A note on the absurd law of large numbers in economics

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Abstract

Let Γ be a Borel probability measure on $\mathbb{R}$ and $(T, \mathcal{C}, Q)$ a nonatomic probability space. Define $\mathcal{H} = \{H \in \mathcal{C} : Q(H) > 0\}$. In some economic models, the following condition is requested. There are a probability space $(\Omega, \mathcal{A}, P)$ and a real process $X = \{X_t : t \in T\}$ satisfying

$$\text{for each } H \in \mathcal{H}, \text{ there is } A_H \in \mathcal{A} \text{ with } P(A_H) = 1 \text{ such that}$$
$$t \mapsto X(t, \omega) \text{ is measurable and } Q\big(\{t : X(t, \omega) \in \cdot\} \mid H\big) = \Gamma(\cdot) \text{ for } \omega \in A_H.$$

Such a condition fails if P is countably additive, $\mathcal{C}$ countably generated and Γ non trivial. Instead, as shown in this note, it holds for any $\mathcal{C}$ and Γ under a finitely additive probability P. Also, X can be taken to have any given distribution.

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1 Introduction and result

Let $(T, \mathcal{C}, Q)$ and $(\Omega, \mathcal{A}, P)$ be probability spaces and $X : T \times \Omega \to \mathbb{R}$ a real stochastic process, indexed by T and defined on $(\Omega, \mathcal{A}, P)$. Denote by $X_t(\cdot) = X(t, \cdot)$ and $X^\omega(\cdot) = X(\cdot, \omega)$ the X-sections with respect to $t \in T$ and $\omega \in \Omega$. Since X is a process, $X_t : \Omega \to \mathbb{R}$ is measurable for fixed $t \in T$.

In various economic frameworks, T is the set of agents and X_t the individual risk of agent $t \in T$. The process X is i.i.d., in the sense that $X_{t_1}, \ldots, X_{t_n}$ are i.i.d. random variables for all $n \geq 1$ and all distinct $t_1, \ldots, t_n \in T$. Also, T is viewed as "very large" and this is formalized by assuming Q nonatomic.

Let Γ denote the distribution common to the X_t. So, Γ is a Borel probability measure on $\mathbb{R}$ such that $X_t \sim \Gamma$ for all $t \in T$. Define also

$$\mathcal{H} = \{H \in \mathcal{C} : Q(H) > 0\}.$$

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The informal idea underlying most economic models is that, for large T, individual risks disappear in the aggregate. To make this intuition precise, it is assumed that

$$X^\omega \text{ is measurable and } \; Q(X^\omega \in \cdot) = \Gamma(\cdot) \; \text{ for } P\text{-almost all } \omega \in \Omega. \tag{1}$$

Moreover, condition (1) is often strengthened as follows

$$\begin{gathered} \text{for each } H \in \mathcal{H}, \text{ there is } A_H \in \mathcal{A} \text{ with } P(A_H) = 1 \text{ such that} \\ X^\omega \text{ is measurable and } \; Q\big(X^\omega \in \cdot \mid H\big) = \Gamma(\cdot) \; \text{ for } \omega \in A_H. \end{gathered} \tag{2}$$

In economics, condition (1) is usually called *law of large numbers* or else *no aggregate uncertainty.* Condition (2) was first emphasized by Feldman and Gilles in [4]. As each $H \in \mathcal{H}$ is a coalition of agents, following the suggestion of an anonymous referee, (2) may be called *coalitional aggregate certainty.* This note focus on (2).

It is not hard to prove that, when $\mathcal{C}$ is countably generated, condition (2) implies that Γ is 0-1 valued; see Section 1 of [4] and Theorem 4.2 of [7] (to make the paper self-contained, a proof is also given in Remark 2). Thus, to get (2) with non trivial Γ and countably generated $\mathcal{C}$, an extension of $(T, \mathcal{C}, Q)$ is to be involved.

One (interesting) approach is to look for reasonable extensions, that is, extensions which grant (2) and some other properties, such as a form of Fubini's theorem. This route is followed by [6], [7], [8]. In Theorem 2.8 of [7], condition (2) is shown to be true if X is *essentially pairwise independent* and measurable with respect to a *Fubini extension* of the product σ-field. Conditions for such an X to exist are given in [6] and [8]. These conditions require $(T, \mathcal{C}, Q)$ to be extended if Γ is non trivial and $\mathcal{C}$ countably generated.

A different route, closer to the ideas of [5], is taken in this note. On one hand, we aim to avoid extensions of $(T, \mathcal{C}, Q)$ and to obtain *any given* distribution for X. Thus, we do not require X i.i.d., but we allow $X \sim \mathcal{P}$ for any consistent set $\mathcal{P}$ of finite dimensional distributions (see Section 2 for precise definitions). On the other hand, we content ourselves with proving consistency of (2) with $X \sim \mathcal{P}$.

