the proof. □

Theorem 8 provides a necessary and sufficient condition for an ESM with bounded density to exist. This condition looks practically usable, but still requires to select the sequence (A_n). However, under some assumptions on Ω and if the ESM is requested an additional requirement, there is essentially a unique choice for (A_n). Further, such a choice is usually known.

Theorem 9. *Let Ω be a topological space, $\mathcal{A}$ the Borel σ-field and L a convex cone of real random variables. Suppose $E_{P_0}|X| < \infty$ for all $X \in L$ and*

$$\Omega = \cup_n B_n,$$

where (B_n) is an increasing sequence of open sets with compact closure. Then, condition (c*) *holds with $A_n = B_n$ if and only if there is an ESM P such that*

$$\sup_{\omega\in\Omega} f(\omega) < \infty \;\; \textit{and} \;\; \inf_{\omega\in K} f(\omega) > 0 \tag{6}$$
$$\textit{for each compact } K \in \mathcal{A} \textit{ with } P_0(K) > 0,$$

where f is a density of P with respect to P_0.

Proof. Let P be an ESM satisfying (6). Since B_n has compact closure,

$$v_n := \inf_{\omega\in B_n} f(\omega) > 0 \quad \text{whenever } P_0(B_n) > 0.$$

Letting $k_n = v_n^{-1} \sup_{\omega\in\Omega} f(\omega)$, it follows that

$$\begin{aligned} E_{P_0}\big(X\, I_{B_n}\big) &\le E_{P_0}\big(X^+\, I_{B_n}\big) = E_P\big\{X^+\,(1/f)\, I_{B_n}\big\} \\ &\le (1/v_n)\, E_P(X^+) \le (1/v_n)\, E_P(X^-) \le k_n\, E_{P_0}(X^-) \end{aligned}$$

for all $X \in L$. Conversely, suppose (c*) holds with $A_n = B_n$. It can be assumed $k_n \ge 1$ for all n. Define $Q(A) = E_{P_0}\big(h\, I_A\big)$ for all $A \in \mathcal{A}$, where

$$h = \sum_{n=1}^{\infty} \frac{u}{k_n\, 2^n}\, I_{B_n} \quad \text{with } u > 0 \text{ a normalizing constant.}$$

Such h is bounded, strictly positive and lower semi-continuous (for the B_n are open). Thus, $\inf_{\omega\in K} h(\omega) > 0$ whenever K is compact and nonempty. Arguing as in the proof of Theorem 8, there is an ESM P such that $(u+1)^{-1}Q \le P \le Q + u\,P_0$. Fix a density g of P with respect to P_0 and define $A = \{(u+1)^{-1}h \le g \le h+u\}$ and $f = I_A\, g + I_{A^c}$. Then, f satisfies condition (6). Since $P_0(A) = 1$, further, f is still a density of P with respect to P_0. $\square$

As an example, if $\Omega = \{\omega_1, \omega_2, \ldots\}$ is countable, there is an ESM with bounded density if and only if condition (c*) holds with $A_n = \{\omega_1, \ldots, \omega_n\}$. Or else, if $\Omega = \mathbb{R}^d$, there is an ESM satisfying (6) if and only if condition (c*) holds with A_n the ball of center 0 and radius n. We finally note that condition (6) is not so artificial. It holds, for instance, whenever f is bounded, strictly positive and lower semi-continuous.

5. Examples

In this Section, L is a linear space. Up to minor changes, however, most examples could be adapted to a convex cone L. Recall that, since L is a linear space, $E_P(X) = 0$ whenever $X \in L$ and P is an ESFA or an ESM.

Example 10. (Finite dimensional spaces). Let $X_1, \ldots, X_d$ be real random variables on $(\Omega, \mathcal{A}, P_0)$. Is there a σ-additive probability $P \in \mathbb{P}_0$ such that

$$P \sim P_0, \quad E_P|X_j| < \infty \quad \text{and} \quad E_P(X_j) = 0 \quad \text{for all } j\ ?$$

The question looks natural and the answer is intuitive as well. Such a P exists if and only if $L \cap L_0^+ = \{0\}$, that is (NA) holds, with

$$L = \text{ linear space generated by } X_1, \ldots, X_d.$$

This is a known result. It follows from [9, Theorem 2.4] and a (nice) probabilistic argument is in [13]. However, to our knowledge, such result does not admit elementary proofs. We now deduce it as an immediate consequence of Corollary 7.

