FINITELY ADDITIVE MIXTURES OF PROBABILITY MEASURES

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Abstract. Let D be a linear space of real bounded functions and $P : D \to \mathbb{R}$ a coherent functional. Also, let $\mathcal{Q}$ be a collection of coherent functionals on D. Under mild conditions, there is a finitely additive probability Π on the power set of $\mathcal{Q}$ such that $P(f) = \int_{\mathcal{Q}} Q(f)\,\Pi(dQ)$ for each $f \in D$. This fact has various consequences and such consequences are investigated in this paper. Three types of results are provided: (i) Existence of common extensions satisfying certain properties, (ii) Finitely additive mixtures of extreme points, (iii) Countably additive mixtures.

1. Introduction

Let Ω be a set and $l^\infty(\Omega)$ the collection of real *bounded* functions on Ω. Let D be a linear subspace of $l^\infty(\Omega)$ and $P : D \to \mathbb{R}$ a coherent functional. (If D includes the constants, coherence just means that P is linear positive and $P(1) = 1$; see Section 2). Also, fix a collection $\mathcal{Q}$ of coherent functionals on D and denote by $\Sigma(\mathcal{Q})$ the σ-field over $\mathcal{Q}$ generated by the maps $Q \mapsto Q(f)$ for all $f \in D$. Then,

$$P(f) \geq \inf_{Q \in \mathcal{Q}} Q(f) \quad \text{for each } f \in D \tag{1}$$

if and only if

$$P(f) = \int_{\mathcal{Q}} Q(f)\,\Pi(dQ) \quad \text{for all } f \in D \tag{2}$$

and some finitely additive probability Π on $\Sigma(\mathcal{Q})$; see Lemma 1.

The equivalence between (1) and (2) is the starting point of this paper. Indeed, even if apparently innocuous, such equivalence has various consequences and this paper investigates (some of) them. Three types of results are provided.

- *Common extensions* (Theorem 3). Let P_i be a coherent functional on a linear subspace $D_i \subset l^\infty(\Omega)$, where i ranges over some index set I. A common extension of $(P_i : i \in I)$ is a coherent functional P on $l^\infty(\Omega)$ such that $P = P_i$ on D_i for each $i \in I$. Here, we give conditions for the existence of a common extension satisfying some additional properties. For instance, fix $F \subset l^\infty(\Omega)$ and, for each $f \in F$, take $G_f \subset l^\infty(\Omega)$. Then, we give conditions for the existence of a common extension P such that

$$P(f) \leq P(g) \quad \text{for all } f \in F \text{ and } g \in G_f.$$

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In particular, if $f \in F$ and $a \le b$ are given constants, one obtains $a \le P(f) \le b$ choosing a suitable G_f; see Remark (ii). Moreover, instead of such inequalities, the common extension P can be asked to satisfy any other requirement which is preserved under mixtures; see Remark (iii).

- *Finitely additive mixtures of extreme points* (Theorem 9). Take $\mathcal{Q}$ to be the set of extreme points of a collection $\mathcal{R}$ of coherent functionals on D. If $\mathcal{R}$ is convex and closed under pointwise convergence (see Section 4) condition (2) amounts to $P \in \mathcal{R}$. Apart from the quoted conditions on $\mathcal{R}$, this result does not require any other assumption.
- *Countably additive mixtures* (Theorem 11 and its consequences). Suppose that P and each $Q \in \mathcal{Q}$ are integrals with respect to σ-additive probability measures. In this case, if condition (2) holds, it is natural to investigate wether Π can be taken to be σ-additive. Hence, conditions for Π to be σ-additive are given. Such conditions are then applied to disintegrability and invariant probability measures. The same conditions can be also exploited to get simple proofs of some classical known results. One first proves condition (1), so that P is a finitely additive mixture of the elements of $\mathcal{Q}$, and then shows that Π can be taken to be σ-additive. This strategy actually works in Example 12 and Theorem 17.

2. Basics on de Finetti's coherence

Let $\mathcal{F}$ be a field of subsets of Ω and μ a finitely additive probability (f.a.p.) on $\mathcal{F}$. A function $f \in l^\infty(\Omega)$ is μ-integrable if $f_n \xrightarrow{\mu} f$ for some sequence f_n of $\mathcal{F}$-simple functions; see [6, Chap. 4]. In that case, $\int f\,d\mu := \lim_n \int f_n\,d\mu$ where the μ-integral of a simple function is defined in the obvious way. In what follows, $\int f\,d\mu$ is also denoted by $\mu(f)$.

For any topological space S, the Borel σ-field on S is denoted by $\mathcal{B}(S)$. We also adopt the usual convention

$$\inf \emptyset = \infty.$$

Let us turn to de Finetti's coherence principle. Let $F \subset l^\infty(\Omega)$ and $P : F \to \mathbb{R}$. Then, P is *coherent* if

$$\sup \sum_{i=1}^n \alpha_i \{ f_i - P(f_i) \} \ge 0$$

for all $n \ge 1$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $f_1, \ldots, f_n \in F$.

Heuristically, suppose P describes your previsions on the members of F. If P is coherent, it is impossible to make you a sure loser, whatever $\omega \in \Omega$ turns out to be true, by some finite combinations of bets (on $f_1, \ldots, f_n$ with stakes $\alpha_1, \ldots, \alpha_n$).

Here, F is an arbitrary subset of $l^\infty(\Omega)$. Under some assumptions on F, however, coherence reduces to some simpler condition. For instance, if F is a linear space including the constants, P is coherent if and only if it is linear positive and $P(1) = 1$. Similarly, if $F = \{1_A : A \in \mathcal{F}\}$ where $\mathcal{F}$ is a field of subsets of Ω, then P is coherent if and only if $A \mapsto P(1_A)$ is a f.a.p. on $\mathcal{F}$.

If P is coherent, by Hahn-Banach theorem, P can be extended to a coherent functional T on $l^\infty(\Omega)$. Let $\mu(A) = T(1_A)$ for each $A \subset \Omega$. Then, μ is a f.a.p. and

$$T(f) = \int f\,d\mu \quad \text{for all } f \in l^\infty(\Omega).$$

Thus, P is coherent if and only if $P(f) = \int f\,d\mu$, $f \in F$, for some f.a.p. μ on the power set of Ω.