Our result is the following. As in most economic models, suppose $(T, \mathcal{C}, Q)$ is given with Q nonatomic and $\{t\} \in \mathcal{C}$ for all $t \in T$. In addition, fix a Borel probability measure Γ on $\mathbb{R}$ and a consistent set $\mathcal{P}$ of finite dimensional distributions. Note that Γ and $\mathcal{P}$ are now arbitrary and not necessarily connected.

Theorem 1. *If $(T, \mathcal{C}, Q)$, Γ and $\mathcal{P}$ are as above, there are a finitely additive probability space $(\Omega, \mathcal{A}, P)$ and a process $X : T \times \Omega \to \mathbb{R}$ such that $X \sim \mathcal{P}$ and condition* (2) *holds.*

In Theorem 1, Ω is the set of all functions $\omega : T \to \mathbb{R}$ and X the canonical process $X(t, \omega) = \omega(t)$. As first noted by Doob in [2], such an X is not measurable with respect to the product σ-field $\mathcal{C} \otimes \mathcal{G}$ where $\mathcal{G} = \sigma(X_t : t \in T)$. Other related results are Theorem 1 of [3], Proposition 3 of [4] and Propositions 6.1 and 6.4 of [7].

Dating from de Finetti, the finitely additive theory of probability is well founded and developed, even if not prevailing. It finds applications in various fields, ranging from statistics and number theory to economics. The spirit of Theorem 1 is that, in such theory, one can always assume condition (2) and $X \sim \mathcal{P}$ for any Γ and $\mathcal{P}$.

Plainly, as Γ and $\mathcal{P}$ are arbitrary, Theorem 1 may also lead to "absurd" claims. (Incidentally, this explains the title of this note). If $T = [0, \infty)$, for instance, one could take $\Gamma = \delta_{x_0}$ for some $x_0 \in \mathbb{R}$ and $\mathcal{P}$ such that X is a Brownian motion.

However, in the subjective approach, the existence of different probability evaluations (modelling different opinions) should be viewed as a merit. It is a task of the economist to choose Γ and $\mathcal{P}$ in a reasonable way. Once the choice is done, in the economist's view, the question is: can I assume condition (2) and $X \sim \mathcal{P}$? In a finitely additive setting, the answer is: yes, you can, but any other choice of Γ and $\mathcal{P}$ (possibly meaningless or absurd) is consistent with (2) as well.

2 Proof and remarks

In this note, a collection $\mathcal{P}$ of finite dimensional distributions is meant as

$$\mathcal{P} = \{\, \mu(t_1, \ldots, t_n) : n \geq 1,\, t_1, \ldots, t_n \in T\}$$

where each $\mu(t_1, \ldots, t_n)$ is a Borel probability measure on $\mathbb{R}^n$. We write $X \sim \mathcal{P}$ if $X = \{X_t : t \in T\}$ is a real process, indexed by T, satisfying $(X_{t_1}, \ldots, X_{t_n}) \sim \mu(t_1, \ldots, t_n)$ for all $n \geq 1$ and $t_1, \ldots, t_n \in T$. We say that $\mathcal{P}$ is *consistent* in case $X \sim \mathcal{P}$ for some process X. Simple conditions for $\mathcal{P}$ to be consistent are given by the well known Kolmogorov extension theorem.

An atom of Q is a set $C \in \mathcal{C}$ such that $Q(C) > 0$ and $Q(\cdot \mid C)$ is 0-1 valued. If Q has no atoms, it is called *nonatomic*. In case T is a separable metric space and $\mathcal{C}$ the Borel σ-field, Q is nonatomic if and only if $Q\{t\} = 0$ for all $t \in T$. Next, for each $H \subset T$, define the Q-outer measure $Q^*(H) = \inf\{Q(C) : H \subset C \in \mathcal{C}\}$. If $Q^*(H) = 1$, then Q can be extended to a probability measure Q_0 on $\sigma(\mathcal{C} \cup \{H\})$ such that $Q_0(H) = 1$.

We are now able to prove Theorem 1.

Proof of Theorem 1. Let Ω be the set of functions $\omega : T \to \mathbb{R}$ and X the canonical process $X(t, \omega) = \omega(t)$ for all $(t, \omega) \in T \times \Omega$. Let $\mathcal{G}$ be the σ-field on Ω generated by the maps $\omega \mapsto \omega(t)$ for all $t \in T$. Note that $X^\omega = \omega$ for all $\omega \in \Omega$. Also, since $\mathcal{P}$ is consistent, there is a probability measure $\mathbb{P}$ on $\mathcal{G}$ such that $X \sim \mathcal{P}$ under $\mathbb{P}$.

