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ASYMPTOTICS FOR RANDOMLY REINFORCED URNS WITH RANDOM BARRIERS

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Abstract

An urn contains black and red balls. Let Z_n be the proportion of black balls at time n and $0 \leq L < U \leq 1$ random barriers. At each time n, a ball b_n is drawn. If b_n is black and $Z_{n-1} < U$, then b_n is replaced together with a random number B_n of black balls. If b_n is red and $Z_{n-1} > L$, then b_n is replaced together with a random number R_n of red balls. Otherwise, no additional balls are added, and b_n alone is replaced. In this paper, we assume $R_n = B_n$. Then, under mild conditions, it is shown that $Z_n \xrightarrow{a.s.} Z$ for some random variable Z, and

$$D_n := \sqrt{n}\,(Z_n - Z) \longrightarrow \mathcal{N}(0, \sigma^2) \quad \text{conditionally a.s.}$$

where σ^2 is a certain random variance. Almost sure conditional convergence means that

$$P\big(D_n \in \cdot \mid \mathcal{G}_n\big) \xrightarrow{weakly} \mathcal{N}(0,\, \sigma^2) \quad \text{a.s.}$$

where $P\big(D_n \in \cdot \mid \mathcal{G}_n\big)$ is a regular version of the conditional distribution of D_n given the past $\mathcal{G}_n$. Thus, in particular, one obtains $D_n \longrightarrow \mathcal{N}(0, \sigma^2)$ stably. It is also shown that $L < Z < U$ a.s. and Z has non-atomic distribution.

Keywords: Bayesian nonparametrics; Central limit theorem; Random probability measure; Stable convergence; Urn model

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1. Introduction

In recent times, there is a growing interest on randomly reinforced urns. A meaningful version of the latter, introduced in [3] and supported by real applications, is the following.

1.1. Framework

An urn contains $b > 0$ black balls and $r > 0$ red balls. At each time, a ball is drawn and then replaced, possibly together with a random number of balls of the same color. Precisely, for each $n \geq 1$, let b_n denote the ball drawn at time n and Z_n the proportion of black balls in the urn at time n. Then,

- If b_n is black and $Z_{n-1} < U$, where U is a random barrier, b_n is replaced together with a random number $B_n \geq 0$ of black balls;

- If b_n is red and $Z_{n-1} > L$, where $L < U$ is another random barrier, b_n is replaced together with a random number $R_n \geq 0$ of red balls;
- Otherwise, b_n is replaced without additional balls, so that the composition of the urn does not change.

To model such urns, we fix a probability space $(\Omega, \mathcal{A}, P)$ supporting the random variables $(L, U, X_n, B_n, R_n : n \geq 1)$ such that

$$0 \leq L < U \leq 1; \quad X_n \in \{0,1\}; \quad 0 \leq B_n,\, R_n \leq c \quad \text{for some constant } c.$$

We let

$$\mathcal{G}_0 = \sigma(L, U), \quad \mathcal{G}_n = \sigma\big(L, U, X_1, B_1, R_1, \ldots, X_n, B_n, R_n\big), \quad Z_0 = b/(b+r),$$
$$Z_n = \frac{b + \sum_{i=1}^n X_i\, B_i\, I_{\{Z_{i-1}<U\}}}{b + r + \sum_{i=1}^n \big[X_i\, B_i\, I_{\{Z_{i-1}<U\}} + (1-X_i)\, R_i\, I_{\{Z_{i-1}>L\}}\big]}.$$

Further, we assume

$$E(X_{n+1} \mid \mathcal{G}_n) = Z_n \text{ a.s.} \quad \text{and}$$
$$(B_n,\, R_n) \quad \text{independent of} \quad \sigma\big(\mathcal{G}_{n-1},\, X_n\big).$$

Clearly, X_n should be regarded as the indicator of the event $\{\text{black ball at time } n\}$ and Z_n as the proportion of black balls in the urn at time n.

1.2. State of the art

Though the literature on randomly reinforced urns is quite huge, random barriers are not so popular. In other terms, the case $L = 0$ and $U = 1$ is widely investigated (see e.g. [1], [2], [4]-[6], [8], [11]-[15] and references therein) but

$$P\big(\{L > 0\} \cup \{U < 1\}\big) > 0$$

is almost neglected. To our knowledge, the only explicit reference is [3]. In such a paper, the barriers L and U are not random (i.e., they are constant) and (B_n) and (R_n) are independent sequences of i.i.d. random variables. The a.s. convergence of Z_n is investigated, and it is shown that $Z_n \xrightarrow{a.s.} L$ if $E(B_1) < E(R_1)$ and $Z_n \xrightarrow{a.s.} U$ if $E(B_1) > E(R_1)$. Among other applications, this model could be usefully exploited in clinical trials, when a response adaptive design is requested targeting a fixed asymptotic allocation.

1.3. Results

In a sense, this paper deals with the opposite case with respect to [3]. Indeed, while (B_n) and (R_n) are independent sequences in [3], throughout this paper it is assumed that

$$R_n = B_n \quad \text{for each } n \geq 1. \tag{1}$$

Condition (1) looks reasonable in several real applications. Furthermore, under (1), in addition to the a.s. convergence of Z_n, a central limit theorem can be obtained. Precisely, the following two results are proved.

Theorem 1.1. *In the framework of Subsection 1.1, suppose*

$$R_n = B_n \quad \textit{and} \quad \liminf_n E(B_n) > 0.$$

Then,

$$Z_n \xrightarrow{a.s.} Z$$

for some random variable Z such that $L \leq Z \leq U$ and $0 < Z < 1$ a.s.

Theorem 1.2. *In the framework of Subsection 1.1, suppose*

$$R_n = B_n, \quad m := \lim_n E(B_n) > 0 \quad \textit{and} \quad q := \lim_n E(B_n^2).$$

Define

$$D_n = \sqrt{n}\,(Z_n - Z) \quad \textit{and} \quad \sigma^2 = q\,Z\,(1-Z)/m^2,$$

where Z is the a.s. limit of Z_n. Then,

$$D_n \longrightarrow \mathcal{N}(0,\, \sigma^2) \quad \textit{conditionally a.s. with respect to } (\mathcal{G}_n).$$

Moreover, Z has a non-atomic distribution and $L < Z < U$ a.s.