Let $D \subset l^\infty(\Omega)$ be a linear subspace. Is there a f.a.p. μ on the power set of Ω such that $\int f\,d\mu = 0$ for each $f \in D$? Various problems in probability theory can be actually reduced to such a question, possibly requesting μ to be σ additive on $\sigma(D)$ (i.e., on the σ-field generated by D); see [3] and [4]. Anyway, the answer is surprisingly simple: Such a μ is available if and only if

$$\sup f \geq 0 \quad \text{for every } f \in D.$$

Define in fact $P(f) = 0$ for all $f \in D$. Since D is a linear space, the above condition implies that P is coherent. Thus, $0 = P(f) = \int f\,d\mu$, $f \in D$, for some f.a.p. μ.

The previous remark allows to prove the equivalence mentioned in Section 1.

Lemma 1. *Conditions* (1) *and* (2) *are equivalent provided $D \subset l^\infty(\Omega)$ is a linear subspace, P a coherent functional on D and $\mathcal{Q}$ a collection of coherent functionals on D.*

Proof. It is trivial that (2) $\Rightarrow$ (1). Conversely, assume condition (1) and define $Y_f(Q) = P(f) - Q(f)$ for $f \in D$ and $Q \in \mathcal{Q}$. Then, $\{Y_f : f \in D\}$ is a linear space of real bounded functions on $\mathcal{Q}$. By condition (1),

$$\sup_{Q \in \mathcal{Q}} Y_f(Q) = P(f) - \inf_{Q \in \mathcal{Q}} Q(f) \geq 0 \quad \text{for each } f \in D.$$

Thus, there is a f.a.p. Λ on the power set of $\mathcal{Q}$ such that

$$P(f) - \int_{\mathcal{Q}} Q(f)\,\Lambda(dQ) = \int_{\mathcal{Q}} Y_f\,d\Lambda = 0 \quad \text{for all } f \in D.$$

Then, condition (2) holds with Π the restriction of Λ on $\Sigma(\mathcal{Q})$. (Recall that $\Sigma(\mathcal{Q})$ is the σ-field over $\mathcal{Q}$ generated by the maps $Q \mapsto Q(f)$ for all $f \in D$). □

Let D be a linear subspace of $l^\infty(\Omega)$ including the constants and P a coherent functional on D. The inner and outer functionals associated to P (also known as lower and upper previsions [16, 17]) are

$$P_*(f) = \sup\{P(h) : h \in D,\, h \leq f\} \quad \text{and} \quad P^*(f) = \inf\{P(h) : h \in D,\, h \geq f\}$$

where $f \in l^\infty(\Omega)$. They are connected through $P_*(f) = -P^*(-f)$, and each coherent extension T of P to $l^\infty(\Omega)$ satisfies

$$P_* \leq T \leq P^*.$$

A nice feature of coherence is that the above extremes are actually attained. This fact is essentially known (see e.g. [7] and [16, 17]) but we provide a simple proof to make the paper self-contained.

Lemma 2. *Let $D \subset l^\infty(\Omega)$ be a linear subspace including the constants and P a coherent functional on D. For each $f \in l^\infty(\Omega)$, there is a coherent functional T on $l^\infty(\Omega)$ such that $T = P$ on D and $T(f) = P_*(f)$.*

Proof. First note that P^* is a Minkowski functional, namely, $P^*(\lambda\phi) = \lambda P^*(\phi)$ and $P^*(\phi + \psi) \leq P^*(\phi) + P^*(\psi)$ whenever $\phi, \psi \in l^\infty(\Omega)$ and the scalar λ is non-negative. Since $P^* = P$ on D, by Hahn-Banach theorem, there is a linear functional $T : l^\infty(\Omega) \to \mathbb{R}$ such that

$$T = P \text{ on } D, \quad T(-f) = P^*(-f), \quad T \leq P^*.$$

If $h \in l^\infty(\Omega)$ and $h \geq 0$, then

$$T(h) = -T(-h) \geq -P^*(-h) = P_*(h) \geq 0.$$

Thus, T is a linear positive functional such that $T(1) = P(1) = 1$, namely, T is coherent. Finally, $T(f) = -T(-f) = -P^*(-f) = P_*(f)$. $\square$

3. Common extensions

In this section, I is an arbitrary index set and, for each $i \in I$, D_i is a linear subspace of $l^\infty(\Omega)$ and P_i a coherent functional on D_i.

Theorem 3. *Let $F \subset l^\infty(\Omega)$ and, for $f \in F$, let $G_f \subset l^\infty(\Omega)$. Define $\mathcal{Q}$ to be the collection of all coherent functionals Q on $l^\infty(\Omega)$ such that*

$$Q(f) \leq Q(g) \quad \text{for all } f \in F \text{ and } g \in G_f.$$

There is $P \in \mathcal{Q}$ such that $P = P_i$ on D_i for all $i \in I$ if and only if

$$\sum_{i \in J} P_i(f_i) \geq \inf_{Q \in \mathcal{Q}} \sum_{i \in J} Q(f_i) \tag{3}$$

whenever $J \subset I$ is finite and $f_i \in D_i$ for each $i \in J$.

