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The renormalized energy for a system of edge dislocations with multiple Burgers vectors

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Marco Bresciani

Introduction

The plastic behaviour of metals is mainly caused by the presence of *dislocations*, which are 1-dimensional defects of the crystal lattice.

To illustrate an example of such defects, in Figure 1 we consider the simple case of a cubic crystal lattice. In the example, the defect is due to the presence of an extra half-plane of atoms in the crystal lattice, that produces a local distortion of its geometry. The boundary of this extra half-plane of atoms constitutes the *dislocation line*.

At a discrete level, the presence of a dislocation can be described by the following procedure. For simplicity, we consider a section of the lattice which is orthogonal to the dislocation line. We take a closed atom-to-atom path surrounding the defect, the *Burgers circuit*, in the deformed crystal. Then we represent the same path on the section of the perfect crystal in the reference configuration. We see that, in the perfect crystal, the path does not close. The vector required to close the circuit is the *Burgers vector* (see Figure 2). We could equivalently consider a 3-dimensional path and we would obtain the same Burgers vector. Note that this vector lies on a plane that is orthogonal to the dislocation line. Dislocations of *edge* type, which are the ones considered in the present work, are characterized by this property.

In response to a shear stress, dislocations can move through the crystal lattice and their motion leads to the plastic deformation of the metal. The gliding mechanism that allows dislocations to move is roughly sketched in Figure 3.

For a complete treatment of the theory of dislocations, we refer to [2], [16].

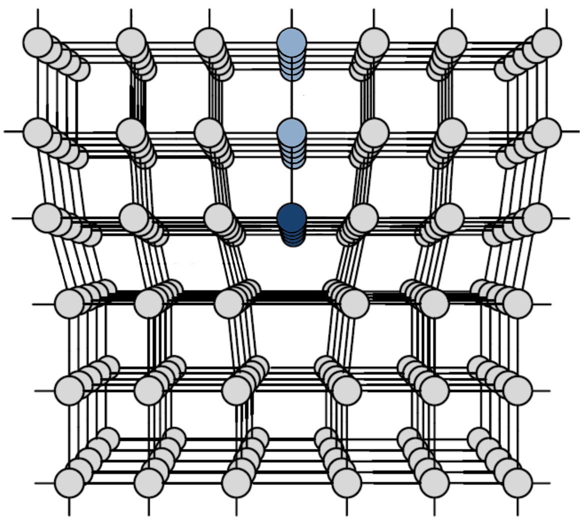


Figure 1: The cubic crystal lattice with the extra half-plane of atoms. The dark atoms form the dislocation line.

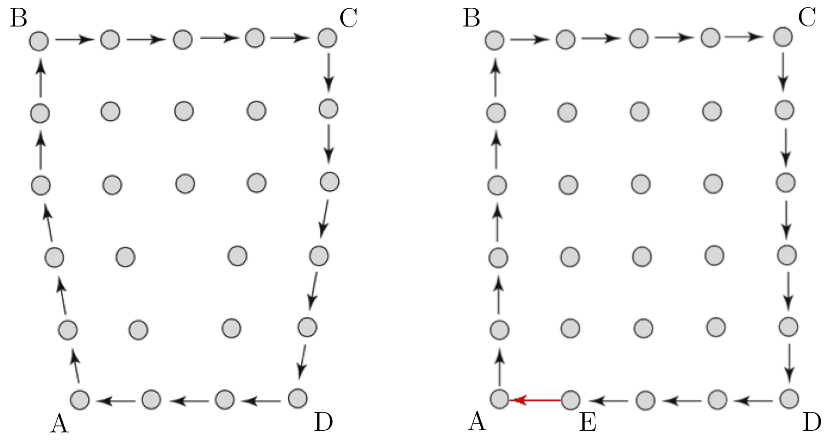


Figure 2: We consider the Burgers circuit $ABCD$ in the deformed crystal. The corresponding path in the perfect crystal does not close, since it ends at E. The Burgers vector is given by EA.

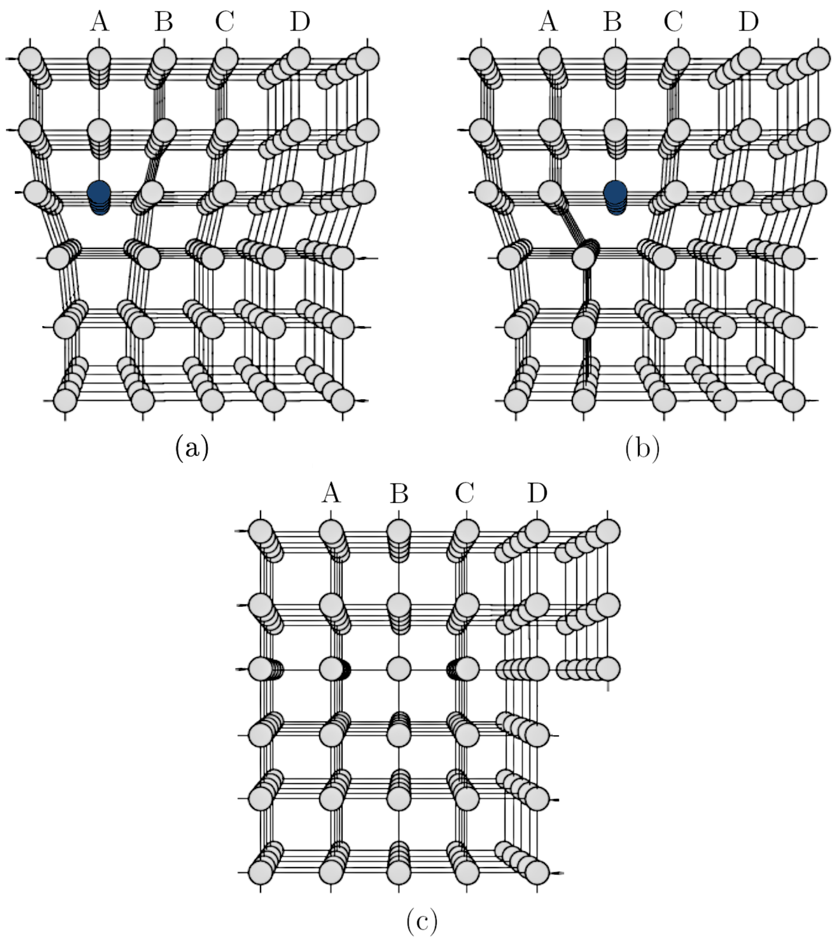


Figure 3: (a) The dislocation is located at A. (b) The breaking of bonds between the atoms and the formation of new ones produces the motion of the defect from A to B. (c) The defect has moved through the lattice, producing a shear deformation of the crystal.

From the mathematical point of view, dislocations have been studied by means of both discrete and continuum models. In this thesis we consider an intermediate scale model, often called *discrete dislocation model*, introduced by Cermelli and Gurtin (see [5]).
In this model a system of a finite number of parallel straight edge dislocations with multiple Burgers vectors is considered. Each defect is identified with a point on a cross section orthogonal to the dislocation lines, so that the model can be reduced to a 2-dimensional setting.
The model is in the framework of linearized elasticity. The strain field is given by a map $\boldsymbol{\beta}$ defined on the cross-section $\Omega \subset \mathbb{R}^2$ and taking values in $\mathbb{R}^{2\times 2}$. In presence of dislocations, the strain field is singular. More precisely, for a system of dislocations located at the points $\mathbf{z}_1, \dots, \mathbf{z}_n \in \Omega$ with Burgers vectors $\mathbf{b}_1, \dots, \mathbf{b}_n \in \mathbb{R}^2$, the equilibrium equations of the system are given by:

$$\begin{cases} \operatorname{div} \mathbb{C}\boldsymbol{\beta} = \mathbf{0} \\ \operatorname{curl} \boldsymbol{\beta} = \sum_{i=1}^n \mathbf{b}_i \delta_{\mathbf{z}_i} \end{cases} \quad \text{in } \Omega, \tag{1}$$

where $\mathbb{C}$ is the elasticity tensor. Since any strain $\boldsymbol{\beta}$ satisfying the second equation does not have finite elastic energy, no variational principle can be applied to determine the equilibrium configurations of the system.
A common strategy to overcome this incompatibility is to adopt the so-called core-region approach. It consists in regularizing the problem by removing small balls centred at the defects and computing the elastic energy stored in the resulting domain.
This approach is employed in [6]. More precisely, given the core-radius $\varepsilon > 0$ and the dislocation density

$$\mu = \frac{1}{n} \sum_{i=1}^n \delta_{\mathbf{z}_i},$$

the energy functional is defined as

$$E_\varepsilon(\mu, \boldsymbol{\beta}) = \int_{\Omega_\varepsilon(\mu)} W(\boldsymbol{\beta}) \,\mathrm{d}\mathbf{x}, \tag{2}$$

where $\Omega_\varepsilon(\mu) = \Omega \backslash (\bigcup_{i=1}^n \overline{B}(\mathbf{z}_i, \varepsilon))$ and $W \colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ is the quadratic elastic energy density given by $W(\boldsymbol{\beta}) = \frac{1}{2}\mathbb{C}\boldsymbol{\beta} : \boldsymbol{\beta}$, for every $\boldsymbol{\beta} \in \mathbb{R}^{2\times 2}$. The admissible strains are required to satisfy the following conditions:

$$\operatorname{curl} \boldsymbol{\beta} = \mathbf{0} \quad \text{in } \Omega_\varepsilon(\mu), \qquad \int_{\partial B(\mathbf{z}_i, \varepsilon)} \boldsymbol{\beta}\, \mathbf{t} \,\mathrm{d}\mathscr{H}^1 = \mathbf{b}_i, \quad i = 1, \dots, n. \tag{3}$$

The energy induced by the system of dislocations is then obtained by minimizing the elastic energy (2) among all strain fields satisfying (3). In [6], the existence of an energy minimizing strain $\boldsymbol{\beta}_\mu^\varepsilon$ is proved. In particular, it is shown that, as $\varepsilon \to 0^+$, the field $\boldsymbol{\beta}_\mu^\varepsilon$ converges in $L^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{\mathbf{z}_1, \dots, \mathbf{z}_n\}; \mathbb{R}^{2\times 2})$ to a solution of system (1). Moreover, an asymptotic estimate of the minimum energy, as $\varepsilon \to 0^+$, is obtained.
Since the typical number of dislocations in a metal is very large, it is natural to study the asymptotic behaviour of the system, as the number of dislocations tends to infinity, performing a homogenization procedure. The ultimate goal would be to gain insight into the macroscopic response of the metal, starting from the knowledge of its microscopic features.
A homogenization analysis of this type is performed in [12]. Here the asymptotic behaviour of the elastic energy, considered as a function of the strain, is studied in terms of Γ-convergence, as the core-radius ε goes to zero and the number of dislocations n_ε goes to

infinity. It is shown that the energy can be decomposed into two terms: the self-energy, concentrated in the core-regions, and the interaction energy, among pairs of dislocations. The self-energy is of order $n_\varepsilon|\log\varepsilon|$, while the interaction energy is of order n_ε^2. The limiting behaviour of the energy depends on the relative size of n_ε and $|\log\varepsilon_n|$. Thus, three regimes are identified: sub-critical ($n_\varepsilon \ll |\log\varepsilon|$), critical ($n_\varepsilon \approx |\log\varepsilon|$), and super-critical ($n_\varepsilon \gg |\log\varepsilon|$). At the limit, the self-energy prevails in the sub-critical regime, while the interaction energy is dominant in the super-critical regime. In the critical regime, a strain-gradient model for plasticity is derived.
As we mentioned, plastic deformations are due to the simultaneous motion of dislocations. Therefore, a dynamical approach in terms of dislocation densities seems to be more appropriate for the study of this problem.
The paper [21] by Mora, Peletier, and Scardia is conceived in this spirit. The aim of this work is to study rate-independent, quasi-static evolutions of systems conserving the total number n of dislocations. Here, the choice is to consider as main variable the dislocation density $\mu \in \mathcal{P}(\Omega)$, which is the most natural variable for dislocation dynamics. It is assumed that all dislocations have the same Burgers vector $\mathbf{b}$ and that they are all contained in a closed rectangle of positive distance from the boundary and well-separated from each other by a distance $r_n \gg \varepsilon_n$, where $\varepsilon_n > 0$ is the core-radius. The circulation condition in (3) is written as:

$$\int_{\partial B(\mathbf{z}_i,\varepsilon_n)} \boldsymbol{\beta}\,\mathbf{t}\;\mathrm{d}\mathscr{H}^1 = \frac{\mathbf{b}}{n}, \quad i = 1,\dots,n. \tag{4}$$

Moreover, precise asymptotic relations between the two scales ε_n and r_n are assumed. Since the number n of dislocations is constant during the evolution, the self-energy is of no relevance, even if n is large. In other words, the evolution is driven by the interaction energy. Hence, the energy is renormalized by considering

$$\mathcal{E}_n(\mu) = \min_{\boldsymbol{\beta}} E_{\varepsilon_n}(\mu,\boldsymbol{\beta}) - \frac{1}{2n^2}\sum_{i=1}^{n}\int_{\Omega_{\varepsilon_n}(\mu)} \mathbb{C}\mathbf{K}_{\mathbf{b}}^{\varepsilon_n}(\mathbf{x};\mathbf{z}_i) : \mathbf{K}_{\mathbf{b}}^{\varepsilon_n}(\mathbf{x};\mathbf{z}_i)\;\mathrm{d}\mathbf{x}. \tag{5}$$