In Theorem 1.2, $\mathcal{N}(a,b)$ denotes the Gaussian law with mean a and variance $b \geq 0$, where $\mathcal{N}(a,0) = \delta_a$. Almost sure conditional convergence is a strong form of stable convergence, introduced in [9]-[10] and involved in [2], [6], [14], [15]. The general definition is discussed in Section 2. In the present case, it means that

$$P\big(D_n \in \cdot \mid \mathcal{G}_n\big)(\omega) \overset{weakly}{\longrightarrow} \mathcal{N}(0,\, \sigma^2(\omega)) \quad \text{for almost all } \omega \in \Omega$$

where $P\big(D_n \in \cdot \mid \mathcal{G}_n\big)$ is a regular version of the conditional distribution of D_n given $\mathcal{G}_n$. Thus, in particular, Theorem 1.2 yields

$$D_n \longrightarrow \mathcal{N}(0,\, \sigma^2) \quad \text{stably;}$$

see Lemma 2.1.

Theorems 1.1-1.2 establish the asymptotics for randomly reinforced urns with random barriers when $R_n = B_n$. The case $R_n \neq B_n$, as well as some other possible developments, are discussed in Section 4.

A last note is that Theorem 1.2 agrees with the result obtained when random barriers are not taken into account. Indeed, if $L = 0$ and $U = 1$, Theorem 1.2 follows from [6, Corollary 3]. Similarly, Theorem 1.1 is quite in line with intuition. Thus, in a sense, Theorems 1.1-1.2 are fairly expected. Despite this fact, their proofs (or at least our proofs) are long and surprisingly involved. Such proofs are delayed to Section 3 after recalling some technical facts in Section 2.

2. Almost sure conditional convergence

Almost sure conditional convergence, introduced in [9]-[10], may be regarded as a strong form of stable convergence. We now make it precise.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and S a metric space. A *kernel* on S (or a *random probability measure* on S) is a measurable collection $N = \{N(\omega) : \omega \in \Omega\}$ of probability measures on the Borel σ-field on S. Measurability means that

$$N(\cdot)(f) = \int f(x)\, N(\cdot)(dx)$$

is a real random variable for each bounded Borel map $f : S \to \mathbb{R}$. To denote such random variable, in the sequel, we will often write $N(f)$ instead of $N(\cdot)(f)$.

For each $n \geq 1$, fix a sub-σ-field $\mathcal{F}_n \subset \mathcal{A}$. Also, let (Y_n) be a sequence of S-valued random variables and N a kernel on S. Say that Y_n converges to N, *conditionally a.s.*

with respect to $(\mathcal{F}_n)$, if

$$E\{f(Y_n) \mid \mathcal{F}_n\} \xrightarrow{a.s.} N(f) \quad \text{for each } f \in C_b(S). \tag{2}$$

If S is Polish, condition (2) has a quite transparent meaning. Suppose in fact S is Polish and fix a regular version $P(Y_n \in \cdot \mid \mathcal{F}_n)$ of the conditional distribution of Y_n given $\mathcal{F}_n$. Then, condition (2) is equivalent to

$$P(Y_n \in \cdot \mid \mathcal{F}_n)(\omega) \xrightarrow{weakly} N(\omega) \quad \text{for almost all } \omega \in \Omega.$$

So far, $(\mathcal{F}_n)$ is an arbitrary sequence of sub-σ-fields. Suppose now that $(\mathcal{F}_n)$ is a filtration, in the sense that $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{A}$ for each n. Then, under a mild measurability condition, almost sure conditional convergence implies stable convergence. This is noted in [9, Section 5] but we give a proof to make the paper self-contained. Let

$$\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n).$$

Lemma 2.1. *Suppose* $(\mathcal{F}_n)$ *is a filtration such that*

$$N(f) \text{ and } Y_n \text{ are } \mathcal{F}_\infty\text{-measurable for all } f \in C_b(S) \text{ and } n \geq 1.$$

If $Y_n \to N$ *conditionally a.s. with respect to* $(\mathcal{F}_n)$*, then* $Y_n \to N$ *stably, that is*

$$E\{N(f) \mid H\} = \lim_n E\{f(Y_n) \mid H\}$$

whenever $f \in C_b(S)$*,* $H \in \mathcal{A}$ *and* $P(H) > 0$*.*

Proof. Let $f \in C_b(S)$ and $H \in \mathcal{A}$. If $H \in \cup_n \mathcal{F}_n$, then $H \in \mathcal{F}_n$ for each sufficiently large n, so that

$$E\{N(f)\, I_H\} = \lim_n E\Big(E\{f(Y_n) \mid \mathcal{F}_n\}\, I_H\Big) = \lim_n E\{f(Y_n)\, I_H\}.$$

Since $\cup_n \mathcal{F}_n$ is a field, by standard arguments one obtains

$$E\{N(f)\, V\} = \lim_n E\{f(Y_n)\, V\} \quad \text{whenever } V \text{ is bounded and } \mathcal{F}_\infty\text{-measurable.}$$

Hence, for arbitrary $H \in \mathcal{A}$, the measurability condition implies

$$E\{N(f)\, I_H\} = E\Big(N(f)\, E(I_H \mid \mathcal{F}_\infty)\Big) = \lim_n E\Big(f(Y_n)\, E(I_H \mid \mathcal{F}_\infty)\Big) = \lim_n E\{f(Y_n)\, I_H\}.$$

Note that the measurability condition of Lemma 2.1 is trivially true if $\mathcal{F}_\infty = \mathcal{A}$.

We refer to [9]-[10] for more on almost sure conditional convergence. Here, for easy of reference, we report three useful facts. The first and the second are already known (see [6, Proposition 1 and Lemma 2] and [10, Theorem 2.2]) while the third is a quick consequence of condition (2). In each of these facts, $(\mathcal{F}_n)$ is a filtration.