Proof. Let $P \in \mathcal{Q}$ be such that $P = P_i$ on D_i for all $i \in I$. Given a finite subset $J \subset I$ and $f_i \in D_i$ for each $i \in J$, one trivially obtains

$$\sum_{i \in J} P_i(f_i) = P\Big(\sum_{i \in J} f_i\Big) \geq \inf_{Q \in \mathcal{Q}} Q\Big(\sum_{i \in J} f_i\Big) = \inf_{Q \in \mathcal{Q}} \sum_{i \in J} Q(f_i).$$

Conversely, under condition (3), one obtains $\mathcal{Q} \neq \emptyset$ (since $\inf \emptyset = \infty$). Let

$$D = \Big\{ \sum_{i \in J} f_i : J \subset I \text{ finite, } f_i \in D_i \text{ for each } i \in J \Big\}.$$

Fix $f \in D$. If $f = \sum_{i \in J_1} f_i$ and $f = \sum_{k \in J_2} g_k$, with $J_1 \subset I$ and $J_2 \subset I$ finite, $f_i \in D_i$ and $g_k \in D_k$ for all $i \in J_1$ and $k \in J_2$, condition (3) yields

$$\sum_{i \in J_1} P_i(f_i) - \sum_{k \in J_2} P_k(g_k) \geq \inf_{Q \in \mathcal{Q}} Q\Big(\sum_{i \in J_1} f_i - \sum_{k \in J_2} g_k\Big) = 0.$$

Similarly, $\sum_{k \in J_2} P_k(g_k) - \sum_{i \in J_1} P_i(f_i) \geq 0$, so that $\sum_{i \in J_1} P_i(f_i) = \sum_{k \in J_2} P_k(g_k)$. Hence, one can define

$$T(f) = \sum_{i \in J} P_i(f_i) \quad \text{for all } f = \sum_{i \in J} f_i \in D.$$

Using (3) again, it follows that T is coherent on D and $T(f) \geq \inf_{Q \in \mathcal{Q}} Q(f)$ for all $f \in D$. By Lemma 1, there is a f.a.p. Π on $\Sigma(\mathcal{Q})$ such that

$$T(f) = \int_{\mathcal{Q}} Q(f)\, \Pi(dQ) \quad \text{for each } f \in D.$$

Finally, take a finitely additive extension Λ of Π to the power set of $\mathcal{Q}$ and define $P(f) = \int_{\mathcal{Q}} Q(f)\, \Lambda(dQ)$ for all $f \in l^\infty(\Omega)$. Then, $P \in \mathcal{Q}$. Further, since $P = T$ on D, one obtains $P(f) = T(f) = P_i(f)$ whenever $i \in I$ and $f \in D_i$. $\square$

Some remarks are in order. Define $\mathcal{Q}$ as in Theorem 3.

(i) $\mathcal{Q}$ may be empty. In this case, however, condition (3) fails because of the convention $\inf \emptyset = \infty$. The same comment applies to the rest of this paper.

(ii) Fix $f \in l^\infty(\Omega)$, two constants $a \le b$, and suppose that $\{f, -f\} \subset F$. Take $G_f = \{f_b\}$ and $G_{-f} = \{-f_a\}$, where f_c denotes the constant function $f_c(\omega) = c$ for all $\omega \in \Omega$. Then, each $Q \in \mathcal{Q}$ satisfies $a \le Q(f) \le b$.

(iii) Theorem 3 is still valid for other choices of $\mathcal{Q}$. Say that a certain property is *preserved under mixtures* if such property holds for $P(\cdot) = \int_{\mathcal{R}} Q(\cdot)\, \Pi(dQ)$ provided it holds for every $Q \in \mathcal{R}$, where $\mathcal{R}$ is any collection of coherent functionals and Π a f.a.p. on the power set of $\mathcal{R}$. Then, Theorem 3 works with $\mathcal{Q}$ the set of all coherent functionals on $l^\infty(\Omega)$ satisfying any set of properties which are preserved under mixtures.

(iv) If $F = \emptyset$, then $\mathcal{Q}$ is the set of all coherent functionals on $l^\infty(\Omega)$ and

$$\inf_{Q\in\mathcal{Q}} \sum_{i\in J} Q(f_i) = \inf_{Q\in\mathcal{Q}} Q\Big(\sum_{i\in J} f_i\Big) = \inf_{\omega\in\Omega} \sum_{i\in J} f_i(\omega).$$

Two special cases are to be stressed. First, condition (3) is connected to the *avoiding sure loss* condition, introduced by Walley [16] in the framework of imprecise probabilities. Second, Theorem 3 reduces to the following result.

Corollary 4. *There is a coherent functional P on $l^\infty(\Omega)$ such that $P = P_i$ on D_i for each $i \in I$ if and only if*

$$\sum_{i\in J} P_i(f_i) \ge \inf_{\omega\in\Omega} \sum_{i\in J} f_i(\omega)$$

whenever $J \subset I$ is finite and $f_i \in D_i$ for all $i \in J$.

Proof. Apply Theorem 3 with $F = \emptyset$. □

It is worth noting that common extensions of linear positive functionals have been studied for a long time. Thus, various characterizations similar to Corollary 4 are already available; see e.g. [13] and references therein.

In turn, Corollary 4 implies a classical result by Guy [11]; see also [6, page 82].

Corollary 5. *Let μ be a f.a.p. on $\mathcal{F}$ and ν a f.a.p. on $\mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ are fields of subsets of Ω. There is a f.a.p. γ on the power set of Ω such that $\gamma = \mu$ on $\mathcal{F}$ and $\gamma = \nu$ on $\mathcal{G}$ if and only if*

$$\mu(A) \le \nu(B) \quad \textit{whenever } A \in \mathcal{F},\ B \in \mathcal{G} \textit{ and } A \subset B. \tag{4}$$

Proof. By Corollary 4, it suffices to see that $\mu(f) \le \nu(g)$ whenever f is a $\mathcal{F}$-simple function, g a $\mathcal{G}$-simple function, and $f \le g$. In fact, $f \le g$ implies $\{f > t\} \subset \{g > t\}$ and $\{g < -t\} \subset \{f < -t\}$ for each $t > 0$. Hence, condition (4) yields

$$\mu(f) = \int_0^\infty \big\{\mu(f > t) - \mu(f < -t)\big\}\, dt \le \int_0^\infty \big\{\nu(g > t) - \nu(g < -t)\big\}\, dt = \nu(g).$$

□

It is well known that μ and ν may fail to admit a common σ-additive extension to $\sigma(\mathcal{F} \cup \mathcal{G})$ even if they are σ-additive and condition (4) holds. We next provide a simple example of this fact. In such example, in addition to (4) and σ-additivity of μ and ν, various other conditions are satisfied. In fact, Ω is a compact metric space and $\mathcal{F}$ and $\mathcal{G}$ are countably generated sub-σ-fields of $\mathcal{B}(\Omega)$ such that $\mathcal{F} \cap \mathcal{G} = \{\emptyset, \Omega\}$.