The fundamental strain field $\mathbf{K}_{\mathbf{b}}(\cdot,\mathbf{z}_0)$ generated by a single dislocation at the point $\mathbf{z}_0$ with Burgers vector $\mathbf{b}$ and its perturbation $\mathbf{K}_{\mathbf{b}}^{\varepsilon}(\cdot,\mathbf{z}_0)$ are defined in (1.9) and (1.32), respectively. The second term at the right hand side of (5) represents the self-energy of the system, which is thus removed from the energy.
The first result proved in [21] is the Γ-convergence, with respect to the narrow topology of probability measures, of the renormalized energy $\mathcal{E}_n$, as $n \to \infty$, to the functional

$$\mathcal{E}(\mu) = \frac{1}{2}\iint_{\Omega\times\Omega} V(\mathbf{y},\mathbf{z})\;\mathrm{d}(\mu\otimes\mu)(\mathbf{y},\mathbf{z}) + \min_{\mathbf{v}} I(\mu,\mathbf{v}),$$

where

$$V(\mathbf{y},\mathbf{z}) = \int_{\Omega} \mathbb{C}\mathbf{K}_{\mathbf{b}}(\mathbf{x};\mathbf{y}) : \mathbf{K}_{\mathbf{b}}(\mathbf{x};\mathbf{z})\;\mathrm{d}\mathbf{x}$$

and $I(\mu,\cdot)$ is an auxiliary functional defined on $H^1(\Omega;\mathbb{R}^2)$. In this limit functional μ represents the density of dislocations at the continuum level, the first term of the energy describes interactions among dislocations, while the second term takes into account the interactions of dislocations with the boundary of the domain Ω. Then the evolution driven by the renormalized energy with a Wasserstein type dissipation with slip-plane confinement is studied.
The aim of this thesis is to extend the Γ-convergence result of [21] to the case of two different Burgers vectors $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^2$ with $\mathbf{b}_1\cdot\mathbf{b}_2 > 0$. Clearly, the case of a finite number

of Burgers vectors with positive pairwise scalar product can be treated exactly in the same way.
Our choice is to decompose the dislocation density as the sum of two measures, describing the position of the two families of dislocations with Burgers vector $\mathbf{b}_1$ and $\mathbf{b}_2$, respectively. Thus, the renormalized energy is now a function of these two measures and the Γ-convergence result is obtained with respect to the product narrow topology. The structure of the Γ-limit that we obtain is analogous to the case of a single Burgers vector, but it contains different potentials describing the interaction between dislocations of possible different species. Indeed, if we set

$$\begin{aligned} V_1(\mathbf{y},\mathbf{z}) &= \int_\Omega \mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}) : \mathbf{K}_1(\mathbf{x};\mathbf{z})\,\mathrm{d}\mathbf{x}, \\ V_2(\mathbf{y},\mathbf{z}) &= \int_\Omega \mathbb{C}\mathbf{K}_2(\mathbf{x};\mathbf{y}) : \mathbf{K}_2(\mathbf{x};\mathbf{z})\,\mathrm{d}\mathbf{x}, \\ V_{1,2}(\mathbf{y},\mathbf{z}) &= \int_\Omega \mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}) : \mathbf{K}_2(\mathbf{x};\mathbf{z})\,\mathrm{d}\mathbf{x}, \end{aligned}$$

where $\mathbf{K}_1 = \mathbf{K}_{\mathbf{b}_1}$ and $\mathbf{K}_2 = \mathbf{K}_{\mathbf{b}_2}$, then the limiting energy for a pair of admissible measures is given by

$$\begin{aligned} \mathcal{E}(\mu^1\mu^2) = &\frac{1}{2}\iint_{\Omega\times\Omega} V_1(\mathbf{y},\mathbf{z})\,\mathrm{d}(\mu^1\otimes\mu^1)(\mathbf{y},\mathbf{z}) \\ &+\frac{1}{2}\iint_{\Omega\times\Omega} V_2(\mathbf{y},\mathbf{z})\,\mathrm{d}(\mu^2\otimes\mu^2)(\mathbf{y},\mathbf{z}) \\ &+\iint_{\Omega\times\Omega} V_{1,2}(\mathbf{y},\mathbf{z})\,\mathrm{d}(\mu^1\otimes\mu^2)(\mathbf{y},\mathbf{z}) + \min_{\mathbf{v}} I(\mu^1,\mu^2,\mathbf{v}), \end{aligned}$$

where $I(\mu^1,\mu^2,\cdot)$ is an auxiliary functional defined on $H^1(\Omega;\mathbb{R}^2)$. In the limit functional μ^1 and μ^2 represent the proportion of the two species of dislocations at the continuum level, the first two terms of the energy describe interaction among dislocations of the same species, while the third term among dislocations of different species. The last term, as before, takes into account the interaction with the boundary.
Our assumptions are the same as in [21]. The confinement hypothesis (that is, the assumption of positive distance of dislocations from the boundary) is due to the fact that we do not impose any boundary condition to the system. Indeed, since the interaction potentials have a logarithmic repulsive behaviour (as already noted in [6]), dislocations can reduce their energy by moving to the boundary. The well-separation hypothesis $r_n \gg \varepsilon_n$, which was already introduced in [12], is physically reasonable, since the typical distance between dislocations (represented by the scale r_n) is much larger than the atomic distance (represented by the scale ε_n). Moreover, we assume the following:

$$\varepsilon_n \to 0, \quad r_n \to 0, \quad \varepsilon_n/r_n^3 \to 0, \quad nr_n^2 \to 0, \qquad \text{as } n\to\infty. \tag{6}$$

The first two conditions are clear. The third condition arises from the choice of imposing the incompatibility condition via the circulation condition (4), rather than as a singularity of the field as in (1). Indeed, the difference between the interaction energy corresponding to the fields $\mathbf{K}_i$ and $\mathbf{K}_i^{\varepsilon_n}$ is of order $\sqrt{\varepsilon_n/r_n^3}$, and the third condition in (6) ensures that it is negligible, as $n\to\infty$. The fourth condition is natural: it says that the total area of the core-regions goes to zero, as $n\to\infty$.
The proof of the Liminf inequality, which is rather straightforward, follows that of the original paper [21]. The proof of the Limsup inequality requires an adaptation of the

geometric construction used in [21] to produce a recovery sequence. The main difficulty is due to the singular character of the interaction potentials, that makes the energy not continuous with respect to narrow convergence and requires to suitably allocate the dislocations to guarantee convergence of the energies. On the other hand, a convenient feature of our setting is the compactness of the space on which the functionals are defined. This is due to the choice of working with probability measures.
A further interesting result that we show in the thesis, is that the class of measures on which the Γ-limit is finite is contained in $H^{-1}(\Omega)$. This fact is consistent with previous works (see [12], [21]) and with the physical observation that dislocations in crystals prefer to distribute along lines, the so-called *dislocation walls*.
A great restriction in our work is given by the assumption $\mathbf{b}_1 \cdot \mathbf{b}_2 > 0$, which is, however, crucial to avoid the creation of dipoles. Removing this hypothesis in our setting is a very challenging problem that, at the moment, seems to be way beyond reach. A first attempt in this direction is proposed in [13], where Burgers vectors of opposite signs are allowed, but some suitable regularizations of the interaction potential are needed to perform the analysis.
It is worth mentioning that the well-separation hypothesis has been overcome, in the setting of [12], in [10] for the sub-critical regime and in [14] for the critical regime. In both cases, the main tool used is an ad hoc version of the so-called *ball construction*. This technique has been introduced in the study of the Ginzburg-Landau functional (see [17], [24]), and consists in constructing a family of growing and merging balls that identifies a family of annuli, where most of the energy is concentrated. The fundamental contribution in [10] and [14] is that, in this construction, the ratio between the radii of the annuli can be controlled, so that the constants of the Korn inequality in these annuli can be uniformly bounded. We expect that a suitable version of the ball construction could possibly be of help also in our setting to remove the well-separation hypothesis.
Other interesting questions could concern the extension to our framework of the results of [21] about quasi-static evolution, as well as the inclusion of creation and annihilation phenomena in the dislocation model.
The thesis is structured as follows. In Chapter 1, the analysis of the variational model is performed, following [6]; the case of multiple Burgers vectors without any assumption on their scalar product is studied and the existence of an energy minimizing strain is proved. In Chapter 2, the renormalized energy is introduced with the assumption of two Burgers vectors $\mathbf{b}_1$,$\mathbf{b}_2$ satisfying $\mathbf{b}_1 \cdot \mathbf{b}_2 > 0$; then, the Γ-convergence result is proved (Theorem 2.1) and the characterization of measures with finite energy is established (Theorem 2.11). In the Appendix, a Korn type inequality for fields on the plane with prescribed curl, taken from [12], is discussed (Theorem A.2).

Notation and preliminaries

We use standard notation for scalars, vectors and matrices.

- For $a, b \in \mathbb{R}$, we set $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$ and we denote by $\lfloor a \rfloor$ the integer part of a, namely $\lfloor a \rfloor = \max\{k \in \mathbb{Z} : k \leq a\}$.
- For $\mathbf{a} \in \mathbb{R}^2$, we denote by $\mathbf{a}^\perp$ the vector obtained from $\mathbf{a}$ by a counter-clockwise rotation of $\pi/2$. Namely, if $\mathbf{a} = (a_1, a_2)$, we set $\mathbf{a}^\perp = (-a_2, a_1)$.
- We denote by $\mathbb{R}^{2\times 2}$ the space of square matrices of order 2 and by $\mathbf{I}$ the identity matrix. The subspaces of symmetric and skew-symmetric matrices are denoted by Sym(2) and Skew(2), respectively.
- For $\mathbf{A} \in \mathbb{R}^{2\times 2}$ we define its symmetric and skew-symmetric part by setting $\mathrm{sym}\mathbf{A} = \left(\mathbf{A} + \mathbf{A}^\top\right)/2$ and $\mathrm{skew}\mathbf{A} = \left(\mathbf{A} - \mathbf{A}^\top\right)/2$.
- For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ with $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$, we define the rank-one matrix $\mathbf{a} \otimes \mathbf{b} = (a_i\, b_j)_{i,j=1,2}$.
- We denote by : the inner product of square matrices. Namely, for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{2\times 2}$ with $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, we have $\mathbf{A} : \mathbf{B} = \sum_{i,j} a_{ij}\, b_{ij}$. We denote by $|\cdot|$ the associated norm, that is, $|\mathbf{A}| = (\mathbf{A} : \mathbf{A})^{1/2}$.
- For $\mathbf{A} \in \mathbb{R}^{2\times 2}$ given by $\mathbf{A} = (a_{ij})$, we set

$$\mathbf{A}^\perp = \begin{pmatrix} -a_{12} & a_{11} \\ -a_{22} & a_{21} \end{pmatrix}.$$

We adopt the common convention of denoting by C a positive constant that can change from line to line and that can be computed in terms of known quantities. When necessary, we are going to underline its dependence on some quantities using parentheses.

For what concerns normal and tangent unit vectors, we fix the following notation. For any bounded set in $\mathbb{R}^2$ with Lipschitz boundary, we always denote by $\mathbf{n}$ the *outward* unit normal on its boundary and we define the unit tangent vector as $\mathbf{t} = \mathbf{n}^\perp$. Thus, for example, given a ball centered at $\mathbf{x}_0$ with radius $r > 0$, we have

$$\mathbf{n}(\mathbf{x}) = \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}, \quad \mathbf{t}(\mathbf{x}) = \frac{(\mathbf{x} - \mathbf{x}_0)^\perp}{|\mathbf{x} - \mathbf{x}_0|},$$

for every $\mathbf{x} \in \partial B(\mathbf{x}_0, r)$.

We recall the following known fact: given a sequence (x_n) in a topological space X and given $x \in X$, then $x_n \to x$, as $n \to \infty$, if and only if for every subsequence (x_{n_k}) there exists a futher subsequence $(x_{n_{k_\ell}})$ such that $x_{n_{k_\ell}} \to x$, as $\ell \to \infty$. Henceforth, we are going to refer to this fact as the *Urysohn property*.

We use standard notation of Measure Theory. For example, we denote the characteristic function of a set $E \subset \mathbb{R}^2$ by χ_E. The spaces of signed and vector-valued Radon measures with bounded variation defined on the Borel subsets of $\Omega \subset \mathbb{R}^2$ are denoted by $\mathcal{M}_{\mathrm{b}}(\Omega)$ and $\mathcal{M}_{\mathrm{b}}(\Omega; \mathbb{R}^2)$, respectively. The set of Borel probability measures is denoted by $\mathcal{P}(\Omega)$. Given $\mu \in \mathcal{M}_{\mathrm{b}}(\Omega)$ and given a continuous map $\mathbf{g} \colon \mathbb{R}^2 \to \mathbb{R}^2$, we denote by $\mathbf{g}^\sharp\mu$ the *push-forward* measure. For the notions of Measure Theory, in particular for the definition of Radon measure, we refer to [7], [11].

We briefly recall the definition of the *narrow topology* on $\mathcal{M}_b(\Omega)$. This is the topology induced by the weak* topology of $\mathrm{rba}(\Omega) = (C_b(\Omega))'$. Thus, given $(\mu_n) \subset \mathcal{M}_b(\Omega)$ and $\mu \in \mathcal{M}_b(\Omega)$, we have that $\mu_n \to \mu$ *narrowly* if and only if

$$\int_\Omega v \, d\mu_n \to \int_\Omega v \, d\mu, \quad \text{as } n \to \infty,$$

for every $v \in C_b(\Omega)$. The properties of narrow convergence and its relation with the weak* convergence in $\mathcal{M}_b(\Omega)$ are treated in [11]. Here we simply recall some basic facts that are going to be used in what follows. Given $(\mu_n), (\nu_n) \subset \mathcal{M}_b(\Omega)$ and $\mu, \nu \in \mathcal{M}_b(\Omega)$, we have the following:

- if $\operatorname{supp} \mu_n \subset K$ for every n, where K is a closed subset of Ω, and $\mu_n \to \mu$ narrowly, as $n \to \infty$, then $\operatorname{supp} \mu \subset K$;
- if $\operatorname{supp} \mu_n \subset K$ for every n, where K is a closed subset of Ω, then $\mu_n \to \mu$ narrowly if and only if
$$\int_\Omega u \, d\mu_n \to \int_\Omega u \, d\mu, \quad \text{as } n \to \infty,$$
for every $u \in C(\Omega)$ which is bounded on K;
- if $\mu_n \to \mu$ narrowly and $\nu_n \to \nu$ narrowly, as $n \to \infty$, then $\mu_n \otimes \nu_n \to \mu \otimes \nu$ narrowly, that is, weakly* in $(C_b(\Omega \times \Omega))'$, as $n \to \infty$.

We use standard notation for Lebesgue and Sobolev spaces and for spaces of functions of bounded variation. For this topics, we refer to [1], [18]. Note that for $E \subset \mathbb{R}^2$ and $1 \leq p \leq \infty$, the space $L^p(\partial E)$ is always intended with respect to the measure $\mathscr{H}^1 \llcorner \partial E$.

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Given $\mathbf{u} \in H^1(\Omega; \mathbb{R}^2)$, we define its symmetric gradient as $\mathrm{E}\mathbf{u} = \operatorname{sym} \mathrm{D}\mathbf{u}$. We call *infinitesimal rigid-body motion* every field $\boldsymbol{\eta} \in H^1(\Omega; \mathbb{R}^2)$ such that $\mathrm{E}\boldsymbol{\eta} = \mathbf{0}$. It is easy to check that such functions are given, for $\mathbf{x} \in \Omega$, by $\boldsymbol{\eta}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ for some $\mathbf{A} \in \mathrm{Skew}(2)$ and $\mathbf{a} \in \mathbb{R}^2$.

In what follows we are going to make an extensive use of the *Korn inequality* that we present in the following form.

Theorem. ***(Korn inequality)*** *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $B \subset \Omega$ a ball. There exists a constant $C > 0$, depending only on Ω, such that, for every field $\mathbf{u} \in H^1(\Omega; \mathbb{R}^2)$ satisfying*

$$\int_B \mathbf{u} \, d\mathbf{x} = \mathbf{0}, \qquad \int_B (\mathrm{D}\mathbf{u} - \mathrm{D}\mathbf{u}^\top) \, d\mathbf{x} = \mathbf{0},$$

we have

$$||\mathbf{u}||_{H^1(\Omega;\mathbb{R}^2)} \leq C ||\mathrm{E}\mathbf{u}||_{L^2(\Omega;\mathbb{R}^{2\times 2})}.$$

For a proof we refer to [23]. In the Appendix we are going to prove an extension of this result which is peculiar of dimension 2 (see Theorem A.2).

