PRICE UNIQUENESS AND FTAP WITH FINITELY ADDITIVE PROBABILITIES

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Abstract. Let L be a linear space of real bounded random variables on the probability space $(\Omega, \mathcal{A}, P_0)$. A finitely additive probability P on $\mathcal{A}$ such that

$$P \sim P_0 \quad \text{and} \quad E_P(X) = 0 \text{ for each } X \in L$$

is called EMFA (equivalent martingale finitely additive probability). In this paper, EMFA's are investigated in case P_0 is atomic. Existence of EMFA's is characterized and a question raised in [3] is answered. Some results of the following type are obtained as well. Let $y \in \mathbb{R}$ and Y a bounded random variable. Then $X_n + y \xrightarrow{a.s.} Y$, for some sequence $(X_n) \subset L$, provided EMFA's exist and $E_P(Y) = y$ for each EMFA P.

1. Introduction

Let $S = (S_t : t \in T)$ be a real process on the measurable space $(\Omega, \mathcal{A})$, where $T \subset \mathbb{R}$ is any index set. Suppose S is adapted to a filtration $\mathcal{F} = (\mathcal{F}_t : t \in T)$ and S_t is a bounded random variable for each $t \in T$. Then, $(S_t, \mathcal{F}_t : t \in T)$ is a martingale, under a probability measure P on $\mathcal{A}$, if and only if $E_P(X) = 0$ for all X in the linear space

$$L(\mathcal{F}, S) = \text{Span}\,\{I_A\,(S_t - S_s) : s, t \in T,\ s < t,\ A \in \mathcal{F}_s\}.$$

Basing on this fact, given *any* linear space L of bounded random variables on $(\Omega, \mathcal{A})$, a probability measure P on $\mathcal{A}$ such that $E_P(X) = 0$ for all $X \in L$ is said to be a martingale measure. Suppose now that, in addition to L, we are given a reference measure P_0 on $\mathcal{A}$. A martingale measure P satisfying $P \sim P_0$ is an equivalent martingale measure (EMM). Similarly, a *finitely additive probability* (f.a.p.) P on $\mathcal{A}$ such that

$$P \sim P_0 \quad \text{and} \quad E_P(X) = 0 \text{ for each } X \in L$$

is an *equivalent martingale f.a.p.* (EMFA). Here, $P \sim P_0$ means that P and P_0 have the same null sets. Also, from an economic point of view, each $X \in L$ should be viewed as the final outcome of some (admissible) investing strategy.

Existence of EMFA's is investigated in [3]. The main results are recalled in Subsection 2.2. Here, we try to motivate EMFA's and we describe the content of this paper.

Quoting from [3], we list some reasons for dealing with EMFA's.

(i) Dating from de Finetti, the finitely additive theory of probability is well founded and developed, even if not prevailing. F.a.p.'s can be always extended to

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the power set and have a solid motivation in terms of coherence. Also, there are problems which can not be solved in the usual countably additive setting, while admit a finitely additive solution. Examples are in conditional probability, convergence in distribution of non measurable random elements, Bayesian statistics, stochastic integration and the first digit problem. See e.g. [2] and references therein. Note also that, in the finitely additive approach, one can clearly use σ-additive probabilities. Merely, one is not obliged to do so.

(ii) Martingale probabilities play a role in various financial frameworks. Their economic motivations, however, do not depend on whether they are σ-additive or not. See e.g. Chapter 1 of [6]. In option pricing, for instance, EMFA's give arbitrage-free prices just as EMM's. Note also that many underlying ideas, in arbitrage price theory, were anticipated by de Finetti and Ramsey.

(iii) It may be that EMM's fail to exist and yet EMFA's are available; see Examples 1, 7 and 9. In addition, existence of EMFA's can be given simple characterizations; see Theorems 2, 4 and 5.

(iv) Each EMFA P can be written as $P = \alpha\, P_1 + (1-\alpha)\, Q$, where $\alpha \in [0, 1)$, P_1 is a pure f.a.p. and Q a probability measure equivalent to P_0; see Theorem 2. Even if one does not like f.a.p.'s, when EMM's do not exist one may be content with an EMFA P whose α is small enough. In other terms, a fraction α of the total mass must be sacrificed for having equivalent martingale probabilities, but the approximation may look acceptable for small α. An extreme situation of this type is exhibited in Example 9. In such example, EMM's do not exist and yet, for each $\epsilon > 0$, there is an EMFA P with $\alpha \leq \epsilon$.

In connection with points (ii)-(iii) above, and to make the notion of EMFA more transparent, we report a simple example from [3].

Example 1. (Example 7 of [3]). Let $\Omega = \{1, 2, \ldots\}$, $\mathcal{A}$ the power set of Ω, and $P_0\{\omega\} = 2^{-\omega}$ for all $\omega \in \Omega$. For each $n \geq 0$, define $D_n = \{n+1, n+2, \ldots\}$. Define also $L = L(\mathcal{F}, S)$, where

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_n = \sigma\big(\{1\}, \ldots, \{n\}\big), \quad S_0 = 1, \quad \text{and}$$
$$S_n(\omega) = \frac{1}{2^n}\, I_{D_n}(\omega) + \frac{\omega^2 + 2\omega + 2}{2^\omega}\,(1 - I_{D_n}(\omega)) \quad \text{for all } \omega \in \Omega.$$

The process S has been introduced in [1]. Loosely speaking, ω could be regarded as a (finite) stopping time and $S_n(\omega)$ as a price at time n. Such a price falls by 50% at each time $n < \omega$. Instead, for $n \geq \omega$, the price is constant with respect to n and depends on ω only.