Up to replacing X_j with $Y_j = \frac{X_j}{1+\sum_{i=1}^d |X_i|}$, it can be assumed $E_{P_0}|X_j| < \infty$ for all j. Let $K = \{X \in L : E_{P_0}|X| = 1\}$, equipped with the L_1-norm. If $L \cap L_0^+ = \{0\}$, then $|\,E_{P_0}(X)\,| < 1$ for each $X \in K$. Since K is compact and $X \mapsto E_{P_0}(X)$ is continuous, $\sup_{X \in K}|\,E_{P_0}(X)\,| < 1$. Thus, condition (b**) holds with $Q = P_0$ and Corollary 7 applies. (Recall that (b**) amounts to (b*) when L is a linear space).

Two remarks are in order. First, if $E_{P_0}|X_j| < \infty$ for all j (so that the X_j should not be replaced by the Y_j) the above argument implies that P can be taken to satisfy $r\,P_0 \leq P \leq s\,P_0$ for some $0 < r \leq s$. Second, Corollary 7 also yields a reasonably simple proof of [9, Theorem 2.6], i.e., the main result of [9].

Example 11. (A question by Rokhlin and Schachermayer). Suppose that $E_{P_0}(X_n) = 0$ for all $n \geq 1$, where the X_n are real bounded random variables. Let L be the linear space generated by the sequence $(X_n : n \geq 1)$ and

$$P_f(A) = E_{P_0}\big(f\,I_A\big), \quad A \in \mathcal{A},$$

where f is a strictly positive measurable function on Ω such that $E_{P_0}(f) = 1$. Choosing P_0, f and X_n suitably, in [20, Example 3] it is shown that

(i) There is a bounded finitely additive measure T on $\mathcal{A}$ such that

$$T \ll P_0, \quad T(A) \geq P_f(A) \quad \text{and} \quad \int X\,dT = 0 \quad \text{for all } A \in \mathcal{A} \text{ and } X \in L;$$

(ii) *No* measurable function $g : \Omega \to [0, \infty)$ satisfies

$$g \geq f \text{ a.s.}, \quad E_{P_0}(g) < \infty \quad \text{and} \quad E_{P_0}\big(g\,X\big) = 0 \text{ for all } X \in L.$$

In [20, Example 3], L is spanned by a (infinite) sequence. Thus, at page 823, the question is raised of whether (i)-(ii) can be realized when L is finite dimensional.

We claim that the answer is no, even if one aims to achieve (ii) alone. Suppose in fact that L is generated by the bounded random variables $X_1, \ldots, X_d$. Since $P_f \sim P_0$ and $E_{P_0}(X) = 0$ for all $X \in L$, then $L \cap L_0^+ = \{0\}$ under P_f as well. Arguing as in Example 10, one obtains $E_Q(X) = 0$, $X \in L$, for some $Q \in \mathbb{P}_0$ such that $r\,P_f \leq Q \leq s\,P_f$, where $0 < r \leq s$. Therefore, a function g satisfying the conditions listed in (ii) is $g = \psi/r$, where ψ is a density of Q with respect to P_0.

