0-1 LAWS FOR REGULAR CONDITIONAL DISTRIBUTIONS

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ABSTRACT. Let $(\Omega, \mathcal{B}, P)$ be a probability space, $\mathcal{A} \subset \mathcal{B}$ a sub-σ-field, and μ a regular conditional distribution for P given $\mathcal{A}$. Necessary and sufficient conditions for $\mu(\omega)(A)$ to be 0-1, for all $A \in \mathcal{A}$ and $\omega \in A_0$, where $A_0 \in \mathcal{A}$ and $P(A_0) = 1$, are given. Such conditions apply, in particular, when $\mathcal{A}$ is a tail sub-σ-field. Let $H(\omega)$ denote the $\mathcal{A}$-atom including the point $\omega \in \Omega$. Necessary and sufficient conditions for $\mu(\omega)(H(\omega))$ to be 0-1, for all $\omega \in A_0$, are also given. If $(\Omega, \mathcal{B})$ is a standard space, the latter 0-1 law is true for various classically interesting sub-σ-fields $\mathcal{A}$, including tail, symmetric, invariant, as well as some sub-σ-fields connected with continuous time processes.

1. INTRODUCTION AND MOTIVATIONS

Let $(\Omega, \mathcal{B}, P)$ be a probability space and $\mathcal{A} \subset \mathcal{B}$ a sub-σ-field. A *regular conditional distribution* (r.c.d.) for P given $\mathcal{A}$ is a mapping $\mu : \Omega \to \mathbb{P}$, where $\mathbb{P}$ denotes the set of probability measures on $\mathcal{B}$, such that $\mu(\cdot)(B)$ is a version of $E(I_B \mid \mathcal{A})$ for all $B \in \mathcal{B}$. A σ-field is *countably generated* (c.g.) in case it is generated by one of its countable subclasses. In the sequel, it is assumed that P admits a r.c.d. given $\mathcal{A}$ and

$$\mu \text{ denotes a } \textit{fixed} \text{ r.c.d., for } P \text{ given } \mathcal{A}, \text{ and } \mathcal{B} \text{ is c.g..} \tag{1}$$

Moreover,

$$H(\omega) = \bigcap_{\omega \in A \in \mathcal{A}} A$$

is the atom of $\mathcal{A}$ including the point $\omega \in \Omega$.

Heuristically, conditioning to $\mathcal{A}$ should mean conditioning to the atom of $\mathcal{A}$ which actually occurs, say $H(\omega)$, and the probability of $H(\omega)$ given $H(\omega)$ should be 1. If this interpretation is agreed, μ should be everywhere *proper*, that is, $\mu(\omega)(A) = I_A(\omega)$ for all $A \in \mathcal{A}$ and $\omega \in \Omega$. Though $\mu(\cdot)(A) = I_A(\cdot)$ a.s. for *fixed* $A \in \mathcal{A}$, however, μ can behave quite inconsistently with properness.

Say that $\mathcal{A}$ is c.g. under P in case the trace σ-field $\mathcal{A} \cap C = \{A \cap C : A \in \mathcal{A}\}$ is c.g. for some $C \in \mathcal{A}$ with $P(C) = 1$. If $\mathcal{A}$ is c.g. under P then

$$\mu(\omega)(A) = I_A(\omega) \quad \text{for all } A \in \mathcal{A} \text{ and } \omega \in A_0, \tag{2}$$

where, here and in what follows, A_0 designates some set of $\mathcal{A}$ with $P(A_0) = 1$. In general, the exceptional set A_0^c can not be removed. Further, (2) implies that $\mathcal{A} \cap A_0$ is c.g.. See Blackwell (1955), Blackwell and Ryll-Nardzewski (1963) and Theorem 1 of Blackwell and Dubins (1975). In other terms, not only everywhere properness is to be weakened into condition (2), but the latter holds if and only if $\mathcal{A}$ is c.g. under P.

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If $\mathcal{A}$ fails to be c.g. under P, various weaker notions of r.c.d. have been investigated. Roughly speaking, in such notions, μ is asked to be everywhere proper but σ-additivity and/or measurability are relaxed. See Berti and Rigo (1999) and references therein. However, not very much is known on r.c.d.'s, regarded in the usual sense, when $\mathcal{A}$ is not c.g. under P (one exception is Seidenfeld et al. (2001)). In particular, when (2) fails, one question is whether some of its consequences are still available.

In this paper, among these consequences, we focus on:

$$\mu(\omega)\big(H(\omega)\big) \in \{0,1\} \quad \text{for all } \omega \in A_0, \tag{3}$$

$$\mu(\omega)(A) \in \{0,1\} \quad \text{for all } A \in \mathcal{A} \text{ and } \omega \in A_0. \tag{4}$$

Note that (4) implies (3) in case $H(\omega) \in \mathcal{A}$ for all $\omega \in A_0$, and that, for (3) to make sense, one needs to assume $H(\omega) \in \mathcal{B}$ for all $\omega \in A_0$.

Both conditions (3) and (4) worth some attention.

Investigating (3) can be seen as a development of the seminal work of Blackwell and Dubins (1975). The conjecture that (3) holds (under mild conditions) is supported by those examples in the literature where $\mathcal{B}$ is c.g.. In these examples, in fact, either $\mu(\omega)\big(H(\omega)\big) = 1$ a.s. or $\mu(\omega)\big(H(\omega)\big) = 0$ a.s.. See for instance Seidenfeld et al. (2001).

Condition (4) seems to have been neglected so far, though it is implicit in some ideas of Dynkin (1978) and Diaconis and Freedman (1981). In any case, (4) holds in a number of real situations and can be attached a clear heuristic meaning. As to the latter, fix $\omega_0 \in \Omega$. Since $\mu(\omega_0)$ comes out by conditioning on $\mathcal{A}$, one could expect that μ is a r.c.d. for $\mu(\omega_0)$ given $\mathcal{A}$, too. Condition (4) grants that this is true, provided $\omega_0 \in A_0$. More precisely, letting $M = \{Q \in \mathbb{P} : \mu \text{ is a r.c.d. for } Q \text{ given } \mathcal{A}\}$, condition (4) is equivalent to

$$\mu(\omega) \in M \quad \text{for all } \omega \in A_0;$$

see Theorem 12.

Since $\mathcal{B}$ is c.g., $P(\mu \neq \nu) = 0$ for any other r.c.d. ν, and this is basic for (3) and (4). For some time, we guessed that $\mathcal{B}$ c.g. is enough for (3) and (4). Instead, as we now prove, some extra conditions are needed. Let

$$\mathcal{N} = \{B \in \mathcal{B} : P(B) = 0\}.$$

Example 1. (A failure of condition (3)**)** Let $\Omega = \mathbb{R}$, $\mathcal{B}$ the Borel σ-field, Q a probability measure on $\mathcal{B}$ vanishing on singletons, and $P = \frac{1}{2}(Q + \delta_0)$. If $\mathcal{A} = \sigma(\mathcal{N})$, then $\mu = P$ a.s. and $H(\omega) = \{\omega\}$ for all ω, so that $H(0) = \{0\} \notin \mathcal{A}$ and $\mu(0)\{0\} = P\{0\} = \frac{1}{2}$.