For matrix-valued fields, the divergence and curl operators are always understood to act row-wise. More precisely, for $\boldsymbol{\beta} \colon \Omega \to \mathbb{R}^{2\times 2}$ with $\boldsymbol{\beta} = (\beta_{ij})$, we set

$$\operatorname{div} \boldsymbol{\beta} = \begin{pmatrix} \partial_1 \beta_{11} + \partial_2 \beta_{12} \\ \partial_1 \beta_{21} + \partial_2 \beta_{22} \end{pmatrix}, \qquad \operatorname{curl} \boldsymbol{\beta} = \begin{pmatrix} \partial_1 \beta_{12} - \partial_2 \beta_{11} \\ \partial_1 \beta_{22} - \partial_2 \beta_{21} \end{pmatrix}.$$

In the following we are going to consider maps $\mathbf{u} \in L^2(\Omega;\mathbb{R}^2)$ which distributional curl is in $L^2(\Omega)$. This means that there exists $v \in L^2(\Omega)$ such that, for every $\varphi \in C_c^\infty(\Omega)$, we have

$$-\int_\Omega \mathbf{u} \cdot \mathrm{D}\varphi^\perp \,\mathrm{d}\mathbf{x} = \int_\Omega v\,\varphi\,\mathrm{d}\mathbf{x}.$$

In this case, we set $\operatorname{curl}\mathbf{u} = v$. The space of such maps $\mathbf{u}$ is usually denoted by $H(\mathrm{curl};\Omega)$ and this is endowed with the norm

$$||\mathbf{u}||_{H(\mathrm{curl};\Omega)} = ||\mathbf{u}||_{L^2(\Omega;\mathbb{R}^2)} + ||\operatorname{curl}\mathbf{u}||_{L^2(\Omega)}.$$

The space $C^1(\overline{\Omega};\mathbb{R}^2)$ is dense in $H(\mathrm{curl};\Omega)$. Moreover the map $\boldsymbol{\varphi} \mapsto \boldsymbol{\varphi}_{|\partial\Omega} \cdot \mathbf{t}$ defined on $C^1(\overline{\Omega};\mathbb{R}^2)$, admits a unique continuous linear extension

$$\boldsymbol{\gamma}_{\mathbf{t}} \colon H(\mathrm{curl};\Omega) \to H^{-1/2}(\partial\Omega)$$

defined by

$${}_{H^{-1/2}(\partial\Omega)}\langle \boldsymbol{\gamma}_{\mathbf{t}}(\mathbf{u}), \zeta\rangle_{H^{1/2}(\partial\Omega)} = \int_\Omega \operatorname{curl}\mathbf{u}\,\tilde{\zeta}\,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{u} \cdot \left(\mathrm{D}\tilde{\zeta}\right)^\perp \mathrm{d}\mathbf{x}$$

where $\tilde{\zeta}$ is a lifting of ζ, that is, a function in $H^1(\Omega)$ such that $\tilde{\zeta} = \zeta$ on $\partial\Omega$ in the sense of traces. As for usual traces, we simply write $\gamma_{\mathbf{t}}(\mathbf{u})$ as $\mathbf{u}\cdot\mathbf{t}$. Furthermore, if the boundary of Ω has several connected components, say $\partial\Omega = \bigcup_{i=1}^m \Gamma_i$, we write

$$\int_{\Gamma_i} \mathbf{u}\cdot\mathbf{t}\,\mathrm{d}\mathscr{H}^1 = {}_{H^{-1/2}(\partial\Omega)}\langle \boldsymbol{\gamma}_{\mathbf{t}}(\mathbf{u}), \chi_{\Gamma_i}\rangle_{H^{1/2}(\partial\Omega)},$$

for $i = 1, \ldots, m$. For details about the space $H(\mathrm{curl};\Omega)$ and about the tangential trace operator, we refer to [9], [15], [20].

In particular, in what follows, we are going to deal with functions $\mathbf{u} \in L^2(\Omega;\mathbb{R}^2)$ such that $\operatorname{curl}\mathbf{u} = 0$ in Ω in the sense of distributions. For such functions, results analogous to that of the classical theory hold. For example, if $\mathbf{u} \in L^2(\Omega;\mathbb{R}^2)$ with $\operatorname{curl}\mathbf{u} = 0$ in Ω is such that the circulation (in the sense specified above) along any closed curve contained in Ω is zero, then it admists a potential. This is substantially the content of Lemma 1.1. Moreover, in the case of simply connected domains, the following weak formulation of the classical *Poincaré Lemma* holds.

Theorem. ***(Weak Poincaré Lemma)*** *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected Lipschitz domain. Let $\mathbf{u} \in L^2(\Omega;\mathbb{R}^2)$ be such that $\operatorname{curl}\mathbf{u} = 0$ in Ω. Then, there exists a potential $f \in H^1(\Omega)$, which is unique up to additive constants, such that $\mathbf{u} = \mathrm{D}f$.*

Note that, for $1 \le p < \infty$, an analogous result holds with $\mathbf{u} \in L^p(\Omega;\mathbb{R}^2)$ and $f \in W^{1,p}(\Omega)$. The proof is based on regularization by convolution and can be found in [15] and [20] (in the references the case $p = 2$ is considered, but the general case with $1 \le p < \infty$ can be proved in the same way).

Finally, we are going to use the notion of Γ-convergence in metric spaces. For this topic, we refer to [3], [8].

Chapter 1

The variational model

In this chapter we introduce a semi-discrete model that describes a system of straight and parallel edge dislocations in an elastic body, as presented in [6]. For more information about the derivation of this model we refer to [5]. We work in the setting of linearized elasticity and we adopt the so-called core-region approach.

1.1 Description of the model

Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded Lipschitz domain representing the cross section of an elastic body whose crystalline structure has a cylindrical symmetry. In the context of linear elasticity, one considers the *displacement*, given by a function $\mathbf{u} \in H^1(\Omega; \mathbb{R}^2)$, with the aim of minimizing the *elastic energy* defined as the functional

$$\mathbf{u} \mapsto \int_\Omega W(\mathrm{D}\mathbf{u}) \, \mathrm{d}\mathbf{x}, \tag{1.1}$$

where

$$W(\boldsymbol{\beta}) = \frac{1}{2} \mathbb{C}\boldsymbol{\beta} : \boldsymbol{\beta}$$

is the *elastic energy density* and $\mathbb{C}$ is the *elasticity tensor*. The tensor $\mathbb{C}$ identifies a linear operator from $\mathbb{R}^{2\times 2}$ in itself that, in the case of an isotropic material as considered here, has the following form:

$$\mathbb{C}\boldsymbol{\beta} = \lambda(\mathrm{tr}\boldsymbol{\beta})\mathbf{I} + \mu(\boldsymbol{\beta} + \boldsymbol{\beta}^\top) = \lambda(\mathrm{tr}\,\mathrm{sym}\boldsymbol{\beta})\mathbf{I} + 2\mu\,\mathrm{sym}\boldsymbol{\beta},$$

where $\lambda, \mu \in \mathbb{R}$ are the *Lamé moduli.* Note that $\mathbb{C}\boldsymbol{\beta} \in \mathrm{Sym}(2)$ for every $\boldsymbol{\beta} \in \mathbb{R}^{2\times 2}$ and that

$$\mathbb{C}\boldsymbol{\beta}_1 : \boldsymbol{\beta}_2 = \mathbb{C}\,\mathrm{sym}\boldsymbol{\beta}_1 : \mathrm{sym}\boldsymbol{\beta}_2 \tag{1.2}$$

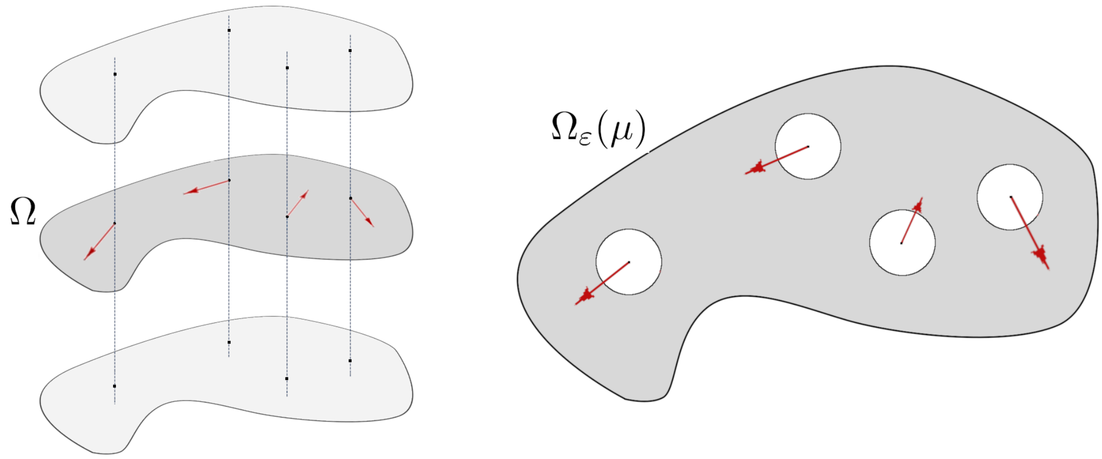
for every $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in \mathbb{R}^{2\times 2}$. Moreover the elasticity tensor is symmetric, i.e., we have

$$\mathbb{C}\,\boldsymbol{\beta}_1 : \boldsymbol{\beta}_2 = \mathbb{C}\,\boldsymbol{\beta}_2 : \boldsymbol{\beta}_1 \tag{1.3}$$

for every $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in \mathbb{R}^{2\times 2}$. In addition to this, we assume the elasticity tensor to be positive definite on symmetric matrices, that is, there exist two constants $C_1, C_2 > 0$ such that

$$C_1|\mathrm{sym}\boldsymbol{\beta}|^2 \le \mathbb{C}\boldsymbol{\beta} : \boldsymbol{\beta} \le C_2|\mathrm{sym}\boldsymbol{\beta}|^2 \tag{1.4}$$

for any $\boldsymbol{\beta} \in \mathbb{R}^{2\times 2}$. It is easy to check that this is equivalent to requiring $\lambda + \mu > 0$ and $\mu > 0$.



(a) (b)

Figure 1.1: (a) The cross section Ω orthogonal to the dislocation lines. (b) The core-regions and the reduced domain $\Omega_\varepsilon(\mu)$.

We consider a system of a finite number n of straight edge dislocations orthogonal to the cross section with Burgers vectors given by $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^2$, respectively. In the model, these defects are identified with the points $\mathbf{z}_1, \ldots, \mathbf{z}_n \in \Omega$ given by the intersection of the dislocation lines with the cross section (see Figure 1.1). The information about the location of the defects is encoded in the *dislocation density*, which is the probability measure defined by

$$\mu = \frac{1}{n} \sum_{i=1}^{n} \delta_{\mathbf{z}_i}.$$

In the presence of dislocations the *strain field* is a map $\boldsymbol{\beta}\colon \Omega \to \mathbb{R}^{2\times 2}$ satisfying the following incompatibility condition:

$$\operatorname{curl} \boldsymbol{\beta} = \sum_{i=1}^{n} \mathbf{b}_i \, \delta_{\mathbf{z}_i}. \tag{1.5}$$

Clearly, for $\boldsymbol{\beta} \in L^2(\Omega; \mathbb{R}^{2\times 2})$, equation (1.5) cannot be satisfied since the right hand side is not in $H^{-1}(\Omega; \mathbb{R}^2)$. On the other hand, by (1.4), it is evident that, in order to minimize the elastic energy (1.1), we have to consider $\boldsymbol{\beta} \in L^2(\Omega; \mathbb{R}^{2\times 2})$. To overcome this problem, we follow the so called *core-region approach* (see [6], [10], [12], [21]).

We introduce a small parameter $\varepsilon > 0$, called the *core-radius*, which should be thought to be comparable with the atomic spacing in the underlying crystal lattice. Then, for each defect, we remove from the domain a ball of radius ε centered at the point $\mathbf{z}_i$ and we consider the effective domain

$$\Omega_\varepsilon(\mu) = \Omega \setminus \left(\bigcup_{i=1}^{n} \overline{B}(\mathbf{z}_i, \varepsilon) \right).$$

Here we implicitly assume that the core-radius is sufficiently small in order to have that $B(\mathbf{z}_i, \varepsilon) \subset\subset \Omega$ for $i = 1, \ldots, n$. Hence, in analogy with (1.1), we introduce the following *energy functional*

$$E_\varepsilon(\mu, \boldsymbol{\beta}) = \int_{\Omega_\varepsilon(\mu)} W(\boldsymbol{\beta}) \, \mathrm{d}\mathbf{x} \tag{1.6}$$

defined for $\boldsymbol{\beta} \in L^2(\Omega_\varepsilon(\mu);\mathbb{R}^{2\times 2})$. The class of *admissible fields* is then given by

$$\begin{aligned}\mathcal{A}_\varepsilon(\mu;\mathbf{b}_1,\dots,\mathbf{b}_n) = \Big\{\boldsymbol{\beta} \in & L^2(\Omega_\varepsilon(\mu)) : \quad \operatorname{curl}\boldsymbol{\beta} = \mathbf{0} \ \text{ in } \Omega_\varepsilon(\mu), \\ & \int_{\partial B(\mathbf{z}_i,\varepsilon)} \boldsymbol{\beta}\,\mathbf{t}\,\mathrm{d}\mathscr{H}^1 = \frac{\mathbf{b}_i}{n} \ \text{ for every } i = 1,\dots,n\Big\}.\end{aligned} \tag{1.7}$$

The condition $\operatorname{curl}\boldsymbol{\beta} = \mathbf{0}$ in (1.7) should be intended in the sense of distributions, namely, for every $\varphi \in C_c^\infty(\Omega_\varepsilon(\mu))$, it is required that

$$\int_{\Omega_\varepsilon(\mu)} \boldsymbol{\beta}(\mathbf{x})\,\mathrm{D}\varphi(\mathbf{x})^\perp\,\mathrm{d}\mathbf{x} = \mathbf{0}.$$