Example 6. Let $\mathcal{I}=[0,1]$, $H=\{(x,y)\in\mathcal{I}^2:x>y\}$ and

$$\varphi(C)=m\{x\in\mathcal{I}:(x,x^2)\in C\}\quad\text{for each } C\in\mathcal{B}(\mathcal{I}^2),$$

where m is the Lebesgue measure on $\mathcal{B}(\mathcal{I})$. Then,

$$\varphi(H)=1\quad\text{and}\quad\varphi\big(A\times\mathcal{I}\big)=m(A)\quad\text{for each } A\in\mathcal{B}(\mathcal{I}).$$

Define

$$\Omega=\mathcal{I}^2,\quad \mathcal{F}=\sigma\Big(\{H,\,A\times\mathcal{I}:A\in\mathcal{B}(\mathcal{I})\}\Big)\quad\text{and}\quad\mathcal{G}=\{\mathcal{I}\times A:A\in\mathcal{B}(\mathcal{I})\}.$$

Then, $\mathcal{F}$ and $\mathcal{G}$ are countably generated sub-σ-field of $\mathcal{B}(\Omega)$ and $\mathcal{F}\cap\mathcal{G}=\{\emptyset,\Omega\}$. Take $\mu=\varphi|\mathcal{F}$ and $\nu(\mathcal{I}\times A)=m(A)$ for all $A\in\mathcal{B}(\mathcal{I})$.

As noted in [15, Example 2], there is a f.a.p. γ on $\mathcal{B}(\Omega)$ such that $\gamma(H)=1$ and $\gamma(A\times\mathcal{I})=\gamma(\mathcal{I}\times A)=m(A)$ for all $A\in\mathcal{B}(\mathcal{I})$. Thus, condition (4) is satisfied. Toward a contradiction, suppose now that μ and ν admit a common σ-additive extension ρ to $\sigma(\mathcal{F}\cup\mathcal{G})=\mathcal{B}(\Omega)$. Then,

$$\int_\Omega(x-y)\,\rho(dx,dy)=\int_0^1 x\,dx-\int_0^1 y\,dy=0.$$

But this is a contradiction, for ρ is σ-additive and $\rho(H)=\mu(H)=1$.

We close this section with a last application of Theorem 3. Next result improves [5, Theorem 7].

Corollary 7. *Let λ be a f.a.p. on a field $\mathcal{E}$ of subsets of Ω and*

$$\mathcal{K}=\{A\in\mathcal{E}:\lambda(A)=1\}.$$

There is a coherent functional P on $l^\infty(\Omega)$ such that

$$P=P_i \text{ on } D_i \text{ for all } i\in I \text{ and } P(1_A)=1 \text{ for each } A\in\mathcal{K}$$

if and only if

$$\sum_{i\in J}P_i(f_i)\geq\inf_{\omega\in A}\sum_{i\in J}f_i(\omega)\tag{5}$$

whenever $A\in\mathcal{K}$, $J\subset I$ is finite, and $f_i\in D_i$ for each $i\in J$.

Proof. Let $\mathcal{Z}$ be the set of all functions $Z:l^\infty(\Omega)\to\mathbb{R}$ satisfying

$$\inf f\leq Z(f)\leq\sup f\quad\text{for each } f\in l^\infty(\Omega).$$

When equipped with the product topology, $\mathcal{Z}$ is compact and convergence on $\mathcal{Z}$ is pointwise convergence. Having noted this fact, assume condition (5). Fix $A\in\mathcal{K}$ and define $\mathcal{Q}_A$ to be the set of those coherent functionals Q on $l^\infty(\Omega)$ such that $Q(1_A)=1$. Because of condition (5) and Theorem 3,

$$\mathcal{M}_A:=\big\{P\in\mathcal{Q}_A:P=P_i\text{ on }D_i\text{ for all }i\in I\big\}$$

is not empty. Thus, $\{\mathcal{M}_A:A\in\mathcal{K}\}$ is a collection of closed subsets of $\mathcal{Z}$ such that

$$\bigcap_{i=1}^n\mathcal{M}_{A_i}=\mathcal{M}_{\cap_{i=1}^n A_i}\neq\emptyset\quad\text{for all } n\geq 1\text{ and } A_1,\ldots,A_n\in\mathcal{K}.$$

Since $\mathcal{Z}$ is compact, it follows that

$$\bigcap_{A\in\mathcal{K}}\mathcal{M}_A\neq\emptyset.$$

This concludes the proof of the "if" part, and the "only if" part is trivial. $\square$

Example 8. (Common absolutely continuous extensions). Let μ, ν, λ be f.a.p.'s on the fields $\mathcal{F}$, $\mathcal{G}$, $\mathcal{E}$, respectively. Define $\mathcal{K}$ as in Corollary 7. There is a f.a.p. γ on the power set of Ω such that

$$\gamma = \mu \text{ on } \mathcal{F}, \quad \gamma = \nu \text{ on } \mathcal{G}, \quad \text{and} \quad \gamma(A) = 0 \text{ whenever } A^c \in \mathcal{K}$$

if and only if

$$\begin{aligned} \mu(A) \leq \nu(B) \quad &\text{whenever } A \in \mathcal{F},\, B \in \mathcal{G} \\ &\text{and } A \cap K \subset B \cap K \text{ for some } K \in \mathcal{K}. \end{aligned} \tag{6}$$

In fact, by Corollary 7, it suffices to see that $\mu(f) \leq \nu(g)$ whenever f is a $\mathcal{F}$-simple function, g is a $\mathcal{G}$-simple function, and $f \leq g$ on K for some $K \in \mathcal{K}$. Under (6), this can be shown exactly as in the proof of Corollary 5.