Lemma 2.2. *Suppose the Y_n are real random variables such that $Y_n \overset{a.s.}{\longrightarrow} Y$. Then,*

$$\sqrt{n}\,(Y_n - Y) \longrightarrow \mathcal{N}(0,U), \quad \textit{conditionally a.s. with respect to } (\mathcal{F}_n),$$

where U is a real random variable, provided

(i) (Y_n) is a uniformly integrable martingale with respect to $(\mathcal{F}_n)$;

(ii) $E\big\{\sup_n \sqrt{n}\,|Y_n - Y_{n-1}|\big\} < \infty$;

(iii) $n \sum_{k\geq n}(Y_k - Y_{k-1})^2 \overset{a.s.}{\longrightarrow} U$.

Lemma 2.3. *Suppose the Y_n are real random variables. If (Y_n) is adapted to $(\mathcal{F}_n)$, $\sum_n n^{-2}E(Y_n^2) < \infty$ and $E\big(Y_{n+1} \mid \mathcal{F}_n\big) \overset{a.s.}{\longrightarrow} U$, for some real random variable U, then*

$$n \sum_{k\geq n} \frac{Y_k}{k^2} \overset{a.s.}{\longrightarrow} U \quad \textit{and} \quad \frac{1}{n}\sum_{k=1}^{n} Y_k \overset{a.s.}{\longrightarrow} U.$$

Lemma 2.4. *Suppose $Y_n \to N$ conditionally a.s. with respect to $(\mathcal{F}_n)$. Define $Q(A) = E\big\{I_A\, V\big\}$ for $A \in \mathcal{A}$, where $V \geq 0$, $E(V) = 1$ and V is $\mathcal{F}_\infty$-measurable. Then $Y_n \to N$, conditionally a.s. with respect to $(\mathcal{F}_n)$, under Q as well.*

Proof. Suppose first $\sup V < \infty$ and define $K_n = V - E(V \mid \mathcal{F}_n)$. Given $f \in C_b(S)$,

$$E_Q\big\{f(Y_n) \mid \mathcal{F}_n\big\} = \frac{E\big\{V\, f(Y_n) \mid \mathcal{F}_n\big\}}{E(V \mid \mathcal{F}_n)} = E\big\{f(Y_n) \mid \mathcal{F}_n\big\} + \frac{E\big\{K_n\, f(Y_n) \mid \mathcal{F}_n\big\}}{E(V \mid \mathcal{F}_n)}, \quad Q\text{-a.s.},$$

where E_Q denotes expectation under Q. Since $\sigma(V) \subset \mathcal{F}_\infty$ and $|K_n| \leq \sup V$ a.s., the martingale convergence theorem (in the version of [7]) implies

$$E(V \mid \mathcal{F}_n) \overset{a.s.}{\longrightarrow} E(V \mid \mathcal{F}_\infty) = V \quad \text{and}$$
$$\Big|E\big\{K_n\, f(Y_n) \mid \mathcal{F}_n\big\}\Big| \leq \sup|f|\, E\big\{|K_n| \mid \mathcal{F}_n\big\} \overset{a.s.}{\longrightarrow} 0.$$

Since $Q(V > 0) = 1$, one obtains $E_Q\big\{f(Y_n) \mid \mathcal{F}_n\big\} \to N(f)$, Q-a.s. This concludes the proof for bounded V. If V is not bounded, it suffices to reply V with $V\, I_{\{V\leq v\}}/E\big(V\, I_{\{V\leq v\}}\big)$ and to take the limit as $v \to \infty$.

3. Proofs

In the sequel, for any events $A_n \in \mathcal{A}$ and $B \in \mathcal{A}$, we say that A_n *is eventually true on* B (or, more briefly, A_n *eventually on* B) whenever

$$P\big\{\omega \in B : \omega \notin A_n \text{ for infinitely many } n\big\} = 0.$$

Assume the conditions of Subsection 1.1 and $R_n = B_n$. Let

$$S_n = b + r + \sum_{i=1}^{n} \big[X_i\, B_i\, I_{\{Z_{i-1}<U\}} + (1 - X_i)\, B_i\, I_{\{Z_{i-1}>L\}}\big]$$

denote the denominator of Z_n, namely, the number of balls in the urn at time n. Also, the filtration $(\mathcal{G}_n)$ is abbreviated by $\mathcal{G}$.

After some (tedious but easy) algebra, one obtains

$$Z_{n+1} - Z_n = Z_n\, H_n + \Delta_{n+1},$$

where

$$H_n = \frac{B_{n+1}}{S_n + B_{n+1}}\,(1 - Z_n)\,\Big(I_{\{Z_n<U\}} - I_{\{Z_n>L\}}\Big),$$
$$\Delta_{n+1} = \frac{B_{n+1}}{S_n + B_{n+1}}\,(X_{n+1} - Z_n)\,\Big((1 - Z_n)I_{\{Z_n<U\}} + Z_n I_{\{Z_n>L\}}\Big).$$

This writing of $Z_{n+1} - Z_n$ is fundamental for our purposes.

3.1. Proof of Theorem 1.1

In this subsection, it is assumed that

$$\liminf_n E(B_n) > 0.$$

Since

$$E\big(X_{n+1} \mid \mathcal{G}_n,\, B_{n+1}\big) = E(X_{n+1} \mid \mathcal{G}_n) = Z_n \quad \text{a.s.},$$

then

$$E(\Delta_{n+1} \mid \mathcal{G}_n) = 0 \quad \text{a.s.}$$

This fact has two useful consequences. First,

$$M_n = \sum_{i=1}^{n} \Delta_i$$

is a $\mathcal{G}$-martingale. Second, (Z_n) is a $\mathcal{G}$-sub-martingale in case $U = 1$. In fact, $U = 1$ implies $H_n \geq 0$, so that

$$E\big\{Z_{n+1} - Z_n \mid \mathcal{G}_n\big\} = E\big\{Z_n H_n + \Delta_{n+1} \mid \mathcal{G}_n\big\} = Z_n\, E(H_n \mid \mathcal{G}_n) \geq 0 \quad \text{a.s.}$$

Similarly, if $L = 0$ then (Z_n) is a $\mathcal{G}$-super-martingale. Therefore, it is not hard to see that Z_n converges a.s. on the set $\{L = 0\} \cup \{U = 1\}$.