Incidentally, Example 1 exhibits also a couple of (perhaps unexpected) facts. Unless $H(\omega) \in \mathcal{A}$ for all $\omega \in A_0$, (4) does not imply (3). Further, it may be that $\mu(\omega)\big(H(\omega)\big) < 1$, for a single point ω ($\omega = 0$ in Example 1), even though $H(\omega) \in \mathcal{B}$ and $\mu(\omega)(A) = I_A(\omega)$ for all $A \in \mathcal{A}$.

Example 2. (A failure of condition (4)**)** Let $\Omega = \mathbb{R}^2$, $\mathcal{B}$ the Borel σ-field, and $P = Q \times Q$ where Q is the $N(0,1)$ law on the real Borel sets. Denoting $\mathcal{G}$ the σ-field on Ω generated by $(x,y) \mapsto x$, a (natural) r.c.d. for P given $\mathcal{G}$ is $\mu((x,y)) = \delta_x \times Q$. With such a μ, condition (4) fails if $\mathcal{A}$ is taken to be $\mathcal{A} = \sigma\big(\mathcal{G} \cup \mathcal{N}\big)$. In fact, μ is also a r.c.d. for P given $\mathcal{A}$, and for all (x,y) one has $\{x\}\times[0,\infty)\in\mathcal{A}$ while

$$\mu((x,y))\Big(\{x\}\times[0,\infty)\Big)=\frac{1}{2}.$$

Note also that (3) holds in this example, since $H(\omega)=\{\omega\}$ and $\mu(\omega)\{\omega\}=0$ for all $\omega\in\Omega$.

This paper provides necessary and sufficient conditions for (3) and (4). Special attention is devoted to the particular case where $\mathcal{A}$ is a *tail sub-σ-field*, i.e., $\mathcal{A}$ is the intersection of a non increasing sequence of *countably generated* sub-σ-fields. The main results are Theorems 3, 4, 8 and 15. Theorem 15 states that (4) is always true whenever $\mathcal{A}$ is a tail sub-σ-field. Theorems 3, 4 and 8 deal with condition (3). One consequence of Theorem 4 is that, when $(\Omega,\mathcal{B})$ is a standard space, (3) holds for various classically interesting sub-σ-fields $\mathcal{A}$, including tail, symmetric, invariant, as well as some sub-σ-fields connected with continuous time processes.

2. When regular conditional distributions are 0-1 on the (appropriate) atoms of the conditioning σ-field

This section deals with condition (3). It is split into three subsections.

2.1. **Basic results.** For any map $\nu:\Omega\rightarrow\mathbb{P}$, we write $\sigma(\nu)$ for the σ-field generated by $\nu(B)$ for all $B\in\mathcal{B}$, where $\nu(B)$ stands for the real function $\omega\mapsto\nu(\omega)(B)$. Since $\mathcal{B}$ is c.g., $\sigma(\nu)$ is c.g., too. In particular, $\sigma(\mu)$ is c.g.. Let

$$\mathcal{A}_P=\{A\subset\Omega:\exists A_1,A_2\in\mathcal{A}\text{ with }A_1\subset A\subset A_2\text{ and }P(A_2-A_1)=0\}$$

be the completion of $\mathcal{A}$ with respect to $P|\mathcal{A}$. The only probability measure on $\mathcal{A}_P$ agreeing with P on $\mathcal{A}$ is still denoted by P. Further, in case $H(\omega)\in\mathcal{B}$ for all ω, we let

$$f_B(\omega)=\mu(\omega)\big(B\cap H(\omega)\big)\quad\text{for }\omega\in\Omega\text{ and }B\in\mathcal{B},$$
$$f=f_\Omega,\quad S=\{f>0\}.$$

We are in a position to state our first characterization of (3).

Theorem 3. *Suppose* (1) *holds and $H(\omega)\in\mathcal{B}$ for all ω. For each $U\in\mathcal{A}$ such that the trace σ-field $\mathcal{A}\cap U$ is c.g., there is $U_0\in\mathcal{A}$ with $U_0\subset U$, $P(U-U_0)=0$ and $f(\omega)=1$ for all $\omega\in U_0$. Moreover, if $S\in\mathcal{A}_P$, then condition* (3) *is equivalent to each of the following conditions (a)-(b):*

- **(a)** *f_B is $\mathcal{A}_P$-measurable for all $B\in\mathcal{B}$;*
- **(b)** *$\mathcal{A}\cap U$ is c.g. for some $U\in\mathcal{A}$ with $U\subset S$ and $P(S-U)=0$.*

Proof. Suppose $\mathcal{A}\cap U$ is c.g. for some $U\in\mathcal{A}$ and define

$$\mathcal{A}_0=\{(A\cap U)\cup F:A\in\mathcal{A},F=\emptyset\text{ or }F=U^c\}.$$

Let $H_0(\omega)$ be the $\mathcal{A}_0$-atom including ω. Then, $\mathcal{A}_0$ is c.g. and $H_0(\omega)=H(\omega)$ for $\omega\in U$. A r.c.d. for P given $\mathcal{A}_0$ is $\mu_0(\omega)=I_U(\omega)\mu(\omega)+I_{U^c}(\omega)\alpha$, where $\alpha(\cdot)=P(\cdot\mid U^c)$ if $P(U)<1$ and α is any fixed element of $\mathbb{P}$ if $P(U)=1$. Since $\mathcal{A}_0$ is c.g., there is $K\in\mathcal{A}_0$ with $P(K)=1$ and $\mu_0(\omega)\big(H_0(\omega)\big)=1$ for all $\omega\in K$. Since $f(\omega)=\mu(\omega)\big(H(\omega)\big)=\mu_0(\omega)\big(H_0(\omega)\big)=1$ for each $\omega\in K\cap U$, it suffices to let $U_0=K\cap U$.

Next, suppose $S\in\mathcal{A}_P$ and take $C,D\in\mathcal{A}$ such that $C\subset S$, $D\subset S^c$ and $P(C\cup D)=1$.

"(3) $\Rightarrow$ (a)". Let $A = A_0 \cap C$, where $A_0 \in \mathcal{A}$ is such that $P(A_0) = 1$ and $f(\omega) \in \{0,1\}$ for all $\omega \in A_0$. Fix $B \in \mathcal{B}$. Since $f_B \leq f = 0$ on D and $f_B = \mu(B)$ on A, one obtains $I_{A\cup D} f_B = I_A f_B = I_A \mu(B)$. Thus, f_B is $\mathcal{A}_P$-measurable.