Example 12. (Example 7 of [5] revisited). Let L be the linear space generated by the random variables $X_1, X_2, \ldots$, where each X_n takes values in $\{-1, 1\}$ and

$$P_0\big(X_1 = x_1, \ldots, X_n = x_n\big) > 0 \quad \text{for all } n \geq 1 \text{ and } x_1, \ldots, x_n \in \{-1, 1\}. \tag{7}$$

Every $X \in L$ can be written as $X = \sum_{j=1}^n b_j X_j$ for some $n \geq 1$ and $b_1, \ldots, b_n \in \mathbb{R}$. By (7),

$$\text{ess sup}(X) = |b_1| + \ldots + |b_n| = \text{ess sup}(-X).$$

Hence, condition (b) is trivially true, and Theorem 2 implies the existence of an ESFA. However, ESM's can fail to exist. To see this, let $P_0(X_n = -1) = (n+1)^{-2}$ and fix $P \in \mathcal{Q}$. Under P_0, the Borel-Cantelli lemma yields $X_n \xrightarrow{a.s.} 1$. Hence, $X_n \xrightarrow{a.s.} 1$ under P as well, and P fails to be an ESM for $E_P(X_n) \to 1$.

This is basically Example 7 of [5]. We now modify such example, preserving the possible economic meaning (provided the X_n are regarded as asset prices) but allowing for ESM's to exist.

Let N be a random variable, independent of the sequence (X_n), with values in $\{1, 2, \ldots\}$. To fix ideas, suppose $P_0(N = n) > 0$ for all $n \geq 1$. Take L to be the collection of X of the type

$$X = \sum_{j=1}^{N} b_j X_j$$

for all real sequences (b_j) such that $\sum_j |b_j| < \infty$. Then, L is a linear space of bounded random variables. Given $n \geq 1$, define L_n to be the linear space spanned by $X_1, \ldots, X_n$. Because of (7) and the independence between N and (X_n), for each $X \in L_n$ one obtains

$$P_0\big(X > 0 \mid N = n\big) > 0 \quad \Longleftrightarrow \quad P_0\big(X < 0 \mid N = n\big) > 0.$$

Hence, condition (NA) holds with $P_0\big(\cdot \mid N = n\big)$ and L_n in the place of P_0 and L. Arguing as in Example 10, it follows that $E_{P_n}(X) = 0$ for all $X \in L_n$ and some $P_n \in \mathbb{P}_0$ such that $P_n \sim P_0\big(\cdot \mid N = n\big)$. Since $P_n(N = n) = 1$, then $E_{P_n}(X) = 0$ for all $X \in L$. Thus, an ESM is $P = \sum_{n=1}^{\infty} 2^{-n} P_n$.

Incidentally, in addition to be an ESM for L, such a P also satisfies

$$E_P\Big(\sum_{j=1}^{N \wedge n} b_j X_j\Big) = 0 \quad \text{for all } n \geq 1 \text{ and } b_1, \ldots, b_n \in \mathbb{R}.$$

Example 13. (No free lunch with vanishing risk). It is not hard to see that $\mathbb{S} \neq \emptyset$ implies

$$\overline{(L - L_0^+) \cap L_\infty} \cap L_\infty^+ = \{0\} \quad \text{with the closure in the norm-topology of } L_\infty.$$

Unlike the bounded case (see the remarks after Theorem 2), however, the converse is not true.

Let Z be a random variable such that $Z > 0$ and $P_0(a < Z < b) > 0$ for all $0 \leq a < b$. Take L to be the linear space generated by $(X_n : n \geq 0)$, where

$$X_0 = Z \sum_{k \geq 0} (-1)^k I_{\{k \leq Z < k+1\}} \quad \text{and}$$

$$X_n = I_{\{Z < n\}} + Z \sum_{k \geq n} (-1)^k I_{\{k + 2^{-n} \leq Z < k+1\}} \quad \text{for } n \geq 1.$$

Also, fix $P \in \mathbb{P}$ such that X_n is P-integrable for each $n \geq 0$ and $P = \delta\, P_1 + (1-\delta)\, Q$ for some $\delta \in [0, 1)$, $P_1 \in \mathbb{P}$ and $Q \in \mathcal{Q}$. From the definition of P-integrability (recalled in Section 2) one obtains

$$E_P(X_n) = P(Z < n) + \sum_{k \geq n} (-1)^k E_P\Big\{Z\, I_{\{k + 2^{-n} \leq Z < k+1\}}\Big\} \quad \text{for } n \geq 1.$$

