ON THE EXISTENCE OF CONTINUOUS PROCESSES WITH GIVEN ONE-DIMENSIONAL DISTRIBUTIONS

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The problem

Throughout, a **process** is meant as a real valued stochastic process indexed by $[0, 1]$. A process is **continuous** if almost all its paths are continuous

Let $\mathcal{P}$ be the collection of Borel probability measures on $\mathcal{R}$, equipped with the weak* topology, and

$\mu : [0, 1] \to \mathcal{P}$ a continuous map

(*) Is there a continuous process X such that $X_t \sim \mu_t$ for each $t \in [0, 1]$?

If yes, say that μ is **presentable**

Motivations

- Problem (*) is connected to gradient flows and certain partial differential equations

- A positive answer to (*) can be regarded as a strong version of Skorohod representation theorem

- By a result of Blackwell and Dubins, there is a process X such that, for each fixed t, $X_t \sim \mu_t$ and almost all X-paths are continuous at t. Despite this fact, however, the answer to (*) may be no:

$$\boxed{\mu_t = (1-t)\,\delta_0 + t\,\delta_1}$$

Remark

Problem (*) is actually an extension problem

In fact, let us extend μ to

$\mathcal{U} = \{\mu_{t_1,\ldots,t_n} : \ n \geq 1, \ t_1, \ldots, t_n \in [0,1]\}$

where each $\mu_{t_1,\ldots,t_n}$ is a Borel probability measure on $\mathcal{R}^n$

There is a continuous process X with $X \sim \mathcal{U}$ if and only if

- $\mathcal{U}$ is consistent (i.e., $Y \sim \mathcal{U}$ for some process Y)
- $Y_s \to Y_t$ in probability as $s \to t$
- $\inf_{\delta>0} \sup_n P(\exists\ s,t \in D_n$ with $|s-t| < \delta$ and $|Y_s - Y_t| > \epsilon) = 0$ for each $\epsilon > 0$, where $D_n = \{j/2^n : j = 0, 1, \ldots, 2^n\}$

The quantile process

Let $F_t(x) = \mu_t(-\infty, x]$ and

$$\boxed{Q_t(\alpha) = \inf\{x \in \mathcal{R} : F_t(x) \geq \alpha\},} \quad t \in [0,1], \ \ \alpha \in (0,1)$$

Q is a process on $((0,1), \mathcal{B}(0,1), \lambda)$, where λ is Lebesgue measure, with finite dimensional distributions

$$\lambda(Q_{t_1} \leq x_1, \ldots, Q_{t_k} \leq x_k) = \min_{1 \leq i \leq k} F_{t_i}(x_i)$$

In particular, $Q_t \sim \mu_t$ for all t so that

Q continuous $\quad \Rightarrow \quad$ μ presentable

Result 1

Letting $J_t = \{\alpha \in (0,1) : F_t(x) = F_t(y) = \alpha \text{ for some } x < y\}$,

Q is continuous $\Leftrightarrow$ $\lambda^*(\cup_t J_t) = 0$

As a consequence, μ is presentable whenever

μ_t is supported by an interval (possibly, by a singleton) for all but countably many t

Result 2

Suppose all μ_t have the same support, say F

If $F = \mathcal{R}$, then μ is presentable by Result 1. Otherwise, since F is closed,

$F^c = \cup_n (a_n, b_n)$

with the (a_n, b_n) pairwise disjoint open intervals. The following statements are equivalent:

- μ is presentable
- Q is continuous
- $F_t(a_n) = F_0(a_n)$ for all $t \in [0, 1]$ and n with $a_n > -\infty$

Work in progress

An intriguing **open problem** is whether

$$\boxed{\mu \text{ presentable } \Rightarrow \; Q \sim X \text{ for some continuous process } X}$$

In a sense, a positive answer would "close" problem (*)

So far, we only have results such as

$$\mu \text{ presentable} + \text{something } \Rightarrow \; Q \sim X \text{ for some continuous } X$$

One is Result 2 above. Another is the following:

Result 3

Q is continuous if there is a process X, defined on some probability space $(\Omega, \mathcal{A}, P)$, such that $X_t \sim \mu_t$ for all t and

$\{X(\omega) : \omega \in \Omega\}$ is an equicontinuous subset of $C[0,1]$

Corollary

μ is presentable if and only if admits the representation

$\mu = \sum_n c_n \, \mu^n$

where: $c_n \geq 0$, $\sum_n c_n = 1$, $\mu^n : [0,1] \to \mathcal{P}$, and the quantile process induced by μ^n is continuous

Result 4

Let $a \geq 1$, $b > 1$, $c > 0$ be constants. Then, $Q \sim X$ for some continuous process X if

$$E\{|Y_s - Y_t|^a\} \leq c\;|s - t|^b$$

for all s, $t \in [0, 1]$ and some process Y such that $Y_t \sim \mu_t$ for all t

This fact is a (simple) consequence of the Chentsov-Kolmogorov criterion

Open problems

There are (at least) three questions

(1) μ presentable $\Rightarrow$ $Q \sim X$ for some continuous process X **?**

This is the main issue (and has been already mentioned)

(2) If $\mu : [0,1] \to \mathcal{P}$ is cadlag,

(**) | Is there a cadlag process X such that $X_t \sim \mu_t$ for each $t \in [0,1]$?

What is a μ which provides a negative answer to (**) ? Now, $\mu_t = (1-t)\,\delta_0 + t\,\delta_1$ no longer works. Just let

$$X_t = 1_{[U,1]}(t)$$

with U uniformly distributed on $(0,1)$

(3) Replace $\mathcal{R}$ with an arbitrary metric space S, i.e., assume

$\mu : [0,1] \to \{$Borel probability measures on $S\}$

continuous and investigate

(***) | Is there a continuous, S-valued process X such that $X_t \sim \mu_t$ for each $t \in [0,1]$?

Among other things, can we conclude that μ is presentable under some reasonable condition on the supports of the μ_t ?

Example

Let $S = \mathcal{R}^2$ and

$\{\pi_t(\cdot|x) : x \in \mathcal{R}\}$

a regular version of the conditional distribution of the second coordinate given the first under μ_t

Then, μ is presentable provided:

- All μ_t have the same marginal on the first coordinate
- $t \mapsto \pi_t(\cdot|x)$ is a continuous map for fixed x
- $\pi_t(\cdot|x)$ is supported by an interval for all t and x