"(a) $\Rightarrow$ (b)". Given any $\alpha \in \mathbb{P}$, define the map $\nu : \Omega \to \mathbb{P}$ by

$$\nu(\omega)(B) = I_S(\omega)\frac{f_B(\omega)}{f(\omega)} + I_{S^c}(\omega)\alpha(B) \quad \omega \in \Omega, B \in \mathcal{B}.$$

Then, (a) implies that $\sigma(\nu) \subset \mathcal{A}_P$. Fix a countable field $\mathcal{B}_0$ generating $\mathcal{B}$, and, for each $B \in \mathcal{B}_0$, take a set $A_B \in \mathcal{A}$ such that $P(A_B) = 1$ and $I_{A_B}\nu(B)$ is $\mathcal{A}$-measurable. Define $U = \big(\bigcap_{B\in\mathcal{B}_0} A_B\big) \cap C$ and note that $U \in \mathcal{A}$, $U \subset S$ and $P(S - U) = 0$. Since $I_U \nu(B)$ is $\mathcal{A}$-measurable for each $B \in \mathcal{B}_0$, it follows that $\sigma(\nu)\cap U \subset \mathcal{A}\cap U$. Since $A\cap U = \{\nu(A) = 1\}\cap U$ for all $A \in \mathcal{A}$, then $\mathcal{A}\cap U \subset \sigma(\nu)\cap U$. Hence, $\mathcal{A} \cap U = \sigma(\nu) \cap U$ is c.g..

"(b) $\Rightarrow$ (3)". By the first assertion of the theorem, since $U \in \mathcal{A}$ and $\mathcal{A}\cap U$ is c.g., there is $U_0 \in \mathcal{A}$ with $U_0 \subset U$, $P(U - U_0) = 0$ and $f = 1$ on U_0. Define $A_0 = U_0 \cup D$ and note that $A_0 \in \mathcal{A}$ and $f \in \{0,1\}$ on A_0. Since $U \subset S$ and $P(S - U) = 0$, one also obtains $P(A_0) = P(U_0) + P(D) = P(S) + P(S^c) = 1$. □

A basic condition for existence of disintegrations is that

$$G = \{(x,y) \in \Omega\times\Omega : H(x) = H(y)\}$$

belongs to $\mathcal{B}\otimes\mathcal{B}$; see Berti and Rigo (1999). Such a condition also plays a role in our main characterization of (3). Let

$$\mathcal{A}^* = \{B \in \mathcal{B} : B \text{ is a union of } \mathcal{A}\text{-atoms}\}.$$

Theorem 4. *If* (1) *holds and* $G \in \mathcal{B}\otimes\mathcal{B}$, *then* $H(\omega) \in \mathcal{B}$ *for all* $\omega \in \Omega$ *and* f_B *is* $\mathcal{A}^*$*-measurable for all* $B \in \mathcal{B}$. *If in addition* $S \in \mathcal{A}_P$, *then each of conditions* (3), *(a) and (b) is equivalent to*

(c) $\mathcal{A} \cap U = \mathcal{A}^* \cap U$ *for some* $U \in \mathcal{A}$ *with* $U \subset S$ *and* $P(S - U) = 0$.

Proof. For $C \subset \Omega\times\Omega$, let $C_\omega = \{u \in \Omega : (\omega,u) \in C\}$ be the ω-section of C. Suppose (1) holds and $G \in \mathcal{B}\otimes\mathcal{B}$. Then, $H(\omega) = G_\omega \in \mathcal{B}$ for all ω. By a monotone class argument, the map $\omega \mapsto \mu(\omega)(C_\omega)$ is $\mathcal{B}$-measurable whenever $C \in \mathcal{B}\otimes\mathcal{B}$. Letting $C = G \cap (\Omega \times B)$, where $B \in \mathcal{B}$, implies that f_B is $\mathcal{B}$-measurable. Since f_B is constant on each $\mathcal{A}$-atom, it is in fact $\mathcal{A}^*$-measurable. (Note that $\mathcal{A}^*$-measurability of f_B does not require $\mathcal{B}$ c.g.). Next, suppose also that $S \in \mathcal{A}_P$. By Theorem 3, conditions (3), (a) and (b) are equivalent. Suppose (c) holds, and fix $B \in \mathcal{B}$ and a Borel set $I \subset \mathbb{R}$. Since $\{f_B \in I\} \in \mathcal{A}^*$, condition (c) yields $\{f_B \in I\} \cap U \in \mathcal{A}$, and $\{f_B \in I\}\cap(S-U) \in \mathcal{A}_P$ due to $(S-U) \in \mathcal{A}_P$ and $P(S-U) = 0$. Thus (assuming $0 \in I$ to fix ideas),

$$\{f_B \in I\} = S^c \cup \big(\{f_B \in I\} \cap (S-U)\big) \cup \big(\{f_B \in I\} \cap U\big) \in \mathcal{A}_P,$$

so that (a) holds. Conversely, suppose (a) holds. For each $B \in \mathcal{B}$, there is $A_B \in \mathcal{A}$ such that $P(A_B) = 1$ and $I_{A_B} f_B$ is $\mathcal{A}$-measurable. Letting $A = \bigcap_{B\in\mathcal{B}_0} A_B$, where $\mathcal{B}_0$ is a countable field generating $\mathcal{B}$, it follows that $A \in \mathcal{A}$, $P(A) = 1$ and $I_A f_B$ is $\mathcal{A}$-measurable for all $B \in \mathcal{B}$. Since $S \in \mathcal{A}_P$ and $P(A) = 1$, there is $U \in \mathcal{A}$ with $U \subset A \cap S$ and $P(S - U) = 0$. Given $B \in \mathcal{A}^*$, on noting that $f_B = I_B f$, one obtains

$$B \cap U = \{I_B f > 0\} \cap U = \{f_B > 0\} \cap U = \big(\{f_B > 0\} \cap A\big) \cap U \in \mathcal{A}.$$

Hence $\mathcal{A} \cap U = \mathcal{A}^* \cap U$, i.e., condition (c) holds. □

We now state a couple of corollaries to Theorem 4. The first covers in particular the case where the $\mathcal{A}$-atoms are the singletons, while the second (and more important) applies to various real situations.

Corollary 5. *Suppose* (1) *holds,* $S \in \mathcal{A}_P$ *and* $\mathcal{A}$, $\mathcal{B}$ *have the same atoms. Then,* (3) *holds if and only if* $\mathcal{A} \cap U = \mathcal{B} \cap U$ *for some* $U \in \mathcal{A}$ *with* $U \subset S$ *and* $P(S-U) = 0$.