4. Finitely additive mixtures of extreme points

Recall that, for any subset H of a linear space, x is an *extreme point* of H if $x \in H$ and $x \neq \alpha y + (1-\alpha)z$ for all $\alpha \in (0,1)$ and y, $z \in H$ with $y \neq z$.

Let $\mathcal{F}$ be a field of subsets of Ω and $D_{\mathcal{F}}$ the linear space of $\mathcal{F}$-simple functions. Also, let $\mathbb{Q}$ be the set of all f.a.p.'s on $\mathcal{F}$. As usual, for any $\mathcal{Q} \subset \mathbb{Q}$, we denote by $\Sigma(\mathcal{Q})$ the σ-field over $\mathcal{Q}$ generated by the maps $Q \mapsto Q(A)$ for all $A \in \mathcal{F}$.

When dealing with $\mathbb{Q}$, there is a natural topology to work with. (A version of such topology has been already used in the proof of Corollary 7). Let $[0,1]^{\mathcal{F}}$ be the set of all functions from $\mathcal{F}$ into $[0,1]$, equipped with the product topology. Then, $[0,1]^{\mathcal{F}}$ is a compact Hausdorff space and convergence on $[0,1]^{\mathcal{F}}$ is setwise convergence. Thus, $\mathbb{Q}$ is a compact Hausdorff space in the relative topology inherited from $[0,1]^{\mathcal{F}}$. Hereafter, this topology is referred to as the *product topology.* Note that $P \mapsto P(f)$ is a continuous map on $\mathbb{Q}$ for any bounded $\mathcal{F}$-measurable function $f : \Omega \to \mathbb{R}$.

Theorem 9. *Let $\mathcal{R} \subset \mathbb{Q}$ and*

$$\mathcal{Q} = \{\textit{extreme points of } \mathcal{R}\}.$$

Fix $P \in \mathbb{Q}$ and suppose $\mathcal{R}$ convex and closed in the product topology. Then, the following statements are equivalent:

- (a) *$P \in \mathcal{R}$;*
- (b) *$P(f) \geq \inf_{Q \in \mathcal{Q}} Q(f)$ for each $f \in D_{\mathcal{F}}$;*
- (c) *There is a f.a.p. Π on $\Sigma(\mathcal{Q})$ such that $P(A) = \int_{\mathcal{Q}} Q(A)\,\Pi(dQ)$ for all $A \in \mathcal{F}$.*

Proof. (a) $\Rightarrow$ (b). By the Krein-Milman theorem,

$$\mathcal{R} = \overline{\text{conv}(\mathcal{Q})}$$

where $\text{conv}(\mathcal{Q})$ is the convex hull of $\mathcal{Q}$. Therefore, if $P \in \mathcal{R}$, one obtains

$$P(f) \geq \inf_{Q \in \mathcal{R}} Q(f) = \inf_{Q \in \mathcal{Q}} Q(f)$$

for each $f \in D_{\mathcal{F}}$, as the map $Q \mapsto Q(f)$ is continuous.

(b) $\Rightarrow$ (c). Just apply Lemma 1 with $D = D_{\mathcal{F}}$.

(c) $\Rightarrow$ (a). Fix $\epsilon > 0$ and a finite subset $\mathcal{V} \subset \mathcal{F}$. In view of (c), there is a finite partition $\{S_1, \ldots, S_m\}$ of $\mathcal{Q}$ such that $S_i \in \Sigma(\mathcal{Q})$ for all i and

$$\Big|P(A) - \sum_{i=1}^{m} \Pi(S_i) Q_i(A)\Big| < \epsilon \quad \text{whenever } A \in \mathcal{V} \text{ and } Q_i \in S_i \text{ for all } i.$$

Take $Q_1 \in S_1, \ldots, Q_m \in S_m$ and define

$$Q_{\mathcal{V},\epsilon} = \sum_{i=1}^{m} \Pi(S_i) Q_i.$$

Since $\mathcal{R}$ is convex, $Q_{\mathcal{V},\epsilon} \in \mathcal{R}$. Hence, it suffices to note that $\mathcal{R}$ is closed and the net $\{Q_{\mathcal{V},\epsilon} : \epsilon > 0,\, \mathcal{V} \subset \mathcal{F} \text{ finite}\}$ converges to P. $\square$

Example 10. (Invariant f.a.p.'s). Let Φ be an arbitrary class of measurable functions from Ω into itself (i.e., $\phi : \Omega \to \Omega$ and $\phi^{-1}(\mathcal{F}) \subset \mathcal{F}$ for all $\phi \in \Phi$) and let $\mathcal{Q}_\Phi$ denote the collection of those f.a.p.'s $P \in \mathbb{Q}$ such that $P \circ \phi^{-1} = P$ for all $\phi \in \Phi$. If P_α is a net in $\mathcal{Q}_\Phi$ and $P_\alpha \to P \in \mathbb{Q}$, then

$$P \circ \phi^{-1}(A) = P\big(\phi^{-1}(A)\big) = \lim_\alpha P_\alpha\big(\phi^{-1}(A)\big) = \lim_\alpha P_\alpha(A) = P(A)$$

for all $A \in \mathcal{F}$ and $\phi \in \Phi$. Hence, $\mathcal{Q}_\Phi$ is (convex and) closed in the product topology.

Letting $\mathcal{R} = \mathcal{Q}_\Phi$ in Theorem 9, it follows that an invariant f.a.p. is a finitely additive mixture of extreme invariant f.a.p.'s, without any assumptions on Φ or $(\Omega, \mathcal{F})$. In a σ-additive framework, instead, the corresponding result requires some conditions; see forthcoming Example 12.

5. Countably additive mixtures

From now on, a σ-additive f.a.p. is called a *probability measure*. Further, $\mathcal{A}$ is a σ-field of subsets of Ω and $\mathbb{P}$ the set of all probability measures on $\mathcal{A}$.