If a f.a.p. P satisfies $E_P(X) = 0$ for all $X \in L(\mathcal{F}, S)$, then

$$1 = E_P(S_0) = E_P(S_n) = \frac{P(D_n)}{2^n} + \sum_{j=1}^{n} \frac{j^2 + 2j + 2}{2^j}\, P\{j\}.$$

Letting $n = 1$ in the above equation yields $P\{1\} = 1/4$. By induction, one obtains $2\,P\{n\} = 1/n(n+1)$ for all $n \geq 1$. Since $\sum_{n=1}^{\infty} P\{n\} = 1/2$, then P is not σ-additive. Thus, EMM's do not exist. Instead, EMFA's are available. Define in fact

$$P = \frac{P_1 + Q}{2}$$

where P_1 and Q are probabilities on $\mathcal{A}$ such that $P_1\{n\} = 0$ and $Q\{n\} = 1/n(n+1)$ for all $n \geq 1$. (Note that Q is σ-additive while P_1 is purely finitely additive). Clearly, $P \sim P_0$. Given $X \in L(\mathcal{F}, S)$, since $S_{n+1} = S_n$ on D_n^c, one obtains

$$X = \sum_{j=0}^{k} b_j \, I_{D_j} \, (S_{j+1} - S_j) \quad \text{for some } k \geq 0 \text{ and } b_0, \ldots, b_k \in \mathbb{R}.$$

Since $D_j = \{j+1\} \cup D_{j+1}$ and $S_{j+1} - S_j = -1/2^{j+1}$ on D_{j+1}, it follows that

$$E_{P_1}(X) = \sum_{j=0}^{k} \frac{b_j}{2^{j+1}} \left\{ \big((j+1)^2 + 2(j+1)\big) \, P_1\{j+1\} - P_1(D_{j+1}) \right\} = -\sum_{j=0}^{k} \frac{b_j}{2^{j+1}}$$

and

$$E_Q(X) = \sum_{j=0}^{k} \frac{b_j}{2^{j+1}} \left\{ \big((j+1)^2 + 2(j+1)\big) \, Q\{j+1\} - Q(D_{j+1}) \right\}$$

$$= \sum_{j=0}^{k} \frac{b_j}{2^{j+1}} \left\{ \frac{(j+1)^2 + 2(j+1)}{(j+1)(j+2)} - \frac{1}{(j+2)} \right\} = \sum_{j=0}^{k} \frac{b_j}{2^{j+1}}.$$

Therefore $E_P(X) = 0$, that is, P is an EMFA.

This paper investigates EMFA's when P_0 is an *atomic* probability measure. There are essentially two reasons for focusing on atomic P_0. One is that atomic models look appropriate in several real situations. The second is the following version of the FTAP (fundamental theorem of asset pricing). Let P_0 be atomic and L any linear space of bounded random variables. Then, existence of EMFA's amounts to

$$\overline{L - L_\infty^+} \cap L_\infty^+ = \{0\} \quad \text{with the closure in the norm-topology;}$$

we refer to Subsection 2.2 for details.

Two types of results are obtained for atomic P_0. First, in Subsection 3.1, existence of EMFA's is given a new characterization. Such a characterization looks practically more useful than the existing ones. A question raised in [3] is answered as well (Example 7). Second, in Subsection 3.3, the following problem is addressed. Suppose EMFA's exist and fix a bounded random variable Y. If

$$E_P(Y) = y \qquad \text{for some } y \in \mathbb{R} \text{ and all EMFA's } P,$$

does $Y - y$ belong to the closure of L in some topology ? Or else, if $E_P(Y) \geq 0$ for all EMFA's P, can Y be approximated by random variables of the form $X + Z$ with $X \in L$ and $Z \geq 0$? Indeed, with EMM's instead of EMFA's, these questions are classical; see [4], [8], [9], [13] and references therein. For instance, if Y is regarded as a contingent claim, $E_P(Y) = y$ for all EMFA's P means that y is the unique arbitrage-free price of Y. Similarly $Y - y \in \overline{L}$, with the closure in a suitable topology, can be seen as a weak form of completeness for the underlying market.

A last remark pertains the choice of L in the examples scattered throughout the paper. Recall that L is regarded as the collection of final outcomes of possible investing strategies. This interpretation makes some sense in Examples 1, 7 and 8. Instead, Examples 9 and 13 aim essentially at exhibiting certain technical phenomena.

2. Known results

2.1. **Notation.** In what follows, L is a linear space of real bounded random variables on the probability space $(\Omega, \mathcal{A}, P_0)$. We let

$$\text{ess sup}(X) = \inf\{a \in \mathbb{R} : P_0(X > a) = 0\} = \inf\{\sup_A X : A \in \mathcal{A},\, P_0(A) = 1\},$$
$$\|X\| = \|X\|_\infty = \text{ess sup}(|X|)$$

for each essentially bounded random variable X.