We next state two lemmas.

Lemma 3.1. *Let $Z_* = \liminf_n Z_n$ and $Z^* = \limsup_n Z_n$. Each of the following statements implies the subsequent:*

(a) $0 < L < U < 1$ *a.s.;*

(b) $0 < Z_* \leq Z^* < 1$ *a.s.;*

(c) $\liminf_n (S_n/n) > 0$ *a.s.;*

(d) M_n *converges a.s.*

Proof. **"(a) $\Longrightarrow$ (b)".** Let $H = \{Z_* = 0,\, L > 0\}$. On the set H, one obtains

$$\sup_n S_n = \infty, \quad \lim_n (Z_{n+1} - Z_n) = 0, \quad Z_n > L \text{ for infinitely many } n.$$

Define $\tau_0 = 0$ and

$$\tau_n = \inf\big\{k : k > \tau_{n-1}, \quad Z_{k-1} > L, \quad Z_k \leq L\big\}.$$

Then, $\tau_n < \infty$ for all n on H. Observe now that $Z_j \geq Z_{j-1}$ whenever $Z_{j-1} \leq L$. Hence, $Z_* = \liminf_n Z_n = \liminf_n Z_{\tau_n}$ on H, which implies the contradiction

$$Z_* \geq \liminf_n Z_{\tau_{n-1}} + \liminf_n (Z_{\tau_n} - Z_{\tau_{n-1}}) = \liminf_n Z_{\tau_{n-1}} \geq L > 0 \quad \text{a.s. on } H.$$

Thus, under (a), one obtains $P(Z_* = 0) = P(H) = 0$. Similarly, $P(Z^* = 1) = 0$.

"(b) $\Longrightarrow$ (c)". Define

$$K_n = \sum_{i=1}^{n} \frac{I_{\{Z_{i-1} < U\}}\big[X_i B_i - Z_{i-1} E(B_i)\big] + I_{\{Z_{i-1} > L\}}\big[(1 - X_i) B_i - (1 - Z_{i-1}) E(B_i)\big]}{i}.$$

Since K_n is a $\mathcal{G}$-martingale and $\sup_n E(K_n^2) < \infty$, then K_n converges a.s. Thus, Kronecker lemma implies $(1/n)\sum_{i=1}^n i\,K_i \xrightarrow{a.s.} 0$, so that

$$\liminf_n \frac{S_n}{n} = \liminf_n \frac{1}{n}\sum_{i=1}^n \Big(I_{\{Z_{i-1}<U\}} Z_{i-1} E(B_i) + I_{\{Z_{i-1}>L\}}\,(1-Z_{i-1})E(B_i)\Big) \text{ a.s.}$$

Since $I_{\{Z_{i-1}<U\}} + I_{\{Z_{i-1}>L\}} \geq 1$, one finally obtains

$$\liminf_n \frac{S_n}{n} \geq \big\{Z_* \wedge (1-Z^*)\big\}\,\liminf_n E(B_n) > 0 \quad \text{a.s.}$$

"(c) $\Longrightarrow$ (d)". Since $0 \leq B_{n+1} \leq c$,

$$E\big\{(M_{n+1}-M_n)^2 \mid \mathcal{G}_n\big\} = E(\Delta_{n+1}^2 \mid \mathcal{G}_n) \leq E\Big(\frac{B_{n+1}^2}{S_n^2} \mid \mathcal{G}_n\Big) = \frac{E(B_{n+1}^2)}{S_n^2} \leq \frac{c^2}{n^2}\,\frac{1}{(S_n/n)^2} \quad \text{a.s.}$$

Thus, $\sum_n E\big\{(M_{n+1}-M_n)^2 \mid \mathcal{G}_n\big\} < \infty$ a.s. by condition (c). It follows that the $\mathcal{G}$-martingale M_n converges a.s.

Lemma 3.2. *If* $\liminf_n (S_n/n) > 0$ *a.s., then* $P(D) = 0$ *where*

$$D = \big\{Z_a \leq L \text{ for infinitely many } a \text{ and } Z_b \geq U \text{ for infinitely many } b\big\}.$$

Proof. On D, there is a sequence (a_n, b_n) such that $a_1 < b_1 < a_2 < b_2 < \ldots$ and

$$Z_{a_n} \leq L, \quad Z_{b_n} \geq U, \quad L < Z_k < U \text{ for each } a_n < k < b_n.$$

Since $H_k = 0$ if $a_n < k < b_n$, then

$$U - L \leq Z_{b_n} - Z_{a_n} = \sum_{k=a_n}^{b_n-1} (Z_{k+1}-Z_k) = \sum_{k=a_n}^{b_n-1} (Z_k H_k + \Delta_{k+1}) = Z_{a_n} H_{a_n} + M_{b_n} - M_{a_n}.$$

Since $\liminf_n (S_n/n) > 0$ a.s., then $\sup_n S_n = \infty$ a.s., which implies $H_n \xrightarrow{a.s.} 0$. Also, by Lemma 3.1, M_n converges a.s. Hence, taking the limit as $n \to \infty$, one obtains $U - L \leq 0$ a.s. on D. Therefore, $P(D) = 0$.

We are now ready to prove a.s. convergence of Z_n.

Since Z_n converges a.s. on the set $\{L = 0\} \cup \{U = 1\}$, it can be assumed $P(0 < L < U < 1) > 0$. In turn, up to replacing P with $P(\cdot \mid 0 < L < U < 1)$, it can be assumed $0 < L < U < 1$ a.s. Then, Lemmas 3.1-3.2 imply $P(D^c) = 1$ and a.s. convergence of M_n. Write

$$Z_n - Z_0 = \sum_{i=0}^{n-1} (Z_{i+1} - Z_i) = \sum_{i=0}^{n-1} Z_i H_i + M_n = K_n + M_n$$

where $K_n = \sum_{i=0}^{n-1} Z_i H_i$. On the set D^c, one has either $Z_i H_i \geq 0$ eventually or $Z_i H_i \leq 0$ eventually. Hence, on D^c, the sequence K_n converges if and only if it is bounded. But K_n is a.s. bounded, since $|K_n| \leq 1 + \sup_k |M_k|$ and M_n converges a.s. Thus, Z_n converges a.s. on D^c. This proves a.s. convergence of Z_n for $P(D^c) = 1$.