Let $\mathcal{H}_0 \subset \mathcal{H}$ be finite. Then, $\mathbb{P}$ can be extended to a probability measure P_0 such that

$$X^\omega \text{ is measurable and } Q\big(X^\omega \in \cdot \mid H\big) = \Gamma(\cdot) \text{ for } H \in \mathcal{H}_0 \text{ and } P_0\text{-almost all } \omega. \tag{3}$$

The proof of (3) is similar to those of Theorem 2.2 of [2] and Proposition 6.1 of [7]. Define

$$A = \{\omega \in \Omega : X^\omega \text{ is measurable and } Q\big(X^\omega \in \cdot \mid H\big) = \Gamma(\cdot) \text{ for all } H \in \mathcal{H}_0\}.$$

It suffices to prove $\mathbb{P}^*(A) = 1$. In turn, for $\mathbb{P}^*(A) = 1$, it suffices $A \neq \emptyset$ and

$$\omega \in A,\, \omega^* \in \Omega,\, \{t : \omega^*(t) \neq \omega(t)\} \text{ countable} \quad \Rightarrow \quad \omega^* \in A. \tag{4}$$

Condition (4) trivially holds as $\mathcal{C}$ includes the singletons (so that X^{ω^*} is measurable) and Q is nonatomic (so that $Q(X^{\omega^*} \neq X^\omega) = 0$). To prove $A \neq \emptyset$, let Π be the partition of T formed by the constituents of the members of $\mathcal{H}_0$. Since $\mathcal{H}_0$ is finite, Π is finite. Fix $K \in \Pi \cap \mathcal{H}$. Since Q is nonatomic, $Q(\cdot \mid K)$ is nonatomic. By Theorem 3.1 of [1], since $Q(\cdot \mid K)$ is nonatomic, there is a measurable function $f_K : T \to \mathbb{R}$ satisfying $Q\big(f_K \in \cdot \mid K\big) = \Gamma(\cdot)$. Define $\omega = f_K$ on K, for all $K \in \Pi \cap \mathcal{H}$, and ω constant otherwise. Then, $X^\omega = \omega$ is measurable. For each $H \in \mathcal{H}_0$, since H is a union of elements of Π and $Q\big(f_K \in \cdot \mid K\big) = \Gamma(\cdot)$, one obtains

$$Q\big(X^\omega \in \cdot \mid H\big) = \sum_{K \in \Pi \cap \mathcal{H}} Q\big(X^\omega \in \cdot \mid K\big) Q(K \mid H) = \sum_{K \in \Pi \cap \mathcal{H}} Q\big(f_K \in \cdot \mid K\big) Q(K \mid H) = \Gamma(\cdot).$$

Therefore $\omega \in A$, and this concludes the proof of (3).

Next, let $\mathcal{A}$ be the power set of Ω and $\mathcal{Z}$ the collection of all $[0, 1]$-valued functions defined on $\mathcal{A}$. For $H \in \mathcal{H}$, define

$$A_H = \{\omega \in \Omega : X^\omega \text{ is measurable and } Q\big(X^\omega \in \cdot \mid H\big) = \Gamma(\cdot)\},$$
$$F_H = \{Z \in \mathcal{Z} : Z \text{ is a finitely additive probability, } Z = \mathbb{P} \text{ on } \mathcal{G},\, Z(A_H) = 1\}.$$

Let $\mathcal{Z}$ be equipped with the product topology. Fix $H \in \mathcal{H}$ and a net $(Z_\alpha) \subset F_H$ such that $Z_\alpha \longrightarrow Z$ for some $Z \in \mathcal{Z}$. Since $Z(A) = \lim_\alpha Z_\alpha(A)$ for all $A \in \mathcal{A}$, then $Z \in F_H$. Hence,

F_H is closed. Let $\mathcal{H}_0 \subset \mathcal{H}$ be finite. By (3), there is a probability measure P_0, defined on a suitable σ-field, such that $P_0 = \mathbb{P}$ on $\mathcal{G}$ and $P_0\big(\bigcap_{H\in\mathcal{H}_0} A_H\big) = 1$. Then $Z \in \bigcap_{H\in\mathcal{H}_0} F_H$, where Z is any finitely additive extension of P_0 to $\mathcal{A}$. (Such a Z is well known to exist, because of Hahn-Banach theorem). Hence, $\{F_H : H \in \mathcal{H}\}$ is a family of closed sets satisfying the finite intersection property. Since $\mathcal{Z}$ is compact, this fact implies

$$\bigcap_{H\in\mathcal{H}} F_H \neq \emptyset.$$

To conclude the proof, it suffices to take any $P \in \bigcap_{H\in\mathcal{H}} F_H$. □

We finally give a couple of remarks.