Let $\mathbb{P}$ denote the set of f.a.p.'s on $\mathcal{A}$ and $\mathbb{P}_0 = \{P \in \mathbb{P} : P \text{ is } \sigma\text{-additive}\}$. In particular, $P_0 \in \mathbb{P}_0$. Given $P, T \in \mathbb{P}$, we write $P \ll T$ if $P(A) = 0$ whenever $A \in \mathcal{A}$ and $T(A) = 0$, and $P \sim T$ if $P \ll T$ and $T \ll P$. We also write

$$E_P(X) = \int X\, dP$$

whenever $P \in \mathbb{P}$ and X is a real bounded random variable.

A f.a.p. P is *pure* if it does not have a non trivial σ-additive part. Precisely, if P is pure and Γ is a σ-additive measure such that $0 \le \Gamma \le P$, then $\Gamma = 0$. By a result of Yosida-Hewitt, any $P \in \mathbb{P}$ can be written as $P = \alpha\, P_1 + (1 - \alpha)\, Q$ where $\alpha \in [0, 1]$, $P_1 \in \mathbb{P}$ is pure and $Q \in \mathbb{P}_0$.

A P_0-*atom* is a set $A \in \mathcal{A}$ with $P_0(A) > 0$ and $P_0(\cdot \mid A) \in \{0, 1\}$; P_0 is *atomic* if there is a countable partition $A_1, A_2, \ldots$ of Ω such that A_n is a P_0-atom for all n.

2.2. **Existence of EMFA's.** We next state a couple of results from [3]. Let

$$\mathbb{M} = \{P \in \mathbb{P} : P \sim P_0 \text{ and } E_P(X) = 0 \text{ for all } X \in L\}$$

be the set of EMFA's. Note that $\mathbb{M} \cap \mathbb{P}_0$ is the set of EMM's.

Theorem 2. *Each $P \in \mathbb{M}$ admits the representation $P = \alpha\, P_1 + (1 - \alpha)\, Q$ where $\alpha \in [0, 1)$, $P_1 \in \mathbb{P}$ is pure, $Q \in \mathbb{P}_0$ and $Q \sim P_0$. Moreover, $\mathbb{M} \ne \emptyset$ if and only if*

$$E_Q(X) \le k \text{ ess sup}(-X), \quad X \in L, \tag{1}$$

for some constant $k > 0$ and $Q \in \mathbb{P}_0$ with $Q \sim P_0$. In particular, under condition (1)*, one obtains*

$$\frac{k\, P_1 + Q}{k + 1} \in \mathbb{M} \quad \textit{for some } P_1 \in \mathbb{P}.$$

Remark 3. In condition (1), Q can be replaced by any $T \in \mathbb{P}$ such that $T \sim P_0$. Precisely, if $E_T(X) \le k \text{ ess sup}(-X)$, $X \in L$, for some $k > 0$ and $T \in \mathbb{P}$ with $T \sim P_0$, then $(k\, P_1 + T)/(k + 1) \in \mathbb{M}$ for some $P_1 \in \mathbb{P}$. This is easily seen by repeating the proof of Theorem 3 of [3] with T in the place of Q.

In addition to characterizing $\mathbb{M} \ne \emptyset$, Theorem 2 provides some information on the weight $1 - \alpha$ of the σ-additive part Q of an EMFA. Indeed, under (1), there is $P \in \mathbb{M}$ such that $\alpha \le k/(k + 1)$. On the other hand, condition (1) is not very helpful in real problems, for it requires to have Q in advance. A characterization independent of Q would be more effective. We will come back to this point in the next section.

We next turn to separation theorems. Write $U - V = \{u - v : u \in U,\, v \in V\}$ whenever U, V are subsets of a linear space. Let

$$L_p = L_p(\Omega, \mathcal{A}, P_0) \quad \text{for all } p \in [0, \infty].$$

We regard L as a subspace of L_∞ and we let $L_p^+ = \{X \in L_p : X \geq 0\}$. Since L_∞ is the dual of L_1, it can be equipped with the weak-star topology $\sigma(L_\infty, L_1)$. Thus, $\sigma(L_\infty, L_1)$ is the topology on L_∞ generated by the maps $X \mapsto E_{P_0}(X\,Y)$ for all $Y \in L_1$.

By a result of Kreps [11] (see also [12]) existence of EMM's amounts to

$$\overline{L - L_\infty^+} \cap L_\infty^+ = \{0\} \quad \text{with the closure in } \sigma(L_\infty, L_1).$$

On the other hand, it is usually argued that the norm topology on L^∞ is geometrically more transparent than $\sigma(L_\infty, L_1)$, and results involving the former are often viewed as superior. Thus, a (natural) question is what happens if the closure is taken in the norm-topology.

Theorem 4. $\mathbb{M} \neq \emptyset$ *if and only if*

$$L_\infty^+ \subset U \cup \{0\} \quad \textit{and} \quad (L - L_\infty^+) \cap U = \emptyset$$
$$\textit{for some norm-open convex set } U \subset L_\infty.$$

In particular, a necessary condition for $\mathbb{M} \neq \emptyset$ *is*

$$\overline{L - L_\infty^+} \cap L_\infty^+ = \{0\} \quad \textit{with the closure in the norm-topology.} \tag{2}$$

If P_0 *is atomic, condition* (2) *is sufficient for* $\mathbb{M} \neq \emptyset$ *as well.*

Condition (2) is essentially the *no free lunch with vanishing risk* condition of [5]. See also [6]. The main difference is that, in [5], L is a suitable class of stochastic integrals (in a fixed time interval and driven by a fixed semi-martingale) and fails to be a linear space. In this paper, instead, L is an arbitrary subspace of L_∞.