Let Z denote the a.s. limit of Z_n. Since $Z_n \xrightarrow{a.s.} 1$ on the set $\{Z < L\}$ and $Z_n \xrightarrow{a.s.} 0$ on the set $\{Z > U\}$, then $L \leq Z \leq U$ a.s.

It remains to see that

$$P(Z = 0) = P(Z = 1) = 0.$$

We just prove $P(Z = 1) = 0$. The proof of $P(Z = 0) = 0$ is quite analogous.

Since $Z \leq U \leq 1$ a.s., then $P(Z = 1) \leq P(U = 1)$. Thus, it can be assumed $P(U = 1) > 0$. In turn, up to replacing P with $P(\cdot \mid U = 1)$, it can be assumed $U = 1$ everywhere. Then, Z_n is a $\mathcal{G}$-sub-martingale, so that

$$Y_n = Z_n/(1 - Z_n)$$

is still a $\mathcal{G}$-sub-martingale. Let $H = \big\{ \sum_n E\big\{Y_{n+1} - Y_n \mid \mathcal{G}_n\big\} < \infty \big\}$. Since Y_n is a positive $\mathcal{G}$-sub-martingale, Y_n converges a.s. (to a real random variable) on the set H. Thus, to get $P(Z = 1) = 0$, it suffices to show that

$$\sum_n E\big\{Y_{n+1} - Y_n \mid \mathcal{G}_n\big\} < \infty \quad \text{a.s. on the set } \{Z = 1\}. \tag{3}$$

To prove (3), let

$$J_n = b + \sum_{i=1}^{n} X_i\, B_i \quad \text{and} \quad L_n = S_n - J_n = r + \sum_{i=1}^{n} (1 - X_i)\, B_i\, I_{\{Z_{i-1} > L\}}$$

be the numbers of black balls and red balls, respectively, in the urn at time n (recall that $U = 1$, so that $Z_{i-1} < U$ is automatically true). On noting that $Y_n = J_n/L_n$, one obtains

$$
\begin{aligned}
E\Big\{Y_{n+1}-Y_n \mid \mathcal{G}_n\Big\} &= -Y_n + E\Big\{\frac{J_n+B_{n+1}}{L_n}\,X_{n+1} \;+\; \frac{J_n}{L_n+I_{\{Z_n>L\}}B_{n+1}}\,(1-X_{n+1}) \mid \mathcal{G}_n\Big\} \\
&= -Y_n + Z_n\,E\Big\{\frac{J_n+B_{n+1}}{L_n} \mid \mathcal{G}_n\Big\} + (1-Z_n)\,E\Big\{\frac{J_n}{L_n+I_{\{Z_n>L\}}B_{n+1}} \mid \mathcal{G}_n\Big\} \\
&= -Y_n\,(1-Z_n) + \frac{Z_n\,E(B_{n+1})}{L_n} + Y_n\,(1-Z_n)\,E\Big\{\frac{L_n}{L_n+I_{\{Z_n>L\}}B_{n+1}} \mid \mathcal{G}_n\Big\} \\
&= \frac{Z_n\,E(B_{n+1})}{L_n} - Z_n\,I_{\{Z_n>L\}}\,E\Big\{\frac{B_{n+1}}{L_n+I_{\{Z_n>L\}}B_{n+1}} \mid \mathcal{G}_n\Big\} \\
&\le \frac{Z_n\,E(B_{n+1})}{L_n} - Z_n\,I_{\{Z_n>L\}}\,E\Big\{\frac{B_{n+1}}{L_n+c} \mid \mathcal{G}_n\Big\} \\
&= \frac{Z_n\,E(B_{n+1})}{L_n} - Z_n\,I_{\{Z_n>L\}}\,\frac{E(B_{n+1})}{L_n+c} \quad \text{a.s.}
\end{aligned}
$$

Since $Z_n \xrightarrow{a.s.} Z$, then $Z_n > L$ eventually on the set $\{Z=1\}$. Hence,

$$
E\Big\{Y_{n+1}-Y_n \mid \mathcal{G}_n\Big\} \le Z_n\,E(B_{n+1})\,\Big(\frac{1}{L_n}-\frac{1}{L_n+c}\Big) \le \frac{c^2}{L_n^2} \quad \text{eventually on } \{Z=1\}.
$$

Next, given $k\in(1,2)$, it is not hard to see that

$$
E\Big\{\frac{J_{n+1}}{L_{n+1}^k}-\frac{J_n}{L_n^k} \mid \mathcal{G}_n\Big\} \le 0 \quad \text{eventually on } \{Z=1\}.
$$

We omit the calculations for they exactly agree with those for proving [13, Lemma A.1(ii)]. Thus, the sequence J_n/L_n^k converges a.s. on $\{Z=1\}$. Furthermore, independence of the B_n yields

$$
\begin{aligned}
\liminf_n \frac{J_n}{n} &= \liminf_n \frac{\sum_{i=1}^n B_i}{n}\,\frac{S_n}{\sum_{i=1}^n B_i}\,Z_n = \liminf_n \frac{\sum_{i=1}^n B_i}{n} \\
&= \liminf_n \frac{\sum_{i=1}^n E(B_i)}{n} \ge \liminf_n E(B_n) > 0 \quad \text{a.s. on } \{Z=1\}.
\end{aligned}
$$

Given any $\gamma<1$, it follows that

$$
\frac{n^\gamma}{L_n^k} = \frac{n^\gamma}{n}\,\frac{n}{J_n}\,\frac{J_n}{L_n^k} \xrightarrow{a.s.} 0 \;\text{ a.s. on } \{Z=1\}.
$$

Thus,

$$
L_n > n^{\gamma/k} \;\text{ eventually on } \{Z=1\}.
$$

Since $k<2$, one can take $\gamma<1$ such that $\gamma/k>1/2$. Therefore, condition (3) holds, and this concludes the proof of Theorem 1.1.