Proof. Since $\mathcal{A}$, $\mathcal{B}$ have the same atoms and $\mathcal{B}$ is c.g.,

$$G = \{(x, y) : x \text{ and } y \text{ are in the same } \mathcal{B}\text{-atom}\} \in \mathcal{B} \otimes \mathcal{B}.$$

Therefore, it suffices applying Theorem 4 and noting that $\mathcal{A}^* = \mathcal{B}$. □

Corollary 6. *If* (1) *holds,* $G \in \mathcal{B} \otimes \mathcal{B}$ *and* $\mathcal{A} \cap C = \mathcal{A}^* \cap C$, *for some* $C \in \mathcal{A}$ *with* $P(C) = 1$, *then condition* (3) *holds.*

Proof. Since $C \in \mathcal{A}$ and $S \cap C \in \mathcal{A}^* \cap C = \mathcal{A} \cap C$, then $S \cap C \in \mathcal{A}$. Since $P(C) = 1$, it follows that $S \in \mathcal{A}_P$ and (c) holds with $U = S \cap C$. Thus, (3) follows from Theorem 4. □

As shown in Blackwell and Dubins (1975), if $(\Omega, \mathcal{B})$ is a standard space (Ω Borel subset of a Polish space and $\mathcal{B}$ the Borel σ-field on Ω), then $G \in \mathcal{B} \otimes \mathcal{B}$ and $\mathcal{A}^* = \mathcal{A}$ for various classically interesting sub-σ-fields $\mathcal{A}$, including tail, symmetric, invariant, as well as some sub-σ-fields connected with continuous time processes. In view of Corollary 6, *condition* (3) *holds in case* $(\Omega, \mathcal{B})$ *is a standard space and* $\mathcal{A}$ *is any one of the above-mentioned sub-*σ*-fields.*

2.2. **Tail sub-σ-fields.** When condition (3) holds, the next step is determining those ω's satisfying $f(\omega) = 1$. Suppose the assumptions of Corollary 6 are in force (so that (3) holds and f is $\mathcal{A}_P$-measurable) and define

$$\mathcal{U} = \{U \in \mathcal{A} : \mathcal{A} \cap U \text{ is c.g.}\} \cup \{\emptyset\}.$$

Since $\mathcal{U}$ is closed under countable unions, some $A \in \mathcal{U}$ meets $P(A) = \sup\{P(U) : U \in \mathcal{U}\}$. By the first assertion in Theorem 3, $P\big(A - \{f = 1\}\big) = 0$. Taking U as in condition (b) and noting that $U \in \mathcal{U}$, one also obtains $P\big(\{f = 1\} - A\big) = P(U - A) = 0$. Therefore, A is the set we are looking for, in the sense that

$$P\big(\{f = 1\} \Delta A\big) = 0.$$

Incidentally, the above remarks provide also a criterion for deciding whether μ is *maximally improper* according to Seidenfeld et al. (2001). Under the assumptions of Corollary 6, in fact, μ is maximally improper precisely when $P(S) = 0$. Hence,

$$\mu \text{ is maximally improper} \quad \Leftrightarrow \quad P(U) = 0 \text{ for all } U \in \mathcal{U}.$$

Some handy description of the members of $\mathcal{U}$, thus, would be useful. Unfortunately, such a description is generally hard to be found. We now discuss a particular case.

Let $\mathcal{A}$ be a *tail sub-*σ*-field*, that is, $\mathcal{A} = \bigcap_{n \geq 1} \mathcal{A}_n$ where $\mathcal{A}_n$ is a *countably generated* σ-field and $\mathcal{B} \supset \mathcal{A}_n \supset \mathcal{A}_{n+1}$ for all $n \geq 1$. As already noted, the assumptions of Corollary 6 hold for such an $\mathcal{A}$ if $(\Omega, \mathcal{B})$ is a standard space. More generally, it is enough that:

Lemma 7. *If* $\mathcal{A}$ *is a tail sub-*σ*-field,* (1) *holds and*

$$\textit{for each } n, \textit{ there is a r.c.d. } \mu_n \textit{ for } P \textit{ given } \mathcal{A}_n,$$

then $G \in \mathcal{B} \otimes \mathcal{B}$ *and* $\mathcal{A} \cap C = \mathcal{A}^* \cap C$ *for some* $C \in \mathcal{A}$ *with* $P(C) = 1$.

Proof. Since $G_n := \{(x, y) \in \Omega \times \Omega : x \text{ and } y \text{ are in the same } \mathcal{A}_n\text{-atom}\} \in \mathcal{A}_n \otimes \mathcal{A}_n$, Proposition 1 of Blackwell and Dubins (1975) implies $G = \bigcup_n G_n \in \mathcal{B} \otimes \mathcal{B}$. For each n, since $\mathcal{A}_n$ is c.g., there is $C_n \in \mathcal{A}_n$ such that $P(C_n) = 1$ and $\mu_n(\omega)(A) = I_A(\omega)$ whenever $A \in \mathcal{A}_n$ and $\omega \in C_n$. Define $C = \bigcup_{n \geq 1} \bigcap_{j \geq n} C_j$ and note that $C \in \mathcal{A}$ and $P(C) = 1$. Fix $B \in \mathcal{A}^*$. Since B is a union of $\mathcal{A}_n$-atoms whatever n is,

$$\lim_n \mu_n(\omega)(B) = I_B(\omega) \quad \text{for all } \omega \in C.$$

Thus, $B \cap C = \{\lim_n \mu_n(B) = 1\} \cap C \in \mathcal{A} \cap C$. □

Each $\mathcal{A}_n$, being c.g., can be written as $\mathcal{A}_n = \sigma(X_n)$ for some $X_n : \Omega \to \mathbb{R}$. Since $\mathcal{A}_n \supset \mathcal{A}_j$ for $j \geq n$, it follows that $\mathcal{A}_n = \sigma(X_n, X_{n+1}, \ldots)$. Thus, $\mathcal{A}$ admits the usual representation

$$\mathcal{A} = \bigcap_n \sigma(X_n, X_{n+1}, \ldots)$$

for some sequence (X_n) of real random variables. In particular,

$$H(\omega) = \{\exists n \geq 1 \text{ such that } X_j = X_j(\omega) \text{ for all } j \geq n\} \in \mathcal{A}$$

so that $\mathcal{A}$ includes its atoms. Note also that a c.g. sub-σ-field is tail while the converse need not be true. In fact, for a σ-field $\mathcal{F}$ to be not c.g., it is enough that $\mathcal{F}$ supports a 0-1 valued probability measure Q such that $Q(F) = 0$ whenever $F \in \mathcal{F}$ and F is an $\mathcal{F}$-atom; see Theorem 1 of Blackwell and Dubins (1975). Thus, for instance, $\mathcal{A} = \bigcap_n \sigma(X_n, X_{n+1}, \ldots)$ is not c.g. in case (X_n) is i.i.d. and X_1 has a non degenerate distribution.

To find usable characterizations of $\mathcal{U}$ is not an easy task. Countable unions of $\mathcal{A}$-atoms belong to $\mathcal{U}$, but generally they are not all the elements of $\mathcal{U}$. For instance, if $\Omega = \mathbb{R}^\infty$ and X_n is the n-th coordinate projection, then

$$U = \{\exists n \geq 1 \text{ such that } X_j = X_n \text{ for all } j \geq n\}$$

is an uncountable union of $\mathcal{A}$-atoms. However, $\mathcal{A} \cap U$ is c.g. since $\mathcal{A} \cap U = \sigma(L) \cap U$ where $L = \limsup_n X_n$.