Since $Z = |X_0|$ is P-integrable, then

$$\begin{aligned}\Big|\sum_{k \geq n} (-1)^k E_P\Big\{Z\, I_{\{k + 2^{-n} \leq Z < k+1\}}\Big\}\Big| &\leq \sum_{k \geq n} E_P\Big\{Z\, I_{\{k \leq Z < k+1\}}\Big\} \\ &= E_P\big\{Z\, I_{\{Z \geq n\}}\big\} \longrightarrow 0 \quad \text{as } n \to \infty.\end{aligned}$$

It follows that

$$\liminf_n E_P(X_n) = \liminf_n P(Z < n) \geq (1 - \delta) \liminf_n Q(Z < n) = (1 - \delta) > 0.$$

Hence $P \notin \mathbb{S}$, and this implies $\mathbb{S} = \emptyset$ since each member of $\mathbb{S}$ should satisfy the requirements asked to P. On the other hand, it is easily seen that

$$\text{ess sup}(X) = \text{ess sup}(-X) = \infty \quad \text{for each } X \in L \text{ with } P_0(X \neq 0) > 0.$$

Thus, $(L - L_0^+) \cap L_\infty = -L_\infty^+$ which trivially implies

$$\overline{(L - L_0^+) \cap L_\infty} \cap L_\infty^+ = \overline{(-L_\infty^+)} \cap L_\infty^+ = (-L_\infty^+) \cap L_\infty^+ = \{0\}.$$

Together with Example 10, the next examples aim to support the results in Section 4. In addition to equivalent martingale measures, in fact, many other existence-problems can be tackled by such results. See also Section 1 of [6].

Example 14. (Stationary Markov chains). Let $S(\mathcal{A})$ be the set of simple functions on $(\Omega, \mathcal{A})$. A kernel on $(\Omega, \mathcal{A})$ is a function K on $\Omega \times \mathcal{A}$ such that $K(\omega, \cdot) \in \mathbb{P}_0$ for $\omega \in \Omega$ and $\omega \mapsto K(\omega, A)$ is measurable for $A \in \mathcal{A}$. A stationary distribution for the kernel K is a (σ-additive) probability $P \in \mathbb{P}_0$ such that $E_P(f) = \int K(\omega, f)\, P(d\omega)$ for all $f \in S(\mathcal{A})$, where

$$K(\omega, f) = \int f(x)\, K(\omega, dx).$$

Let K be a kernel on $(\Omega, \mathcal{A})$. Then, K admits a stationary distribution P, satisfying $P \sim P_0$ and $P \leq s\, P_0$ for some constant $s > 0$, if and only if

$$E_{P_0}\Big\{ I_{A_n}\, \big(K(\cdot, f) - f\big) \Big\} \leq k_n\, E_{P_0}\Big\{ \big(K(\cdot, f) - f\big)^- \Big\} \tag{8}$$

for all $n \geq 1$ and $f \in S(\mathcal{A})$, where $k_n \geq 0$ is a constant, $A_n \in \mathcal{A}$ and $\lim_n P_0(A_n) = 1$. This follows directly from Theorem 8, applied to the linear space

$$L = \big\{ K(\cdot, f) - f : f \in S(\mathcal{A}) \big\}.$$

Condition (8) looks potentially useful, for the usual criteria for the existence of stationary distributions are not very simple to work with. Also, by Theorem 9, if $\Omega = \{\omega_1, \omega_2, \ldots\}$ is countable one can take $A_n = \{\omega_1, \ldots, \omega_n\}$. In this case, condition (8) turns into

$$\sum_{j=1}^{n} P_0\{\omega_j\} \big\{ K(\omega_j, f) - f(\omega_j) \big\} \leq k_n \sum_{j=1}^{\infty} P_0\{\omega_j\} \big\{ K(\omega_j, f) - f(\omega_j) \big\}^-.$$