The boundary integral in (1.7) should be interpreted as a duality pairing in the sense of traces. More explicitly, since curl$\boldsymbol{\beta} = 0$, this means that for every $i = 1,\dots,n$, any admissible field need to satisfy

$$-\int_{\Omega_\varepsilon(\mu)} \boldsymbol{\beta}(\mathbf{x})\,\mathrm{D}\zeta(\mathbf{x})^\perp\,\mathrm{d}\mathbf{x} = \frac{1}{n}\sum_{i=1}^n \mathbf{b}_i$$

for every $\zeta \in H_0^1(\Omega)$ such that $\zeta = 1$ on $B(\mathbf{z}_i,\varepsilon)$. Note the order $1/n$ of the integral incompatibility condition in (1.7) which is due to the choice of working with probability measures.
We remark that, instead of the two conditions in (1.7), we could alternatively require the following circulation condition for the strain:

$$\operatorname{curl}\boldsymbol{\beta} = \frac{1}{2\pi n\varepsilon_n}\sum_{i=1}^n \mathbf{b}_i\mathscr{H}^1 \llcorner \partial B(\mathbf{z}_i,\varepsilon_n) \quad \text{in } \Omega.$$

Sometimes, with an abuse of terminology, we are going to refer to the admissible fields as strain fields. Finally, the equilibrium configurations of the system will be given by the solutions of the following minimization problem:

$$\min_{\boldsymbol{\beta}\in\mathcal{A}_\varepsilon(\mu;\mathbf{b}_1,\dots,\mathbf{b}_n)} E_\varepsilon(\mu,\boldsymbol{\beta}). \tag{1.8}$$

1.2 Structure of admissible strains

In order to proceed, we need to introduce a particular class of functions that are going to be used extensively in our study. For any $\mathbf{z}_0 \in \Omega$ and any $\mathbf{b} \in \mathbb{R}^2$, we set

$$\mathbf{K}_\mathbf{b}(\mathbf{x};\mathbf{z}_0) = \frac{1}{2\pi|\mathbf{x}-\mathbf{z}_0|^2}\mathbf{b}\otimes(\mathbf{x}-\mathbf{z}_0)^\perp + \mathrm{D}\mathbf{v}_\mathbf{b}(\mathbf{x}-\mathbf{z}_0), \tag{1.9}$$

where

$$\mathbf{v}_\mathbf{b}(\mathbf{x}) = -\frac{\mu\log|\mathbf{x}|}{2\pi(\lambda+2\mu)}\mathbf{b}^\perp - \frac{\lambda+\mu}{4\pi(\lambda+2\mu)|\mathbf{x}|^2}\Big((\mathbf{b}\cdot\mathbf{x}^\perp)\mathbf{x} + (\mathbf{b}\cdot\mathbf{x})\mathbf{x}^\perp\Big).$$

Henceforth we will refer to this function as the *fundamental strain*. Indeed the function in (1.9) can be regarded as the deformation of the whole plane due to the presence of a single dislocation with Burgers vector $\mathbf{b}$ located at the point $\mathbf{z}_0$. Clearly we have that

$\mathbf{K_b}(\cdot;\mathbf{z}_0)\in C^\infty(\mathbb{R}^2\setminus\{\mathbf{z}_0\};\mathbb{R}^{2\times2})$. The explicit expressions of the strain (1.9) and of the corresponding stress are given by

$$\begin{aligned}\mathbf{K_b}(\cdot;\mathbf{z}_0)=\frac{1}{2\pi(\lambda+2\mu)\varrho}\Big\{&\mu(\mathbf{b}\cdot\mathbf{e}_\vartheta)\mathbf{e}_\varrho\otimes\mathbf{e}_\varrho+(2\lambda+3\mu)(\mathbf{b}\cdot\mathbf{e}_\varrho)\mathbf{e}_\varrho\otimes\mathbf{e}_\vartheta\\&-\mu(\mathbf{b}\cdot\mathbf{e}_\varrho)\mathbf{e}_\vartheta\otimes\mathbf{e}_\varrho+\mu(\mathbf{b}\cdot\mathbf{e}_\vartheta)\mathbf{e}_\vartheta\otimes\mathbf{e}_\vartheta\Big\}\end{aligned}\tag{1.10}$$

and

$$\begin{aligned}\mathbb{C}\mathbf{K_b}(\cdot;\mathbf{z}_0)=\frac{\mu(\lambda+\mu)}{\pi(\lambda+2\mu)\varrho}\Big\{&(\mathbf{b}\cdot\mathbf{e}_\vartheta)\mathbf{e}_\varrho\otimes\mathbf{e}_\varrho+(\mathbf{b}\cdot\mathbf{e}_\varrho)\mathbf{e}_\varrho\otimes\mathbf{e}_\vartheta\\&+(\mathbf{b}\cdot\mathbf{e}_\varrho)\mathbf{e}_\vartheta\otimes\mathbf{e}_\varrho+(\mathbf{b}\cdot\mathbf{e}_\vartheta)\mathbf{e}_\vartheta\otimes\mathbf{e}_\vartheta\Big\}\end{aligned}\tag{1.11}$$

respectively, where we denoted by (ϱ,ϑ) the polar coordinates centered at $\mathbf{z}_0$ and by $(\mathbf{e}_\varrho,\mathbf{e}_\vartheta)$ the associated frame. From (1.10) and (1.11) we deduce that

$$|\mathbf{K_b}(\mathbf{x};\mathbf{z}_0)|\le\frac{C|\mathbf{b}|}{|\mathbf{x}-\mathbf{z}_0|},\qquad|\mathbb{C}\mathbf{K_b}(\mathbf{x};\mathbf{z}_0)|\le\frac{C|\mathbf{b}|}{|\mathbf{x}-\mathbf{z}_0|}\tag{1.12}$$

for every $\mathbf{x}\neq\mathbf{z}_0$, thus we have $\mathbf{K_b}(\cdot;\mathbf{z}_0)\in L^1_{\text{loc}}(\mathbb{R}^2;\mathbb{R}^{2\times2})$ and $\mathbb{C}\mathbf{K_b}(\cdot;\mathbf{z}_0)\in L^1_{\text{loc}}(\mathbb{R}^2;\mathbb{R}^{2\times2})$. The fundamental strain satisfies the following circulation condition:

$$\int_{\partial B(\mathbf{y}_0,r)}\mathbf{K_b}(\mathbf{x};\mathbf{y}_0)\,\mathbf{t}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})=\mathbf{b}\tag{1.13}$$

for every $r>0$. Moreover, straightforward computations show that

$$\operatorname{div}\mathbb{C}\mathbf{K_b}(\mathbf{x};\mathbf{z}_0)=\mathbf{0},\quad\operatorname{curl}\mathbf{K_b}(\mathbf{x};\mathbf{z}_0)=\mathbf{0}\qquad\text{for every }\mathbf{x}\neq\mathbf{z}_0,\tag{1.14}$$

and actually the field $\mathbf{K_b}(\cdot;\mathbf{z}_0)$ is a distributional solution of the following system:

$$\begin{cases}\operatorname{div}\mathbb{C}\mathbf{K_b}(\cdot;\mathbf{z}_0)=\mathbf{0},\\\operatorname{curl}\mathbf{K_b}(\cdot;\mathbf{z}_0)=\mathbf{b}\,\delta_{\mathbf{z}_0}\end{cases}\quad\text{in }\mathbb{R}^2.\tag{1.15}$$

To prove this, without loss of generality, we can assume that $\mathbf{z}_0=\mathbf{0}$. Consider any $\varphi\in C^\infty_c(\mathbb{R}^2)$ and take $R>0$ such that $\operatorname{supp}\varphi\subset B(\mathbf{0},R)$. We begin with the proof of the first equation (1.15). We have to show that

$$\int_{B(\mathbf{0},R)}\mathbb{C}\mathbf{K_b}(\mathbf{x};\mathbf{z}_0)\,\mathrm{D}\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}=\mathbf{0}.\tag{1.16}$$

By the integrability of the fundamental stress $\mathbb{C}\mathbf{K_b}(\cdot;\mathbf{0})$, we can write

$$\begin{aligned}\int_{B(\mathbf{0},R)}\mathbb{C}\mathbf{K_b}(\mathbf{x};\mathbf{0})\,\mathrm{D}\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}&=\lim_{r\to0^+}\int_{B(\mathbf{0},R)\setminus B(\mathbf{0},r)}\mathbb{C}\mathbf{K_b}(\mathbf{x};\mathbf{0})\,\mathrm{D}\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}\\&=-\lim_{r\to0^+}\int_{B(\mathbf{0},R)\setminus B(\mathbf{0},r)}\operatorname{div}\mathbb{C}\mathbf{K_b}(\mathbf{x};\mathbf{0})\,\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}\\&\quad-\lim_{r\to0^+}\int_{\partial B(\mathbf{0},r)}\mathbb{C}\mathbf{K_b}(\mathbf{x};\mathbf{0})\,\mathbf{n}(\mathbf{x})\,\varphi(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}),\end{aligned}\tag{1.17}$$

where in the last equality we used integration by parts. From (1.14) we deduce that the first integral at the right hand side of (1.17) is zero, while for the second one we have

$$\begin{aligned}\int_{\partial B(\mathbf{0},r)} &\mathbb{C}\mathbf{K}_{\mathbf{b}}(\mathbf{x};\mathbf{0})\,\mathbf{n}(\mathbf{x})\,\varphi(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\ &= \frac{\mu(\lambda+\mu)}{\pi(\lambda+2\mu)}\int_{\partial B(\mathbf{0},r)}\frac{1}{r^3}\left((\mathbf{b}\cdot\mathbf{x}^\perp)\mathbf{x}+(\mathbf{b}\cdot\mathbf{x})\mathbf{x}^\perp\right)\varphi(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\ &= \frac{\mu(\lambda+\mu)}{\pi(\lambda+2\mu)}\int_{\partial B(\mathbf{0},1)}\left((\mathbf{b}\cdot\mathbf{y}^\perp)\mathbf{y}+(\mathbf{b}\cdot\mathbf{y})\mathbf{y}^\perp\right)\varphi(r\mathbf{y})\,\mathrm{d}\mathscr{H}^1(\mathbf{y}),\end{aligned}$$

where in the last equality we performed the change of variables $\mathbf{x} = r\mathbf{y}$. Then, using the Dominated Convergence Theorem, we obtain

$$\begin{aligned}\int_{\partial B(\mathbf{0},1)} &\left((\mathbf{b}\cdot\mathbf{y}^\perp)\mathbf{y}+(\mathbf{b}\cdot\mathbf{y})\mathbf{y}^\perp\right)\varphi(r\mathbf{y})\,\mathrm{d}\mathbf{y}\\ &\to \int_{\partial B(\mathbf{0},1)}\left((\mathbf{b}\cdot\mathbf{y}^\perp)\mathbf{y}+(\mathbf{b}\cdot\mathbf{y})\mathbf{y}^\perp\right)\varphi(\mathbf{0})\,\mathrm{d}\mathbf{y}, \quad \text{as } r\to 0^+,\end{aligned}$$

and the last integral is zero. Indeed, denoting $\mathbf{b} = (b_1, b_2)$ and $\mathbf{y} = (y_1, y_2)$, we have that $(\mathbf{b}\cdot\mathbf{y}^\perp)\mathbf{y} + (\mathbf{b}\cdot\mathbf{y})\mathbf{y}^\perp = y_1y_2(b_2 - b_1)(1,1)$ and $\int_{\partial B(\mathbf{0},1)} y_1y_2\,\mathrm{d}\mathbf{y} = 0$. Thus (1.16) is proved and we move to the proof of the second equation in (1.15). We have to prove that

$$-\int_{B(\mathbf{0},R)}\mathbf{K}_{\mathbf{b}}(\mathbf{x};\mathbf{z}_0)\,\mathrm{D}\varphi(\mathbf{x})^\perp\,\mathrm{d}\mathbf{x} = \mathbf{b}\,\varphi(\mathbf{0}) \tag{1.18}$$

for every $\varphi \in C_c^\infty(\Omega)$ with $\operatorname{supp}\varphi \subset B(\mathbf{0}, R)$. Recalling (1.9), it is sufficient to show that

$$-\int_{B(\mathbf{0},R)}\frac{1}{2\pi|\mathbf{x}|^2}\left(\mathbf{b}\otimes\mathbf{x}^\perp\right)\mathrm{D}\varphi(\mathbf{x})^\perp\,\mathrm{d}\mathbf{x} = \mathbf{b}\,\varphi(\mathbf{0}). \tag{1.19}$$

As before, we write

$$\begin{aligned}\int_{B(\mathbf{0},R)} &\frac{1}{2\pi|\mathbf{x}|^2}\left(\mathbf{b}\otimes\mathbf{x}^\perp\right)\mathrm{D}\varphi(\mathbf{x})^\perp\,\mathrm{d}\mathbf{x}\\ &= \lim_{r\to0^+}\int_{B(\mathbf{0},R)\setminus B(\mathbf{0},r)}\frac{1}{2\pi|\mathbf{x}|^2}\left(\mathbf{b}\otimes\mathbf{x}^\perp\right)\mathrm{D}\varphi(\mathbf{x})^\perp\,\mathrm{d}\mathbf{x}\\ &= -\lim_{r\to0^+}\int_{B(\mathbf{0},R)\setminus B(\mathbf{0},r)}\operatorname{curl}\left(\frac{1}{2\pi|\mathbf{x}|^2}\left(\mathbf{b}\otimes\mathbf{x}^\perp\right)\right)\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}\\ &\quad-\lim_{r\to0^+}\int_{\partial B(\mathbf{0},r)}\frac{1}{2\pi|\mathbf{x}|^2}\left(\mathbf{b}\otimes\mathbf{x}^\perp\right)\mathbf{t}(\mathbf{x})\,\varphi(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}),\end{aligned} \tag{1.20}$$

where we used integration by parts. Again (1.14) entails that the first integral at the right hand side of (1.20) is zero. For the second one we have

$$\begin{aligned}\int_{\partial B(\mathbf{0},r)}\frac{1}{2\pi|\mathbf{x}|^2}\left(\mathbf{b}\otimes\mathbf{x}^\perp\right)\mathbf{t}(\mathbf{x})\,\varphi(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) &= \int_{\partial B(\mathbf{0},r)}\frac{1}{2\pi r}\mathbf{b}\,\varphi(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\ &= \int_{\partial B(\mathbf{0},1)}\frac{1}{2\pi}\mathbf{b}\,\varphi(r\mathbf{y})\,\mathrm{d}\mathscr{H}^1(\mathbf{y}),\end{aligned}$$

where in the last line we set $\mathbf{x} = r\mathbf{y}$. Finally, using again the Dominated Convergence Theorem, we get

$$\int_{\partial B(\mathbf{0},1)}\frac{1}{2\pi}\mathbf{b}\,\varphi(r\mathbf{y})\,\mathrm{d}\mathscr{H}^1(\mathbf{y}) \to \int_{\partial B(\mathbf{0},1)}\frac{1}{2\pi}\mathbf{b}\,\varphi(0)\,\mathrm{d}\mathscr{H}^1(\mathbf{y}) = \mathbf{b}\,\varphi(\mathbf{0}), \quad \text{as } r\to0^+,$$

which gives (1.19).
Now we are going to use the fundamental strains to study the class of admissible strains. This turns out to have an affine structure as stated in the following result.