3.2. Proof of Theorem 1.2

In this subsection, it is assumed that

$$m := \lim_n E(B_n) > 0 \quad \text{and} \quad q := \lim_n E(B_n^2).$$

By Theorem 1.1, $Z_n \xrightarrow{a.s.} Z$ for some random variable Z such that $L \le Z \le U$ and $0 < Z < 1$ a.s.

On noting that

$$0 < Z_* = Z = Z^* < 1 \quad \text{a.s.},$$

the same argument used after Lemma 3.2 yields $\sum_n Z_n |H_n| < \infty$ a.s. Since $Z > 0$ a.s., it follows that

$$\sum_n |H_n| < \infty \quad \text{a.s.} \tag{4}$$

Define

$$T_n = \prod_{i=1}^{n-1} (1 + H_i) \quad \text{and} \quad W_n = \frac{Z_n}{T_n}.$$

Condition (4) implies $T_n \xrightarrow{a.s.} T$, for some real random variable $T > 0$, so that

$$W_n \xrightarrow{a.s.} Z/T := W.$$

Our next goal is to show that $\sqrt{n}\,(W_n - W)$ converges conditionally a.s. To this end, we first fix the asymptotic behavior of S_n.

Lemma 3.3. *$S_n/n \xrightarrow{a.s.} m$.*

Proof. Let $Q_n = \sum_{i=1}^n B_i \left[X_i I_{\{Z_{i-1} \ge U\}} + (1 - X_i) I_{\{Z_{i-1} \le L\}} \right]$. Since $Z_i \xrightarrow{a.s.} Z$, then $1 - Z_i > (1 - Z)/2$ eventually. Moreover, $I_{\{Z_{i-1} \ge U\}} + I_{\{Z_{i-1} \le L\}} = |I_{\{Z_{i-1} < U\}} - I_{\{Z_{i-1} > L\}}|$. Therefore,

$$\begin{aligned} B_i \left[X_i I_{\{Z_{i-1} \ge U\}} + (1 - X_i) I_{\{Z_{i-1} \le L\}} \right] &\le B_i \left| I_{\{Z_{i-1} < U\}} - I_{\{Z_{i-1} > L\}} \right| \\ = |H_{i-1}| \frac{S_{i-1} + B_i}{1 - Z_{i-1}} &\le \frac{2\,|H_{i-1}|}{1 - Z} (S_{i-1} + B_i) \quad \text{eventually.} \end{aligned}$$

By condition (4) and Kronecker lemma,

$$\frac{Q_n}{S_n} \le \frac{2}{1 - Z} \frac{1}{S_n} \sum_{i=1}^n |H_{i-1}| S_{i-1} + \frac{2\,c}{1 - Z} \frac{1}{S_n} \sum_{i=1}^n |H_{i-1}| \xrightarrow{a.s.} 0.$$

Hence,

$$\frac{Q_n}{n} = \frac{Q_n}{S_n}\,\frac{S_n}{n} \leq \frac{Q_n}{S_n}\,\frac{r+b+nc}{n} \xrightarrow{a.s.} 0.$$

On noting that $(1/n)\sum_{i=1}^n B_i \xrightarrow{a.s.} m$, one finally obtains

$$\frac{S_n}{n} = \frac{r+b}{n} - \frac{Q_n}{n} + \frac{\sum_{i=1}^n B_i}{n} \xrightarrow{a.s.} m.$$

In view of the next lemma, we recall that

$$\sigma^2 = q\,Z\,(1-Z)/m^2.$$

Lemma 3.4.

$$\sqrt{n}\,(W_n - W) \longrightarrow \mathcal{N}\big(0,\,\sigma^2/T^2\big) \quad \textit{conditionally a.s. with respect to } \mathcal{G}.$$

Proof. First note that W_n can be written as

$$W_n = Z_1 + \sum_{i=1}^{n-1} \frac{\Delta_{i+1}}{T_{i+1}}.$$

Thus, W_n is a $\mathcal{G}$-martingale and the obvious strategy would be applying Lemma 2.2 to $Y_n = W_n$. However, conditions (i)-(ii)-(iii) are not easy to check with $Y_n = W_n$. Accordingly, we adopt an approximation procedure.

Given $\epsilon > 0$, define

$$W_n^{(\epsilon)} = Z_1 + \sum_{i=1}^{n-1} \frac{\Delta_{i+1}\,I_{A_i}}{\epsilon \vee T_{i+1}} \quad \text{where } A_i = \{2\,S_i > i\,m\}.$$

For fixed $\epsilon > 0$, $W_n^{(\epsilon)}$ is still a $\mathcal{G}$-martingale and

$$\sup_n E\big\{(W_n^{(\epsilon)})^2\big\} \leq 1 + \frac{1}{\epsilon^2}\sum_{i=1}^\infty E\big\{\Delta_{i+1}^2\,I_{A_i}\big\} \leq 1 + \frac{c^2}{\epsilon^2}\sum_{i=1}^\infty E\Big\{\frac{I_{A_i}}{S_i^2}\Big\} \leq 1 + \Big(\frac{2\,c}{m\,\epsilon}\Big)^2 \sum_{i=1}^\infty \frac{1}{i^2}.$$

Hence, $W_n^{(\epsilon)} \xrightarrow{a.s.} W^{(\epsilon)}$ for some random variable $W^{(\epsilon)}$. Since $S_n/n \xrightarrow{a.s.} m$, the events A_n are eventually true, so that

$$W - W_n = \sum_{i\geq n} \frac{\Delta_{i+1}}{T_{i+1}} = \sum_{i \geq n} \frac{\Delta_{i+1}\,I_{A_i}}{\epsilon \vee T_{i+1}} = W^{(\epsilon)} - W_n^{(\epsilon)} \quad \text{eventually on } \{T > \epsilon\}.$$

Therefore, it suffices to show that, for fixed $\epsilon > 0$,

$$\sqrt{n}\,\big\{W_n^{(\epsilon)} - W^{(\epsilon)}\big\} \longrightarrow \mathcal{N}\big(0,\,\sigma^2/(\epsilon \vee T)^2\big) \quad \text{conditionally a.s. with respect to } \mathcal{G}.$$