One more remark concerns the classical *no-arbitrage* condition

$$P_0(X > 0) > 0 \quad \Longleftrightarrow \quad P_0(X < 0) > 0 \quad \text{for each } X \in L, \tag{3}$$

or equivalently $(L - L_0^+) \cap L_0^+ = \{0\}$. Since $L \subset L_\infty$, such condition can be written as $(L - L_\infty^+) \cap L_\infty^+ = \{0\}$. Hence, $\mathbb{M} \neq \emptyset$ implies no-arbitrage (just apply Theorem 4). Instead, it may be that $\mathbb{M} = \emptyset$ and yet P_0 is atomic and condition (3) holds; see Example 8. Thus, (2) implies (3) but not conversely, even if P_0 is atomic.

It is still open whether condition (2) implies $\mathbb{M} \neq \emptyset$ for arbitrary $P_0 \in \mathbb{P}_0$. However, (2) is equivalent to $\mathbb{M} \neq \emptyset$ when P_0 is atomic. This is a first reason for paying special attention to the latter case. A second (and more important) reason is that atomic models are suitable in various real situations. Accordingly, in the sequel we focus on atomic P_0.

3. New results

In this section, $(\Omega, \mathcal{A}, P_0)$ is an *atomic* probability space. Everything is well understood if P_0 has finitely many atoms only (such a case can be reduced to that of Ω finite). Thus, the P_0-atoms are assumed to be *infinitely many.* Let $A_1, A_2, \ldots$ be a countable partition of Ω such that A_n is a P_0-atom for each n. Also, $X|A_n$ denotes the a.s.-constant value of the random variable X on A_n.

3.1. Existence of EMFA's in the atomic case.

Theorem 2 gives a general characterization of existence of EMFA's. As already noted, however, a characterization not involving Q would be more usable in real problems. In case P_0 is atomic, one such characterization is actually available.

Theorem 5. *Let P_0 be atomic. Suppose that, for each $n \geq 1$, there is a constant $k_n > 0$ satisfying*

$$X|A_n \leq k_n \text{ ess sup}(-X) \quad \text{for each } X \in L. \tag{4}$$

Letting $\beta = \inf_n k_n$, for each $\alpha \in \big(\frac{\beta}{1+\beta},\, 1\big)$ one obtains

$$\alpha\, P_1 + (1-\alpha)\, Q \in \mathbb{M} \quad \text{for some } P_1 \in \mathbb{P} \text{ and } Q \in \mathbb{P}_0 \text{ with } Q \sim P_0.$$

Moreover, condition (4) is necessary for $\mathbb{M} \neq \emptyset$ (so that $\mathbb{M} \neq \emptyset$ if and only if (4) holds).

Proof. Suppose first $\mathbb{M} \neq \emptyset$. Fix $P \in \mathbb{M}$, $n \geq 1$ and $X \in L$. Since $E_P(X) = 0$,

$$P(A_n)\, X|A_n \leq P(A_n)\, X^+|A_n \leq E_P(X^+)$$
$$= E_P(X) + E_P(X^-) = E_P(X^-) \leq \text{ess sup}(-X).$$

Therefore, condition (4) holds with $k_n = 1/P(A_n)$. Conversely, suppose (4) holds. Fix any sequence $(q_n : n \geq 1)$ satisfying $q_n > 0$ for all n, $\sum_n q_n = 1$ and $\sum_n (q_n/k_n) < \infty$. For each $A \in \mathcal{A}$, define

$$I(A) = \{n : P_0(A \cap A_n) > 0\} \quad \text{and} \quad Q(A) = \frac{\sum_{n \in I(A)} (q_n/k_n)}{\sum_n (q_n/k_n)}.$$

Then, $Q \in \mathbb{P}_0$ and $Q \sim P_0$. Also, for each $X \in L$, condition (4) yields

$$E_Q(X) = \sum_n Q(A_n)\, X|A_n \leq \text{ess sup}(-X) \sum_n Q(A_n)\, k_n = \frac{\text{ess sup}(-X)}{\sum_n (q_n/k_n)}.$$

Thus, condition (1) holds with $k = \big\{\sum_n (q_n/k_n)\big\}^{-1}$. By Theorem 2, there is $P_1 \in \mathbb{P}$ such that $(k\, P_1 + Q)/(k+1) \in \mathbb{M}$. Finally, fix $\alpha \in \big(\frac{\beta}{1+\beta},\, 1\big)$. Condition (4) remains true if the k_n are replaced by arbitrary constants $k_n^* \geq k_n$. Thus, it can be assumed $\sup_n k_n = \infty$. In this case, it suffices to note that

$$k = \frac{1}{\sum_n (q_n/k_n)} = \frac{\alpha}{1-\alpha}$$

for a suitable choice of $(q_n : n \geq 1)$. □

It is not hard to see that condition (4) can be written as

$$\sup_{X \in L^*} X|A_n < \infty \quad \text{for each } n \geq 1, \text{ where}$$
$$L^* = \{X \in L : X \geq -1 \text{ a.s.}\}.$$

Thus, (4) can be given the following interpretation. Let $K \subset L$ be any set of final outcomes of possible investing strategies. Roughly speaking, if K is uniformly bounded from below (on all of Ω) then K can not be unbounded from above on some atom A_n. This is a viability condition for the market.