Lemma 1.1. ***(Structure of admissible strains)*** *For every pair of admissible strains* $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in \mathcal{A}_\varepsilon(\mu; \mathbf{b}_1, \dots, \mathbf{b}_n)$ *there exists a function* $\mathbf{u} \in H^1(\Omega_\varepsilon(\mu); \mathbb{R}^2)$ *such that*

$$\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2 = \mathrm{D}\mathbf{u} \quad \textit{in } \Omega_\varepsilon(\mu).$$

Proof. It is clearly sufficient to show that every $\boldsymbol{\beta} \in L^2(\Omega_\varepsilon(\mu); \mathbb{R}^{2\times 2})$ satisfying

$$\operatorname{curl} \boldsymbol{\beta} = 0 \quad \text{in } \Omega_\varepsilon(\mu) \tag{1.21}$$

in the sense of distributions and

$$\int_{\partial B(\mathbf{z}_i,\varepsilon)} \boldsymbol{\beta}(\mathbf{x})\, \mathbf{t}(\mathbf{x})\, \mathrm{d}\mathscr{H}^1(\mathbf{x}) = \mathbf{0}, \qquad i = 1, \dots, n, \tag{1.22}$$

in the sense of traces, is the gradient of a function $\mathbf{u} \in H^1(\Omega_\varepsilon(\mu); \mathbb{R}^2)$. Firstly we prove that this is true in every open set compactly contained in $\Omega_\varepsilon(\mu)$.
Given $\delta > 0$ small enough, we define $\Omega' = \{\mathbf{x} \in \Omega_\varepsilon(\mu) : d(\mathbf{x}, \partial\Omega_\varepsilon(\mu)) > \delta\}$, so that $\Omega' \subset\subset \Omega_\varepsilon(\mu)$. Using (1.21) and (1.22), it is easy to see that

$$\int_{\partial B(\mathbf{z}_i,\varepsilon+\delta)} \boldsymbol{\beta}(\mathbf{x})\, \mathbf{t}(\mathbf{x})\, \mathrm{d}\mathscr{H}^1(\mathbf{x}) = \mathbf{0}, \qquad i = 1, \dots, n \tag{1.23}$$

in the sense of traces. We consider a sequence of standard mollifiers (ρ_k) and we define the regularized functions $\boldsymbol{\beta}_k = \boldsymbol{\beta} * \rho_k \in C^\infty(\overline{\Omega'}; \mathbb{R}^{2\times 2})$. By the usual properties of the mollification, for any k we have

$$\operatorname{curl} \boldsymbol{\beta}_k(\mathbf{x}) = -\int_{\Omega_\varepsilon(\mu)} \boldsymbol{\beta}(\mathbf{y})\, \mathrm{D}\rho_k(\mathbf{x} - \mathbf{y})^\perp\, \mathrm{d}\mathbf{y}, \qquad \text{for every } \mathbf{x} \in \Omega'.$$

From (1.21) it follows that

$$\operatorname{curl} \boldsymbol{\beta}_k = \mathbf{0} \quad \text{in } \Omega', \qquad \text{for } k \gg 1 \tag{1.24}$$

in the classical sense. Moreover, we also have $\boldsymbol{\beta}_k \to \boldsymbol{\beta}$ in $L^2(\Omega'; \mathbb{R}^{2\times 2})$ as $k \to \infty$ which, combined with (1.21) and (1.24), says that

$$\boldsymbol{\beta}_k \to \boldsymbol{\beta} \quad \text{in } H(\mathrm{curl}; \Omega'), \quad \text{as } k \to \infty.$$

Therefore, by the continuity of the tangential trace operator in $H(\mathrm{curl}; \Omega')$, we deduce that $\boldsymbol{\beta}_k \mathbf{t} \to \boldsymbol{\beta}\, \mathbf{t}$ in $H^{-1/2}(\partial\Omega'; \mathbb{R}^2)$, as $k \to \infty$, and in particular

$$\int_{\partial B(\mathbf{z}_i,\varepsilon+\delta)} \boldsymbol{\beta}_k(\mathbf{x})\, \mathbf{t}(\mathbf{x})\, \mathrm{d}\mathscr{H}^1(\mathbf{x}) \to \int_{\partial B(\mathbf{z}_i,\varepsilon+\delta)} \boldsymbol{\beta}(\mathbf{x})\, \mathbf{t}(\mathbf{x})\, \mathrm{d}\mathscr{H}^1(\mathbf{x}), \qquad \text{as } k \to \infty \tag{1.25}$$

for $i = 1, \dots, n$. If we denote by $\mathbf{b}^i_k \in \mathbb{R}^2$ the left hand side in the previous limit, then from (1.23) and (1.25) we have that $\mathbf{b}^i_k \to \mathbf{0}$ as $k \to \infty$. For $i = 1, \dots, n$ we consider the fundamental strains $\mathbf{K}_{\mathbf{b}^i_k}(\cdot; \mathbf{z}_i)$ which, analogously to (1.13) and (1.14), satisfy

$$\operatorname{curl} \mathbf{K}_{\mathbf{b}^i_k}(\cdot; \mathbf{z}_i) = \mathbf{0} \qquad \text{in } \Omega' \tag{1.26}$$

and

$$\int_{\partial B(\mathbf{z}_j,\varepsilon+\delta)} \mathbf{K}_{\mathbf{b}_k^i}(\mathbf{x};\mathbf{z}_i)\,\mathbf{t}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) = \delta_{ij}\mathbf{b}_k^i \tag{1.27}$$

for every k and for $j = 1,\dots,n$. Here δ_{ij} denotes the usual *Kronecker delta symbol* defined by $\delta_{ij} = 1$, if $i = j$, and by $\delta_{ij} = 0$, otherwise. We now focus on the field $\boldsymbol{\beta}_k - \sum_{i=1}^n \mathbf{K}_{\mathbf{b}_k^i}(\cdot;\mathbf{z}_i) \in C^\infty(\overline{\Omega'};\mathbb{R}^{2\times2})$. From (1.24) and (1.26) we know that this field has zero curl on Ω' for $k \gg 1$. Moreover, recalling the definition of $\mathbf{b}_k^i$ and using (1.27), we have that its circulation along any closed curve in Ω' is zero. Therefore, by the classical theory, we know that this field admits a potential, that is, there exists a function $\mathbf{u}_k \in C^\infty(\overline{\Omega'};\mathbb{R}^2)$ such that

$$\boldsymbol{\beta}_k - \sum_{i=1}^n \mathbf{K}_{\mathbf{b}_k^i}(\cdot;\mathbf{z}_i) = \mathrm{D}\mathbf{u}_k \qquad \text{in } \Omega' \tag{1.28}$$

for $k \gg 1$. It is clear that the function $\mathbf{u}_k$ can be chosen to have zero mean on a fixed ball $B \subset\subset \Omega'$. For $i = 1,\dots,n$, using (1.12), we have that

$$|\mathbf{K}_{\mathbf{b}_k^i}(\mathbf{x};\mathbf{z}_i)| \le \frac{C|\mathbf{b}_k^i|}{|\mathbf{x}-\mathbf{z}_i|} \le \frac{C|\mathbf{b}_k^i|}{\varepsilon+\delta} \tag{1.29}$$

for every $\mathbf{x} \in \Omega'$. Since $\mathbf{b}_k^i \to \mathbf{0}$ as $k \to \infty$, we deduce from (1.29) that $\mathbf{K}_{\mathbf{b}_k^i}(\cdot;\mathbf{z}_i) \to \mathbf{0}$ uniformly on Ω' and hence in $L^2(\Omega';\mathbb{R}^{2\times2})$, as $k \to \infty$. Thus, recalling (1.28), we have

$$||\mathrm{D}\mathbf{u}_k||_{L^2(\Omega';\mathbb{R}^{2\times2})} \le ||\boldsymbol{\beta}_k||_{L^2(\Omega';\mathbb{R}^{2\times2})} + \sum_{i=1}^n ||\mathbf{K}_{\mathbf{b}_k^i}(\cdot;\mathbf{z}_i)||_{L^2(\Omega';\mathbb{R}^{2\times2})} \le C$$

and applying the Poincaré-Wirtinger inequality we obtain

$$||\mathbf{u}_k||_{L^2(\Omega';\mathbb{R}^2)} = \left\|\mathbf{u}_k - \fint_B \mathbf{u}_k\,\mathrm{d}\mathbf{x}\right\|_{L^2(\Omega';\mathbb{R}^2)} \le C||\mathrm{D}\mathbf{u}_k||_{L^2(\Omega';\mathbb{R}^{2\times2})} \le C.$$

Hence, by weak compactness, there exist a subsequence $(\mathbf{u}_{k_\ell})$ and a function $\mathbf{u} \in H^1(\Omega';\mathbb{R}^2)$ such that $\mathbf{u}_{k_\ell} \rightharpoonup \mathbf{u}$ in $H^1(\Omega';\mathbb{R}^2)$, as $\ell \to \infty$. On the other hand, passing to the limit in (1.28), as $\ell \to \infty$, we see that $\mathrm{D}\mathbf{u}_{k_\ell} \to \boldsymbol{\beta}$ in $L^2(\Omega';\mathbb{R}^{2\times2})$, as $\ell \to \infty$. Therefore, we conclude that $\boldsymbol{\beta} = \mathrm{D}\mathbf{u}$ a.e. in Ω', as claimed.

Now we extend the result to the whole domain $\Omega_\varepsilon(\mu)$. Consider a sequence of open sets $(\Omega_h) \subset \Omega_\varepsilon(\mu)$ such that $\Omega_h \subset\subset \Omega_{h+1} \subset\subset \Omega_\varepsilon(\mu)$ and $\Omega_\varepsilon(\mu) = \bigcup_{h=1}^\infty \Omega_h$. Using the previous arguments, we can show that, for every h, there exists $\mathbf{u}_h \in H^1(\Omega_h;\mathbb{R}^2)$ such that $\boldsymbol{\beta} = \mathrm{D}\mathbf{u}_h$ a.e. in Ω_h and $\fint_B \mathbf{u}_h\,\mathrm{d}\mathbf{x} = \mathbf{0}$, where B is a fixed ball with $B \subset\subset \Omega_1$. In this case, for every h we have $\mathrm{D}\mathbf{u}_{h+1} = \mathrm{D}\mathbf{u}_h$ a.e. in Ω_h, hence $\mathbf{u}_{h+1} = \mathbf{u}_h + \mathbf{a}$ a.e. in Ω_h for some constant $\mathbf{a} \in \mathbb{R}^2$. Using the zero mean condition on B, we deduce that $\mathbf{u}_{h+1} = \mathbf{u}_h$ a.e. in Ω_h. Therefore, if we set $\mathbf{u}(\mathbf{x}) = \mathbf{u}_h(\mathbf{x})$ for $\mathbf{x} \in \Omega_h$, then the resulting function $\mathbf{u}$ is well-defined with $\mathbf{u} \in L^2_{\mathrm{loc}}(\Omega_\varepsilon(\mu);\mathbb{R}^2)$ and $\mathrm{D}\mathbf{u} = \boldsymbol{\beta}$ in $\Omega_\varepsilon(\mu)$ in the sense of distributions. Since $\Omega_\varepsilon(\mu)$ is a Lipschitz domain, this implies that $\mathbf{u} \in L^2(\Omega_\varepsilon(\mu);\mathbb{R}^2)$ (see Corollary at p.23 in [19] or Theorem 7.6 in [22]), so that $\mathbf{u} \in H^1(\Omega_\varepsilon(\mu);\mathbb{R}^2)$, as desired. This fact can be also proved directly by means of the following truncation argument. Set $\mathbf{u} = (u_1, u_2)$ and define $u_1^M = (u_1 \wedge M) \vee (-M)$ where $M > 0$. Then $u_1^M \in H^1(\Omega_\varepsilon(\mu))$ with $\mathrm{D}u_1^M = \mathrm{D}u_1\chi_{\{|u_1|<M\}}$ and by the Poincaré-Wirtinger inequality we have

$$\left\|u_1^M - \fint_B u_1^M(\mathbf{x})\,\mathrm{d}\mathbf{x}\right\|_{L^2(\Omega_\varepsilon(\mu))} \le C||\mathrm{D}u_1^M||_{L^2(\Omega_\varepsilon(\mu);\mathbb{R}^2)} \le C||\boldsymbol{\beta}||_{L^2(\Omega_\varepsilon(\mu);\mathbb{R}^{2\times2})}.$$

Thus, since $u_1^M \to u_1$ a.e. and $\fint_B u_1^M \to \fint_B u_1 = 0$ as $M \to +\infty$, by the Fatou Lemma we obtain

$$\begin{aligned}\int_{\Omega_\varepsilon(\mu)} |u_1|^2 \,\mathrm{d}\mathbf{x} &= \int_{\Omega_\varepsilon(\mu)} \left| u_1 - \fint_B u_1 \,\mathrm{d}\mathbf{y} \right|^2 \mathrm{d}\mathbf{x} \\ &\leq \liminf_{M\to+\infty} \int_{\Omega_\varepsilon(\mu)} \left| u_1^M - \fint_B u_1^M \,\mathrm{d}\mathbf{y} \right|^2 \mathrm{d}\mathbf{x} \\ &\leq C \int_{\Omega_\varepsilon(\mu)} |\boldsymbol{\beta}|^2 \,\mathrm{d}\mathbf{x} < +\infty\end{aligned}$$

that is, $u_1 \in L^2(\Omega_\varepsilon(\mu))$. Applying the same argument to u_2, we finally conclude that $\mathbf{u} \in L^2(\Omega_\varepsilon(\mu); \mathbb{R}^2)$. □

1.3 Existence and uniqueness of the minimizer

We are now ready to prove the existence of solutions for the minimization problem (1.8). Since our assumptions ensure the convexity of the functional $E_\varepsilon(\mu, \cdot)$, we have the following minimality criterion.