Remark 2. *If $(\Omega, \mathcal{A}, P)$ is a (countably additive) probability space, condition (2) holds and $\mathcal{C}$ is countably generated, then Γ is 0-1 valued.* Take in fact a countable field $\mathcal{F}$ such that $\mathcal{C} = \sigma(\mathcal{F})$. By (2) and $\mathcal{F}$ countable, there is $A \in \mathcal{A}$ with $P(A) = 1$ and

$$X^\omega \text{ is measurable and } Q\big(X^\omega \in \cdot \mid H\big) = \Gamma(\cdot) \text{ for all } H \in \mathcal{F}\cap\mathcal{H} \text{ and } \omega \in A.$$

Fix $\omega \in A$ and a Borel set $B \subset \mathbb{R}$. Since $\mathcal{F}$ is a field and $\mathcal{C} = \sigma(\mathcal{F})$, it follows that $Q\big(H \cap \{X^\omega \in B\}\big) = \Gamma(B)\, Q(H)$ for all $H \in \mathcal{C}$. Letting $H = T$ yields $Q(X^\omega \in B) = \Gamma(B)$. Hence, for $H = \{X^\omega \in B\}$, one obtains $\Gamma(B) = Q(X^\omega \in B) = \Gamma(B)\, Q(X^\omega \in B) = \Gamma(B)^2$.

Remark 3. Suppose that, rather than a single law Γ, we are given a collection $\{\Gamma_H : H \in \mathcal{H}\}$ of Borel probability measures on $\mathbb{R}$. Replacing Γ with $\{\Gamma_H : H \in \mathcal{H}\}$, condition (2) turns into

$$\begin{aligned} &\text{for each } H \in \mathcal{H}, \text{ there is } A_H \in \mathcal{A} \text{ with } P(A_H) = 1 \text{ such that} \\ &X^\omega \text{ is measurable and } Q\big(X^\omega \in \cdot \mid H\big) = \Gamma_H(\cdot) \text{ for } \omega \in A_H. \end{aligned} \tag{2*}$$

Condition (2*) looks (to us) a reasonable extension of (2). Roughly speaking, for each coalition $H \in \mathcal{H}$, there is no aggregate uncertainty on H but the compensation of individual risks may depend on H. Moreover, for suitable $\{\Gamma_H : H \in \mathcal{H}\}$, condition (2*) can be realized under a countably additive P without extending $(T, \mathcal{C}, Q)$.

Suppose $(T, \mathcal{C}, Q)$ and $\mathcal{P}$ are as in Theorem 1 and $\{\Gamma_H : H \in \mathcal{H}\}$ is of the form

$$\Gamma_H(\cdot) = Q\big(f \in \cdot \mid H\big), \quad H \in \mathcal{H}, \text{ for some measurable function } f : T \to \mathbb{R}.$$

Then, there are a (countably additive) probability space $(\Omega, \mathcal{A}, P)$ and a process $X : T \times \Omega \to \mathbb{R}$ such that $X \sim \mathcal{P}$ and condition (2) holds.*

Such result can be proved by the same argument of Theorem 1. As an example, if $T \subset \mathbb{R}$ and $\mathcal{C}$ is the Borel σ-field, it applies to $\Gamma_H(B) = Q(B \cap T \mid H)$ where $B \subset \mathbb{R}$ is a Borel set (just take $f : T \to \mathbb{R}$ the inclusion map). This choice of $\{\Gamma_H : H \in \mathcal{H}\}$ is tempting in a few situations, for instance when $T = [0, 1]$ and Q is Lebesgue measure.

References

[1] Berti P., Pratelli L., Rigo P. (2007) Skorohod representation on a given probability space, *Prob. Theo. Rel. Fields*, 137, 277-288.

[2] Doob J.L. (1937) Stochastic processes depending on a continuous parameter, *Trans. Amer. Math. Soc.*, 42, 107-140.

[3] Dubins L.E. (1999) Paths of finitely additive Brownian motion need not be bizarre, *Seminaire de probabilites*, 33, 395-396.

[4] Feldman M., Gilles C. (1985) An expository note on individual risk without aggregate uncertainty, *J. Econ. Theory*, 35, 26-32.

[5] Judd K.L. (1985) The law of large numbers with a continuum of IID random variables, *J. Econ. Theory*, 35, 19-25.

[6] Podczeck K. (2010) On existence of rich Fubini extensions, *Econom. Theory*, 45, 1-22.

[7] Sun Y.N. (2006) The exact law of large numbers via Fubini extension and characterization of insurable risks, *J. Econ. Theory*, 126, 31-69.

[8] Sun Y.N., Zhang Y. (2009) Individual risk and Lebesgue extension without aggregate uncertainty, *J. Econ. Theory*, 144, 432-443.