Another such condition, called *no-arbitrage of the first kind* (say NA_1), is investigated in [10]. NA_1 is stated in terms of a nonnegative semi-martingale S, to be viewed as the discounted price process of a financial asset. It turns out that NA_1 amounts to the existence of a f.a.p. $P \in \mathbb{P}$ such that: (i) P is σ-additive when restricted to certain sub-σ-fields; (ii) P makes S a local martingale; (iii) $P \sim P_0$. We refer to [10] for the precise statement of NA_1 in terms of S. Here we note that, when adapted to the present framework, NA_1 can be written as:

(NA$_1$) For each $Z \in L_0^+$, $P_0(Z > 0) > 0$, there is a constant $a \in (0,1)$ such that

$$P_0\big\{a\,(X+1) < Z\big\} > 0 \quad \text{whenever } X \in L^*.$$

Actually, NA$_1$ is a no-arbitrage condition for the collection L of possible final outcomes. In particular, NA$_1$ is stronger than (3).

If $P \in \mathbb{M}$, $Z \in L_0^+$ and $P_0(Z > 0) > 0$, then $E_P(Z) > 0$ and $E_P\{a\,(X+1)\} = a$ for all $a > 0$ and $X \in L^*$. Thus, $\mathbb{M} \neq \emptyset$ implies NA$_1$. A (natural) question is whether the converse holds as well. This is actually true when P_0 is atomic.

Corollary 6. *If P_0 is atomic,*

$$NA_1 \quad \Longleftrightarrow \quad \mathbb{M} \neq \emptyset \quad \Longleftrightarrow \quad \overline{L - L_\infty^+} \cap L_\infty^+ = \{0\}$$

with the closure in the norm-topology.

Proof. By Theorem 4, since P_0 is atomic, $\mathbb{M} \neq \emptyset$ amounts to $\overline{L - L_\infty^+} \cap L_\infty^+ = \{0\}$. It has been noted above that $\mathbb{M} \neq \emptyset$ implies NA$_1$. Hence, we have only to prove that NA$_1$ implies $\mathbb{M} \neq \emptyset$. Suppose that NA$_1$ holds. Fix $n \geq 1$. By NA$_1$, applied to $Z = I_{A_n}$, there is a constant $a_n \in (0,1)$ such that $P_0\big\{a_n\,(Y+1) < I_{A_n}\big\} > 0$ for all $Y \in L^*$. Since $a_n\,(Y+1) \geq 0 = I_{A_n}$ a.s. on A_n^c, it follows that

$$Y|A_n < (1/a_n) - 1 \quad \text{for all } Y \in L^*.$$

Define $k_n = (1/a_n) - 1$ and fix $X \in L$ with $P_0(X \neq 0) > 0$. If ess sup$(-X) \leq 0$, then $X \in L_0^+$ and $P_0(X > 0) > 0$, and NA$_1$ fails for $Z = X$. (In fact, $X/a \in L^*$ and $a\big\{(X/a) + 1\big\} > X$ for each $a > 0$). Thus, ess sup$(-X) > 0$. Let $Y = X/\text{ess sup}(-X)$. Since $Y \in L^*$, one obtains $X|A_n < k_n\,\text{ess sup}(-X)$. Thus, Theorem 5 yields $\mathbb{M} \neq \emptyset$. □

3.2. **Examples.** In view of Theorem 2, a sufficient (but not necessary) condition for $\mathbb{M} \neq \emptyset$ is

$$\text{ess sup}\,(X) \leq k\,\text{ess sup}(-X), \quad X \in L, \tag{5}$$

for some constant $k > 0$. In the atomic case, condition (5) can be seen as a uniform version of (4). Indeed, (5) amounts to (4) and $\sup_n k_n < \infty$.

If $\lim_n X|A_n = 0$ for all $X \in L$, condition (5) implies $\mathbb{M} \cap \mathbb{P}_0 \neq \emptyset$ (that is, existence of EMM's); see Example 5 of [3]. An open problem is whether condition (5) alone yields $\mathbb{M} \cap \mathbb{P}_0 \neq \emptyset$. We now prove that the answer is no.

Example 7. Let $\Omega = \{-1,1\}^\infty$ and $X_n : \Omega \to \{-1,1\}$ the n-th coordinate map, $n \geq 1$. Take $\mathcal{A} = \sigma(X_1, X_2, \ldots)$ and L the linear space generated by the sequence (X_n). Also, take P_0 such that (X_n) is independent with $P_0(X_n = -1) = 1/(n+1)^2$. Define $A = \cup_{n\geq 1} \cap_{k \geq n} \{X_k = 1\}$. Since $\sum_n P_0(X_n = -1) < \infty$, then $P_0(A) = 1$. Thus, P_0 is atomic (for A is countable) and $E_Q|X_n - 1| \to 0$ for each $Q \in \mathbb{P}_0$ with $Q \sim P_0$. In particular, no $Q \in \mathbb{P}_0$ satisfies $Q \sim P_0$ and $E_Q(X_n) = 0$ for all n. However, condition (5) holds with $k = 1$. Fix in fact $X \in L$, say $X = \sum_{i=1}^n b_i X_i$ for some $n \geq 1$ and $b_1, \ldots, b_n \in \mathbb{R}$. Since $P_0(X_1 = x_1, \ldots, X_n = x_n) > 0$ for all $x_1, \ldots, x_n \in \{-1,1\}$, one obtains

$$\text{ess sup}\,(X) = |b_1| + \ldots + |b_n| = \text{ess sup}\,(-X).$$

Even if P_0 is atomic, it may be that $\mathbb{M} = \emptyset$ and yet the classical no-arbitrage condition (3) is satisfied. We prove this fact by adapting Example 6 of [3].