Another possibility could be selecting a subclass $\mathbb{Q} \subset \mathbb{P}$ and showing that $U \in \mathcal{U}$ if and only if $U \in \mathcal{A}$ and $Q(U) = 0$ for each $Q \in \mathbb{Q}$. We do not know whether some (non trivial) characterization of this type is available. Here, we just note that

$$\mathbb{Q}_0 = \{Q \in \mathbb{P} : (X_n) \text{ is i.i.d. and } X_1 \text{ has a non degenerate distribution, under } Q\}$$

does not work (though the "only if" implication is true, in view of Theorem 1 of Blackwell and Dubins (1975)). As an example, $U := \{X_n \to 0\} \notin \mathcal{U}$ even though $U \in \mathcal{A}$ and $Q(U) = 0$ for all $Q \in \mathbb{Q}_0$. To see that $U \notin \mathcal{U}$, let X_n be the n-th coordinate projection on $\Omega = \mathbb{R}^\infty$, and let P_U be a probability measure on the Borel sets of Ω which makes (X_n) independent and each X_n uniformly distributed on $(0, \frac{1}{n})$. Then $P_U(U) = 1$ and, when restricted to $\mathcal{A} \cap U$, P_U is a 0-1 probability measure such that $P_U(H(\omega)) = 0$ for each $\omega \in U$. Hence, Theorem 1 of Blackwell and Dubins (1975) implies that $\mathcal{A} \cap U$ is not c.g..

A last note is that $P(S)$ can assume any value between 0 and 1. For instance, take $U \in \mathcal{U}$ and $P_1, P_2 \in \mathbb{P}$ such that: (i) $P_1(U) = P_2(U^c) = 1$; (ii) P_2 is 0-1 on $\mathcal{A}$ with $P_2(H(\omega)) = 0$ for all ω. Define $P = uP_1 + (1 - u)P_2$ where $u \in (0, 1)$. A r.c.d. for P given $\mathcal{A}$ is $\mu(\omega) = I_U(\omega)\mu_1(\omega) + I_{U^c}(\omega)P_2$, where μ_1 denotes a r.c.d. for P_1 given $\mathcal{A}$. Since $U \in \mathcal{U}$, Theorem 3 implies $\mu_1(\omega)\big(H(\omega)\big) = 1$ for P_1-almost all $\omega \in U$. Thus, $P(S) = P(U) = u$.

2.3. **Miscellaneous results.** A weaker version of (3) lies in asking $\mu(\cdot)\big(H(\cdot)\big)$ to be 0-1 over a set of $\mathcal{A}^*$, but not necessarily of $\mathcal{A}$, that is

(3*) There is $B_0 \in \mathcal{A}^*$ with $P(B_0) = 1$ and $\mu(\omega)\big(H(\omega)\big) \in \{0, 1\}$ for all $\omega \in B_0$.

Suitably adapted, the proofs of Theorems 3 and 4 yield a characterization of (3*) as well. Recall $\mathcal{N} = \{B \in \mathcal{B} : P(B) = 0\}$ and note that

$$\sigma(\mathcal{A} \cup \mathcal{N}) = \{B \in \mathcal{B} : \mu(B) = I_B \text{ a.s.}\}.$$

Theorem 8. *Suppose* (1) *holds and* $G \in \mathcal{B} \otimes \mathcal{B}$. *Then, condition (3*) implies* $S \in \sigma(\mathcal{A} \cup \mathcal{N})$. *Moreover, if* $S \in \sigma(\mathcal{A} \cup \mathcal{N})$, *then*

$$(3^*) \quad \Leftrightarrow \quad (b^*) \quad \Leftrightarrow \quad (c^*)$$

where

(b*) $\mathcal{A} \cap V$ *is c.g. for some* $V \in \mathcal{A}^*$ *with* $V \subset S$ *and* $P(S - V) = 0$;

(c*) $\mathcal{A} \cap V = \mathcal{A}^* \cap V$ *for some* $V \in \mathcal{A}^*$ *with* $V \subset S$ *and* $P(S - V) = 0$.

Proof. If (3*) holds, then $\mu(S) = 1$ on $B_0 \cap S$, and since $P(B_0) = 1$ one obtains

$$E\big(\mu(S) I_{S^c}\big) = P(S) - E\big(\mu(S) I_{B_0} I_S\big) = P(S) - E\big(I_{B_0} I_S\big) = 0.$$

Thus, $\mu(S) = I_S$ a.s., that is, $S \in \sigma(\mathcal{A} \cup \mathcal{N})$. Next, suppose that $S \in \sigma(\mathcal{A} \cup \mathcal{N})$.

"(3*) $\Rightarrow$ (c*)". Define $V = B_0 \cap S$ and note that $B \cap V = \{\mu(B) = 1\} \cap V \in \mathcal{A} \cap V$ for all $B \in \mathcal{A}^*$.

"(c*) $\Rightarrow$ (b*)". Fix $\alpha \in \mathbb{P}$ and define $\nu(\omega)(B) = I_V(\omega) \frac{f_B(\omega)}{f(\omega)} + I_{V^c}(\omega)\alpha(B)$ for all $\omega \in \Omega$ and $B \in \mathcal{B}$. Then, $\sigma(\nu) \subset \mathcal{A}^*$. Further, $\nu(\omega)\big(H(\omega)\big) = 1$ for all $\omega \in V$, so that $B \cap V = \{\nu(B) = 1\} \cap V$ for all $B \in \mathcal{A}^*$. Hence, (c*) implies that $\mathcal{A} \cap V = \mathcal{A}^* \cap V = \sigma(\nu) \cap V$ is c.g..

"(b*) $\Rightarrow$ (3*)". Let $\mathcal{A}_0 = \{(A \cap V) \cup F : A \in \mathcal{A}, F = \emptyset \text{ or } F = V^c\}$. Since $\mu(V) = \mu(S) = I_S = I_V$ a.s., for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ one obtains

$$E\big(I_A I_V \mu(B)\big) = E\big(I_A \mu(B \cap V)\big) = P\big((A \cap V) \cap B\big).$$

So, $\mu_0(\omega) = I_V(\omega)\mu(\omega) + I_{V^c}(\omega)\alpha$ is a r.c.d. for P given $\mathcal{A}_0$, where $\alpha(\cdot) = P(\cdot \mid V^c)$ if $P(V) < 1$ and α is any fixed element of $\mathbb{P}$ if $P(V) = 1$. Since $\mathcal{A}_0$ is c.g., there is $K \in \mathcal{A}_0$ with $P(K) = 1$ and $\mu_0(\omega)\big(H_0(\omega)\big) = 1$ for all $\omega \in K$, where $H_0(\omega)$ denotes the $\mathcal{A}_0$-atom including ω. Hence, it suffices to let $B_0 = (K \cap V) \cup S^c$ and noting that $H_0(\omega) = H(\omega)$ and $\mu_0(\omega) = \mu(\omega)$ for all $\omega \in V$. $\square$

One consequence of Theorem 8 is that, if (1) holds and $G \in \mathcal{B} \otimes \mathcal{B}$, then condition (3*) is equivalent to $\mu(S) = I_S$ a.s. and $P(0 < f \leq \frac{1}{2}) = 0$. In fact,

$$A \cap \{f > \frac{1}{2}\} = \{\mu(A) > \frac{1}{2}\} \cap \{f > \frac{1}{2}\} \quad \text{for all } A \in \mathcal{A},$$

so that $\mathcal{A} \cap \{f > \frac{1}{2}\} = \sigma(\mu) \cap \{f > \frac{1}{2}\}$ is c.g.. Hence, if $P(0 < f \leq \frac{1}{2}) = 0$, condition (b*) holds with $V = \{f > \frac{1}{2}\}$.