Let $P \in \mathbb{P}$ and $\mathcal{Q} \subset \mathbb{P}$. Some classical results state that, under suitable assumptions, P is a σ-additive mixture of the elements of $\mathcal{Q}$. An obvious strategy for proving such results is as follows. One first proves that

$$P(f) \geq \inf_{Q \in \mathcal{Q}} Q(f) \quad \text{for each } \mathcal{A}\text{-simple function } f, \tag{7}$$

so that Lemma 1 yields $P(\cdot) = \int_{\mathcal{Q}} Q(\cdot)\, \Pi(dQ)$ for some f.a.p. Π on $\Sigma(\mathcal{Q})$. Then, one shows that Π is (or can be taken to be) a probability measure.

In this section, we aim to realize this program. Some new results, as well as simple proofs of known facts, are obtained.

For any $\mathcal{Q} \subset \mathbb{P}$, define

$$\mathcal{G} = \{A \in \mathcal{A} : Q(A) \in \{0,1\} \text{ for each } Q \in \mathcal{Q}\}$$

and call $\Sigma_0(\mathcal{Q})$ the σ-field over $\mathcal{Q}$ generated by the maps $Q \mapsto Q(A)$ for all $A \in \mathcal{G}$.

Theorem 11. *Given $P \in \mathbb{P}$ and $\mathcal{Q} \subset \mathbb{P}$, suppose $P(\cdot) = \int_{\mathcal{Q}} Q(\cdot)\, \Pi(dQ)$ for some f.a.p. Π on $\Sigma(\mathcal{Q})$. Let Π_0 be the restriction of Π on $\Sigma_0(\mathcal{Q})$. Then, Π_0 is a probability measure determined by $P|\mathcal{G}$. Moreover, Π is a probability measure determined by P provided, for each $A \in \mathcal{A}$, there is an $\mathcal{A}$-measurable map $h_A : \Omega \to [0,1]$ such that*

$$Q\big\{\omega \in \Omega : h_A(\omega) = Q(A)\big\} = 1 \quad \textit{for all } Q \in \mathcal{Q}. \tag{8}$$

Proof. First note that $\Sigma_0(\mathcal{Q})$ consists of sets of the form $\{Q \in \mathcal{Q} : Q(A) = 1\}$ for some $A \in \mathcal{G}$. By definition of $\mathcal{G}$, one also obtains

$$\Pi\{Q \in \mathcal{Q} : Q(A) = 1\} = \int_{\mathcal{Q}} Q(A)\,\Pi(dQ) = P(A) \quad \text{for each } A \in \mathcal{G}.$$

Thus, Π_0 is determined by $P|\mathcal{G}$. Fix a sequence $H_n \in \Sigma_0(\mathcal{Q})$ and take $A_n \in \mathcal{G}$ such that $H_n = \{Q \in \mathcal{Q} : Q(A_n) = 1\}$. Since $\cup_n H_n = \{Q \in \mathcal{Q} : Q(\cup_n A_n) = 1\}$,

$$\sum_n \Pi(H_n) = \sum_n P(A_n) \geq P(\cup_n A_n) = \Pi(\cup_n H_n)$$

where $\geq$ depends on P is σ-additive. Hence, Π_0 is a probability measure.

To conclude the proof, it suffices to see that $\Sigma(\mathcal{Q}) = \Sigma_0(\mathcal{Q})$ under condition (8). Let $H \in \Sigma(\mathcal{Q})$. By a routine argument, there is a countable class $\mathcal{C} \subset \mathcal{A}$ such that $H \in \Sigma_{\mathcal{C}}(\mathcal{Q})$, where $\Sigma_{\mathcal{C}}(\mathcal{Q})$ is the σ-field over $\mathcal{Q}$ generated by $Q \mapsto Q(C)$ for all $C \in \mathcal{C}$. Write $\mathcal{C} = \{A_1, A_2, \ldots\}$ and define

$$g(Q) = \big(Q(A_1), Q(A_2), \ldots\big) \quad \text{for all } Q \in \mathcal{Q}.$$

Then, $g : \mathcal{Q} \to \mathbb{R}^\infty$ and $\Sigma_{\mathcal{C}}(\mathcal{Q})$ can be written as $\Sigma_{\mathcal{C}}(\mathcal{Q}) = \big\{\{g \in B\} : B \in \mathcal{B}(\mathbb{R}^\infty)\big\}$. In particular, $H = \{g \in B\}$ for some $B \in \mathcal{B}(\mathbb{R}^\infty)$. Define

$$h(\omega) = \big(h_{A_1}(\omega), h_{A_2}(\omega), \ldots\big) \quad \text{for all } \omega \in \Omega.$$

For each $Q \in \mathcal{Q}$, condition (8) implies

$$Q\big(h = g(Q)\big) = Q\big(h_{A_i} = Q(A_i) \text{ for all } i\big) = 1.$$

Thus, letting $A = \{h \in B\}$, one obtains

$$Q(A) = Q\big(h \in B,\, h = g(Q)\big) = 1_H(Q).$$

Hence, $A \in \mathcal{G}$ and $H = \{Q \in \mathcal{Q} : Q(A) = 1\} \in \Sigma_0(\mathcal{Q})$. □

Theorem 11 improves (and is inspired by) Proposition (3.4) of [9].

A few applications of Theorem 11 are discussed in the next three examples. We first note that condition (8) holds true automatically if some sub-σ-field $\mathcal{D} \subset \mathcal{G}$ is sufficient for $\mathcal{Q}$. We recall that $\mathcal{D}$ is *sufficient* for $\mathcal{Q}$ if, for each $A \in \mathcal{A}$, there is a $\mathcal{D}$-measurable map $h_A : \Omega \to [0, 1]$ such that

$$\int_D h_A\, dQ = Q(A \cap D) \quad \text{for all } D \in \mathcal{D} \text{ and } Q \in \mathcal{Q}.$$

In other terms, h_A is a version of $E_Q(1_A|\mathcal{D})$ for all $Q \in \mathcal{Q}$.