Proposition 1.2. *Let $\boldsymbol{\beta} \in \mathcal{A}_\varepsilon(\mu; \mathbf{b}_1, \cdots, \mathbf{b}_n)$. Then $\boldsymbol{\beta}$ is a solution of the minimization problem (1.8) if and only if $\boldsymbol{\beta}$ is a weak solution of the following Neumann problem*

$$\begin{cases} \operatorname{div} \mathbb{C}\boldsymbol{\beta} = \mathbf{0} & \text{in } \Omega_\varepsilon(\mu), \\ \mathbb{C}\boldsymbol{\beta}\,\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega_\varepsilon(\mu). \end{cases} \tag{1.30}$$

Proof. By (1.4), the quadratic form W is positive semidefinite and thus the functional $E_\varepsilon(\mu, \cdot)$ is convex. Moreover, the class of admissible fields is also convex. Hence, by standard arguments in the Calculus of Variations, we know that $\boldsymbol{\beta}$ is a minimizer if and only if it satisfies the Euler-Lagrange equations. By Lemma 1.1, any admissible field is of the form $\boldsymbol{\beta} + \mathrm{D}\mathbf{w}$ for some $\mathbf{w} \in H^1(\Omega_\varepsilon(\mu); \mathbb{R}^2)$. Hence, for a real parameter t, we compute

$$E_\varepsilon(\mu, \boldsymbol{\beta} + t\mathrm{D}\mathbf{w}) = \int_{\Omega_\varepsilon(\mu)} W(\boldsymbol{\beta}) \,\mathrm{d}\mathbf{x} + t \int_{\Omega_\varepsilon(\mu)} \mathbb{C}\boldsymbol{\beta} : \mathrm{D}\mathbf{w} \,\mathrm{d}\mathbf{x} + t^2 \int_{\Omega_\varepsilon(\mu)} W(\mathrm{D}\mathbf{w}) \,\mathrm{d}\mathbf{x},$$

where we used (1.3). Thus we easily deduce that the Euler-Lagrange equations take the following form;

$$\forall \mathbf{w} \in H^1(\Omega_\varepsilon(\mu); \mathbb{R}^2), \quad \int_{\Omega_\varepsilon(\mu)} \mathbb{C}\boldsymbol{\beta} : \mathrm{D}\mathbf{w} \,\mathrm{d}\mathbf{x} = 0, \tag{1.31}$$

which is the weak formulation of (1.30). □

Remark 1.3. Consider any strain field $\boldsymbol{\beta}$. Note that, by (1.2), for every infinitesimal rigid-body motion $\boldsymbol{\eta}$, we have $E_\varepsilon(\mu, \boldsymbol{\beta} + \mathrm{D}\boldsymbol{\eta}) = E_\varepsilon(\mu, \boldsymbol{\beta})$. Hence it is clear that we cannot expect uniqueness of solutions for problem (1.8). Note that this fact is consistent with the form (1.31) of the Euler-Lagrange equations.

In the particular case of a single dislocation in the whole plane, the solutions of (1.30) are explicitly known in the literature (see [16]). For $\Omega = \mathbb{R}^2$, we consider a defect located at the point $\mathbf{z}_0$ and with Burgers vector given by $\mathbf{b}$. Then, using the previous notation, we have $\mu = \delta_{\mathbf{z}_0}$ and $\Omega_\varepsilon(\mu) = \mathbb{R}^2 \setminus \overline{B}(\mathbf{z}_0, \varepsilon)$. We define

$$\mathbf{K}_\mathbf{b}^\varepsilon(\mathbf{x}; \mathbf{z}_0) = \mathbf{K}_\mathbf{b}(\mathbf{x}; \mathbf{z}_0) + \varepsilon^2 \mathrm{D}\mathbf{w}_\mathbf{b}(\mathbf{x} - \mathbf{z}_0), \tag{1.32}$$

where

$$\mathbf{w}_{\mathbf{b}}(\mathbf{x}) = \frac{\lambda_1+\lambda_2}{4\pi(\lambda_1+2\lambda_2)|\mathbf{x}|^4}\Big\{(\mathbf{b}\cdot\mathbf{x}^\perp)\mathbf{x}+(\mathbf{b}\cdot\mathbf{x})\mathbf{x}^\perp\Big\}$$

while the fundamental strain $\mathbf{K}_{\mathbf{b}}(\cdot;\mathbf{z}_0)$ has been introduced in (1.9). Clearly, $\mathbf{w}_{\mathbf{b}} \in C^\infty(\mathbb{R}^2\setminus\{\mathbf{0}\};\mathbb{R}^2)$ and it easy to check that

$$|\mathbf{w}_{\mathbf{b}}(\mathbf{x})| \le \frac{C|\mathbf{b}|}{|\mathbf{x}|^2}, \qquad |\mathrm{D}\mathbf{w}_{\mathbf{b}}(\mathbf{x})| \le \frac{C|\mathbf{b}|}{|\mathbf{x}|^3}, \qquad |\mathbb{C}\mathrm{D}\mathbf{w}_{\mathbf{b}}(\mathbf{x})| \le \frac{C|\mathbf{b}|}{|\mathbf{x}|^3} \tag{1.33}$$

for every $\mathbf{x}\neq\mathbf{z}_0$. Then, a direct computation shows that

$$\begin{cases}\operatorname{div}\mathbb{C}\mathbf{K}^\varepsilon_{\mathbf{b}}(\cdot;\mathbf{z}_0)=\mathbf{0} & \text{in } \mathbb{R}^2\setminus\overline{B}(\mathbf{z}_0,\varepsilon),\\ \mathbb{C}\mathbf{K}^\varepsilon_{\mathbf{b}}(\cdot;\mathbf{z}_0)\mathbf{n}=\mathbf{0} & \text{on } \partial B(\mathbf{z}_0,\varepsilon)\end{cases} \tag{1.34}$$

in the classical sense. Note that the field in (1.32) satisfies the two conditions in (1.7). Indeed, from (1.13) and (1.15), we trivially have that

$$\int_{\partial B(\mathbf{y}_0,r)} \mathbf{K}^\varepsilon_{\mathbf{b}}(\mathbf{x};\mathbf{z}_0)\,\mathbf{t}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})=\mathbf{b} \tag{1.35}$$

and

$$\operatorname{curl}\mathbf{K}^\varepsilon_{\mathbf{b}}(\cdot,\mathbf{z}_0)=\mathbf{0} \qquad \text{in } \mathbb{R}^2\setminus\overline{B}(\mathbf{z}_0,\varepsilon). \tag{1.36}$$

Moreover, it also satisfies

$$\int_{\Omega_\varepsilon(\mu)}\Big(\mathbf{K}^\varepsilon_{\mathbf{b}}(\mathbf{x};\mathbf{z}_0)-\mathbf{K}^\varepsilon_{\mathbf{b}}(\mathbf{x};\mathbf{z}_0)^\top\Big)\,\mathrm{d}\mathbf{x}=\mathbf{0}. \tag{1.37}$$

However, the field $\mathbf{K}^\varepsilon_{\mathbf{b}}(\cdot;\mathbf{z}_0)$ is not admissible since it is not in $L^2(\mathbb{R}^2\setminus\overline{B}(\mathbf{z}_0,\varepsilon);\mathbb{R}^{2\times2})$, so that its energy is not finite.
Now we go back to the general case of n dislocations in the bounded domain $\Omega\subset\mathbb{R}^2$. Set, for simplicity, $\mathbf{K}_i=\mathbf{K}_{\mathbf{b}_i}$, $\mathbf{w}_i=\mathbf{w}_{\mathbf{b}_i}$ and $\mathbf{K}^\varepsilon_i=\mathbf{K}^\varepsilon_{\mathbf{b}_i}$. From (1.35) and (1.36), we have that

$$\frac{1}{n}\sum_{i=1}^n\mathbf{K}^\varepsilon_i(\cdot;\mathbf{z}_i)\in\mathcal{A}_\varepsilon(\mu;\mathbf{b}_1,\cdots,\mathbf{b}_n),$$

hence, by Lemma 1.1, every admissible field has the form

$$\boldsymbol{\beta}=\frac{1}{n}\sum_{i=1}^n\mathbf{K}^\varepsilon_i(\cdot;\mathbf{z}_i)+\mathrm{D}\mathbf{u} \tag{1.38}$$

for some $\mathbf{u}\in H^1(\Omega_\varepsilon(\mu);\mathbb{R}^2)$. Hence, using (1.3) and (1.38), we can write the energy corresponding to an admissible field $\boldsymbol{\beta}$ as

$$\begin{aligned}E_\varepsilon(\mu,\boldsymbol{\beta})=&\frac{1}{2n^2}\sum_{i,j=1}^n\int_{\Omega_\varepsilon(\mu)}\mathbb{C}\mathbf{K}^\varepsilon_i(\mathbf{x};\mathbf{z}_i):\mathbf{K}^\varepsilon_j(\mathbf{x};\mathbf{z}_j)\,\mathrm{d}\mathbf{x}\\&+\int_{\Omega_\varepsilon(\mu)}W(\mathrm{D}\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}+\frac{1}{n}\sum_{i=1}^n\int_{\Omega_\varepsilon(\mu)}\mathbb{C}\mathbf{K}^\varepsilon_i(\mathbf{x};\mathbf{z}_i):\mathrm{D}\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Applying integration by parts to the last integral at the right hand side and recalling (1.34), we obtain that

$$E_\varepsilon(\mu,\boldsymbol{\beta})=\frac{1}{2n^2}\sum_{i,j=1}^n\int_{\Omega_\varepsilon(\mu)}\mathbb{C}\mathbf{K}^\varepsilon_i(\mathbf{x};\mathbf{z}_i):\mathbf{K}^\varepsilon_j(\mathbf{x};\mathbf{z}_j)\,\mathrm{d}\mathbf{x}+I_\varepsilon(\mu,\mathbf{u}), \tag{1.39}$$

where we have introduced the auxiliary functional

$$I_\varepsilon(\mu,\mathbf{v}) = \int_{\Omega_\varepsilon(\mu)} W(\mathrm{D}\mathbf{v}(\mathbf{x}))\,\mathrm{d}\mathbf{x} + \frac{1}{n}\sum_{i=1}^{n}\int_{\partial\Omega_\varepsilon(\mu)} \mathbf{v}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_i^\varepsilon(\mathbf{x};\mathbf{z}_i)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) \tag{1.40}$$

defined for $\mathbf{v}\in H^1(\Omega_\varepsilon(\mu);\mathbb{R}^2)$. Therefore, from (1.39), we see that solving (1.8) is equivalent to solving the following minimization problem:

$$\min_{\mathbf{u}\in H^1(\Omega_\varepsilon(\mu);\mathbb{R}^2)} I_\varepsilon(\mu,\mathbf{u}). \tag{1.41}$$

Recalling Remark 1.3, we deduce from (1.39) that the functional $I_\varepsilon(\mu,\cdot)$ is invariant with respect to infinitesimal rigid-body motion. This suggests to look for a minimizer in the class

$$Y_\mu^\varepsilon(\Omega) = \left\{\mathbf{v}\in H^1(\Omega_\varepsilon(\mu);\mathbb{R}^2) : \int_{B_0}\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \mathbf{0},\ \int_{B_0}(\mathrm{D}\mathbf{v}(\mathbf{x}) - \mathrm{D}\mathbf{v}(\mathbf{x})^\top)\,\mathrm{d}\mathbf{x} = \mathbf{0}\right\}, \tag{1.42}$$

where $B_0 \subset\subset \Omega_\varepsilon(\mu)$ is a fixed ball. Moreover, the conditions in (1.42) are needed in order to guarantee the coerciveness of $I_\varepsilon(\mu,\cdot)$ and, in turn, the existence of a minimizer.

Lemma 1.4. ***(Minimization of the auxiliary functional)*** *There exists a unique minimizer $\mathbf{u}_\mu^\varepsilon\in H^1(\Omega_\varepsilon(\mu);\mathbb{R}^2)$ of the functional $I_\varepsilon(\mu,\cdot)$ in the class $Y_\mu^\varepsilon(\Omega)$.*

Proof. First of all we note that, for $i=1,\dots,n$, the function $\mathbb{C}\mathbf{K}_i^\varepsilon(\cdot,\mathbf{z}_i)$ is bounded on $\partial\Omega_\varepsilon(\mu)$. Indeed, for every $\mathbf{x}\in\partial\Omega_\varepsilon(\mu)$ we have $|\mathbf{x}-\mathbf{z}_i|\geq\varepsilon$ for $i=1,\dots,n$, hence, from (1.12), (1.32) and (1.33), we easily deduce that

$$|\mathbb{C}\mathbf{K}_i^\varepsilon(\mathbf{x};\mathbf{z}_i)| \leq |\mathbb{C}\mathbf{K}_i^\varepsilon(\mathbf{x};\mathbf{z}_i)| + \varepsilon^2|\mathbb{C}\mathrm{D}\mathbf{w}_i(\mathbf{x}-\mathbf{z}_i)| \leq \frac{C|\mathbf{b}_i|}{\varepsilon} \tag{1.43}$$

for every $\mathbf{x}\in\partial\Omega_\varepsilon(\mu)$. Taking into account this fact, it is easy to see that the functional $I_\varepsilon(\mu,\cdot)$ is continuous on $H^1(\Omega_\varepsilon(\mu);\mathbb{R}^2)$. Moreover, by (1.4), it is also convex, and hence it is weakly lower semicontinuous on $H^1(\Omega_\varepsilon(\mu);\mathbb{R}^2)$. Take any $\mathbf{v}\in Y_\mu^\varepsilon(\Omega)$. Using (1.4) and the Korn inequality, we have that

$$\int_{\Omega_\varepsilon(\mu)} W(\mathrm{D}\mathbf{v}(\mathbf{x}))\,\mathrm{d}\mathbf{x} \geq C||\mathbf{v}||^2_{H^1(\Omega_\varepsilon(\mu);\mathbb{R}^2)}. \tag{1.44}$$

Futhermore, using (1.43) and the trace inequality, we have

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^{n}\int_{\partial\Omega_\varepsilon(\mu)} \mathbf{v}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_i^\varepsilon(\mathbf{x};\mathbf{z}_i)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\right| \\ \leq \frac{1}{n}\sum_{i=1}^{n}||\mathbb{C}\mathbf{K}_i^\varepsilon(\cdot;\mathbf{z}_i)||_{L^\infty(\partial\Omega_\varepsilon(\mu);\mathbb{R}^{2\times2})}\int_{\partial\Omega_\varepsilon(\mu)}|\mathbf{v}(\mathbf{x})|\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) \\ \leq C||\mathbf{v}||_{L^2(\partial\Omega_\varepsilon(\mu);\mathbb{R}^2)} \leq C||\mathbf{v}||_{H^1(\Omega_\varepsilon(\mu);\mathbb{R}^2)},\end{aligned} \tag{1.45}$$

where the constant $C>0$ depends on ε, n and, the Burgers vectors. Thus, combining (1.44) and (1.45), we obtain that there exists two constants $C_1, C_2>0$ such that

$$I_\varepsilon(\mu,\mathbf{v}) \geq C_1||\mathbf{v}||^2_{H^1(\Omega_\varepsilon(\mu);\mathbb{R}^2)} - C_2||\mathbf{v}||_{H^1(\Omega_\varepsilon(\mu);\mathbb{R}^2)} \tag{1.46}$$

for every $\mathbf{v}\in Y_\mu^\varepsilon(\Omega)$, that is the functional $I_\varepsilon(\mu,\cdot)$ is weakly coercive on $Y_\mu^\varepsilon(\Omega)$. Therefore, by the Direct Method, we have the existence of a minimizer in this class. To prove its uniqueness, suppose that $\mathbf{u}_1, \mathbf{u}_2 \in Y^\varepsilon_\mu(\Omega)$ are two minimizers. Then they solve the following Euler-Lagrange equations:

$$\forall \mathbf{v} \in Y^\varepsilon_\mu(\Omega), \quad \int_{\Omega_\varepsilon(\mu)} \mathbb{C}\mathrm{D}\mathbf{u}_1 : \mathrm{D}\mathbf{v}\, \mathrm{d}\mathbf{x} + \frac{1}{n}\sum_{i=1}^n \int_{\partial\Omega_\varepsilon(\mu)} \mathbf{v}(\mathbf{x}) \cdot \mathbb{C}\mathbf{K}_i(\mathbf{x};\mathbf{z}_i)\mathbf{n}(\mathbf{x})\, \mathrm{d}\mathscr{H}^1(\mathbf{x}) = 0, \tag{1.47}$$

$$\forall \mathbf{v} \in Y^\varepsilon_\mu(\Omega), \quad \int_{\Omega_\varepsilon(\mu)} \mathbb{C}\mathrm{D}\mathbf{u}_2 : \mathrm{D}\mathbf{v}\, \mathrm{d}\mathbf{x} + \frac{1}{n}\sum_{i=1}^n \int_{\partial\Omega_\varepsilon(\mu)} \mathbf{v}(\mathbf{x}) \cdot \mathbb{C}\mathbf{K}_i(\mathbf{x};\mathbf{z}_i)\mathbf{n}(\mathbf{x})\, \mathrm{d}\mathscr{H}^1(\mathbf{x}) = 0. \tag{1.48}$$