Finally, we give one more condition for (3). Though seemingly simple, it is hard to be tested in real problems.

Proposition 9. *If* (1) *holds and* $H(\omega) \in \mathcal{B}$ *for all* ω, *a sufficient condition for* (3) *is*

$$\mu(x)\big(H(y)\big) = 0 \quad \text{whenever } H(x) \neq H(y). \tag{5}$$

Proof. As stated in the forthcoming Lemma 10, since $\sigma(\mu)$ is c.g. and μ is also a r.c.d. for P given $\sigma(\mu)$, there is a set $T \in \sigma(\mu)$ such that $P(T) = 1$ and $\mu(\omega)\big(\mu = \mu(\omega)\big) = 1$ for each $\omega \in T$. Let $A_0 = T$ and fix $\omega \in S$. Then, $\mu(\omega) = \mu(x)$ if $x \in H(\omega)$ (since $\sigma(\mu) \subset \mathcal{A}$) and $\mu(\omega) \neq \mu(x)$ if $x \notin H(\omega)$ since in the latter case (5) yields

$$\mu(x)\big(H(\omega)\big) = 0 < f(\omega) = \mu(\omega)\big(H(\omega)\big).$$

Thus, $H(\omega) = \{\mu = \mu(\omega)\}$. If $\omega \in T \cap S = A_0 \cap S$, this implies

$$\mu(\omega)\big(H(\omega)\big) = \mu(\omega)\big(\mu = \mu(\omega)\big) = 1.$$

□

3. When regular conditional distributions are 0-1 on the conditioning σ-field

In this section, condition (4) is shown to be true whenever $\mathcal{A}$ is a tail sub-σ-field. Moreover, two characterizations of (4) and a result in the negative (i.e., a condition for (4) to be false) are given.

We begin by recalling a few simple facts about $\sigma(\mu)$.

Lemma 10. *If* (1) *holds, then $\sigma(\mu)$ is c.g., μ is a r.c.d. for P given $\sigma(\mu)$, and there is a set $T \in \sigma(\mu)$ with $P(T) = 1$ and*

$$\mu(\omega)\big(\mu = \mu(\omega)\big) = 1 \quad \textit{for all } \omega \in T.$$

Moreover,

$$\mathcal{A} = \sigma\Big(\sigma(\mu) \cup (\mathcal{A} \cap \mathcal{N})\Big).$$

Proof. Since $\sigma(\mu) \subset \mathcal{A}$, μ is a r.c.d. given $\sigma(\mu)$. Since $\mathcal{B}$ is c.g., $\sigma(\mu)$ is c.g. with atoms of the form $\{\mu = \mu(\omega)\}$. Hence, there is $T \in \sigma(\mu)$ with $P(T) = 1$ and $\mu(\omega)\big(\mu = \mu(\omega)\big) = 1$ for all $\omega \in T$. Finally, since

$$A = \big(\{\mu(A) = 1\} \cap \{\mu(A) = I_A\}\big) \cup \big(A \cap \{\mu(A) \neq I_A\}\big)$$

for all $A \in \mathcal{A}$, it follows that $\mathcal{A} \subset \sigma\big(\sigma(\mu) \cup (\mathcal{A} \cap \mathcal{N})\big) \subset \mathcal{A}$. □

By Lemma 10, $\mu(\omega)$ is 0-1 on $\sigma(\mu)$ for each $\omega \in T$. Since $\mathcal{A} = \sigma\big(\sigma(\mu) \cup (\mathcal{A} \cap \mathcal{N})\big)$, condition (4) can be written as

$$\mu(\omega)(A) \in \{0, 1\} \quad \text{for all } \omega \in A_0 \text{ and } A \in \mathcal{A} \text{ with } P(A) = 0.$$

In particular, (4) *holds whenever P is atomic on $\mathcal{A}$*, in the sense that there is a countable partition $\{A_1, A_2, \ldots\}$ of Ω satisfying $A_j \in \mathcal{A}$ and $P(A \cap A_j) \in \{0, P(A_j)\}$ for all $j \geq 1$ and $A \in \mathcal{A}$. In this case, in fact, $\mu(\omega) \ll P$ for each ω in some set $C \in \mathcal{A}$ with $P(C) = 1$.

Slightly developing the idea underlying Example 2, we next give a sufficient condition for (4) to be false.

Proposition 11. *Suppose* (1) *holds and $P\big(\mu = \mu(\omega)\big) = 0$ for all ω. Then,*

$$F = \{\omega : \mu(\omega) \textit{ is not 0-1 on } \mathcal{B}\} \quad \textit{and} \quad F_0 = \{\omega : \mu(\omega) \textit{ is non-atomic on } \mathcal{B}\}$$

belong to $\sigma(\mu)$. Moreover, if $\mathcal{N} \subset \mathcal{A}$, then

$$\mu(\omega) \textit{ is not 0-1 on } \mathcal{A} \textit{ for each } \omega \in F \cap T, \textit{ and}$$

$$\mu(\omega) \textit{ is non-atomic on } \mathcal{A} \textit{ for each } \omega \in F_0 \cap T$$

with T as in Lemma 10. In particular, condition (4) *fails if $P(F) > 0$.*

Proof. Since $\mathcal{B}$ is c.g., it is clear that $F \in \sigma(\mu)$, while $F_0 \in \sigma(\mu)$ is from Dubins and Freedman (1964) (see 2.13, p. 1214). Suppose now that $\mathcal{N} \subset \mathcal{A}$. Let $\omega \in F \cap T$. Since $\omega \in F$, there is $B_\omega \in \mathcal{B}$ with $\mu(\omega)(B_\omega) \in (0,1)$. Define $A_\omega = B_\omega \cap \{\mu = \mu(\omega)\}$. Since $\mathcal{N} \subset \mathcal{A}$ and $P(A_\omega) \leq P\big(\mu = \mu(\omega)\big) = 0$, then $A_\omega \in \mathcal{A}$. Since $\omega \in T$,

$$\mu(\omega)\big(A_\omega\big) = \mu(\omega)\big(B_\omega\big) \in (0,1)$$

so that $\mu(\omega)$ is not 0-1 on $\mathcal{A}$. Finally, fix $\omega \in F_0 \cap T$ and $\epsilon > 0$. Since $\omega \in F_0$, there is a finite partition $\{B_{1,\omega}, \ldots, B_{n,\omega}\}$ of Ω such that $B_{i,\omega} \in \mathcal{B}$ and $\mu(\omega)(B_{i,\omega}) < \epsilon$ for all i. As above, letting $A_{i,\omega} = B_{i,\omega} \cap \{\mu = \mu(\omega)\}$, one obtains $A_{i,\omega} \in \mathcal{A}$ and $\mu(\omega)(A_{i,\omega}) = \mu(\omega)(B_{i,\omega}) < \epsilon$. Hence, $\mu(\omega)$ is non-atomic on $\mathcal{A}$ since $\mu(\omega)\big(\mu \neq \mu(\omega)\big) = 0$. □

Even if $\mathcal{N}$ is not contained in $\mathcal{A}$, Proposition 11 applies at least to $\mathcal{A}' = \sigma(\mathcal{A} \cup \mathcal{N})$. Under mild conditions, μ is even non-atomic on $\mathcal{A}'$ with probability $P(F_0)$. Thus, a lot of r.c.d.'s give rise to a failure of (4) on some sub-σ-field $\mathcal{A}'$. Since we are conditioning to $\mathcal{A}$ (and not to $\mathcal{A}'$), this fact is not essential. On the other hand, it suggests that (4) is a rather delicate condition.