Example 12. (Invariant probability measures). Let Φ be a collection of measurable maps of Ω into itself and $\mathcal{Q}_\Phi$ the set of those probability measures $P \in \mathbb{P}$ such that $P \circ \phi^{-1} = P$ for all $\phi \in \Phi$. Define

$$\mathcal{Q} = \{\text{extreme points of } \mathcal{Q}_\Phi\}.$$

Suppose Φ is separable (i.e., $\mathcal{Q}_\Phi = \mathcal{Q}_{\Phi_0}$ for some countable $\Phi_0 \subset \Phi$) and Ω is an universally measurable subset of a Polish space equipped with $\mathcal{A} = \mathcal{B}(\Omega)$. Then, for each $P \in \mathcal{Q}_\Phi$, there is a unique probability measure Π on $\Sigma(\mathcal{Q})$ such that $P(\cdot) = \int_{\mathcal{Q}} Q(\cdot)\,\Pi(dQ)$; see [14, Theorem 1]. We just make two remarks.

First, this result admits the following simple proof. Let

$$\mathcal{D} = \big\{A \in \mathcal{A} :\ P\big(A \Delta \phi^{-1} A\big) = 0 \text{ for all } \phi \in \Phi \text{ and } P \in \mathcal{Q}_\Phi\big\}.$$

Then, $\mathcal{D} \subset \mathcal{G}$ and, since Φ is separable, $\mathcal{D}$ is sufficient for $\mathcal{Q}_\Phi$; see [14, Lemma 4]. Thus, condition (8) holds. By Theorem 11, it remains only to show that each $P \in \mathcal{Q}_\Phi$ satisfies condition (7). This can be actually done, with a reasonable effort, exploiting the assumptions on $(\Omega, \mathcal{A})$. We omit the explicit calculations.

Second, the assumptions on $(\Omega, \mathcal{A})$ are required only to check condition (7) and are superfluous to apply Theorem 11. Hence, the following result is available.

Proposition 13. *Let $(\Omega, \mathcal{A}, P)$ be any (σ-additive) probability space. Take Φ and $\mathcal{Q}$ as in Example 12 and suppose Φ separable. Then, $P(\cdot) = \int_\mathcal{Q} Q(\cdot)\,\Pi(dQ)$ for some (unique) probability measure Π on $\Sigma(\mathcal{Q})$ if and only if P meets condition* (7).

Example 14. (Exchangeable probability measures). This is just a special case of Example 12. Given any measurable space $(S, \mathcal{E})$, let

$$\Omega = S^\infty, \quad \mathcal{A} = \mathcal{E}^\infty, \quad \Phi = \{\text{finite permutations of } S^\infty\}.$$

Then, the elements of $\mathcal{Q}_\Phi$ are the exchangeable probability measures on $\mathcal{E}^\infty$ and

$$\mathcal{Q} = \{\text{extreme points of } \mathcal{Q}_\Phi\} = \{\text{i.i.d. probability measures on } \mathcal{E}^\infty\}.$$

Let $P \in \mathbb{P}$. By Proposition 13, since Φ is countable, P is a σ-additive mixture of i.i.d. probability measures if and only if it satisfies condition (7). Note that, since $(S, \mathcal{E})$ is arbitrary, it may be that some $P \in \mathcal{Q}_\Phi$ does not admit the representation $P(\cdot) = \int_\mathcal{Q} Q(\cdot)\,\Pi(dQ)$ with Π a probability measure on $\Sigma(\mathcal{Q})$; see [9]. Hence, condition (7) may fail for some $P \in \mathcal{Q}_\Phi$.

Example 15. (Disintegrability). Let $\mathcal{H} \subset \mathcal{A}$ be a partition of Ω. Given $P \in \mathbb{P}$, a pair (α, β) is a disintegration for P on $\mathcal{H}$ if

- For each $H \in \mathcal{H}$, $\alpha(\cdot|H)$ is a f.a.p. on $\mathcal{A}$ such that $\alpha(H|H) = 1$;
- β is a f.a.p. on the power set of $\mathcal{H}$;
- $P(A) = \int_\mathcal{H} \alpha(A|H)\,\beta(dH)$ for all $A \in \mathcal{A}$;

see [1], [2], [8], [10]. Here, we also assume $\alpha(\cdot|H) \in \mathbb{P}$ for each $H \in \mathcal{H}$.

Let $\sigma(\alpha)$ be the σ-field over $\mathcal{H}$ generated by the maps $H \mapsto \alpha(A|H)$ for all $A \in \mathcal{A}$. Only the restriction of β on $\sigma(\alpha)$ plays a role in the above definition. Thus, since $P \in \mathbb{P}$, a (natural) question is whether β is a probability measure on $\sigma(\alpha)$. For instance, if each $\alpha(\cdot|H)$ is 0-1 valued, Theorem 11 implies that β is a probability measure on $\sigma(\alpha)$. (Just let $\mathcal{Q} = \big\{\alpha(\cdot|H) : H \in \mathcal{H}\big\}$ and note that $\mathcal{G} = \mathcal{A}$). To get a more general answer, a certain extension P_0 of P is to be involved.

Each $S \subset \mathcal{H}$ can be identified with a subset of Ω, denoted by S^*. Let $\mathcal{A}_0$ be the σ-field on Ω generated by $A \cap S^*$ for all $A \in \mathcal{A}$ and $S \in \sigma(\alpha)$. On noting that $\alpha(H|H) = 1$ and $C \cap H \in \mathcal{A}$ for each $C \in \mathcal{A}_0$ and $H \in \mathcal{H}$, one can define $\alpha_0(C|H) = \alpha\big(C \cap H|H\big)$ and

$$P_0(C) = \int_\mathcal{H} \alpha_0(C|H)\,\beta(dH) \quad \text{for all } C \in \mathcal{A}_0.$$

Proposition 16. *β is σ-additive on $\sigma(\alpha)$ if and only if P_0 is σ-additive on $\mathcal{A}_0$.*