Example 15. (Equivalent probability measures with given marginals). Let

$$\Omega = \Omega_1 \times \Omega_2 \quad \text{and} \quad \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$$

where $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are measurable spaces. Fix a (σ-additive) probability T_i on $\mathcal{A}_i$ for $i = 1, 2$. Is there a σ-additive probability $P \in \mathbb{P}_0$ such that

$$P \sim P_0 \quad \text{and} \quad P\big(\cdot \times \Omega_2\big) = T_1(\cdot), \; P\big(\Omega_1 \times \cdot\big) = T_2(\cdot)\ ? \tag{9}$$

Again, the question looks natural (to us). Nevertheless, as far as we know, such a question has been neglected so far. For instance, the well known results by Strassen [22] do not apply here, for $\mathcal{Q}$ fails to be closed in any reasonable topology on $\mathbb{P}_0$. However, a possible answer can be manufactured through the results in Section 4.

Let M_i be a class of bounded measurable functions on Ω_i, $i = 1, 2$. Suppose each M_i is both a linear space and a determining class, in the sense that, if μ and ν are (σ-additive) probabilities on $\mathcal{A}_i$ then

$$\mu = \nu \quad \Longleftrightarrow \quad E_\mu(f) = E_\nu(f) \text{ for all } f \in M_i.$$

Define L to be the class of random variables X on $\Omega = \Omega_1 \times \Omega_2$ of the type

$$X(\omega_1, \omega_2) = \big\{f(\omega_1) - E_{T_1}(f)\big\} + \big\{g(\omega_2) - E_{T_2}(g)\big\}$$

for all $f \in M_1$ and $g \in M_2$. Then, L is a linear space of bounded random variables. Furthermore, there is $P \in \mathbb{P}_0$ satisfying (9) if and only if L admits an ESM. In turn, by Lemma 6, the latter fact amounts to $E_Q(X) \le k\, E_Q(X^-)$ for all $X \in L$ and some $Q \in \mathcal{Q}$ and $k \ge 0$.

To fix ideas, we discuss a particular case. Let

$$R_1(\cdot) = P_0(\cdot \times \Omega_2), \quad R_2(\cdot) = P_0(\Omega_1 \times \cdot), \quad R = R_1 \times R_2 \quad \text{and} \quad T = T_1 \times T_2.$$

Thus, R_1 and R_2 are the marginals of P_0, R and T are product probabilities on $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ and R has the same marginals as P_0. Then, condition (9) holds for some $P \in \mathbb{P}_0$ provided

$$R \ll P_0, \quad T_i \ll R_i \quad \text{and} \quad R_i \le b_i\, T_i$$

for $i = 1, 2$ and some constants $b_1 > 0$ and $b_2 > 0$.

Define in fact $P^* = (1/2)\,(P_0 + T)$. Then, P^* has marginals $R_i^* = (1/2)\,(R_i + T_i)$ for $i = 1, 2$. Furthermore,

$$P^* \sim P_0, \quad T_i \le 2\, R_i^* \le (1 + b_i)\, T_i, \quad R_1^* \times R_2^* \ll P^*.$$

Thus, up to replacing P_0 with P^*, it can be assumed

$$R \ll P_0 \quad \text{and} \quad a_i\, T_i \le R_i \le b_i\, T_i$$

where $i = 1, 2$ and both $a_i > 0$ and $b_i > 0$ are constants. Under such assumptions, take a density of R with respect to P_0, say f, and define

$$c = E_{P_0}(f \vee 1) \quad \text{and} \quad Q(A) = (1/c)\, E_{P_0}\big\{(f \vee 1)\, I_A\big\} \quad \text{for } A \in \mathcal{A}.$$

Observe now that $E_T(X) = 0$ for all $X \in L$ (since T has marginals T_1 and T_2) and

$$(b_1\, b_2)^{-1} R \le T \le (a_1\, a_2)^{-1} R.$$

By Corollary 7, applied with R in the place of P_0, one obtains $E_R(X) \le u\, E_R(X^-)$, $X \in L$, for some constant $u \ge 0$. Given $Y \in L$, it follows that

$$\begin{aligned} E_{P_0}(Y) &= E_R(Y) \le u\, E_R(Y^-) = u\, E_{P_0}\big(f\, Y^-\big) \\ &\le u\, E_{P_0}\big\{(f \vee 1)\, Y^-\big\} = c\, u\, E_Q(Y^-) \end{aligned}$$

where the first equality is because P_0 and R have the same marginals. Letting $h = 1/(f \vee 1)$ and noting that $h \le 1$, one also obtains

$$(1/c)\, E_{P_0}(Y) = E_Q\big(h\, Y\big) = E_Q\big(h\, Y^+\big) - E_Q\big(h\, Y^-\big) \ge E_Q\big(h\, Y^+\big) - E_Q(Y^-).$$