If P is invariant under a countable collection of measurable transformations and $\mathcal{A}$ is the corresponding invariant sub-σ-field, then (4) holds; see Maitra (1977). This well known fact is generalized by our first characterization of (4).

Theorem 12. *Suppose* (1) *holds and let* $M = \{Q \in \mathbb{P} : \mu$ *is a r.c.d. for* Q *given* $\mathcal{A}\}$. *Then,* Q *is an extreme point of* M *if and only if* $Q \in M$ *and* Q *is 0-1 on* $\mathcal{A}$, *and in that case* $Q = \mu(\omega)$ *for some* $\omega \in \Omega$. *Moreover, for each* $\omega \in T$ *(with* T *as in Lemma 10), the following statements are equivalent:*

(i) $\mu(\omega)(A) \in \{0,1\}$ *for all* $A \in \mathcal{A}$;

(ii) $\mu(\omega)$ *is an extreme point of* M;

(iii) $\mu(\omega) \in M$.

In particular, condition (4) *holds if and only if, for some* $A_0 \in \mathcal{A}$ *with* $P(A_0) = 1$,

$$\mu(\omega) \in M \quad \textit{for all } \omega \in A_0.$$

Proof. Fix $Q \in M$. If $Q(A) \in (0,1)$ for some $A \in \mathcal{A}$, then

$$Q(\cdot) = Q(A)Q(\cdot \mid A) + (1 - Q(A))Q(\cdot \mid A^c),$$

and Q is not extreme since $Q(\cdot \mid A)$ and $Q(\cdot \mid A^c)$ are distinct elements of M. Suppose now that $Q = uQ_1 + (1-u)Q_2$, where $u \in (0,1)$ and $Q_1 \neq Q_2$ are in M. Since two elements of M coincide if and only if they coincide on $\mathcal{A}$, there is $A \in \mathcal{A}$ with $Q_1(A) \neq Q_2(A)$, and this implies $Q(A) \in (0,1)$. Hence, $Q \in M$ is extreme if and only if it is 0-1 on $\mathcal{A}$. In particular, if Q is extreme then it is 0-1 on the c.g. σ-field $\sigma(\mu)$, so that $Q\big(\mu = \mu(\omega)\big) = 1$ for some $\omega \in \Omega$; see Theorem 1 of Blackwell and Dubins (1975). Thus,

$$Q(B) = \int \mu(x)(B)\, Q(dx) = \mu(\omega)(B) \quad \text{for all } B \in \mathcal{B}.$$

This concludes the proof of the first part. As to the second one, fix $\omega \in T$, and let A and B denote arbitrary elements of $\mathcal{A}$ and $\mathcal{B}$, respectively. Since $\omega \in T$,

$$\int_A \mu(x)(B)\, \mu(\omega)(dx) = \int_{A \cap \{\mu = \mu(\omega)\}} \mu(x)(B)\, \mu(\omega)(dx) = \mu(\omega)(A)\mu(\omega)(B).$$

"(i) ⇒ (ii)". By what already proved, it is enough showing that $\mu(\omega) \in M$, and this depends on $\mu(\omega)(A \cap B) = \mu(\omega)(A)\mu(\omega)(B) = \int_A \mu(x)(B)\, \mu(\omega)(dx)$.

"(ii) $\Rightarrow$ (iii)". Obvious.

"(iii) $\Rightarrow$ (i)". Under (iii), $\mu(\omega)(A\cap B) = \int_A \mu(x)(B)\,\mu(\omega)(dx) = \mu(\omega)(A)\mu(\omega)(B)$, and letting $B = A$ yields $\mu(\omega)(A) = \mu(\omega)(A)^2$. □

Next characterization of (4) stems from a result of Fremlin (1981, Lemma 2A, p. 391).

Theorem 13 (Fremlin). *Let X be an Hausdorff topological space, $\mathcal{F}$ a σ-field on X including the open sets, Q a complete Radon probability measure on $\mathcal{F}$, and $\mathcal{C}_0$ a class of pairwise disjoint Q-null elements of $\mathcal{F}$. Then,*

$$\bigcup_{C\in\mathcal{C}} C \in \mathcal{F} \quad \textit{for all } \mathcal{C} \subset \mathcal{C}_0 \quad \Longleftrightarrow \quad Q\big(\bigcup_{C\in\mathcal{C}_0} C\big) = 0.$$

Say that P is *perfect* in case each $\mathcal{B}$-measurable function $h : \Omega \to \mathbb{R}$ meets $P(h \in I) = 1$ for some real Borel set $I \subset h(\Omega)$. For P to be perfect, it is enough that Ω is an universally measurable subset of a Polish space and $\mathcal{B}$ the Borel σ-field on Ω. In the present framework, since $\mathcal{B}$ is c.g., Theorem 13 applies precisely when P is perfect. We are now able to state our second characterization of (4). It is of possible theoretical interest even if of little practical use.

Theorem 14. *Suppose* (1) *holds and P is perfect, define*

$$\mathcal{A}(\omega) = \{A \in \mathcal{A} : \mu(\omega)(A) \in \{0,1\}\} \quad \omega \in \Omega,$$

and let Γ_0 denote the class of those σ-fields $\mathcal{G} \subset \mathcal{A}$ with $\mathcal{G} \neq \mathcal{A}$. Then, condition (4) *holds if and only if*

$$\bigcup_{\mathcal{G}\in\Gamma} \{\omega : \mathcal{A}(\omega) = \mathcal{G}\} \in \mathcal{A}_P \qquad \textit{for all } \Gamma \subset \Gamma_0. \tag{6}$$

Proof. If $\mu(\omega)$ is 0-1 on $\mathcal{A}$ for all $\omega \in A_0$, where $A_0 \in \mathcal{A}$ and $P(A_0) = 1$, then (6) follows from

$$\{\omega : \mathcal{A}(\omega) = \mathcal{G}\} \subset A_0^c \quad \text{for all } \mathcal{G} \in \Gamma_0.$$

