Basic ideas underlying conglomerability and disintegrability

Patrizia Berti

Dipartimento di Matematica Pura ed Applicata "G. Vitali", Universita' di Modena e Reggio-Emilia, via Campi 213/B, 41100 Modena, Italy. patrizia.berti@unimore.it

Enrique Miranda

Departamento de Estadística e I.O. y D.M., Universidad de Oviedo C-Federico García Lorca, 18, 33007 Oviedo, Spain. mirandaenrique@uniovi.es

Pietro Rigo

Corresponding author. Dipartimento di Matematica "F. Casorati", Universita' di Pavia, via Ferrata 1, 27100 Pavia, Italy. pietro.rigo@unipv.it

---

Abstract

The basic mathematical theory underlying the notions of conglomerability and disintegrability is reviewed. Both the precise and the imprecise cases are concerned.

Keywords: Coherence, conglomerability, disintegrability, finitely additive probability, lower prevision, natural extension.

---

1. Introduction

In time, conglomerability and disintegrability have been investigated by various authors, in different frameworks and under different assumptions. This produced a slightly chaotic situation with risk of misunderstandings. As a result, conglomerability/disintegrability may look much more involved than they are.

For instance, it is not always clear whether or not conglomerability implies disintegrability; see e.g. [1, page 206]. Instead, this issue is quite simple: It depends on the class of random variables where the various probability evaluations are defined. Or else, sometimes, one speaks about "conglomerability according to X" and "conglomerability according to Y", where X and Y are different authors, raising the doubt of the existence of more than one notion of conglomerability. There is only one notion, instead, which is merely applied in different settings. (The situation is more involved in the so called "imprecise case", introduced below).

This paper does not include new results but aims to bring out and make precise the basic ideas underlying conglomerability and disintegrability. We focus on the general (mathematical) theory more than on results concerning specific problems.

So far, we referred to the so called *precise* case. Roughly speaking, "precise" means that probability evaluations are (at least) finitely additive. Recently, however, there is a growing interest on the *imprecise* case, where finite additivity is not required and weaker types of probability evaluations come into play. Accordingly, in the second part of this paper, conglomerability/disintegrability are discussed in the imprecise case. There is however a notable difference with the precise case. Indeed, the general theory of conglomerability/disintegrability is essentially understood in the precise case, while it is still in progress in the imprecise case.

To deal with conglomerability/disintegrability implies some choices about the ingredients of the problem. In this paper, for the reasons explained in Subsection 2.1, we restrict to bounded random variables. At least in the precise case, probability measures are finitely additive and conditioning is based on de Finetti's coherence principle. Finally, integrals are meant in Dunford-Schwartz's sense [2]; see Subsection 2.1 again.

2. Basics

2.1. Integral and notational conventions

Throughout, Ω is a non-empty set and Π a partition of Ω. For any set I, we let $\mathcal{P}(I)$ denote the power set of I and $l^\infty(I)$ the class of real bounded functions on I. Moreover, the abbreviation *f.a.p.* stands for *finitely additive probability*.

A random variable is a real function on Ω (no measurability constraints are required). In this paper, for convenience, we restrict to *bounded* random variables. The main reason for that is the integral representation. In fact, a coherent function on a class $\mathcal{D}$ of bounded random variables can be written as the integral with respect to a finitely additive probability, but this useful fact is no longer true if $\mathcal{D}$ includes unbounded random variables; see [3], [4] and references therein. It should be stressed, however, that conglomerability/disintegrability can be developed for unbounded random variables as well. It suffices to broaden the notion of integral; see [4] again.

Since we restrict to bounded integrands, and since any bounded function is the uniform limit of simple functions, integrals with respect to f.a.p.'s are meant in the obvious way; see e.g. [2]. Fix in fact $X \in l^\infty(\Omega)$ and a f.a.p. μ on the σ-field generated by X. If X is simple, say $X = \sum_i a_i \, 1_{A_i}$, then $\int_\Omega X \, d\mu = \sum_i a_i \, \mu(A_i)$. Otherwise, $\int_\Omega X \, d\mu = \lim_n \int_\Omega X_n \, d\mu$ where X_n is a sequence of simple functions such that $X_n \to X$ uniformly.

We always denote by $\mathcal{D} \subset l^\infty(\Omega)$ a class of bounded random variables. In addition, three notational conventions are adopted. First, a set and its indicator are denoted by the same symbol. Thus, if $A \subset \Omega$, then A also designates the indicator function of the set A. Second, if $\mathcal{A}$ is a collection of subsets of Ω, we write $\mathcal{D} = \mathcal{A}$ (or $\mathcal{D} \supset \mathcal{A}$) to mean that $\mathcal{D}$ coincides with (or $\mathcal{D}$ includes) the class of indicators of the members of $\mathcal{A}$. Third, for each $S \subset \Pi$, we let S^* denote the subset of Ω obtained as union of the elements of S, namely, $S^* = \bigcup_{H \in S} H$. Roughly speaking, S and S^* are essentially the same set, but S is a subset of Π while S^* a subset of Ω.

2.2. Coherent conditional probabilities

Some claims in this subsection are without proofs. We refer to [5] and [6] for the latter.

A conditional bounded random variable, $X|H$, is the restriction of $X \in l^\infty(\Omega)$ to a (non-empty) subset $H \subset \Omega$. As usual, if $H = \Omega$, we write X instead of $X|\Omega$. Let $\mathcal{C}$ be any class of conditional bounded random variables and let P be a real function on $\mathcal{C}$. Then, P is *coherent* if, for all $n \geq 1$, $c_1, \ldots, c_n \in \mathbb{R}$ and $X_1|H_1, \ldots, X_n|H_n \in \mathcal{C}$, one obtains

$$\sup G|H \geq 0$$

where

$$G = \sum_{i=1}^n c_i \, H_i \left\{ X_i - P(X_i|H_i) \right\} \quad \text{and} \quad H = \bigcup_{i=1}^n H_i. \tag{1}$$

This is de Finetti's coherence principle. Indeed, for an arbitrary class $\mathcal{C}$ of conditional bounded random variables, de Finetti's ideas have been realized by various authors independently; see [5], [6], [7] and references therein.

Some more remarks are in order.

A coherent function P is also called a *prevision*. It is called a *conditional probability* when each element of $\mathcal{C}$ is of the form $A|H$, with $A, H \subset \Omega$ and $H \neq \emptyset$.

Since $c_1, \ldots, c_n$ are arbitrary constants, one also obtains

$$\inf G|H = -\sup -G|H \leq 0$$

whenever P is coherent and G and H are given by (1).

If P is coherent, the map $X \mapsto P(X|H)$ is a linear positive functional supported by H. More precisely, fix a non empty subset $H \subset \Omega$ and define $\mathcal{C}_H = \{X \in l^\infty(\Omega) : X|H \in \mathcal{C}\}$. Then,

$$\sup X|H \geq P(X|H) \geq \inf X|H \quad \text{and} \quad P\big(aX + bY|H\big) = a\,P(X|H) + b\,P(Y|H)$$

whenever $a, b \in \mathbb{R}$ and $X, Y, aX + bY$ belong to $\mathcal{C}_H$. In particular,

$$P(H|H) = 1 \qquad \text{provided } H \in \mathcal{C}_H.$$

Coherence of P admits various characterizations under some assumptions on $\mathcal{C}$. We just mention three cases. Let $\mathcal{A}$ be a field of subsets of Ω. Then,

(i) If $\mathcal{C} = \mathcal{A}$, then P is coherent if and only if it is a f.a.p. on $\mathcal{A}$;

(ii) If $\mathcal{C} = \mathcal{D}$, where $\mathcal{D} \subset l^\infty(\Omega)$ is a linear space including the constants, P is coherent if and only if it is a linear positive functional on $\mathcal{D}$ such that $P(\Omega) = 1$;

(iii) If $\mathcal{C} = \{A|H : A,\, H \in \mathcal{A},\, H \neq \emptyset\}$, then P is coherent if and only if
– $P(\cdot|H)$ is a f.a.p. on $\mathcal{A}$ and $P(H|H) = 1$ for every $H \in \mathcal{A} \setminus \{\emptyset\}$, and
– $P(A \cap B|C) = P(A|B \cap C)\, P(B|C)$ for all $A,\, B,\, C \in \mathcal{A}$ with $B \cap C \neq \emptyset$.

In case (iii), following Dubins [8], a coherent function P is also called a *full conditional probability.*

Let $\widetilde{\mathcal{C}} = \{X|H : X \in l^\infty(\Omega),\, \emptyset \neq H \subset \Omega\}$ be the class of all conditional bounded random variables. If P is coherent, then P can be coherently extended to $\widetilde{\mathcal{C}}$.