Hence, $E_Q\big(h\, Y^+\big) \le (u + 1)\, E_Q(Y^-)$. Finally, let $A_n = \{f \le n\}$. On noting that $n\, h \ge 1$ on A_n, one obtains

$$\begin{aligned} E_Q\big(I_{A_n}\, Y\big) &\le E_Q\big(I_{A_n}\, Y^+\big) \le n\, E_Q\big\{I_{A_n}\, h\, Y^+\big\} \\ &\le n\, E_Q\big(h\, Y^+\big) \le n\,(u + 1)\, E_Q(Y^-). \end{aligned}$$

Further, $\lim_n Q(A_n) = 1$. By Theorem 8, applied with Q in the place of P_0, there is $P \in \mathbb{P}_0$ such that $P \sim Q$ and $E_P(X) = 0$ for all $X \in L$. Since $Q \sim P_0$, such a P satisfies condition (9).

Example 16. (Conditional moments). Let $\mathcal{G} \subset \mathcal{A}$ be a sub-σ-field and U, V real random variables satisfying $E_{P_0}\big\{|U|^k + |V|^k\big\} < \infty$ for some integer $k \geq 1$. Suppose we need a σ-additive probability $P \in \mathbb{P}_0$ such that

$$P \sim P_0, \quad E_P\big\{|U|^k + |V|^k\big\} < \infty, \quad \text{and}$$
$$E_P\big(U^j \mid \mathcal{G}\big) = E_P\big(V^j \mid \mathcal{G}\big) \text{ a.s. for } 1 \leq j \leq k.$$

For such a P to exist, it suffices to prove condition (c*) for the linear space L generated by $I_A(U^j - V^j)$ for all $A \in \mathcal{G}$ and $1 \leq j \leq k$. In this case in fact, by Theorem 8, there is an ESM P such that $P \leq s\,P_0$ for some constant $s > 0$. Hence, $E_P\big\{|U|^k + |V|^k\big\} \leq s\,E_{P_0}\big\{|U|^k + |V|^k\big\} < \infty$, and $E_P\big(U^j - V^j \mid \mathcal{G}\big) = 0$ a.s. follows from $E_P\big\{I_A\,(U^j - V^j)\big\} = 0$ for all $A \in \mathcal{G}$.

Let C be the collection of $X \in L$ of the form $X = \sum_{j=1}^k q_j\,(U^j - V^j)$ with $q_1, \ldots, q_k$ rational numbers. Define

$$W = \sup_{X \in C} \frac{E_{P_0}(X \mid \mathcal{G})}{E_{P_0}(X^- \mid \mathcal{G})},$$

with the conventions $0/0 = 0$ and $x/0 = \text{sgn}(x) \cdot \infty$ if $x \neq 0$, and suppose

$$P_0(W < \infty) = 1.$$

Fix $X \in L$. Then, X can be written as $X = \sum_{j=1}^k Y_j\,(U^j - V^j)$ where $Y_1, \ldots, Y_k$ are $\mathcal{G}$-measurable random variables. Basing on this fact and $P_0(W < \infty) = 1$, one obtains $E_{P_0}(X \mid \mathcal{G}) \leq W\,E_{P_0}(X^- \mid \mathcal{G})$ a.s. Let $A_n = \{W \leq n\}$. Since $A_n \in \mathcal{G}$,

$$\begin{aligned} E_{P_0}\big(X\,I_{A_n}\big) &= E_{P_0}\big\{I_{A_n}\,E_{P_0}(X \mid \mathcal{G})\big\} \leq E_{P_0}\big\{I_{A_n}\,W\,E_{P_0}(X^- \mid \mathcal{G})\big\} \\ &\leq n\,E_{P_0}\big\{E_{P_0}(X^- \mid \mathcal{G})\big\} = n\,E_{P_0}(X^-). \end{aligned}$$

On noting that $\lim_n P_0(A_n) = P_0(W < \infty) = 1$, thus, condition (c*) holds.