In turn, since $W_n^{(\epsilon)}$ is a uniformly integrable $\mathcal{G}$-martingale, it suffices to check conditions (ii)-(iii) of Lemma 2.2 with $Y_n = W_n^{(\epsilon)}$ and $U = \sigma^2/(\epsilon \vee T)^2$. As to (ii),

$$E\Big\{\big(\sup_n \sqrt{n}\,|W_n^{(\epsilon)} - W_{n-1}^{(\epsilon)}|\big)^4\Big\} \leq \sum_n n^2 E\big\{(W_n^{(\epsilon)} - W_{n-1}^{(\epsilon)})^4\big\}$$
$$\leq \frac{1}{\epsilon^4} \sum_n n^2 E\big\{\Delta_n^4\, I_{A_{n-1}}\big\} \leq \frac{c^4}{\epsilon^4} \sum_n n^2 E\Big\{\frac{I_{A_{n-1}}}{S_{n-1}^4}\Big\} \leq \Big(\frac{2\,c}{m\,\epsilon}\Big)^4 \sum_n \frac{n^2}{(n-1)^4} < \infty.$$

We next turn to condition (iii). We have to prove

$$n \sum_{k\geq n} \big(W_k^{(\epsilon)} - W_{k-1}^{(\epsilon)}\big)^2 = n \sum_{k\geq n} \frac{I_{A_{k-1}}\, \Delta_k^2}{(\epsilon \vee T_k)^2} \xrightarrow{a.s.} \sigma^2/(\epsilon \vee T)^2.$$

Since $T_k \xrightarrow{a.s.} T$, the above condition reduces to

$$n \sum_{k\geq n} I_{A_{k-1}}\, \Delta_k^2 \xrightarrow{a.s.} \sigma^2. \tag{5}$$

Since $1 - Z_k > (1-Z)/2$ eventually and $\sum_k |H_k| < \infty$ a.s., Abel summation formula yields

$$n \sum_{k\geq n} I_{A_{k-1}}\, \frac{B_k^2}{(S_{k-1}+B_k)^2} \Big(I_{\{Z_{k-1}\geq U\}} + I_{\{Z_{k-1}\leq L\}}\Big)$$
$$= n \sum_{k\geq n} I_{A_{k-1}}\, \frac{B_k}{S_{k-1}+B_k}\, \frac{|H_{k-1}|}{1-Z_{k-1}}$$
$$\leq \frac{2\,c\,n}{1-Z} \sum_{k\geq n} I_{A_{k-1}}\, \frac{|H_{k-1}|}{S_{k-1}} \leq \frac{4\,c\,n}{m(1-Z)} \sum_{k\geq n} \frac{|H_{k-1}|}{k-1} \xrightarrow{a.s.} 0.$$

Hence, to get (5), it suffices to prove that

$$n \sum_{k\geq n} I_{A_{k-1}}\, \frac{B_k^2}{(S_{k-1}+B_k)^2}\, (X_k - Z_{k-1})^2 \xrightarrow{a.s.} \sigma^2.$$

Finally, such condition follows from Lemma 2.3 if $E(V_{n+1} \mid \mathcal{G}_n) \xrightarrow{a.s.} \sigma^2$, where

$$V_n = n^2 I_{A_{n-1}}\, \frac{B_n^2}{(S_{n-1}+B_n)^2}\, (X_n - Z_{n-1})^2.$$

In fact,

$$E(V_{n+1} \mid \mathcal{G}_n) = I_{A_n}\, (n+1)^2 E\Big\{\frac{B_{n+1}^2}{(S_n + B_{n+1})^2}\, (X_{n+1} - Z_n)^2 \mid \mathcal{G}_n\Big\}$$
$$\leq (n+1)^2 E\Big\{\frac{B_{n+1}^2}{S_n^2}\, (X_{n+1} - Z_n)^2 \mid \mathcal{G}_n\Big\}$$
$$= \frac{(n+1)^2}{S_n^2}\, E\big\{B_{n+1}^2\, (X_{n+1} - Z_n)^2 \mid \mathcal{G}_n\big\} = \frac{(n+1)^2}{S_n^2}\, E(B_{n+1}^2)\, E\big\{(X_{n+1} - Z_n)^2 \mid \mathcal{G}_n\big\}$$
$$= \frac{(n+1)^2}{S_n^2}\, E(B_{n+1}^2)\, Z_n\, (1 - Z_n) \xrightarrow{a.s.} \frac{q\, Z\, (1-Z)}{m^2} = \sigma^2.$$

Since the events A_n are eventually true, one similarly obtains

$$\begin{aligned} E(V_{n+1} \mid \mathcal{G}_n) &\geq I_{A_n}\, (n+1)^2 E\Big\{ \frac{B_{n+1}^2}{(S_n+c)^2}\, (X_{n+1}-Z_n)^2 \mid \mathcal{G}_n \Big\} \\ &= I_{A_n}\, \frac{(n+1)^2}{(S_n+c)^2}\, E\big\{ B_{n+1}^2\, (X_{n+1}-Z_n)^2 \mid \mathcal{G}_n \big\} \xrightarrow{a.s.} \sigma^2. \end{aligned}$$

Hence, $E(V_{n+1} \mid \mathcal{G}_n) \xrightarrow{a.s.} \sigma^2$. This proves condition (5) and concludes the proof of the lemma.

Theorem 1.2 is a quick consequence of Lemma 3.4. Define in fact

$$D_n = \sqrt{n}\,(Z_n - Z) \quad \text{and} \quad F_n = \prod_{i=n}^{\infty} (1 + H_i).$$

Because of Lemma 3.4,

$$\sqrt{n}\,\big(F_n\, Z_n - Z\big) = T\,\sqrt{n}\,(W_n - W) \longrightarrow \mathcal{N}(0,\, \sigma^2)$$

conditionally a.s. with respect to $\mathcal{G}$.