One consequence of such extension theorem is the integral representation. Suppose in fact P is coherent and $\mathcal{C} = \mathcal{D}$ for some $\mathcal{D} \subset l^\infty(\Omega)$. Take a coherent extension Q of P to $l^\infty(\Omega)$ and define $\mu(A) = Q(A)$ for all $A \subset \Omega$. Then, μ is a f.a.p. on $\mathcal{P}(\Omega)$ and it is straightforward to see that

$$P(X) = \int_\Omega X\, d\mu \quad \text{for all } X \in \mathcal{D}.$$

Finally, de Finetti's coherence principle admits a (nice) betting interpretation. A bet on $X|H$ with stake $c \in \mathbb{R}$ can be thought of as follows. Let $\omega \in \Omega$. If $\omega \notin H$, the bet is called off. Otherwise, if $\omega \in H$, one pays the price $c\, P(X|H)$ to receive $c\, X(\omega)$. Suppose now that P describes your feelings on the elements of $\mathcal{C}$. If you bet on $X_1|H_1, \ldots, X_n|H_n$ with stakes $c_1, \ldots, c_n$, your gain is $G|H$ where G and H are given by (1) (recall that all bets are called off on H^c). If you are not coherent, $\sup G|H < 0$ for some choice of $X_1|H_1, \ldots, X_n|H_n$ and $c_1, \ldots, c_n$. Thus, if you are not coherent, a bookie can find a finite combination of bets which makes you a sure loser.

2.3. Conglomerability and disintegrability

In this subsection, $\mathcal{D}$ is an arbitrary subclass of $l^\infty(\Omega)$, Π a partition of Ω and P a coherent function on

$$\mathcal{C} = \big\{X|H : X \in \mathcal{D},\, H \in \Pi \text{ or } H = \Omega\big\}.$$

As usual, we write $P(X)$ instead of $P(X|\Omega)$ whenever $X \in \mathcal{D}$.

(C) P is Π-*conglomerable* (or simply conglomerable) if

$$\inf_{H \in \Pi} P(X|H) \leq P(X) \leq \sup_{H \in \Pi} P(X|H) \quad \text{for each } X \in \mathcal{D}.$$

Condition (C) was first discussed by de Finetti in [9]. It is not hard to find examples where (C) fails. The following is a classical one.

Example 1. (P. Lévy). *Let* $\Omega = \mathbb{N} \times \mathbb{N}$*,* $\mathcal{D} = \mathcal{P}(\Omega)$ *and*

$$G_n = \{n\} \times \mathbb{N}, \quad H_n = \mathbb{N} \times \{n\}, \quad A = \{(i,j) : i \leq j\}.$$

Take a f.a.p. Q on $\mathcal{P}(\Omega)$ such that $Q\{\omega\} = 0$, $Q(G_n) > 0$ and $Q(H_n) > 0$ for all $\omega \in \Omega$ and $n \in \mathbb{N}$. If P is a coherent extension of Q, then

$$P(A|G_n) = Q(A \cap G_n)/Q(G_n) = 1 \quad \text{and} \quad P(A|H_n) = Q(A \cap H_n)/Q(H_n) = 0$$

for all n. Hence, P fails to be conglomerable in at least one of the partitions $\{G_n : n \in \mathbb{N}\}$ and $\{H_n : n \in \mathbb{N}\}$. ♦

We next turn to disintegrability.

(D) P is Π-*disintegrable* (or simply disintegrable) if there is a f.a.p. μ on $\mathcal{P}(\Pi)$ such that
$$P(X) = \int_{\Pi} P(X|H)\,\mu(dH) \quad \text{for each } X \in \mathcal{D}.$$

Condition (D) trivially holds if Π is finite (the so called theorem of total probability) and it is quite natural to investigate it for an arbitrary partition Π.

It is quite obvious that (D) implies (C). Under the present conditions (recall that $\mathcal{D}$ is arbitrary) the converse is not true. As discussed in Section 3, however, (C) implies (D) whenever $\mathcal{D}$ is a linear space.

Two other points are to be stressed.

(i) The role played by μ. Roughly speaking, (D) means that $P(X)$ can be reconstructed as a weighted mean of the conditional expectations $P(X|H)$. For doing this, however, the maps $H \mapsto P(X|H)$ are to be integrated for every $X \in \mathcal{D}$. Let $\mathcal{S}$ be the σ-field on Π generated by $H \mapsto P(X|H)$ for all $X \in \mathcal{D}$. Suppose P is disintegrable and

$$S^* \in \mathcal{D} \quad \text{for all } S \in \mathcal{S}. \tag{2}$$

Given $S \in \mathcal{S}$ and $H \in \Pi$, either $S^* \cap H = \emptyset$ and $P(S^*|H) = 0$ or $S^* \cap H = H$ and $P(S^*|H) = 1$. It follows that

$$\mu(S) = \int_{\Pi} P(S^*|H)\,\mu(dH) = P(S^*).$$

In other terms, if (2) holds, μ agrees with P on $\mathcal{S}$ and, with a slight abuse of notation, P can be regarded as a f.a.p. on $\mathcal{S}$. Thus, the maps $H \mapsto P(X|H)$ are P-integrable and (D) reduces to

$$P(X) = \int_{\Pi} P(X|H)\,P(dH) \quad \text{for each } X \in \mathcal{D}.$$

To summarize, under (2), there is no need to involve μ in condition (D). But in general condition (2) does not hold, and this is why μ comes into play.

(ii) Conglomerability/disintegrability for a coherent function Q on $\mathcal{D}$. As defined above, (C)-(D) require a coherent function P on $\mathcal{C}$. Sometimes, however, we are not given P but only a coherent function Q on $\mathcal{D}$. In that case,

Q is said to be conglomerable or disintegrable provided it admits a coherent extension P to $\mathcal{C}$ that is conglomerable or disintegrable. To develop this point further, a definition is needed.

A Π-*strategy* is a coherent function σ on $\mathcal{D}\times\Pi$. Equivalently, if $H,\, XH \in \mathcal{D}$ whenever $H \in \Pi$ and $X \in \mathcal{D}$ (as assumed in the rest of this subsection), a Π-strategy is a map $\sigma : \mathcal{D}\times\Pi \to \mathbb{R}$ such that $\sigma(H|H) = 1$ and $\sigma(\cdot|H)$ is a coherent function on $\mathcal{D}$ for every $H \in \Pi$.

Let Q be a coherent function on $\mathcal{D}$. To investigate conglomerability/disintegrability of Q, the first step is to select a Π-strategy σ and to define

$$P_\sigma(X) = Q(X) \quad \text{and} \quad P_\sigma(X|H) = \sigma(X|H) \quad \text{for all } X \in \mathcal{D} \text{ and } H \in \Pi.$$

Such a P_σ is an extension of Q to $\mathcal{C}$. Thus, to proceed further, coherence of P_σ is to be checked. Due to the simple structure of $\mathcal{C}$, it is an easy consequence of de Finetti's coherence principle that P_σ is coherent if and only if

$$Q(XH) = \sigma(X|H)\, Q(H) \quad \text{for all } X \in \mathcal{D} \text{ and } H \in \Pi;$$

see e.g. [10, Corollary 1.3] and [11, Corollary 2.6]. This fact has (at least) two consequences. First, if $Q(H) > 0$ for all $H \in \Pi$, the only choice of σ that makes P_σ coherent is $\sigma(X|H) = Q(XH)/Q(H)$. On the other hand, if $Q(H) = 0$ for all $H \in \Pi$, then P_σ is coherent for each Π-strategy σ. Second, P_σ is coherent if $Q(X) = \int_\Pi \sigma(X|H)\, \mu(dH)$, $X \in \mathcal{D}$, for some f.a.p. μ on $\mathcal{P}(\Pi)$. Hence, after selecting σ, the checking of coherence of P_σ can be skipped if one aims to Π-disintegrability of Q. In fact, coherence of P_σ is automatic if one is able to prove that $Q(X) = \int_\Pi \sigma(X|H)\, \mu(dH)$. Instead, the checking of coherence can not be eluded if one aims to Π-conglomerability of Q. In fact, it may be that P_σ is not coherent (so that Π-conglomerability of P_σ does not make sense) and yet

$$\inf_{H\in\Pi} \sigma(X|H) \le Q(X) \le \sup_{H\in\Pi} \sigma(X|H) \quad \text{for each } X \in \mathcal{D};$$

see Example 3.2 of [12].

Π-disintegrability of Q has been investigated in various papers for various partitions Π; see e.g. [13], [14], [15] and references therein. A related open problem is mentioned in Section 6.

2.4. Classical conditional probabilities

Classical (Kolmogorovian) conditional probabilities may fail to be coherent. This is quite expected, for classical conditional probabilities are essentially arbitrary on a null set. However, under some assumptions, classical conditional probabilities can be made coherent. In that case, they are also countably additive and Π-disintegrable, where Π is the partition in the atoms of the conditioning σ-field. In a nutshell, this is the heart of the matter.

A seminal paper on classical conditional probabilities is [16]. Further information can be drawn from [13], [17], [18], [19]. Here, we just give a quick summary of the issue.