Conversely, suppose (6) holds. Let X be the partition of Ω in the atoms of $\mathcal{B}$. The elements of $\mathcal{B}$ are unions of elements of X, so that $\mathcal{B}$ can be regarded as a σ-field on X. Let $(X, \mathcal{F}, Q)$ be the completion of $(X, \mathcal{B}, P)$. Since $\mathcal{B}$ is c.g., under a suitable distance, X is separable metric and $\mathcal{B}$ the corresponding Borel σ-field; see Blackwell (1955). Since P is perfect, P is Radon by a result of Sazonov (Theorem 12 of Sazonov (1965)), so that Q is Radon, too. Next, define $C_{\mathcal{G}} = \{\omega : \mathcal{A}(\omega) = \mathcal{G}\}$ for $\mathcal{G} \in \Gamma_0$, $U_A = \{\omega : \mu(\omega)(A) \in (0,1)\}$ for $A \in \mathcal{A}$, and $U = \{\omega : \mathcal{A}(\omega) \neq \mathcal{A}\}$ (all regarded as subsets of X). For each $\mathcal{G} \in \Gamma_0$ there is $A \in \mathcal{A}$ with $C_{\mathcal{G}} \subset U_A$. Since $U_A \in \mathcal{A}$ and $P(U_A) = 0$, then $C_{\mathcal{G}} \in \mathcal{F}$ and $Q(C_{\mathcal{G}}) = 0$. Hence, $\mathcal{C}_0 = \{C_{\mathcal{G}} : \mathcal{G} \in \Gamma_0\}$ is a collection of pairwise disjoint Q-null elements of $\mathcal{F}$ satisfying $U = \bigcup_{\mathcal{G}\in\Gamma_0} C_{\mathcal{G}}$. By (6), Theorem 13 yields $Q(U) = 0$. Finally, since $U \in \mathcal{A}_P$, $Q(U) = 0$ implies $U \subset A$ for some $A \in \mathcal{A}$ with $P(A) = 0$. Thus, to get (4), it suffices to let $A_0 = A^c$. □

Finally, by a martingale argument, we prove that (4) holds when $\mathcal{A}$ is a tail sub-σ-field. This is true, in addition, even though $\mathcal{B}$ fails to be c.g..

Theorem 15. *Let $\mathcal{A} = \bigcap_{n\geq 1} \mathcal{A}_n$, where $\mathcal{B} \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \ldots$ and $\mathcal{A}_n$ is a c.g. σ-field for each n. Given a r.c.d. μ, for P given $\mathcal{A}$, there is a set $A_0 \in \mathcal{A}$ such that $P(A_0) = 1$ and $\mu(\omega)(A) \in \{0,1\}$ for all $A \in \mathcal{A}$ and $\omega \in A_0$.*

Proof. First recall that a probability measure $Q \in \mathbb{P}$ is 0-1 on $\mathcal{A}$ if (and only if) $\sup_{A\in\mathcal{A}_n} |Q(A\cap B) - Q(A)Q(B)| \to 0$, as $n \to \infty$, for all $B \in \mathcal{A}_1$. Also, given any field $\mathcal{F}_n$ such that $\mathcal{A}_n = \sigma(\mathcal{F}_n)$, the "sup" can be taken over $\mathcal{F}_n$, i.e.,

$$\sup_{A\in\mathcal{A}_n} |Q(A\cap B) - Q(A)Q(B)| = \sup_{A\in\mathcal{F}_n} |Q(A\cap B) - Q(A)Q(B)|.$$

Now, since the $\mathcal{A}_n$ are c.g., there are countable fields $\mathcal{F}_n$ satisfying $\mathcal{A}_n = \sigma(\mathcal{F}_n)$ for all n. Let

$$V_n^B(\omega) = \sup_{A\in\mathcal{F}_n} \Big|\mu(\omega)(A\cap B) - \mu(\omega)(A)\mu(\omega)(B)\Big|, \quad n \geq 1, B \in \mathcal{A}_1, \omega \in \Omega.$$

Since $\mathcal{F}_n$ is countable, V_n^B is an $\mathcal{A}$-measurable random variable for all n and B. It is enough proving that

$$V_n^B \to 0 \quad \text{a.s., as } n \to \infty, \text{ for all } B \in \mathcal{A}_1. \tag{7}$$

Suppose in fact (7) holds and define

$$A_0 = \{\omega : \lim_n V_n^B(\omega) = 0 \text{ for each } B \in \mathcal{F}_1\}.$$

Since $\mathcal{F}_1$ is countable, $A_0 \in \mathcal{A}$ and (7) implies $P(A_0) = 1$. Fix $\omega \in A_0$. Since $\mathcal{A}_1 = \sigma(\mathcal{F}_1)$, given $B \in \mathcal{A}_1$ and $\epsilon > 0$, there is $C \in \mathcal{F}_1$ such that $\mu(\omega)(B\Delta C) < \epsilon$. Hence,

$$V_n^B(\omega) \leq \sup_{A\in\mathcal{F}_n} \Big|\mu(\omega)(A\cap B) - \mu(\omega)(A\cap C)\Big| +$$

$$+ \sup_{A\in\mathcal{F}_n} \Big|\mu(\omega)(A\cap C) - \mu(\omega)(A)\mu(\omega)(C)\Big| + \sup_{A\in\mathcal{F}_n} \Big|\mu(\omega)(A)\mu(\omega)(C) - \mu(\omega)(A)\mu(\omega)(B)\Big|$$

$$\leq V_n^C(\omega) + 2\mu(\omega)(B\Delta C) < V_n^C(\omega) + 2\epsilon \quad \text{for all } n.$$

Since $\omega \in A_0$ and $C \in \mathcal{F}_1$, it follows that

$$\limsup_n V_n^B(\omega) \leq 2\epsilon + \limsup_n V_n^C(\omega) = 2\epsilon \quad \text{for all } B \in \mathcal{A}_1 \text{ and } \epsilon > 0.$$

Therefore, $\mu(\omega)$ is 0-1 on $\mathcal{A}$. It remains to check condition (7). Fix $B \in \mathcal{A}_1$, take any version of $E(I_B \mid \mathcal{A}_n)$ and define $Z_n = E(I_B \mid \mathcal{A}_n) - \mu(B)$. Then, $|Z_n| \leq 2$ a.s. for all n, and the martingale convergence theorem yields $Z_n \to 0$ a.s.. Further, for fixed $n \geq 1$ and $A \in \mathcal{F}_n$, one obtains

$$\Big|E(I_A I_B \mid \mathcal{A}) - E(I_A \mid \mathcal{A})\, E(I_B \mid \mathcal{A})\Big|$$

$$= \Big|E\Big(I_A E(I_B \mid \mathcal{A}_n) \mid \mathcal{A}\Big) - E\Big(I_A E(I_B \mid \mathcal{A}) \mid \mathcal{A}\Big)\Big|$$

$$= |E(I_A Z_n \mid \mathcal{A})| \leq E(|Z_n| \mid \mathcal{A}) \quad \text{a.s.}.$$

Since $\mathcal{F}_n$ is countable, it follows that

$$V_n^B = \sup_{A\in\mathcal{F}_n} \Big|E(I_A I_B \mid \mathcal{A}) - E(I_A \mid \mathcal{A})\, E(I_B \mid \mathcal{A})\Big| \leq E(|Z_n| \mid \mathcal{A}) \to 0 \quad \text{a.s.}.$$

□

As noted in Subsection 2.2, a tail sub-σ-field includes its atoms so that (4) implies (3). Thus, by Theorem 15, condition (3) holds provided $\mathcal{A}$ is a tail sub-σ-field and P admits a r.c.d. μ given $\mathcal{A}$, even if the other assumptions of Lemma 7 fail. (In fact, such assumptions grant something more than (3)).

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