Subtracting (1.47) and (1.48) and then choosing $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$, we obtain

$$\int_{\Omega_\varepsilon(\mu)} \mathbb{C}\mathrm{D}(\mathbf{u}_1 - \mathbf{u}_2) : \mathrm{D}(\mathbf{u}_1 - \mathbf{u}_2)\, \mathrm{d}\mathbf{x} = 0,$$

from which, using (1.4), we deduce $\mathrm{E}(\mathbf{u}_1 - \mathbf{u}_2) = \mathbf{0}$. Thus, $\mathbf{u}_1$ and $\mathbf{u}_2$ differ for an infinitesimal rigid-body motion and, by (1.42), we conclude that $\mathbf{u}_1 = \mathbf{u}_2$. ☐

Remark 1.5. In order to conclude that $\mathbf{u}^\varepsilon_\mu$ is also a solution of (1.41), we have to show that

$$\inf_{\mathbf{u}\in H^1(\Omega_\varepsilon(\mu);\mathbb{R}^2)} I_\varepsilon(\mu, \mathbf{u}) = \inf_{\mathbf{u}\in Y^\varepsilon_\mu(\Omega)} I_\varepsilon(\mu, \mathbf{u}). \tag{1.49}$$

To prove this, we take any $\mathbf{u} \in H^1(\Omega_\varepsilon(\mu);\mathbb{R}^2)$ and we show that there exist a competitor in $Y^\varepsilon_\mu(\Omega)$ with the same energy. We set

$$\mathbf{A} = \frac{1}{2}\fint_{B_0} (\mathrm{D}\mathbf{u}(\mathbf{y})^\top - \mathrm{D}\mathbf{u}(\mathbf{y}))\, \mathrm{d}\mathbf{y}, \qquad \mathbf{a} = -\mathbf{A}\fint_{B_0} \mathbf{y}\, \mathrm{d}\mathbf{y} - \fint_{B_0} \mathbf{u}(\mathbf{y})\, \mathrm{d}\mathbf{y}.$$

Then $\mathbf{A} \in \mathrm{Skew}(2)$, so that $\boldsymbol{\eta}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ defines an infinitesimal rigid-body motion. If we consider $\mathbf{v} = \mathbf{u} + \boldsymbol{\eta}$, then we can easily check that $\mathbf{v} \in Y^\varepsilon_\mu(\Omega)$. Moreover, we have that $I_\varepsilon(\mu, \mathbf{v}) = I_\varepsilon(\mu, \mathbf{u})$ and this proves (1.49).

Finally, given a solution of (1.41), a solution of (1.8) is trivially obtained. In the proof of the following result, we briefly resume the whole argument used.

Theorem 1.6. ***(Existence and uniqueness of the minimizer)*** *The problem (1.8) has a unique solution $\boldsymbol{\beta}^\varepsilon_\mu \in \mathcal{A}_\varepsilon(\mu; \mathbf{b}_1, \ldots, \mathbf{b}_n)$ satisfying*

$$\int_{\Omega_\varepsilon(\mu)} \left(\boldsymbol{\beta}^\varepsilon_\mu(\mathbf{x}) - \boldsymbol{\beta}^\varepsilon_\mu(\mathbf{x})^\top\right)\, \mathrm{d}\mathbf{x} = \mathbf{0}. \tag{1.50}$$

Proof. Consider the unique solution $\mathbf{u}^\varepsilon_\mu \in H^1(\Omega_\varepsilon(\mu);\mathbb{R}^2)$ of (1.41) given by Lemma 1.4. Looking at the decomposition (1.38), we define

$$\boldsymbol{\beta}^\varepsilon_\mu = \frac{1}{n}\sum_{i=1}^n \mathbf{K}_i(\cdot;\mathbf{z}_i) + \mathrm{D}\mathbf{u}^\varepsilon_\mu.$$

By (1.39) and the minimality of $\mathbf{u}^\varepsilon_\mu$ we clearly have that $\boldsymbol{\beta}^\varepsilon_\mu$ is a minimizer of the functional $E_\varepsilon(\mu, \cdot)$ in the class of admissible fields. Moreover, combining (1.37) and (1.42), we see that the field $\boldsymbol{\beta}^\varepsilon_\mu$ satisfies the condition (1.50).

To prove uniqueness, suppose that $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in \mathcal{A}_\varepsilon(\mu; \mathbf{b}_1, \dots, \mathbf{b}_n)$ are two minimizers satisfying condition (1.50). By Proposition 1.2, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ solve the Euler-Lagrange equations (1.31). If we subtract the two corresponding expressions, then we obtain

$$\forall \mathbf{w} \in H^1(\Omega_\varepsilon(\mu); \mathbb{R}^2), \quad \int_{\Omega_\varepsilon(\mu)} \mathbb{C}(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2) : \mathrm{D}\mathbf{w} \, \mathrm{d}\mathbf{x} = 0. \tag{1.51}$$

By Lemma 1.1, we have that $\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2 = \mathrm{D}\mathbf{u}$ for some $\mathbf{u} \in H^1(\Omega_\varepsilon(\mu); \mathbb{R}^2)$. Hence, choosing $\mathbf{w} = \mathbf{u}$ in (1.51), we obtain

$$\int_{\Omega_\varepsilon(\mu)} \mathbb{C}\mathrm{D}\mathbf{u} : \mathrm{D}\mathbf{u} \, \mathrm{d}\mathbf{x} = 0,$$

from which, by (1.4), we deduce that $\mathrm{E}\mathbf{u} = \mathbf{0}$. Therefore $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ differ for a constant skew-symmetric matrix but, by (1.50), it has to be the zero matrix. □

Chapter 2

The Γ-convergence of the renormalized energy

In this chapter we introduce the renormalized energy and we compute its Γ-limit, as the total number of dislocations goes to infinity. This is a slight generalization of the analogous statement obtained in [21] and constitutes the original contribution of this work. Here the case of two different Burgers vectors is considered, under the hypothesis that their scalar product is positive, so that the energy is a function of two measures, each encoding the information about the location of the defects corresponding to one Burgers vector.

2.1 The main result

In this chapter we restrict ourselves to the situation in which there are only two possible Burgers vectors $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^2$ with the assumption that $\mathbf{b}_1 \cdot \mathbf{b}_2 > 0$. In this case we have two families of dislocations, each corresponding to one of them, located respectively at the points $\{\mathbf{y}_i : \ i = 1, \dots, N\}$ and $\{\mathbf{z}_i : \ i = 1, \dots, M\}$ with $n = N + M$.
Since we are interested in the behaviour of the system as the total number n of dislocations goes to infinity, we introduce the following setting. We consider two sequences of positive integers (N_n) and (M_n) giving the number of dislocations with Burgers vector $\mathbf{b}_1$ and $\mathbf{b}_2$, respectively, so that $n = N_n + M_n$ for every n. Moreover we denote by $0 < \mathsf{m} < 1$ the asymptotic proportion of the family of dislocations with Burgers vector $\mathbf{b}_1$, namely we assume that $N_n/n \to \mathsf{m}$ and $M_n/n \to 1 - \mathsf{m}$, as $n \to \infty$. We introduce also two sequences of positive real parameters (ε_n) and (r_n) representing the core-radius and the minimum distance between the defects. Moreover we fix a positive real parameter r_0 and we assume the following:

- **(Confinement)** All the dislocations are located in a fixed open set $\Omega_0 \subset\subset \Omega$, with $d(\overline{\Omega}_0, \partial\Omega) \geq r_0$;
- **(Well-separation)** For every n, the distance between any pair of dislocations is at least r_n;
- **(Asymptotic relations)** The following relations hold:

$$\varepsilon_n \to 0, \qquad r_n \to 0, \qquad \varepsilon_n / r_n^3 \to 0, \qquad n r_n^2 \to 0, \qquad \text{as } n \to \infty. \tag{2.1}$$

We can assume that $\varepsilon_n < r_0/2$ for every n. Moreover we choose the ball B_0 in (1.42) in order to have $d(\mathbf{x}, \partial\Omega) < r_0/2$ for every $\mathbf{x} \in B_0$. Taking into account the confinement hypothesis, we introduce the following space of measures

$$\mathcal{X}(\Omega) = \{\mu \in \mathcal{M}^+_{\mathrm{b}}(\Omega) : \ \mathrm{supp}\mu \subseteq \overline{\Omega}_0, \ \mu(\Omega) \leq 1\}$$

and we endow it with the topology induced by the weak* topology of $\mathrm{rba}(\Omega)$, that is by the narrow topology of bounded Radon measures. It is easy to see that $\mathcal{X}(\Omega)$ is isomorphic to $\mathcal{M}^+_1(\overline{\Omega}_0) = \{\mu \in \mathcal{M}^+_{\mathrm{b}}(\overline{\Omega}_0) : \ \mu(\overline{\Omega}_0) \leq 1\}$ with respect to the weak* topology. This, as a bounded subset of $\mathrm{rba}(\overline{\Omega}_0) \cong \Big(C_b(\overline{\Omega}_0)\Big)'$, is weakly* compact by the Banach-Alaoglu Theorem; moreover, since $\overline{\Omega}_0$ is compact so that $C_b(\overline{\Omega}_0)$ is separable, it is weakly* metrizable. Thus we get that also $\mathcal{X}(\Omega)$ is compact, hence closed, and metrizable.
For each n, we define the two sets of empirical measures

$$X^1_n = \left\{ \frac{1}{n} \sum_{i=1}^{N_n} \delta_{\mathbf{y}_i} : \ \mathbf{y}_i \in \overline{\Omega}_0 \right\}, \quad X^2_n = \left\{ \frac{1}{n} \sum_{i=1}^{M_n} \delta_{\mathbf{z}_i} : \ \mathbf{z}_i \in \overline{\Omega}_0 \right\} \tag{2.2}$$

and we introduce the family of the dislocation densities satisfying the well-separation condition

$$X_n = \left\{ \frac{1}{n} \sum_{i=1}^{n} \delta_{\mathbf{x}_i} : \ \mathbf{x}_i \in \overline{\Omega}_0, \ |x_i - x_j| \geq r_n \ \forall i \neq j \right\}.$$

Now, consider two empirical measures $\mu^1 \in X^1_n$ and $\mu^2 \in X^2_n$ for a certain n. We recall from (1.39) that, highlighting the dependence on the Burgers vector, the minimum energy is given by

$$\begin{aligned}
E_{\varepsilon_n}(\mu^1 + \mu^2, \boldsymbol{\beta}^{\varepsilon}_{\mu^1+\mu^2}) = {} & \frac{1}{2n^2} \sum_{i=1}^{N_n} \int_{\Omega_{\varepsilon_n}(\mu^1+\mu^2)} \mathbb{C}\mathbf{K}^{\varepsilon_n}_1(\mathbf{x};\mathbf{y}_i) : \mathbf{K}^{\varepsilon_n}_1(\mathbf{x};\mathbf{y}_i)\, \mathrm{d}\mathbf{x} \\
& + \frac{1}{2n^2} \sum_{i=1}^{M_n} \int_{\Omega_{\varepsilon_n}(\mu^1+\mu^2)} \mathbb{C}\mathbf{K}^{\varepsilon_n}_2(\mathbf{x};\mathbf{z}_i) : \mathbf{K}^{\varepsilon_n}_2(\mathbf{x};\mathbf{z}_i)\, \mathrm{d}\mathbf{x} \\
& + \frac{1}{2n^2} \sum_{i=1}^{N_n} \sum_{i\neq j} \int_{\Omega_{\varepsilon_n}(\mu^1+\mu^2)} \mathbb{C}\mathbf{K}^{\varepsilon_n}_1(\mathbf{x};\mathbf{y}_i) : \mathbf{K}^{\varepsilon_n}_1(\mathbf{x};\mathbf{y}_j)\, \mathrm{d}\mathbf{x} \\
& + \frac{1}{2n^2} \sum_{i=1}^{M_n} \sum_{i\neq j} \int_{\Omega_{\varepsilon_n}(\mu^1+\mu^2)} \mathbb{C}\mathbf{K}^{\varepsilon_n}_2(\mathbf{x};\mathbf{z}_i) : \mathbf{K}^{\varepsilon_n}_2(\mathbf{x};\mathbf{z}_j)\, \mathrm{d}\mathbf{x} \\
& + \frac{1}{n^2} \sum_{i=1}^{N_n} \sum_{j=1}^{M_n} \int_{\Omega_{\varepsilon_n}(\mu^1+\mu^2)} \mathbb{C}\mathbf{K}^{\varepsilon_n}_1(\mathbf{x};\mathbf{y}_i) : \mathbf{K}^{\varepsilon_n}_2(\mathbf{x};\mathbf{z}_j)\, \mathrm{d}\mathbf{x} \\
& + I_{\varepsilon_n}(\mu^1 + \mu^2, \mathbf{u}^{\varepsilon_n}_{\mu^1+\mu^2})
\end{aligned}$$

where $\boldsymbol{\beta}^{\varepsilon}_{\mu^1+\mu^2}$ and $\mathbf{u}^{\varepsilon}_{\mu^1+\mu^2}$ are defined as in Theorem 1.6 and Lemma 1.4, respectively. The *renormalized energy* is obtained from the right hand side in the equation above by removing the *self-energy*, which is given by the first two terms, and is defined as a functional acting on the pair of measures (μ^1, μ^2).
Therefore we define the functionals

$$\mathcal{F}_n \colon \mathcal{X}(\Omega) \times \mathcal{X}(\Omega) \to \mathbb{R} \cup \{+\infty\}$$

and

$$\mathcal{G}_n \colon \mathcal{X}(\Omega) \times \mathcal{X}(\Omega) \to \mathbb{R} \cup \{+\infty\}$$