Let $\mathcal{G} \subset \mathcal{A}$ be σ-fields on Ω and Q a countably additive probability on $\mathcal{A}$. A *regular conditional distribution* (for Q given $\mathcal{G}$) is a map K on $\Omega\times\mathcal{A}$ such that

- $K(\omega, \cdot)$ is a countably additive probability on $\mathcal{A}$ for fixed $\omega \in \Omega$,
- $\omega \mapsto K(\omega, A)$ is $\mathcal{G}$-measurable for fixed $A \in \mathcal{A}$,
- $Q(A \cap B) = \int_B K(\omega, A)\, Q(d\omega)$ for $A \in \mathcal{A}$ and $B \in \mathcal{G}$.

A regular conditional distribution K is *proper* if there is a set $B_0 \in \mathcal{G}$ such that $Q(B_0) = 1$ and

$$K(\omega, B) = \delta_\omega(B) \quad \text{for all } \omega \in B_0 \text{ and } B \in \mathcal{G}$$

where δ_ω denotes the unit mass at ω.

A regular conditional distribution can fail to exist and can fail to be proper whenever it exists. However, suppose Q admits a proper regular conditional distribution K. Suppose also that $\Pi \subset \mathcal{G}$ where Π is the partition of Ω in the atoms of $\mathcal{G}$. For each $H \in \Pi$, select a point $\omega_H \in H$ and define

$$P(A) = Q(A) \text{ and } P(A|H) = \begin{cases} K(\omega_H,\, A) & \text{if } \omega_H \in B_0, \\ \delta_{\omega_H}(A) & \text{if } \omega_H \notin B_0, \end{cases}$$

where $A \in \mathcal{A}$ and $H \in \Pi$.

Then, P is coherent, Π-disintegrable, and $P(\cdot|H)$ is countably additive for all $H \in \Pi \cup \{\Omega\}$.

3. Conglomerability versus disintegrability

Let $P : \mathcal{C} \to \mathbb{R}$ be coherent, where $\mathcal{C} = \big\{X|H : X \in \mathcal{D},\, H \in \Pi \text{ or } H = \Omega\big\}$ with $\mathcal{D} \subset l^\infty(\Omega)$ and Π a partition of Ω.

If $\mathcal{D}$ is a linear space, then (C) implies (D). This fact was firstly proved by Dubins [8, Theorem 1] when $\mathcal{D} = l^\infty(\Omega)$ and subsequently in [11, Theorem 3.1] for an arbitrary linear space $\mathcal{D}$. See also [4] for the case of unbounded random variables. Here, we report a short proof of [11, Theorem 3.1]. Such a proof highlights that the result is nothing but a consequence of de Finetti's coherence principle. The following (obvious) remark may help: If $\mathcal{Y} \subset l^\infty(\Pi)$ is a linear space, the null functional on $\mathcal{Y}$ is coherent if and only if $\sup Y \geq 0$ for each $Y \in \mathcal{Y}$; see also [20, Lemma 1].

Theorem 1. *Let $\mathcal{D}$ be a linear space. Then, P is Π-conglomerable if and only if it is Π-disintegrable.*

Proof. The "if" part is trivial. Suppose P is Π-conglomerable and define

$$Y_X(H) = P(X|H) - P(X) \quad \text{for all } X \in \mathcal{D} \text{ and } H \in \Pi.$$

Since $\mathcal{D}$ is a linear space, $\mathcal{Y} := \{Y_X : X \in \mathcal{D}\}$ is a linear space of bounded functions on Π. Since P is Π-conglomerable, $\sup_{H\in\Pi} Y_X(H) \geq 0$ for each $X \in \mathcal{D}$. Hence, the null functional on $\mathcal{Y}$ is coherent, and this implies that

$$0 = \int_\Pi Y_X(H)\, \mu(dH) = \int_\Pi P(X|H)\, \mu(dH) - P(X)$$

for all $X \in \mathcal{D}$ and some f.a.p. μ on $\mathcal{P}(\Pi)$. □

In general, if $\mathcal{D}$ is arbitrary, (D) implies (C) while the converse is not true. Indeed, it may be that (C) holds and (D) fails even if $\mathcal{D}$ is "very large" (but it is not a linear space). In the next example, P is conglomerable, $\mathcal{D} = \mathcal{P}(\Omega)$, and yet P is not disintegrable.

Example 2. (Example 3.3 of [12]). *Let* $\Omega = \mathbb{N}$, $\mathcal{D} = \mathcal{P}(\Omega)$ *and*

$$\Pi = \big\{\{2n-1,\, 2n\} : n \in \mathbb{N}\big\}.$$

Let $E = \{2n : n \in \mathbb{N}\}$ *be the set of even integers,* S *the collection of* $H \in \Pi$ *such that* $H = \{2^n - 1,\, 2^n\}$ *for some* $n \in \mathbb{N}$ *(note that* S *is a subset of* Π*) and*

$$\Omega_1 = E^c \cap S^*,\ \Omega_2 = E \cap S^*,\ \Omega_3 = E^c \cap (S^*)^c,\ \Omega_4 = E \cap (S^*)^c.$$

For each $A \subset \Omega$, *define*

$$A_0 = \{2n-1 : n \in \mathbb{N},\, 2n \in A\} \quad \text{and} \quad A^0 = \{2n : n \in \mathbb{N},\, 2n-1 \in A\}$$

and note that $(A_0)^0 = A \cap E$ *and* $(A^0)_0 = A \cap E^c$. *Define also*

$$P(A) = \sum_{i=1}^{4} \big\{\rho_i(A \cap \Omega_i) + \nu_i(A \cap \Omega_i)\big\} \quad \text{for all } A \subset \Omega,$$

where ρ_i *and* ν_i *are measures on* $\mathcal{P}(\Omega_i)$ *such that* ρ_i *is* σ*-additive,* ν_i *is finitely additive, and*

- $\rho_1\{\omega\} > 0$ *and* $\nu_1\{\omega\} = 0$ *for each* $\omega \in \Omega_1$, $\rho_1(\Omega_1) = 1/12$, $\nu_1(\Omega_1) = 1/6$;
- $\rho_2(A) = (1/2)\,\rho_1(A_0)$ *and* $\nu_2(A) = (1/4)\,\nu_1(A_0)$ *for each* $A \subset \Omega_2$;
- $\rho_3\{\omega\} > 0$ *and* $\nu_3\{\omega\} = 0$ *for each* $\omega \in \Omega_3$, $\rho_3(\Omega_3) = 1/12$, $\nu_3(\Omega_3) = 1/3$;
- $\rho_4(A) = (1/2)\,\rho_3(A_0)$ *and* $\nu_4(A) = (5/8)\,\nu_3(A_0)$ *for each* $A \subset \Omega_4$.

Since $P(H) > 0$ *for each* $H \in \Pi$, *the only coherent extension of* P *to* $\mathcal{C}$ *(still denoted by* P*) is* $P(A|H) = P(A \cap H)/P(H)$ *for all* $A \subset \Omega$ *and* $H \in \Pi$. *Such a* P *fails to be* Π*-disintegrable. In fact, since* $P(S^*) = 1/3$, $P(E|H) = 1/3$ *for each* $H \in \Pi$ *and condition* (2) *holds, one obtains*

$$\begin{aligned}\int_\Pi P(E \cap S^*|H)\,P(dH) &= \int_S P(E|H)\,P(dH)\\ &= (1/3)\,P(S^*) = 1/9 > 1/12 = P(E \cap S^*).\end{aligned}$$

It remains to show that P *is* Π*-conglomerable. It suffices to prove*

$$P(A) \geq \inf_{H \in \Pi} P(A|H) \quad \text{for all } A \subset \Omega. \tag{3}$$

In fact, under (3), *taking complements yields* $P(A) \leq \sup_{H\in\Pi} P(A|H)$ *for all* $A \subset \Omega$. *Fix* $A \subset \Omega$. *To prove* (3), *it can be assumed* $A \neq \Omega$ *and* $A \cap H \neq \emptyset$ *for* *each* $H \in \Pi$*. If* $H = \{2n-1, 2n\}$*, then* $P(A|H)$ *takes the values* $1/3$, $2/3$, 1 *according to* $A \cap H = \{2n\}$, $A \cap H = \{2n-1\}$, $A \cap H = H$*, respectively. If* $A \supset E^c$*, then*

$$P(A) \geq P(E^c) = 2/3 = \inf_{H \in \Pi} P(A|H)$$

where the last equality is because $A \neq \Omega$*. Otherwise, if* $A \cap E^c \neq E^c$*, then* $\inf_{H \in \Pi} P(A|H) = 1/3$*. Since* $A \cap H \neq \emptyset$ *for each* $H \in \Pi$*, one obtains*

$$(A \cap \Omega_1)^0 \cup (A \cap \Omega_2) = \Omega_2 \quad \textit{and} \quad (A \cap \Omega_3)^0 \cup (A \cap \Omega_4) = \Omega_4.$$

Further, $P(A \cap \Omega_i) \geq P\Big((A \cap \Omega_i)^0\Big)$ *if* $i = 1$ *or* $i = 3$*. It follows that*

$$\begin{aligned} P(A) &= \sum_{i=1}^{4} P(A \cap \Omega_i) \\ &\geq P\Big((A \cap \Omega_1)^0\Big) + P(A \cap \Omega_2) + P\Big((A \cap \Omega_3)^0\Big) + P(A \cap \Omega_4) \\ &\geq P(\Omega_2) + P(\Omega_4) = 1/3 = \inf_{H \in \Pi} P(A|H). \end{aligned}$$

Thus, condition (3) *holds, that is,* P *is* Π*-conglomerable.* ♦

We close this section by showing that (D) amounts to a stronger form of (C) provided $\mathcal{D}$ is large enough (but not necessarily a linear space).