Example 8. Take $(\Omega, \mathcal{A}, P_0)$ and (X_n) as in previous Example 7. Define

$$S_0 = 0, \quad S_n = \sum_{i=1}^n X_i, \quad \mathcal{F}_n = \sigma(S_0, S_1, \ldots, S_n), \quad L = L(\mathcal{F}, S),$$

where $L(\mathcal{F}, S)$ has been defined in Section 1. Since $P_0\big(X_1 = x_1, \ldots, X_n = x_n\big) > 0$ for all n and $x_1, \ldots, x_n \in \{-1, 1\}$, the no-arbitrage condition (3) applies. However, $\mathbb{M} = \emptyset$. Suppose in fact $P \in \mathbb{M}$. Since $E_P(X) = 0$ for all $X \in L$, it is easily seen that (X_n) is i.i.d. under P with $P(X_1 = 1) = P(X_1 = -1) = 1/2$; see Example 6 of [3]. Let $Q_0 \in \mathbb{P}_0$ be the unique σ-additive probability on $\mathcal{A}$ which makes (X_n) i.i.d. with $Q_0(X_1 = 1) = Q_0(X_1 = -1) = 1/2$. Then, $Q_0 = P$ on $\cup_n \mathcal{F}_n$. By Theorem 2,

$$Q_0 = P \geq (1 - \alpha)\, Q \quad \text{on } \cup_n \mathcal{F}_n$$

for some $\alpha < 1$ and $Q \in \mathbb{P}_0$ such that $Q \sim P_0$. Since $Q, Q_0 \in \mathbb{P}_0$ and $\cup_n \mathcal{F}_n$ is a field, it follows that $Q_0 \geq (1 - \alpha)\, Q$ on $\sigma\big(\cup_n \mathcal{F}_n\big) = \mathcal{A}$. Hence, $P_0 \sim Q \ll Q_0$. But this is a contradiction, for $P_0(A) = Q_0(A^c) = 1$ where $A = \cup_{n \geq 1} \cap_{k \geq n} \{X_k = 1\}$.

Our last example has been discussed in point (iv) of Section 1.

Example 9. Let $\Omega = \{1, 2, \ldots\}$. Take $\mathcal{A}$ to be the power set and $P_0\{\omega\} = 2^{-\omega}$ for all $\omega \in \Omega$. Define $T = (2P_0 + P^* - \delta_1)/2$, where $P^* \in \mathbb{P}$ is any pure f.a.p. and δ_1 the point mass at 1. Since $P^*\{\omega\} = 0$ for all $\omega \in \Omega$, then $T\{1\} = 0$ and $T \in \mathbb{P}$. Let $B = \{2, 3, \ldots\}$ and define L to be the linear space generated by $\{I_A - T(A)\, I_B : A \subset B\}$. If $P \in \mathbb{P}$ satisfies $E_P(X) = 0$ for all $X \in L$, then

$$P\{n, n+1, \ldots\} = T\{n, n+1, \ldots\}\, P(B) \geq \frac{P(B)}{2} \quad \text{for all } n > 1.$$

Thus, $P \notin \mathbb{P}_0$ as far as $P(B) > 0$, so that $\mathbb{M} \cap \mathbb{P}_0 = \emptyset$. On the other hand,

$$P_\epsilon := \epsilon\, T + (1 - \epsilon)\, \delta_1 \in \mathbb{M} \quad \text{for all } \epsilon \in (0, 1).$$

In fact, $P_\epsilon\{\omega\} > 0$ for all $\omega \in \Omega$ (so that $P_\epsilon \sim P_0$) and

$$E_{P_\epsilon}(X) = \epsilon\, E_T(X) + (1 - \epsilon)\, X(1) = 0 \quad \text{for all } X \in L.$$

To sum up, in this example, EMM's do not exist and yet, for each $\epsilon > 0$, there is $P \in \mathbb{M}$ such that $\alpha(P) \leq \epsilon$. Here, $\alpha(P)$ denotes the weight of the pure part of P, in the sense that $P = \alpha(P)\, P_1 + (1 - \alpha(P))\, Q$ for some pure f.a.p. P_1 and $Q \in \mathbb{P}_0$ with $Q \sim P_0$.

3.3. **Superhedging and attainability type results.** Suppose $\mathbb{M} \neq \emptyset$ and fix $Y \in L_\infty$. If $E_P(Y) = y$ for some $y \in \mathbb{R}$ and all $P \in \mathbb{M}$, does $Y - y$ belong to the closure of L in some reasonable topology ? Or else, if $E_P(Y) \geq 0$ for all $P \in \mathbb{M}$, can Y be approximated by random variables of the form $X + Z$ with $X \in L$ and $Z \in L_\infty^+$? Up to replacing EMFA's with EMM's, questions of this type are classical; see [4], [8], [9], [13] and references therein. Indeed, regarding Y as a contingent claim, $E_P(Y) = y$ for all $P \in \mathbb{M}$ means that y is the unique arbitrage-free price of Y. Similarly $Y - y \in \overline{L}$, with the closure in a suitable topology, can be seen as a weak form of completeness for the underlying market.