As a concrete example, suppose $k = 1$. Then, $C = \{q\,(U - V) : q \text{ rational}\}$ and $P_0(W < \infty) = 1$ can be written as

$$\frac{E_{P_0}(U - V \mid \mathcal{G})}{E_{P_0}\big\{(U - V)^- \mid \mathcal{G}\big\}} < \infty \quad \text{and} \quad \frac{E_{P_0}(U - V \mid \mathcal{G})}{E_{P_0}\big\{(U - V)^+ \mid \mathcal{G}\big\}} > -\infty \quad \text{a.s.}$$

or equivalently

$$\frac{|\,E_{P_0}(U - V \mid \mathcal{G})\,|}{E_{P_0}\big\{|U - V| \mid \mathcal{G}\big\}} < 1 \quad \text{a.s.}$$

Under such condition, one obtains $E_P\big(U \mid \mathcal{G}\big) = E_P\big(V \mid \mathcal{G}\big)$ a.s. for some $P \in \mathbb{P}_0$ such that $P \sim P_0$.

A last remark is in order. Suppose that $P_0(W < \infty) = 1$ is weakened into $P_0(W < \infty) > 0$. Then, $E_P\big(U^j \mid \mathcal{G}\big) = E_P\big(V^j \mid \mathcal{G}\big)$ a.s., $1 \leq j \leq k$, for some $P \in \mathbb{P}_0$ such that $P \ll P_0$. In fact, letting $Q(\cdot) = P_0(\cdot \mid W < \infty)$, the above argument implies $E_Q\big(X\,I_{A_n}\big) \leq n\,E_Q(X^-)$ for all $n \geq 1$ and $X \in L$. Hence, P can be taken such that $P \sim Q \ll P_0$.

Example 17. (Translated Brownian motion). Let

$$S_t = B_t - \int_0^t Y_s\, ds,$$

where $B = (B_t : 0 \leq t \leq 1)$ is a standard Brownian motion and $Y = (Y_t : 0 \leq t \leq 1)$ a real measurable process on $(\Omega, \mathcal{A}, P_0)$. Suppose that almost all Y-paths satisfy

$$\int_0^1 |Y_t|\, dt \leq b \quad \text{and} \quad Y = 0 \text{ on } [\delta, 1]$$

with $b > 0$ and $\delta \in (0, 1)$ constants. Then, for any sequence $0 = t_1 < t_2 < t_3 < \ldots$ with $\sup_n t_n = 1$, there is a σ-additive probability $P \in \mathbb{P}_0$ such that

$$r\, P_0 \leq P \leq s\, P_0 \quad \text{and} \quad E_P(S_{t_n}) = 0 \tag{10}$$

for all $n \geq 1$ and some constants $0 < r \leq s$.

We next prove (10). Let L be the linear space generated by $\{S_{t_{j+1}} - S_{t_j} : j \geq 1\}$ and let $n_0 \geq 1$ be such that $t_j \geq \delta$ for all $j > n_0$. Fix $X \in L$, say

$$X = \sum_{j=1}^{n} c_j\, (S_{t_{j+1}} - S_{t_j}) \quad \text{where } c_1, \ldots, c_n \in \mathbb{R} \text{ and } c_j \neq 0 \text{ for some } j.$$

Then,

$$|\, E_{P_0}(X)\, | = \Big|\, \sum_{j=1}^{n\wedge n_0} c_j\, E_{P_0}\Big(\int_{t_j}^{t_{j+1}} Y_s\, ds\Big)\, \Big| \leq b \sum_{j=1}^{n\wedge n_0} |c_j|.$$