Let $\mathcal{S}$ be the σ-field on Π generated by the maps $H \mapsto P(X|H)$ for all $X \in \mathcal{D}$, and let T be a coherent extension of P to $\mathcal{C} \cup \{X|S^* : X \in \mathcal{D}, S \in \mathcal{S}\}$. Suppose that

$$S^* \in \mathcal{D} \text{ and } XS^* \in \mathcal{D} \quad \text{for all } S \in \mathcal{S} \text{ and } X \in \mathcal{D}. \tag{4}$$

A (natural) strengthening of (C) is to require $T(\cdot|S^*)$ to be conglomerable for all $S \in \mathcal{S}$ with $P(S^*) > 0$. On the other hand, since $T(\cdot|S^*)$ is actually supported by S^*, conglomerability should be requested on S (and not on Π). On noting that $T(X|S^*) = P(XS^*)/P(S^*)$ if $P(S^*) > 0$, this leads to the condition

$$P(S^*) \inf_{H \in S} P(X|H) \leq P(XS^*) \leq P(S^*) \sup_{H \in S} P(X|H) \tag{5}$$

for all $X \in \mathcal{D}$ and $S \in \mathcal{S}$.

Condition (5) turns out to be exactly the right one. The next result slightly improves [10, Theorem 1.6].

Theorem 2. *Under condition* (4)*,* P *is* Π*-disintegrable if and only if it satisfies condition* (5)*.*

Proof. Since (4) implies (2), Π-disintegrability reduces to

$$P(X) = \int_{\Pi} P(X|H) \, P(dH)$$

for all $X \in \mathcal{D}$; see point (i) of Subsection 2.3. Hence, if P is Π-disintegrable, condition (5) follows from

$$P(XS^*) = \int_\Pi P(XS^*|H)\, P(dH) = \int_S P(X|H)\, P(dH)$$

for all $X \in \mathcal{D}$ and $S \in \mathcal{S}$. Conversely, suppose (5) holds and fix $X \in \mathcal{D}$. Given $\epsilon > 0$, since $H \mapsto P(X|H)$ is P-integrable (because of (4)) there is a finite partition $\{S_1, \ldots, S_n\} \subset \mathcal{S}$ of Π such that

$$\int_\Pi P(X|H)\, P(dH) < \epsilon + \sum_{i=1}^n P(S_i^*) \inf_{H \in S_i} P(X|H).$$

Thus, condition (5) yields

$$\int_\Pi P(X|H)\, P(dH) - \epsilon < \sum_{i=1}^n P(S_i^*) \inf_{H \in S_i} P(X|H) \leq \sum_{i=1}^n P(XS_i^*) = P(X).$$

Hence, $\int_\Pi P(X|H)\, P(dH) \leq P(X)$. The opposite inequality can be proved by exactly the same argument. □

4. Simultaneous conglomerability on several partitions

In this section, $\mathcal{A}$ is a σ-field of subsets of Ω and P is a f.a.p. on $\mathcal{A}$. Also, $\mathbb{B}$ denotes the collection of all countable partitions Π of Ω such that $\Pi \subset \mathcal{A}$.

By a classical result of Yosida and Hewitt [21], P can be uniquely written as

$$P = \alpha\, P_1 + (1 - \alpha)\, P_2,$$

where $\alpha \in [0,1]$, P_1 is a purely finitely additive f.a.p. and P_2 a σ-additive f.a.p. In the sequel, to stress the dependence on P, we write $\alpha(P)$ instead of α. We also recall that a f.a.p. Q on $\mathcal{A}$ is purely finitely additive if, for each $\epsilon > 0$, there is $\Pi \in \mathbb{B}$ such that $\sum_{H \in \Pi} Q(H) < \epsilon$.

Let P^* be a coherent extension of P to the class of all conditional bounded random variables. If $A \in \mathcal{A}$ and $\Pi \in \mathbb{B}$, then

$$\begin{aligned} P(A) &\geq \sum_{H \in \Pi} P(A \cap H) = \sum_{H \in \Pi} P^*(A|H)\, P(H) \\ &\geq \big\{1 - \alpha(P)\big\} \sum_{H \in \Pi} P^*(A|H)\, P_2(H) \geq \big\{1 - \alpha(P)\big\} \inf_{H \in \Pi} P^*(A|H) \end{aligned}$$

where P_2 is the σ-additive part of P. Taking complements, one obtains

$$P(A) - \big\{1 - \alpha(P)\big\} \sup_{H \in \Pi} P^*(A|H) \leq \alpha(P).$$

It follows that

$$\beta(P^*) := \sup_{A \in \mathcal{A}, \Pi \in \mathbb{B}} \Big\{P(A) - \sup_{H \in \Pi} P^*(A|H)\Big\} \leq \alpha(P).$$

The number $\beta(P^*)$ quantifies the degree of non-conglomerability of P^* on the partitions belonging to $\mathbb{B}$. In particular, $\beta(P^*) = 0$ if and only if P^* is Π-conglomerable for every $\Pi \in \mathbb{B}$. Usually, $\beta(P^*)$ is called the *extent of non-conglomerability of* P. The terminology may look inappropriate, for $\beta(P^*)$ apparently depends on P^* and not on P. This is not the case, however, thanks to a (nice) result of Schervish-Seidenfeld-Kadane [1]; see also [22].

Theorem 3. *If P is a f.a.p. on a σ-field $\mathcal{A}$, then*

$$\beta(P^*) = \alpha(P)$$

for every coherent extension P^* of P.

Theorem 3 can be strengthened under the assumption that P takes infinitely many values. Suppose in fact this is true and define $\mathbb{B}_0 = \{\Pi \in \mathbb{B} : P(H) > 0$ for each $H \in \Pi\}$. Then, [1, Theorem 3.1] implies that

$$\alpha(P) = \sup_{A \in \mathcal{A}, \Pi \in \mathbb{B}_0} \Big\{ P(A) - \sup_{H \in \Pi} \frac{P(A \cap H)}{P(H)} \Big\}.$$

Trivially, $\beta(P^*) = 0$ whenever P is σ-additive. de Finetti has over and over maintained that Π-conglomerability on a "wide" class of (countable) partitions Π implies σ-additivity. de Finetti's insight is endorsed by Theorem 3. In fact, if P^* is Π-conglomerable on every $\Pi \in \mathbb{B}$, then $\alpha(P) = \beta(P^*) = 0$, which clearly amounts to σ-additivity of P.

We finally mention an analogous of Theorem 3 which holds for an arbitrary (infinite) cardinal k. Say that P is k-additive if $P(\cup_i A_i) = \sup_i P(A_i)$ whenever $\cup_i A_i \in \mathcal{A}$, (A_i) is an increasing collection of elements of $\mathcal{A}$, and the cardinality of (A_i) is less than or equal to k. Under some conditions (including k not weakly inaccessible), if P is not k-additive then P fails to be Π-conglomerable in some partition Π such that $\text{card}(\Pi) \le k$; see [23].

5. The imprecise case

In recent decades, there has been a growing interest in extending some notions and results from probability theory to deal better with situations where the information is vague or scarce. This has given rise to a number of mathematical models that are often gathered under the common term *imprecise probabilities*; see [24] for a review.

5.1. Conditional lower previsions

A first extension of de Finetti's coherence principle to the imprecise case was established by Peter Williams [7].

Definition 1. *[Williams' coherence] Let $\mathcal{C}$ be any class of conditional bounded random variables and $\underline{P}$ a real function on $\mathcal{C}$. Then, $\underline{P}$ is* coherent *if, for all $n \geq 1$, real numbers $c_i \geq 0$ and $X_i|H_i \in \mathcal{C}$ $(i = 0, 1, \ldots, n)$, one obtains*

$$\sup G|H \geq 0$$

where

$$G = \sum_{i=1}^{n} c_i\, H_i \left\{ X_i - \underline{P}(X_i|H_i) \right\} - c_0\, H_0 \left\{ X_0 - \underline{P}(X_0|H_0) \right\} \text{ and } H = \bigcup_{i=0}^{n} H_i.$$

A coherent function $\underline{P}$ is also called a *coherent conditional lower prevision.* It is called a coherent conditional lower *probability* if each element of $\mathcal{C}$ is of the form $A|H$, with $A, H \subset \Omega$ and $H \neq \emptyset$. Further, it is called a *coherent lower prevision* when $H = \Omega$ for every $X|H \in \mathcal{C}$.