In the sequel, L_∞ is equipped with the norm-topology. Accordingly, for each $H \subset L_\infty$, $\overline{H}$ denotes the closure of H in the norm-topology.

Theorem 10. *Suppose P_0 atomic, $\mathbb{M} \neq \emptyset$, and fix $Y \in L_\infty$. Then,*

(i) $Y \in \overline{L - L_\infty^+} \quad \Longleftrightarrow \quad E_P(Y) \leq 0$ *for each* $P \in \mathbb{M}$,

(ii) $Y \in \bigcap_{P \in \mathbb{M}} \overline{L}^P \quad \Longleftrightarrow \quad E_P(Y) = 0$ *for each* $P \in \mathbb{M}$,

where $\overline{L}^P$ denotes the closure of L in the $L_1(P)$-topology. In addition, if $E_P(Y) = 0$ for each $P \in \mathbb{M}$, then $X_n \xrightarrow{a.s.} Y$ for some sequence $(X_n) \subset L$.

Proof. First note that "$\Longrightarrow$" is obvious in both (i) and (ii). Suppose $Y \notin \overline{L - L_\infty^+}$. Fix $A \in \mathcal{A}$ with $P_0(A) > 0$ and define

$$U = \overline{L - L_\infty^+}, \quad V = \{\alpha I_A + (1-\alpha) Y : 0 \leq \alpha \leq 1\}.$$

Then, $U \cap V = \emptyset$. In fact, $I_A \notin U$ because of $\mathbb{M} \neq \emptyset$ and Theorem 4. If $\alpha I_A + (1-\alpha) Y \in U$ for some $\alpha < 1$, there are $(X_n) \subset L$ and $(Z_n) \subset L_\infty^+$ such that $X_n - Z_n \xrightarrow{L_\infty} \alpha I_A + (1-\alpha) Y$, which in turn implies

$$\frac{X_n - (Z_n + \alpha I_A)}{1 - \alpha} \xrightarrow{L_\infty} Y.$$

But this is a contradiction, as $Y \notin U$. Next, since U and V are convex and closed with V compact, some linear (continuous) functional $\Phi : L_\infty \to \mathbb{R}$ satisfies

$$\inf_{f \in V} \Phi(f) > \sup_{f \in U} \Phi(f).$$

It is routine to verify that Φ is positive and $\Phi(1) > 0$. Hence, $\Phi(f) = \Phi(1)\, E_{P_A}(f)$ for all $f \in L_\infty$ and some $P_A \in \mathbb{P}$ with $P_A \ll P_0$. Since L is a linear space and $\sup_{f \in L} \Phi(f) \leq \sup_{f \in U} \Phi(f) < \infty$, then $\Phi = 0$ on L. To sum up, P_A satisfies $P_A \ll P_0$, $P_A(A) > 0$, $E_{P_A}(Y) > 0$, and $E_{P_A}(X) = 0$ for all $X \in L$. It follows that

$$P := \sum_n \frac{1}{2^n} P_{A_n} \in \mathbb{M} \quad \text{and} \quad E_P(Y) > 0. \tag{6}$$

This concludes the proof of (i). Suppose now that $E_P(Y) = 0$ for all $P \in \mathbb{M}$. By (i), there are sequences $(X_n) \subset L$ and $(Z_n) \subset L_\infty^+$ such that $X_n - Z_n \xrightarrow{L_\infty} Y$. For each $P \in \mathbb{M}$, since $Z_n \in L_\infty^+$ and $E_P(X_n) = E_P(Y) = 0$, one obtains

$$E_P|X_n - Y| \leq E_P|X_n - Z_n - Y| + E_P(Z_n)$$
$$= E_P|X_n - Z_n - Y| - E_P(X_n - Z_n - Y) \leq 2\,\|X_n - Z_n - Y\| \longrightarrow 0.$$

This proves (ii). Finally, take $P \in \mathbb{M}$, say $P = \alpha\, P_1 + (1-\alpha)\, Q$ where $\alpha \in [0,1)$, $P_1 \in \mathbb{P}$, $Q \in \mathbb{P}_0$ and $Q \sim P_0$. Arguing as above,

$$E_Q(Z_n) \leq \frac{E_P(Z_n)}{1-\alpha} \leq \frac{\|X_n - Z_n - Y\|}{1-\alpha} \longrightarrow 0.$$

Thus, $Z_{n_j} \xrightarrow{a.s.} 0$ and $X_{n_j} = Z_{n_j} + (X_{n_j} - Z_{n_j}) \xrightarrow{a.s.} Y$ for some subsequence (n_j). □

Remark 11. In the above proof, P_0 atomic is used only in definition (6), to get an EMFA P such that $E_P(Y) > 0$ starting from the collection

$$\mathcal{P} = \{P_A : A \in \mathcal{A},\, P_0(A) > 0\}.$$

If the P_A were σ-additive, such a P could be obtained without assuming atomicity of P_0. In fact, the model $\mathcal{P}$ is dominated (by P_0) so that, by Halmos-Savage theorem, $\mathcal{P}$ is equivalent to some countable subset $\{P_{A_1}, P_{A_2}, \ldots\} \subset \mathcal{P}$. See e.g. Theorem

5.2.3 of [6] and Theorem 1.61 of [7]. But this classical argument does not work here, for the P_A need not be σ-additive.