Proof. By a standard monotone argument, the map $H \mapsto \alpha_0(C|H)$ is $\sigma(\alpha)$-measurable for all $C \in \mathcal{A}_0$. Also, for fixed $H \in \mathcal{H}$, $\alpha_0(\cdot|H)$ is a probability measure on $\mathcal{A}_0$ (recall that $\alpha(\cdot|H) \in \mathbb{P}$). Thus, P_0 is σ-additive on $\mathcal{A}_0$ if β is σ-additive on $\sigma(\alpha)$. Conversely, suppose P_0 is σ-additive on $\mathcal{A}_0$. Define $\mathcal{Q} = \big\{\alpha_0(\cdot|H) : H \in \mathcal{H}\big\}$ and

$$h_C(\omega) = \sum_{H \in \mathcal{H}} 1_H(\omega)\,\alpha_0(C|H) \quad \text{for all } \omega \in \Omega \text{ and } C \in \mathcal{A}_0.$$

For fixed $C \in \mathcal{A}_0$, the map $h_C : \Omega \to [0,1]$ is $\mathcal{A}_0$-measurable, and clearly

$$\alpha_0\Big(\{\omega \in \Omega : h_C(\omega) = \alpha_0(C|H)\} \mid H\Big) = \alpha(H|H) = 1 \quad \text{for all } H \in \mathcal{H}.$$

An application of Theorem 11 concludes the proof. □

We close the paper with a classical result. In what follows, Ω is a compact metric space, $\mathcal{A} = \mathcal{B}(\Omega)$, and $\mathbb{P}$ is equipped with the usual topology of weak convergence of probability measures, i.e., the weakest topology on $\mathbb{P}$ which makes continuous the maps $P \mapsto P(f)$ for all $f \in C(\Omega)$. We need the following (well known) fact.

(*) If $\mathcal{R} \subset \mathbb{P}$ is convex and closed, there is a strictly convex continuous function ψ on $\mathcal{R}$. Take one such ψ and define

$$\widehat{\psi} = \inf\{\varphi :\, \varphi \in L,\, \varphi \geq \psi\}$$

where L is the linear space of affine continuous functions on $\mathcal{R}$. Then, for each $Q \in \mathcal{R}$, one obtains $\widehat{\psi}(Q) = \psi(Q)$ if and only if Q is an extreme point of $\mathcal{R}$. See e.g. [12].

Theorem 17. *Suppose Ω is a compact metric space and $\mathcal{R} \subset \mathbb{P}$ is convex and closed. Let $\mathcal{Q}$ be the set of extreme points of $\mathcal{R}$. Then, each $P \in \mathcal{R}$ admits the representation $P(\cdot) = \int_{\mathcal{Q}} Q(\cdot)\,\Pi(dQ)$ for some probability measure Π on $\Sigma(\mathcal{Q})$.*

Proof. Take a strictly convex continuous function ψ on $\mathcal{R}$ and define L and $\widehat{\psi}$ as in (*). For each map φ on $\mathcal{R}$, denote by $\varphi_0 = \varphi|\mathcal{Q}$ the restriction of φ on $\mathcal{Q}$. Note also that $\mathcal{R}$ is a compact metric space (for Ω is a compact metric space).

Fix $P \in \mathcal{R}$ and define $T(\varphi_0) = \varphi(P)$ for all $\varphi \in L$. Since $\mathcal{R} = \overline{\text{conv}(\mathcal{Q})}$,

$$\varphi(P) \geq \inf_{Q \in \mathcal{Q}} \varphi(Q) \quad \text{for each } \varphi \in L.$$

This inequality implies that T is a coherent functional on $\{\varphi_0 : \varphi \in L\}$. Further,

$$\widehat{\psi}(P) = \inf\{\varphi(P) :\, \varphi \in L,\, \varphi \geq \psi\} = \inf\{T(\varphi_0) :\, \varphi \in L,\, \varphi_0 \geq \psi_0\} = T^*(\psi_0).$$

By Lemma 2, there is a f.a.p. Γ on $\Sigma(\mathcal{Q})$ such that

$$\varphi(P) = \int_{\mathcal{Q}} \varphi(Q)\,\Gamma(dQ) \text{ for all } \varphi \in L \quad \text{and} \quad \widehat{\psi}(P) = \int_{\mathcal{Q}} \psi(Q)\,\Gamma(dQ).$$

By definition of $\widehat{\psi}$, one also obtains $\widehat{\psi}(P) = \int_{\mathcal{Q}} \widehat{\psi}(Q)\,\Gamma(dQ)$.

Regard Γ as a f.a.p. on $\Sigma(\mathcal{R})$ such that $\Gamma(\mathcal{Q}) = 1$. Since $\mathcal{R}$ is compact Hausdorff, by Riesz theorem, there is a probability measure Π on $\Sigma(\mathcal{R})$ such that $\Pi(h) = \Gamma(h)$ for all $h \in C(\mathcal{R})$. Let $h_f(Q) = Q(f)$ for $f \in C(\Omega)$ and $Q \in \mathcal{R}$. Since $h_f \in L$,

$$P(f) = h_f(P) = \Gamma(h_f) = \Pi(h_f) = \int_{\mathcal{R}} Q(f)\,\Pi(dQ) \quad \text{for each } f \in C(\Omega).$$

Hence, to conclude the proof, it suffices to see that $\Pi(\mathcal{Q}) = 1$. In fact,

$$\Pi(\psi) = \Gamma(\psi) = \Gamma(\widehat{\psi}) = \widehat{\psi}(P) = \inf\{\Pi(\varphi) :\, \varphi \in L,\, \varphi \geq \psi\} \geq \Pi(\widehat{\psi})$$

where the second equality is because $\Gamma(\widehat{\psi} = \psi) = \Gamma(\mathcal{Q}) = 1$; see (*). Since $\psi \leq \widehat{\psi}$ and $\Pi(\psi) \geq \Pi(\widehat{\psi})$, one finally obtains $\Pi(\mathcal{Q}) = \Pi(\widehat{\psi} = \psi) = 1$. □

Theorem 17 is obviously known and our only goal is to provide it with a simple proof. Indeed, Theorem 17 holds under more general assumptions. By a result of Winkler [18], for any metric space Ω, it suffices to assume $\mathcal{R}$ a convex closed set of *tight* probability measures. However, the crucial step of Winkler's proof is the case where Ω is compact. Once this is done, to prove the theorem in the general case is not too hard.

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