As shown in [7], coherent conditional lower previsions are related to full conditional probabilities, as defined in Subsection 2.2, in the sense that the latter can be seen as (linear) previsions that are coherent in the sense above; see also [25, Section 3.1].

Fix $H \subset \Omega$, $H \neq \emptyset$, and define $\mathcal{C}_H = \{X \in l^\infty(\Omega) : X|H \in \mathcal{C}\}$. If $\underline{P}$ is coherent, then

$$\begin{aligned} &\inf X|H \leq \underline{P}(X|H) \leq \sup X|H, \quad \underline{P}(\lambda X|H) = \lambda\, \underline{P}(X|H) \\ &\text{and } \ \underline{P}(X+Y|H) \geq \underline{P}(X|H) + \underline{P}(Y|H) \end{aligned}$$

whenever $\lambda > 0$ and X, Y, λX, $X + Y$ belong to $\mathcal{C}_H$.

A real function $\underline{P}$ on $\mathcal{C}$ is coherent if and only if

$$\underline{P}(X|H) = \inf_{P \in \mathcal{M}(\underline{P})} P(X|H) \quad \text{for all } X|H \in \mathcal{C} \tag{6}$$

where $\mathcal{M}(\underline{P}) = \{P \text{ prevision} : P \geq \underline{P} \text{ on } \mathcal{C}\}$; see [7]. Condition (6) allows to attach coherent conditional lower previsions a Bayesian sensitivity analysis interpretation. One more consequence of (6) is that if $\{\underline{P}_\lambda : \lambda \in \Lambda\}$ is any family of coherent conditional lower previsions, then its lower envelope

$$\underline{Q}(X|H) = \inf_{\lambda \in \Lambda} \underline{P}_\lambda(X|H)$$

is still coherent.

As in the precise case, a coherent conditional lower prevision can be coherently extended to $\{X|H : X \in l^\infty(\Omega), \emptyset \neq H \subset \Omega\}$, the set of all conditional bounded random variables. In particular, a coherent lower probability on $\mathcal{P}(\Omega)$ can be extended to a coherent lower prevision on $l^\infty(\Omega)$. However, and unlike the precise case, the extension is not unique in general. For this reason, we distinguish between lower previsions (the expectation operators) and the less informative lower probabilities (their restrictions to indicators).

The betting interpretation mentioned in Subsection 2.2 is essentially preserved in the imprecise case, but with some modifications. This point is discussed in the next subsection, for it requires the notion of coherent set of desirable gambles.

5.2. Desirability and coherent lower previsions

The theory sketched above can be developed introducing coherent sets of *desirable* bounded random variables, also called *gambles* in this context.

Definition 2. *A subset $\mathcal{D}$ of $l^\infty(\Omega)$ is called a* coherent set of desirable gambles *when it satisfies the following properties:*

- $0 \notin \mathcal{D}$,
- $X \gneq 0 \Rightarrow X \in \mathcal{D}$,
- $X \in \mathcal{D}, \lambda > 0 \Rightarrow \lambda X \in \mathcal{D}$,
- $X, Y \in \mathcal{D} \Rightarrow X + Y \in \mathcal{D}$.

We refer to [24, Chapter 1] and [26, Section 3.7] for more on desirable gambles. Here we just note that, if the members of $\mathcal{D}$ are regarded as gains from transactions, the above axioms may be seen as rationality criteria for the acceptability of such transactions.

Given a coherent set $\mathcal{D}$ of desirable gambles, a coherent lower prevision on $l^\infty(\Omega)$ can be obtained by means of the formula

$$\underline{P}(X) = \sup\{p : X - p \in \mathcal{D}\}. \tag{7}$$

Conversely, given a coherent lower prevision $\underline{P}$ on $l^\infty(\Omega)$, the set of gambles

$$\left\{X : \underline{P}(X) > 0 \ \text{ or } \ X \gneq 0\right\} \tag{8}$$

is coherent and induces $\underline{P}$ via equation (7). The correspondence between coherent lower previsions and coherent sets of desirable gambles is not one-to-one, however, as different coherent sets of desirable gambles may be associated with the same coherent lower prevision (the set in (8) is just one of them). In this sense, desirable gambles are a more informative model.

We are now able to attach a (nice) interpretation to coherent lower previsions.

In the precise case, given $X \in l^\infty(\Omega)$, a coherent assessment $P(X)$ can be regarded as a *fair* price for X, in the sense that we are disposed to accept both transactions $\{X - P(X)\}$ and $\{P(X) - X\}$. In fact, we are willing to accept $c\,\{X - P(X)\}$ where the constant c has arbitrary sign.

In the imprecise case, instead, we should distinguish between the acceptable *buying* prices and the acceptable *selling* prices for X. The first are those prices p such that $\{X - p\}$ is considered to be a desirable transaction (i.e. it belongs to a coherent set of desirable gambles) while the second are those prices p' such that $\{p' - X\}$ is considered desirable.

With this in mind, formula (7) says that the coherent lower prevision $\underline{P}(X)$ is the supremum acceptable buying price for X. Similarly, the upper prevision $\overline{P}(X) := -\underline{P}(-X)$ would be the infimum acceptable selling price for X. Since buying X is equivalent to selling $-X$, however, all the assessments can be made in terms of lower previsions.

The lower and upper previsions for X do not agree in general. If $\underline{P}(X) < \overline{P}(X)$, there are some prices for which we would neither buy nor sell X. When $\underline{P}(X) = \overline{P}(X)$, this common value is the prevision or fair price for X.

These ideas extend similarly to the conditional case, as discussed in Subsection 5.1.

Next, exploiting Definition 2, Williams' notion of coherence for lower previsions (Definition 1) can be also given a betting interpretation, analogous to that of De Finetti's coherence principle. Fix in fact a coherent set $\mathcal{D}$ of desirable gambles and define $\underline{P}(X|H) = \sup\{p : H(X-p) \in \mathcal{D}\}$ for all $X \in l^\infty(\Omega)$ and $H \subset \Omega$ with $H \neq \emptyset$. By the properties of $\mathcal{D}$, the transaction $H(X-p)$ is acceptable for $p < \underline{P}(X|H)$ while it is not for $p > \underline{P}(X|H)$. In addition, a positive linear combination of acceptable transactions is still acceptable, as it is any transaction more profitable than an acceptable one. It follows that $\sup G|H \geq 0$, where

$$G = \sum_{i=1}^{n} c_i\, H_i \left\{X_i - \underline{P}(X_i|H_i)\right\} - c_0\, H_0 \left\{X_0 - \underline{P}(X_0|H_0)\right\} \quad \text{and} \quad H = \bigcup_{i=0}^{n} H_i$$

are as in Definition 1. In fact, if $\sup G|H < 0$, there is $\delta > 0$ such that

$$c_0\, H_0 \left\{X_0 - \underline{P}(X_0|H_0) - \delta\right\} \geq \sum_{i=1}^{n} c_i\, H_i \left\{X_i - \underline{P}(X_i|H_i) + \delta\right\}.$$

Define the transactions

$$T_0 = c_0\, H_0 \left\{X_0 - \underline{P}(X_0|H_0) - \delta\right\}, \quad T = \sum_{i=1}^{n} c_i\, H_i \left\{X_i - \underline{P}(X_i|H_i) + \delta\right\},$$

and note that T is acceptable, being a positive linear combination of acceptable transactions. Therefore, we get a contradiction:

- If $c_0 = 0$, then $T \leq 0$. Thus, $T \notin \mathcal{D}$ contrary to its acceptability.
- If $c_0 > 0$, since $T_0 \geq T$ and T is acceptable, T_0 is acceptable as well. Thus, $\underline{P}(X_0|H_0) + \delta$ is an acceptable buying price for $X_0|H_0$, even if strictly greater than the supremum acceptable buying price $\underline{P}(X_0|H_0)$.

Finally, the interpretation underlying coherence is finitary, in the sense that a positive finite combination of acceptable transations is required to be acceptable, but nothing is requested about infinite sums of acceptable transactions. Walley argued that an infinite sum of acceptable transactions should be acceptable in one particular case: when no two of the transactions can act simultaneously, because they involve different events of a partition Π of Ω. In fact, given a partition Π, a coherent set $\mathcal{D}$ of desirable gambles is called Π-*conglomerable* if

$$HX \in \mathcal{D} \cup \{0\} \text{ for each } H \in \Pi \Rightarrow X \in \mathcal{D} \cup \{0\}. \tag{9}$$

Condition (9) was called the *conglomerative principle* by Walley [27, Section 6]. We refer to [28, Section 3] and [29, Section 6] for a study of this notion.

5.3. Conglomerability for conditional lower previsions

The first studies of conglomerability for conditional lower previsions were made by Peter Walley [26]. These studies motivated a notion of coherence, stronger than the one in Definition 1, referred to as Walley-coherence in this paper. Let Π be a partition of Ω and $\underline{P}$ a real function on

$$\mathcal{C} = \{X|H : X \in l^{\infty}(\Omega), H \in \Pi \text{ or } H = \Omega\}.$$

The notions introduced below are easily extendable to the case where X ranges in a (possibly non-linear) subset $\mathcal{D}$ of $l^{\infty}(\Omega)$; see [31] and [26].