If $L < Z < U$ a.s., then $L < Z_n < U$ eventually, which in turn implies $F_n = 1$ and $D_n = \sqrt{n}\,\big(F_n\, Z_n - Z\big)$ eventually. Thus $D_n \longrightarrow \mathcal{N}(0,\, \sigma^2)$, conditionally a.s. with respect to $\mathcal{G}$, provided $L < Z < U$ a.s.

Lemma 3.5. $P(L < Z < U) = 1$.

Proof. We just prove $P(Z = L) = 0$. The proof of $P(Z = U) = 0$ is the same. Since $P(Z = L = 0) \leq P(Z = 0) = 0$, it suffices to show that $P(Z = L > 0) = 0$. Let $H = \{Z = L > 0\}$. Toward a contradiction, suppose $P(H) > 0$ and define $Q(\cdot) = P(\cdot \mid H)$. Since $\sqrt{n}\,(Z_n - L)$ is $\mathcal{G}_n$-measurable and

$$D_n = \sqrt{n}\,(Z_n - Z) \leq \sqrt{n}\,(Z_n F_n - Z) \quad \text{if } F_n \geq 1,$$

then

$$\begin{aligned} I_{\{\sqrt{n}\,(Z_n - L) \leq 0\}} &= Q\Big(D_n \leq 0 \mid \mathcal{G}_n\Big) \geq Q\Big(F_n \geq 1,\, \sqrt{n}\,(Z_n F_n - Z) \leq 0 \mid \mathcal{G}_n\Big) \\ &\geq Q\Big(\sqrt{n}\,(Z_n F_n - Z) \leq 0 \mid \mathcal{G}_n\Big) - Q(F_n < 1 \mid \mathcal{G}_n) \quad \text{a.s.} \end{aligned}$$

Since $Z_n < U$ eventually on H, then $F_n \geq 1$ eventually on H. Hence, the martingale convergence theorem in [7] yields $Q(F_n < 1 \mid \mathcal{G}_n) \xrightarrow{Q-a.s.} 0$. By Lemma 2.4 and $\sigma^2 > 0$

a.s., it follows that

$$Q\Big(\sqrt{n}\,(Z_nF_n - Z) \leq 0 \mid \mathcal{G}_n\Big) \overset{Q-a.s.}{\longrightarrow} \mathcal{N}(0,\,\sigma^2)\big((-\infty, 0]\big) = 1/2.$$

Thus, $I_{\{\sqrt{n}\,(Z_n-L)\leq 0\}} \overset{Q-a.s.}{\longrightarrow} 1$, namely, $Z_n \leq L$ eventually on H, which implies the contradiction $Z_n \overset{a.s.}{\longrightarrow} 1$ on H. Thus, $P(H) = 0$.

It remains only to show that Z has non-atomic distribution. This follows from the same argument of Lemma 3.5. Suppose in fact $P(Z = z) > 0$ for some $z \in (0, 1)$ and define $Q(\cdot) = P(\cdot \mid Z = z)$. Then, on the complement of a Q-null set, one obtains the contradiction $\sigma^2 = q\,z\,(1 - z)/m^2 > 0$ and

$$\delta_{\sqrt{n}\,(Z_n-z)}(\cdot) = Q\Big(D_n \in \cdot \mid \mathcal{G}_n\Big) \overset{weakly}{\longrightarrow} \mathcal{N}(0,\,\sigma^2).$$

This concludes the proof of Theorem 1.2.

4. Concluding remarks

Some hints for future research, partly suggested by an anonymous referee, are listed in this section.

- The assumption $R_n = B_n$ makes sense in a number of real applications. But clearly Theorems 1.1 and 1.2 would be much improved if such assumption could be relaxed. In case $L = 0$ and $U = 1$, actually, $R_n = B_n$ may be weakened into

  $$E(R_n) = E(B_n) \quad \text{for all } n \geq 1;$$

  see [6, Cor. 3]. Also, up to minor complications, various points in the proofs of Theorems 1.1 and 1.2 seem to run assuming only that $E(R_n) = E(B_n)$. Thus, we *conjecture* that Theorems 1.1 and 1.2 are still valid if $R_n = B_n$ is replaced by $E(R_n) = E(B_n)$.
- Let $\overline{X}_n = (1/n)\sum_{i=1}^n X_i$ and $C_n = \sqrt{n}\,(\overline{X}_n - Z_n)$. If $L = 0$ and $U = 1$, as shown in [6, Cor. 3], one obtains

  $$C_n \longrightarrow \mathcal{N}\big(0,\,\sigma^2 - Z(1 - Z)\big) \quad \text{stably.}$$

  Thus, the asymptotic behavior of C_n, or even of the pair (C_n, D_n), could be investigated in the general case $0 \leq L < U \leq 1$. Again, this could be (tentatively) performed assuming $E(R_n) = E(B_n)$ instead of $R_n = B_n$.

- Since $\mathcal{G}_0 = \sigma(L, U)$ and $(B_n,\, R_n)$ is independent of $\sigma\big(\mathcal{G}_{n-1},\, X_n\big)$, the barriers L and U could be assumed to be constants (i.e., non random). Such assumption has not been made, however, for it does not simplify any point in the proofs of Theorems 1.1 and 1.2. Rather, L and U could be made dependent on the time. In other terms, L and U could be replaced by L_n and U_n such that $L_n \xrightarrow{a.s.} L$ and $U_n \xrightarrow{a.s.} U$.
- Let μ be the probability distribution of Z. Even if $L = 0$ and $U = 1$, just very little about μ is known; see [1]. Some information on μ, possibly in the general case $0 \leq L < U \leq 1$, would be a major step forward. For instance, under which conditions μ is absolutely continuous with respect to Lebesgue measure? Or else, is it possible to (explicitly) connect μ with the distribution of (L, U)?
- A last (obvious) improvement is considering multicolor urns instead of 2-color urns. Indeed, most additional problems arising in the multicolor case are of the notational type. Similarly, the framework in Subsection 1.1 could be generalized. For instance, $B_n \vee R_n \leq c$ could be replaced by a suitable moment condition. Or else, (B_n, R_n) independent of $\sigma\big(\mathcal{G}_{n-1},\, X_n\big)$ could be replaced by (B_n, R_n) conditionally independent of X_n given $\mathcal{G}_{n-1}$.

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