Define

$$u = \sqrt{\sum_{j=1}^{n} c_j^2\, (t_{j+1} - t_j)}, \quad V = \sum_{j=1}^{n} (c_j/u)\, \{B_{t_{j+1}} - B_{t_j}\}, \quad a = \min\{t_{j+1} - t_j : 1 \leq j \leq n_0\}.$$

On noting that $(1/u^2) \sum_{j=1}^{n\wedge n_0} c_j^2\, (t_{j+1} - t_j) \leq 1$, one obtains

$$X/u = V - (1/u) \sum_{j=1}^{n\wedge n_0} c_j\, \sqrt{t_{j+1} - t_j}\; \frac{\int_{t_j}^{t_{j+1}} Y_s\, ds}{\sqrt{t_{j+1} - t_j}}$$

$$\leq V + \sqrt{(1/u^2) \sum_{j=1}^{n\wedge n_0} c_j^2\, (t_{j+1} - t_j) \sum_{j=1}^{n\wedge n_0} \frac{\big(\int_{t_j}^{t_{j+1}} Y_s\, ds\big)^2}{t_{j+1} - t_j}}$$

$$\leq V + \sqrt{(1/a) \sum_{j=1}^{n\wedge n_0} \Big(\int_{t_j}^{t_{j+1}} |Y_s|\, ds\Big)^2}$$

$$\leq V + \sqrt{(1/a) \Big(\sum_{j=1}^{n\wedge n_0} \int_{t_j}^{t_{j+1}} |Y_s|\, ds\Big)^2} \leq V + \sqrt{b^2/a}.$$

On the other hand,

$$u^2 \geq a \sum_{j=1}^{n\wedge n_0} c_j^2 \geq \frac{a}{n \wedge n_0} \Big(\sum_{j=1}^{n\wedge n_0} |c_j|\Big)^2 \geq \frac{a}{b^2 n_0} \Big(E_{P_0}(X)\Big)^2.$$

Thus,

$$E_{P_0}(X^-) = u\,E_{P_0}\big\{(X/u)^-\big\} \geq u\,E_{P_0}\big\{\big(V + \sqrt{b^2/a}\,\big)^-\big\}$$
$$\geq \sqrt{\frac{a}{b^2 n_0}}\,E_{P_0}\big\{\big(V + \sqrt{b^2/a}\,\big)^-\big\}\,|\,E_{P_0}(X)\,|.$$

Since V has standard normal distribution under P_0, then $E_{P_0}\big\{\big(V+\sqrt{b^2/a}\,\big)^-\big\} > 0$. Therefore, to get condition (10), it suffices to apply Corollary 7 with

$$k = \frac{b\,\sqrt{n_0}}{\sqrt{a}\,E_{P_0}\big\{\big(V + \sqrt{b^2/a}\,\big)^-\big\}}.$$

Finally, we make two remarks. Fix a filtration $\mathcal{G} = (\mathcal{G}_t : 0 \leq t \leq 1)$, satisfying the usual conditions, and suppose that B is a standard Brownian motion with respect to $\mathcal{G}$ as well. A conclusion much stronger than (10) can be drawn if $\int_0^1 Y_t^2\,dt < \infty$ a.s., Y is $\mathcal{G}$-adapted, and the process

$$Z_t = \exp\Big(\int_0^t Y_s\,dB_s - (1/2)\int_0^t Y_s^2\,ds\Big)$$

is a $\mathcal{G}$-martingale. In this case, in fact, Girsanov theorem implies that S is a standard Brownian motion with respect to $\mathcal{G}$ under Q, where $Q(A) = E_{P_0}\big(Z_1\,I_A\big)$ for $A \in \mathcal{A}$. Unlike Girsanov theorem, however, condition (10) holds even if Y is not $\mathcal{G}$-adapted and/or Z fails to be a $\mathcal{G}$-martingale.

The second remark is that the above argument applies under (various) different assumptions. For instance, such an argument works if B is replaced by any symmetric α-stable Levy process. Or else, if the constant δ is replaced by a (suitable) $(0,1)$-valued random variable.

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