For $X \in l^{\infty}(\Omega)$, define

$$G(X|\Pi) = \sum_{H \in \Pi} H\big\{X - \underline{P}(X|H)\big\}.$$

Also, let us call $\underline{P}_1$ the restriction of $\underline{P}$ to $l^{\infty}(\Omega)$ and $\underline{P}_{\Pi}$ the restriction of $\underline{P}$ to $\{X|H : X \in l^{\infty}(\Omega), H \in \Pi\}$. Then, $\underline{P}$ is *Walley-coherent* if:

(a) $\underline{P}_1$ and $\underline{P}_{\Pi}$ are each of them coherent in the sense of Definition 1,

(b) $\underline{P}\big(H(X - \underline{P}(X|H))\big) = 0$ for every $X|H \in \mathcal{C}$,

(c) $\underline{P}\big(G(X|\Pi)\big) \geq 0$ for every $X \in l^{\infty}(\Omega)$.

Condition (b) is usually called *generalised Bayes' rule*. Under (b), condition (a) could be replaced by asking $\underline{P}$ to be a coherent conditional lower prevision. In fact, conditions (a)-(b) imply that $\underline{P}$ is coherent in Williams' sense [7, 32].

The motivation behind condition (c) lies in Walley's conglomerative principle (condition (9)): if we are disposed to accept the transaction $H\big\{X - \underline{P}(X|H)\big\}$, no matter which $H \in \Pi$ turns out to occur, then their sum $G(X|\Pi)$ should also be viewed as acceptable, for only one of its building blocks will come into play. See also [29, 30].

Let $\underline{P}_1$ be a coherent lower prevision on $l^{\infty}(\Omega)$. According to Walley, $\underline{P}_1$ is Π-*conglomerable* if there exists a coherent conditional lower prevision $\underline{P}_{\Pi}$ on $\{X|H : X \in l^{\infty}(\Omega), H \in \Pi\}$ such that $\underline{P}_1$ and $\underline{P}_{\Pi}$ are Walley-coherent. More precisely, the function $\underline{P}$ on $\mathcal{C}$ given by

$$\underline{P}(X) = \underline{P}_1(X) \quad \text{and} \quad \underline{P}(X|H) = \underline{P}_{\Pi}(X|H), \quad X \in l^{\infty}(\Omega),\ H \in \Pi,$$

is Walley-coherent. This means that the unconditional model $\underline{P}_1$ can be updated to a conditional one and this conditional model satisfies Walley's notion of coherence with respect to $\underline{P}_1$.

Conglomerability can be characterized in a number of ways, some of which are summarized in the following theorem. Given $H \subset \Omega$ and a coherent lower prevision $\underline{P}_1$ on $l^\infty(\Omega)$, let us define the *conditional natural extension* of $\underline{P}_1$ to be the function $\underline{E}_H$ on $\{X|H : X \in l^\infty(\Omega)\}$ given by

$$\underline{E}_H(X|H) = \begin{cases} \inf X|H & \text{if } \underline{P}_1(H) = 0 \\ \inf\{P_1(HX)/P_1(H) : P_1 \geq \underline{P}_1,\ P_1 \text{ prevision}\} & \text{if } \underline{P}_1(H) > 0. \end{cases}$$

We also let $\underline{E}_\Pi(X|H) = \underline{E}_H(X|H)$ for each $H \in \Pi$. Such $\underline{E}_\Pi$, also called conditional natural extension of $\underline{P}_1$, is a coherent conditional lower prevision on $\{X|H : X \in l^\infty(\Omega), H \in \Pi\}$.

Theorem 4. *[26, Chapter 6] Let $\underline{P}_1$ be a coherent lower prevision on $l^\infty(\Omega)$. The following are equivalent:*

(i) $\underline{P}_1$ is Π-conglomerable;

(ii) If $X \in l^\infty(\Omega)$ and $\underline{P}_1(XH) \geq 0$ for each $H \in \Pi$ such that $\underline{P}_1(H) > 0$, then $\underline{P}_1(X) \geq 0$;

(iii) $\underline{P}_1$ is Walley-coherent with its conditional natural extension $\underline{E}_\Pi$.

Theorem 4-(ii) underlines the aforementioned interpretation of conglomerability. It takes into account Walley's *updating principle* [26, Section 6.1.6], that renders equivalent the desirability of X conditional on the observation of H with the desirability of the contingent random variable XH.

Theorem 4-(ii) also implies that only those $H \in \Pi$ with positive lower probability matter for Walley's notion of conglomerability. This is not equivalent to considering all $H \in \Pi$. As a trivial example, take a prevision P_1 on $l^\infty(\Omega)$ such that $P_1(H) = 0$ for all $H \in \Pi$. Then, P_1 is Π-conglomerable by Theorem 4-(ii). On the other hand, it is not true that $P_1(X) \geq 0$ if $P_1(XH) \geq 0$ for all $H \in \Pi$. If X is the constant -1, for instance, $P_1(X) < 0$ even if $P_1(XH) = 0$ for all $H \in \Pi$.

Thus, since at most countably many disjoint events may have positive lower probability, to investigate conglomerability we may restrict to countable partitions. This is one of the main differences with the precise case (recall the comments at the end of Section 4).

We next briefly mention *disintegrability* of coherent lower previsions. Let $\underline{P}_1$ be a coherent lower prevision on $l^\infty(\Omega)$ and $\underline{E}_\Pi$ its conditional natural extension. Define

$$\underline{P}(X|\Pi) = \sum_{H\in\Pi} H\, \underline{E}_\Pi(X|H) \quad \text{for } X \in l^\infty(\Omega).$$

In line with the precise theory (Subsection 2.3) it is reasonable to say that $\underline{P}_1$ is Π-disintegrable if

$$\underline{P}_1(X) = \underline{P}_1\big(\underline{P}(X|\Pi)\big) \quad \text{for all } X \in l^\infty(\Omega).$$

However, despite $l^\infty(\Omega)$ being a linear space, Π-conglomerability does not amount to Π-disintegrability. In fact, the coherence of $\underline{P}_1$ and the definition of $\underline{E}_\Pi$ yield

$$\underline{P}_1\big(X - \underline{P}_1(X)\big) = 0 \quad \text{and} \quad \underline{P}_1\big(H(X - \underline{E}_\Pi(X|H))\big) = 0$$

for all $X \in l^\infty(\Omega)$ and $H \in \Pi$. Therefore, Theorem 4-(iii) implies that $\underline{P}_1$ is Π-conglomerable if and only if $\underline{P}_1\big(X - \underline{P}(X|\Pi)\big) \geq 0$ for all $X \in l^\infty(\Omega)$, or equivalently $\underline{P}_1(X) \geq \underline{P}_1\big(\underline{P}(X|\Pi)\big)$ for all $X \in l^\infty(\Omega)$. But the equality need not hold in general; see e.g. the prevision in [26, Example 6.6.10].

Finally, Walley's notion of conglomerability applies, in particular, to previsions P_1 with domain $l^\infty(\Omega)$. Once again, Π-conglomerability of P_1 is equivalent to Walley-coherence of P_1 with its conditional natural extension $\underline{E}_\Pi$. However, $\underline{E}_\Pi$ need not be a precise model, namely, it may be that

$$\underline{E}_H(X|H) \neq -\underline{E}_H(-X|H) = \overline{E}_H(X|H) \quad \text{for some } X \in l^\infty(\Omega) \text{ and } H \in \Pi.$$

It follows from its definition that $\underline{E}_H$ is precise when $P_1(H) > 0$ (in which case it is given by Bayes' rule) and $\underline{E}_H$ is imprecise when $P_1(H) = 0$. In fact, there are examples where Π-conglomerable previsions are not Walley-coherent with any conditional prevision [26, Example 6.6.10]. However, if $P_1(H) > 0$ for every $H \in \Pi$, one obtains

$$P_1 \text{ is } \Pi\text{-conglomerable} \quad \Leftrightarrow \quad P_1(X) = P_1\big(\underline{P}(X|\Pi)\big) \text{ for each } X \in l^\infty(\Omega).$$

We refer to [25, Section 4] for some results on the connections between disintegrability and Walley's notion of conglomerability for previsions P_1 with domain $l^\infty(\Omega)$. In particular, under some assumptions on the cardinality of Ω, these notions become equivalent when they are required to hold with respect to *all* partitions (what we call full conglomerability in Subsection 5.6).

5.4. Mathematical properties

As shown in [26, Section 2.6], the class of coherent lower previsions on $l^\infty(\Omega)$ is closed under convex combinations, point-wise limits, and lower envelopes. However, these properties do not necessarily hold for the subclass of Π-conglomerable lower previsions.