As regards part (ii) of Theorem 10, a question is whether $E_P(Y) = 0$ for all $P \in \mathbb{M}$ implies $Y \in \overline{L}$. The answer is generally no, while it is yes if $\mathbb{M}$ is rich enough. We finally prove these two facts. To this end, the following lemma is useful.

Lemma 12. *Let P_0 be atomic and $P \in \mathbb{P}$. If $P \ll P_0$ and $P(A_n) = P_0(A_n)$ for all n, then $P = P_0$.*

Proof. Fix $A \in \mathcal{A}$ and $n \geq 1$. If $P_0(A \cap A_n) = 0$, then $P(A \cap A_n) = 0 = P_0(A \cap A_n)$. If $P_0(A \cap A_n) > 0$, then $P_0(A^c \cap A_n) = 0$, and thus

$$P(A \cap A_n) = P(A_n) = P_0(A_n) = P_0(A \cap A_n).$$

It follows that $P(A) \geq \sum_{i=1}^n P(A \cap A_i) = \sum_{i=1}^n P_0(A \cap A_i)$. As $n \to \infty$, one obtains $P(A) \geq P_0(A)$. Finally, taking complements yields $P = P_0$. □

Example 13. Let L be the linear space generated by $\{I_{A_n} - P_0(A_n) : n \geq 1\}$ and

$$Y = \frac{I_A}{P_0(A)} - \frac{I_{A^c}}{P_0(A^c)} \quad \text{where } A = \cup_{n=1}^{\infty} A_{2n}.$$

Each $P \in \mathbb{M}$ meets $P \ll P_0$ and $P(A_n) = P_0(A_n)$ for all n. Thus, Lemma 12 yields $\mathbb{M} = \{P_0\}$. Further, $E_{P_0}(Y) = 0$. However, $Y \notin \overline{L}$. Fix in fact $X \in L$. Since $X = x$ a.s. on the set $\big(\cup_{i=1}^n A_i\big)^c$, for some $n \geq 1$ and $x \in \mathbb{R}$, one obtains

$$\|Y - X\| = \sup_i |(Y - X)|A_i| \geq \sup_{i>n} |(Y - x)|A_i|$$
$$= \max\Big\{\Big|\frac{1}{P_0(A)} - x\Big|, \Big|\frac{1}{P_0(A^c)} + x\Big|\Big\} \geq \frac{1}{P_0(A)} \wedge \frac{1}{P_0(A^c)}.$$

Suppose condition (5) holds. Arguing as in the proof of the next corollary, for each $P \in \mathbb{P}$ such that $P \ll P_0$, there is $P^* \in \mathbb{M}$ of the form $P^* = \gamma P + (1 - \gamma) \widetilde{P}$ where $\gamma > 0$ and $\widetilde{P} \in \mathbb{P}$. Thus, a plenty of EMFA's are available under (5).

Corollary 14. *Suppose P_0 atomic, condition* (5) *holds, and fix $Y \in L_\infty$. Then, $Y \in \overline{L}$ if and only if $E_P(Y) = 0$ for all $P \in \mathbb{M}$.*

Proof. By (5), $\mathbb{M} \neq \emptyset$. Suppose $E_P(Y) = 0$ for all $P \in \mathbb{M}$. By part (i) of Theorem 10, one obtains $X_n - Z_n \xrightarrow{L_\infty} Y$ for some sequences $(X_n) \subset L$ and $(Z_n) \subset L_\infty^+$. Fix $P \in \mathbb{P}$ such that $P \ll P_0$ and define $T = (P + P_0)/2$. By (5),

$$E_T(X) \leq \text{ess sup}(X) \leq k \,\text{ess sup}(-X), \quad X \in L,$$

for some constant $k > 0$. Since $T \sim P_0$, Remark 3 yields

$$P^* := \frac{2k\, P_1 + P + P_0}{2(k+1)} = \frac{k\, P_1 + T}{k+1} \in \mathbb{M} \quad \text{for some } P_1 \in \mathbb{P}.$$

As $P^* \in \mathbb{M}$, arguing as in the proof of Theorem 10 one obtains

$$E_P(Z_n) \leq 2(k+1)\, E_{P^*}(Z_n) = -2(k+1)\, E_{P^*}(X_n - Z_n - Y)$$
$$\leq 2(k+1)\, \|X_n - Z_n - Y\| \longrightarrow 0.$$

Since P is arbitrary (as far as $P \in \mathbb{P}$ and $P \ll P_0$) it follows that $Z_n \longrightarrow 0$ in the weak topology of L_∞. In turn, $Z_n \longrightarrow 0$ weakly implies $\|Z_n^*\| \longrightarrow 0$ for some sequence (Z_n^*) of convex combinations of (Z_n), say

$$Z_n^* = \sum_{i=n}^{m_n} b_{i,n} Z_i$$

where $n \leq m_n < \infty$, $b_{i,n} \geq 0$ and $\sum_{i=n}^{m_n} b_{i,n} = 1$. Let $X_n^* = \sum_{i=n}^{m_n} b_{i,n} X_i$. Then, $X_n^* \in L$ and

$$\|X_n^* - Y\| \leq \|Z_n^*\| + \sum_{i=n}^{m_n} b_{i,n} \|X_i - Z_i - Y\| \longrightarrow 0.$$

This concludes the proof of the "if" part, while the "only if' is trivial. □

Example 7 exhibits a situation where condition (5) holds, so that Corollary 14 applies.

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