First of all, the point-wise limit of a sequence of Π-conglomerable lower previsions need not be Π-conglomerable; see [33, Example 1] and [26, Example 6.6.7]. Similarly, a convex combination of Π-conglomerable models need not be Π-conglomerable; see [1, Example 4.1].

The situation is slightly more delicate for lower envelopes. In fact, the lower envelope of a family of Π-conglomerable lower previsions is still Π-conglomerable [26, Theorem 6.9.3], but there are Π-conglomerable lower previsions that are not dominated by any Π-conglomerable prevision.

Example 3. (Example 6.6.9 of [26]). *Let* $\Omega = \mathbb{N} \cup -\mathbb{N}$ *and* $\Pi = \{\{-n, n\} : n \in \mathbb{N}\}$. *Define* $\underline{P}_1 = \min\{Q_1, Q_2\}$, *where* Q_1 *and* Q_2 *are previsions on* $l^\infty(\Omega)$ *such that*

$$Q_1(\mathbb{N}) = 1/2 = Q_1(-\mathbb{N}), \quad Q_1\{n\} = 2^{-(n+1)}, \quad Q_1\{-n\} = 0,$$
$$Q_2(\mathbb{N}) = 1/2 = Q_2(-\mathbb{N}), \quad Q_2\{n\} = 0, \quad Q_2\{-n\} = 3^{-n},$$

for all $n \in \mathbb{N}$*. Fix* $X \in l^\infty(\Omega)$*. If* $H = \{-n, n\}$*, then*

$$\underline{P}_1(XH) = \min\{Q_1(XH), Q_2(XH)\} = \min\{2^{-(n+1)}X(n), 3^{-n}X(-n)\}.$$

Hence, $\underline{P}_1(XH) \geq 0$ *for all* $H \in \Pi$ *if and only if* $X \geq 0$*, which in turn implies* $\underline{P}_1(X) \geq 0$*. By Theorem 4-(ii),* $\underline{P}_1$ *is* Π*-conglomerable.*

However, even if $\underline{P}_1$ *is* Π*-conglomerable, no prevision* P *dominating* $\underline{P}_1$ *is* Π*-conglomerable. To prove this fact, note that any such* P *can be expressed as* $P = \lambda Q_1 + (1-\lambda)Q_2$ *for some* $\lambda \in [0, 1]$*. If* $\lambda = 0$*, then* $P = Q_2$ *and* Q_2 *is not* Π*-conglomerable, since* $Q_2(\mathbb{N}) = 1/2$ *and* $Q_2(\mathbb{N}|H) = 0$ *for all* $H \in \Pi$*. On the other hand, if* $\lambda > 0$*, then*

$$P\big(\{n\}|\{-n, n\}\big) = \frac{P\{n\}}{P\{n\} + P\{-n\}} = \frac{\lambda}{\lambda + 2\,(1-\lambda)\,(2/3)^n} \longrightarrow 1$$

as $n \to \infty$*. Define* $C = \{m \in \mathbb{N} \cup -\mathbb{N} : m \geq -n^*\}$*, where* $n^* \in \mathbb{N}$ *is such that* $P\big(\{n\}|\{-n, n\}\big) \geq 1 - (\lambda/4)$ *for each* $n \geq n^*$*. Then,*

$$P(C) = \lambda Q_1(C) + (1-\lambda)Q_2(C) \leq 1 - \frac{\lambda}{2} \quad \text{and} \quad P(C|H) \geq 1 - \frac{\lambda}{4} \text{ for all } H \in \Pi,$$

so that P *is not* Π*-conglomerable.* ♦

We conclude this subsection remarking that the envelopes of the set of conglomerable extensions of a strategy and a prior probability are characterized in [34], while the notions of conglomerability/disintegrability for the so-called full T-conditional measures are investigated in [35]. Full T-conditional measures arise in possibility theory [36], and constitute an extension of full conditional probabilities to this framework. Again, a formula for the envelopes of conglomerable extensions is established.

5.5. Conglomerable natural extension

The term *natural extension* is used in imprecise probability theory to refer to the procedure of determining the closest model to a given one that satisfies some properties. For instance, if a real function $\underline{P}_1$ on $l^\infty(\Omega)$ admits a dominating coherent lower prevision, the natural extension of $\underline{P}_1$ is the smallest dominating coherent lower prevision, or equivalently the lower envelope of $\mathcal{M}(\underline{P}_1)$:

$$\underline{E}(X) = \min\{P(X) : P \in \mathcal{M}(\underline{P}_1)\} \quad \text{for each } X \in l^\infty(\Omega).$$

Moreover, $\underline{P}_1$ is coherent if and only if coincides with its natural extension.

In a similar vein, given a coherent lower prevision $\underline{P}_1$ on $l^\infty(\Omega)$, its Π-*conglomerable natural extension* is the smallest Π-conglomerable lower prevision $\underline{Q}_1$ such that $\underline{Q}_1 \geq \underline{P}_1$. This was investigated in some detail in [28, 33]. However, the conglomerable natural extension suffers from a number of problems.

The first problem is its existence: it is not trivial to characterise those coherent lower previsions that are not Π-conglomerable but admit a Π-conglomerable natural extension. In [33, Section 3], a number of necessary or sufficient conditions for the existence of the conglomerable natural extension were established, but none of them is simultaneously necessary *and* sufficient. This contrasts with the natural extension considered above, that exists if and only if the credal set $\mathcal{M}(\underline{P}_1)$ is non-empty (in which case, $\underline{P}_1$ is said *to avoid sure loss*).

On the other hand, when the Π-conglomerable natural extension exists, there is not an easy way to compute it. As deduced from Example 3, it can not be generally obtained as a lower envelope of Π-conglomerable previsions. Another approach would be considering Walley's notion of natural extension of unconditional and conditional assessments [26, Section 8.1]. However, this natural extension is not Π-conglomerable in general, and produces only a conservative approximation of the Π-conglomerable natural extension [28, Example 5]. What this means is that, even if the notion of Walley-coherence is taking conglomerability into account, it is not doing so in its full extent, and that to model conglomerability in the imprecise case it is better to take conglomerability for sets of desirable gambles (i.e., condition (9)) as primary notion, as in [30]. This approach also has the advantage that it allows us to overcome the restriction of positive lower probabilities that eventually leads to the consideration of countable partitions, as discussed after Theorem 4.

If we stick to the notion established by Walley for lower previsions, we can instead consider an iterative approximation to the conglomerable natural extension, by means of an increasing sequence of coherent lower previsions obtained by means of Walley's notion of natural extension. Nevertheless, this sequence may not stabilise in a finite number of steps [33, Example 4]. A number of sufficient conditions for its convergence were established in [28, 33].

5.6. Simultaneous conglomerability with respect to several partitions

Given a family $\mathbb{D}$ of partitions of Ω, a coherent lower prevision $\underline{P}_1$ on $l^\infty(\Omega)$ is $\mathbb{D}$-conglomerable if it is Π-conglomerable for every $\Pi \in \mathbb{D}$. This means that, for each $\Pi \in \mathbb{D}$, there exists a coherent conditional lower prevision $\underline{P}_\Pi$ on $\{X|H : X \in l^\infty(\Omega), H \in \Pi\}$ such that $\underline{P}_1$ and $\underline{P}_\Pi$ are Walley-coherent.

This notion has been considered in [37], [28, Section 7] and [25, Section 6.1]. In particular, $\underline{P}_1$ is said to be *fully conglomerable* if it is Π-conglomerable for *every* partition Π of Ω. As can be expected, fully conglomerable previsions are quite related to countably additive ones.

Theorem 5. *[26, Theorems 6.9.1 and 6.9.2] Let P be a prevision on $l^\infty(\Omega)$. If P is countably additive, then it is fully conglomerable. The converse holds when the restriction of P to $\mathcal{P}(\Omega)$ takes an infinite number of values.*

Theorem 5 is analogous to Theorem 3 in the special case $\mathcal{A} = \mathcal{P}(\Omega)$. When making the comparison, recall that Walley's notion of conglomerability (the one discussed in this subsection) involves countable partitions only, since only conditioning events with positive probability are considered.

The situation is less clear for coherent lower previsions. The problem was studied recently in [25]. In addition to the family of fully conglomerable coherent lower previsions, denoted by M in what follows, two other subfamilies were considered:

$$M_1 = \{\text{lower envelopes of countably additive previsions}\},$$

$$M_2 = \{\text{lower envelopes of fully conglomerable previsions}\}.$$

It holds that

$$M_1 \subset M_2 \subset M,$$

the inclusions being strict. Moreover, some continuity and σ-additivity properties for coherent lower previsions were also studied. Consider the conditions:

(a) $X_n \to X \Rightarrow \underline{P}(X_n) \to \underline{P}(X)$,

(b) $X_n \downarrow X \Rightarrow \underline{P}(X_n) \downarrow \underline{P}(X)$,

(c) $X_n \downarrow 0 \Rightarrow \underline{P}(X_n) \downarrow 0$,

(d) $X_n \uparrow X \Rightarrow \underline{P}(X_n) \uparrow \underline{P}(X)$,

(e) $\underline{P}\left(\sum_n X_n\right) \geq \sum_n \underline{P}(X_n)$ if $X_n \in l^\infty(\Omega)$ for each n and $\sum_n X_n \in l^\infty(\Omega)$.

The connection between full conglomerability and such conditions is summarised in the following graph:

$$\begin{array}{ccccccc}
(a) & \Rightarrow & (d) & \Rightarrow & \underline{P} \in M_1 & & \\
 & & & \swarrow & & \searrow & \\
 & & (b) & & & & \underline{P} \in M_2 \\
 & & \Downarrow & & & \swarrow & \\
(c) & \Leftarrow & (e) & \Rightarrow & \underline{P} \in M. & &
\end{array}$$

It should be noted that, although there are a number of sufficient conditions for full conglomerability, none of them is simultaneously sufficient *and* necessary.

6. Concluding remarks

In this paper, to make the exposition clearer, the precise case (PC) and the imprecise case (IC) are handled separately and PC is discussed before IC. However, the paper could have been organized differently, starting with IC and arriving to PC. This alternative formulation has some merits. In fact, PC can be seen as a particular case of IC. Furthermore, conglomerability arises quite naturally within IC, via the notion of coherent set of desirable gambles from Subsection 5.2.

To begin with PC, in turn, can be motivated by essentially two reasons. The first is of the historical type. Not only PC was developed long before IC, but the latter was strongly affected by the former. The second reason is that, as noted in Section 1, conglomerability/disintegrability are quite settled within PC. There are of course some open problems (one is mentioned below) but the general theory seems to be essentially understood. On the contrary, as regards

IC, conglomerability/disintegrability are still in progress and some work is to be done.

We next state the (announced) *open problem* on PC. Let $\Omega = [0,1]$, $\mathcal{D}$ the Borel σ-field and Q the Lebesgue measure on $\mathcal{D}$. An intriguing question, raised by Dubins and Prikry in [15], is whether or not Q is Π-disintegrable on *every* Borel partition Π of $[0,1]$.

Let us turn now to IC. Here, de Finetti's theory has been extended mostly in two ways, by Peter Williams [7] and by Peter Walley [26]. The difference between the two approaches lies precisely in the conglomerative principle, that allows to consider the infinite sum of acceptable transactions involving different events of a partition of Ω. In Williams' approach this sum need not be acceptable, while for Walley it should be. Then, Walley's notion of a conglomerable lower prevision means it can be updated into a conditional model while at the same time satisfying the conglomerative principle.

As we have seen, some of the mathematical properties from PC do not extend straightforwardly to IC. For instance, conglomerability cannot be characterised by conditions of countable additivity or continuity. Also, it is not easy to accommodate conglomerability together with the notion of natural extension, which is one of the pillars of the theory of coherent lower previsions and is the tool for making conservative inferences. Thus, although the use of conglomerability has again been recently advocated [29], some work remains in order to make it fully operational.

Acknowledgements

The research in this paper has been supported by the Spanish Ministry of Economy and Competitiveness [grant number TIN2014-59543-P]. We also thank two anonymous referees for their useful remarks.

References

[1] M.J. Schervish, T. Seidenfeld, J. Kadane, The extent of non-conglomerability of finitely additive probabilities, Z. Wahrsch. Verw. Gebiete, 66 (1984) 205-226.

[2] N. Dunford, J.T. Schwartz, Linear Operators, Part I: General Theory, Wiley-Interscience, New York, 1958.

[3] P. Berti, E. Regazzini, P. Rigo, Strong previsions of random elements, Stat. Methods Appl. 10 (2001) 11-28.

[4] M.J. Schervish, T. Seidenfeld, J. Kadane, On the equivalence of conglomerability and disintegrability for unbounded random variables, Stat. Methods Appl. 23 (2014) 501-518.

[5] E. Regazzini, Finitely additive conditional probabilities, Rend. Sem. Mat. Fis. Milano 55 (1985) 69-89. Corrections in: Rend. Sem. Mat. Fis. Milano 57 (1987) 599.

[6] E. Regazzini, de Finetti's coherence and statistical inference, Ann. Statist., 15 (1987) 845-864.

[7] P. Williams, Notes on conditional previsions, Technical Report, University of Sussex, 1975. Reprinted in: Int. J. App. Reasoning 44 (2007) 366-383.

[8] L.E. Dubins, Finitely additive conditional probabilities, conglomerability and disintegrations, Ann. Probab. 3 (1975) 88-99.

[9] B. de Finetti, Sulla proprieta' conglomerativa delle probabilita' subordinate, Rend. Real. Inst. Lomb. 63 (1930) 414-418.

[10] P. Berti, E. Regazzini, P. Rigo, Coherent statistical inference and Bayes theorem, Ann. Statist. 19 (1991) 366-381.

[11] P. Berti, P. Rigo, Weak disintegrability as a form of preservation of coherence, J. Italian Stat. Soc. 1 (1992) 161-181.

[12] P. Berti, P. Rigo, Conglomerabilita', disintegrabilita' e coerenza, Serie Ricerche Teoriche n. 11, Dipartimento Statistico, Universita' di Firenze, 1989.

[13] P. Berti, P. Rigo, Sufficient conditions for the existence of disintegrations, J. Theore. Probab. 12 (1999) 75-86.

[14] P. Berti, P. Rigo, On coherent conditional probabilities and disintegrations, Ann. Math. Artif. Intell. 35 (2002) 71-82.

[15] L.E. Dubins, K. Prikry, On the existence of disintegrations, in: Seminaire de Probab. XXIX, Lect. Notes in Math., Springer, 1613 (1995) 248-259.

[16] D. Blackwell, L.E. Dubins, On existence and nonexistence of proper, regular, conditional distributions, Ann. Probab. 3 (1975) 741-752.

[17] P. Berti, P. Rigo, 0-1 laws for regular conditional distributions, Ann. Probab. 35 (2007) 649-662.

[18] D. Blackwell, On a class of probability spaces, in: Proc. Third Berkeley Symp. Math. Stat. Prob., University of California Press, 1955 pp.1-6.

[19] D. Blackwell, C. Ryll-Nardzewski, Non-existence of everywhere proper conditional distributions, Ann. Math. Statist. 34 (1963) 223-225.

[20] D. Heath, W.D. Sudderth, On finitely additive priors, coherence and extended admissibility, Ann. Statist., 6 (1978) 333-345.

[21] K. Yosida, E. Hewitt, Finitely additive measures, Trans. Amer. Math. Soc., 72 (1952) 46-66.

[22] A. Zame, Conglomerability and finite partitions, Proc. Amer. Math. Soc. 102 (1988) 165-168.

[23] M.J. Schervish, T. Seidenfeld, J. Kadane, Nonconglomerability for countably additive measures that are not k-additive, Rev. Symb. Logic, 10 (2017) 284-300.

[24] T. Augustin, F. Coolen, G. de Cooman, M.C.M. Troffaes, Introduction to imprecise probabilities, Wiley, Chichester, 2014.

[25] E. Miranda, M. Zaffalon, Full conglomerability, J. Stat. Theo. Practice, in press (2017). doi: http://dx.doi.org/10.1080/15598608.2017.1295890

[26] P. Walley, Statistical reasoning with imprecise probabilities, Chapman and Hall, London, 1991.

[27] P. Walley, Towards a unified theory of imprecise probability, Int. J. Appr. Reas. 24(2000) 125-148.

[28] E. Miranda, M. Zaffalon, G. de Cooman, Conglomerable natural extension, Int. J. Appr. Reas. 53 (2012) 1200-1227.

[29] M. Zaffalon, E. Miranda, Probability and time, Art. Intellig. 198 (2013) 1-51.

[30] E. Miranda, M. Zaffalon, Conglomerable coherence, Int. J. Appr. Reas. 54 (2013) 1322-1350.

[31] E. Miranda, G. de Cooman, Coherence and independence for non-linear spaces, Tech. Rep. 05/10, Rey Juan Carlos University, 2005.

[32] R. Pelessoni, P. Vicig, Williams coherence and beyond, Int. J. Appr. Reas., 50 (2009) 612-626.

[33] E. Miranda, M. Zaffalon, On the problem of computing the conglomerable natural extension, Int. J. Appr. Reas. 56 (2015) 1-27.

[34] D. Petturiti, B. Vantaggi, Envelopes of conditional probabilities extending a strategy and a prior probability, Int. J. Appr. Reas. 81 (2017) 160-182.

[35] G. Coletti, D. Petturiti, Finitely maxitive conditional possibilities, Bayesian-like inference, disintegrability and conglomerability, Fuzz. Sets and Syst. 284 (2016) 31-55.

[36] D. Dubois, H. Prade, Possibility theory, Plenum Press, New York, 1988.

[37] G. de Cooman, F. Hermans, Coherent immediate prediction: bridging two theories of imprecise probability, Art. Intellig. 172 (2008) 1400-1427.