as follows: if $\mu^1 \in X_n^1$ and $\mu^2 \in X_n^2$ are such that $\mu^1 + \mu^2 \in X_n$, then we set

$$\begin{aligned}\mathcal{F}_n(\mu^1, \mu^2) = &\frac{1}{2n^2} \sum_{i=1}^{N_n} \sum_{i \neq j} \int_{\Omega_{\varepsilon_n}(\mu^1+\mu^2)} \mathbb{C}\mathbf{K}_1^{\varepsilon_n}(\mathbf{x}; \mathbf{y}_i) : \mathbf{K}_1^{\varepsilon_n}(\mathbf{x}; \mathbf{y}_j)\, \mathrm{d}\mathbf{x} \\ &+ \frac{1}{2n^2} \sum_{i=1}^{M_n} \sum_{i \neq j} \int_{\Omega_{\varepsilon_n}(\mu^1+\mu^2)} \mathbb{C}\mathbf{K}_2^{\varepsilon_n}(\mathbf{x}; \mathbf{z}_i) : \mathbf{K}_2^{\varepsilon_n}(\mathbf{x}; \mathbf{z}_j)\, \mathrm{d}\mathbf{x} \\ &+ \frac{1}{n^2} \sum_{i=1}^{N_n} \sum_{j=1}^{M_n} \int_{\Omega_{\varepsilon_n}(\mu^1+\mu^2)} \mathbb{C}\mathbf{K}_1^{\varepsilon_n}(\mathbf{x}; \mathbf{y}_i) : \mathbf{K}_2^{\varepsilon_n}(\mathbf{x}; \mathbf{z}_j)\, \mathrm{d}\mathbf{x}\end{aligned}$$

and

$$\mathcal{G}_n(\mu^1, \mu^2) = I_{\varepsilon_n}(\mu^1 + \mu^2, \mathbf{u}^{\varepsilon_n}_{\mu^1+\mu^2})$$

otherwise we define $\mathcal{F}_n(\mu^1, \mu^2) = \mathcal{G}_n(\mu^1, \mu^2) = +\infty$. The functional $\mathcal{F}_n$ represents the pairwise interaction between dislocations, and we call it the *interaction energy*, while $\mathcal{G}_n$ takes into account the interaction of dislocations with the boundary. Finally, the renormalized energy is defined as

$$\mathcal{E}_n \colon \mathcal{X}(\Omega) \times \mathcal{X}(\Omega) \to \mathbb{R} \cup \{+\infty\}$$

given by

$$\mathcal{E}_n(\mu^1, \mu^2) = \mathcal{F}_n(\mu^1, \mu^2) + \mathcal{G}_n(\mu^1, \mu^2).$$

As we mentioned, we are going to study its behaviour, as $n \to \infty$, in the sense of Γ-convergence. To this aim, it is convenient to rewrite the interaction energy for a pair of admissible empirical measure, using the notation above, as

$$\begin{aligned}\mathcal{F}_n(\mu^1, \mu^2) = &\frac{1}{2} \iint_{\Omega\times\Omega} \left(\int_{\Omega_{\varepsilon_n}(\mu^1+\mu^2)} \mathbb{C}\mathbf{K}_1^{\varepsilon_n}(\mathbf{x}; \mathbf{y}) : \mathbf{K}_1^{\varepsilon_n}(\mathbf{x}; \mathbf{z})\, \mathrm{d}\mathbf{x} \right) \mathrm{d}(\mu^1 \boxtimes \mu^1)(\mathbf{y}, \mathbf{z}) \\ &+ \frac{1}{2} \iint_{\Omega\times\Omega} \left(\int_{\Omega_{\varepsilon_n}(\mu^1+\mu^2)} \mathbb{C}\mathbf{K}_2^{\varepsilon_n}(\mathbf{x}; \mathbf{y}) : \mathbf{K}_2^{\varepsilon_n}(\mathbf{x}; \mathbf{z})\, \mathrm{d}\mathbf{x} \right) \mathrm{d}(\mu^2 \boxtimes \mu^2)(\mathbf{y}, \mathbf{z}) \\ &+ \iint_{\Omega\times\Omega} \left(\int_{\Omega_{\varepsilon_n}(\mu^1+\mu^2)} \mathbb{C}\mathbf{K}_1^{\varepsilon_n}(\mathbf{x}; \mathbf{y}) : \mathbf{K}_2^{\varepsilon_n}(\mathbf{x}; \mathbf{z})\, \mathrm{d}\mathbf{x} \right) \mathrm{d}(\mu^1 \otimes \mu^2)(\mathbf{y}, \mathbf{z}),\end{aligned}$$

where we defined $\mu^1 \boxtimes \mu^1 = \sum_{i=1}^{N_n} \sum_{j\neq i} \delta_{(\mathbf{y}_i, \mathbf{y}_j)}$ and analogously $\mu^2 \boxtimes \mu^2 = \sum_{i=1}^{M_n} \sum_{j\neq i} \delta_{(\mathbf{z}_i, \mathbf{z}_j)}$. This expression suggests to introduce the *interaction potentials*

$$V_1, V_2, V_{1,2} \colon \Omega \times \Omega \to \mathbb{R} \cup \{+\infty\}$$

defined as

$$V_1(\mathbf{y}, \mathbf{z}) = \begin{cases} \int_\Omega \mathbb{C}\mathbf{K}_1(\mathbf{x}; \mathbf{y}) : \mathbf{K}_1(\mathbf{x}; \mathbf{z})\, \mathrm{d}\mathbf{x} & \mathbf{y} \neq \mathbf{z}, \\ +\infty & \mathbf{y} = \mathbf{z}, \end{cases}$$

$$V_2(\mathbf{y}, \mathbf{z}) = \begin{cases} \int_\Omega \mathbb{C}\mathbf{K}_2(\mathbf{x}; \mathbf{y}) : \mathbf{K}_2(\mathbf{x}; \mathbf{z})\, \mathrm{d}\mathbf{x} & \mathbf{y} \neq \mathbf{z}, \\ +\infty & \mathbf{y} = \mathbf{z}, \end{cases}$$

$$V_{1,2}(\mathbf{y}, \mathbf{z}) = \begin{cases} \int_\Omega \mathbb{C}\mathbf{K}_1(\mathbf{x}; \mathbf{y}) : \mathbf{K}_2(\mathbf{x}; \mathbf{z})\, \mathrm{d}\mathbf{x} & \mathbf{y} \neq \mathbf{z}, \\ +\infty & \mathbf{y} = \mathbf{z}. \end{cases}$$

The limiting interaction energy will be given by the functional

$$\mathcal{F} \colon \mathcal{X}(\Omega) \times \mathcal{X}(\Omega) \to \mathbb{R} \cup \{+\infty\}$$

defined for a pair of measures (ν^1, ν^2) by setting

$$\begin{aligned}\mathcal{F}(\nu^1,\nu^2) = &\frac{1}{2}\iint_{\Omega\times\Omega} V_1(\mathbf{y},\mathbf{z})\,\mathrm{d}(\nu^1\otimes\nu^1)(\mathbf{y},\mathbf{z})\\ &+\frac{1}{2}\iint_{\Omega\times\Omega} V_2(\mathbf{y},\mathbf{z})\,\mathrm{d}(\nu^2\otimes\nu^2)(\mathbf{y},\mathbf{z})\\ &+\iint_{\Omega\times\Omega} V_{1,2}(\mathbf{y},\mathbf{z})\,\mathrm{d}(\nu^1\otimes\nu^2)(\mathbf{y},\mathbf{z})\end{aligned} \tag{2.3}$$

if $\nu^1(\Omega) = \mathsf{m}$ and $\nu^2(\Omega) = 1 - \mathsf{m}$, and $\mathcal{F}(\nu^1, \nu^2) = +\infty$ otherwise. For what concerns $\mathcal{G}_n$, we recall from Lemma 1.4 and Remark 1.5 that it is defined, for a pair of admissible measures as above, as

$$\mathcal{G}_n(\mu^1,\mu^2) = \min\Big\{I_{\varepsilon_n}(\mu^1+\mu^2,\mathbf{v}) :\ \mathbf{v}\in H^1(\Omega_{\varepsilon_n}(\mu^1+\mu^2);\mathbb{R}^2)\Big\},$$

where

$$\begin{aligned}I_{\varepsilon_n}(\mu^1+\mu^2,\mathbf{v}) = &\int_{\Omega_{\varepsilon_n}(\mu^1+\mu^2)} W(\mathrm{D}\mathbf{v}(\mathbf{x}))\,\mathrm{d}\mathbf{x}\\ &+\frac{1}{n}\sum_{i=1}^{N_n}\int_{\partial\Omega_{\varepsilon_n}(\mu^1+\mu^2)} \mathbf{v}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1^{\varepsilon_n}(\mathbf{x};\mathbf{y}_i)\mathbf{n}(x)\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\ &+\frac{1}{n}\sum_{i=1}^{M_n}\int_{\partial\Omega_{\varepsilon_n}(\mu^1+\mu^2)} \mathbf{v}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_2^{\varepsilon_n}(\mathbf{x};\mathbf{z}_i)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}).\end{aligned}$$

Again, if we rewrite the energy by highlighting the dependence on the empirical measures, that is,

$$\begin{aligned}I_{\varepsilon_n}(\mu^1+\mu^2,\mathbf{v}) = &\int_{\Omega_{\varepsilon_n}(\mu^1+\mu^2)} W(\mathrm{D}\mathbf{v}(\mathbf{x}))\,\mathrm{d}\mathbf{x}\\ &+\int_\Omega\left(\int_{\partial\Omega_{\varepsilon_n}(\mu^1+\mu^2)} \mathbf{v}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1^{\varepsilon_n}(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\right)\mathrm{d}\mu^1(\mathbf{y})\\ &+\int_\Omega\left(\int_{\partial\Omega_{\varepsilon_n}(\mu^1+\mu^2)} \mathbf{v}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_2^{\varepsilon_n}(\mathbf{x};\mathbf{z})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(x)\right)\mathrm{d}\mu^2(\mathbf{z})\end{aligned}$$

then we can guess the form of the corresponding limiting energy. This turns out to be given, for a pair of measures (ν^1, ν^2), as the infimum of the following auxiliary functional

$$\begin{aligned}I(\nu^1,\nu^2,\mathbf{v}) = &\int_\Omega W(\mathrm{D}\mathbf{v}(\mathbf{x}))\,\mathrm{d}\mathbf{x}\\ &+\int_\Omega\left(\int_{\partial\Omega} \mathbf{v}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\right)\mathrm{d}\nu^1(\mathbf{y})\,+\\ &+\int_\Omega\left(\int_{\partial\Omega} \mathbf{v}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_2(\mathbf{x};\mathbf{z})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\right)\mathrm{d}\nu^2(\mathbf{z})\end{aligned} \tag{2.4}$$

defined for $\mathbf{v} \in H^1(\Omega; \mathbb{R}^2)$. More explicitly, we define

$$\mathcal{G}\colon \mathcal{X}(\Omega)\times\mathcal{X}(\Omega)\to\mathbb{R}\cup\{+\infty\}$$

by setting

$$\mathcal{G}(\nu^1,\nu^2) = \inf\Big\{I(\nu^1,\nu^2,\mathbf{v}) :\ \mathbf{v}\in H^1(\Omega;\mathbb{R}^2)\Big\} \tag{2.5}$$

if $\nu^1(\Omega) = \mathsf{m}$ and $\nu^2(\Omega) = 1 - \mathsf{m}$, and $\mathcal{G}(\nu^1, \nu^2) = +\infty$ otherwise. As we will see later, this infimum turns out to be actually a minimum.
Thus the limiting energy is defined as

$$\mathcal{E}\colon \mathcal{X}(\Omega) \times \mathcal{X}(\Omega) \to \mathbb{R} \cup \{+\infty\}, \qquad \mathcal{E}(\nu^1, \nu^2) = \mathcal{F}(\nu^1, \nu^2) + \mathcal{G}(\nu^1, \nu^2).$$

We are now ready to present the main result. We are going to assume the following:

$$\text{there exists a point } \mathbf{x}_0 \in \Omega_0 \text{ such that, given the homothety } \omega_\vartheta \text{ centered at } \mathbf{x}_0 \text{ with parameter } 0 < \vartheta < 1, \text{ we have } \omega_\vartheta(\Omega_0) \subset\subset \Omega_0. \tag{2.6}$$

Note that this assumption is satisfied, for example, if Ω_0 is convex.

Theorem 2.1. ***(Γ-convergence)*** *Assume (2.1) and (2.6). Then the renormalized energy $\mathcal{E}_n$ Γ-converges to the functional $\mathcal{E}$, as $n \to \infty$, with respect to the product topology of $\mathcal{X}(\Omega) \times \mathcal{X}(\Omega)$.*

Note that, since the topology of $\mathcal{X}(\Omega)$ is metrizable, we can use the sequential notion of Γ-convergence. As we are going to see, assumption (2.6) is needed in the construction of the recovery sequence in the Limsup inequality.

2.2 Preliminary results

Before proving the Γ-convergence theorem, we need some preliminary results. The first one states some properties of the interaction potentials that appear in the expression (2.3) of the limiting interaction energy. Before presenting it, we note that, by (1.9), we have

$$\left| \int_\Omega \mathbb{C}\mathbf{K}_i(\mathbf{x};\mathbf{y}) : \mathbf{K}_j(\mathbf{x};\mathbf{z})\, \mathrm{d}\mathbf{x} \right| \le C\, |\mathbf{b}_i|\, |\mathbf{b}_j| \int_\Omega \frac{1}{|\mathbf{x}-\mathbf{y}|\, |\mathbf{x}-\mathbf{z}|}\, \mathrm{d}\mathbf{x} < +\infty$$

for $\mathbf{y} \neq \mathbf{z}$ and $i, j = 1, 2$, so that the interaction potentials are well-defined.

Lemma 2.2. ***(Properties of the interaction potentials)*** *Let V be one of the interaction potentials V_1, V_2 and $V_{1,2}$. Then V is continuous on $\Omega \times \Omega$ and the following estimates hold:*

(i) there exists two constants $C > 0$ and $L > 0$ such that for every $\mathbf{y}, \mathbf{z} \in \Omega$ with $\mathbf{y} \neq \mathbf{z}$

$$|V(\mathbf{y}, \mathbf{z})| \le C(1 + \log L - \log |\mathbf{y} - \mathbf{z}|); \tag{2.7}$$

(ii) for every open set $\Omega' \subset\subset \Omega$ there exists two positive constants $C' > 0$ and $R' > 0$, both depending on Ω', such that for every $\mathbf{y}, \mathbf{z} \in \Omega'$ with $0 < |\mathbf{y} - \mathbf{z}| < R'$

$$V(\mathbf{y}, \mathbf{z}) \ge C'(1 - \log |\mathbf{y} - \mathbf{z}|). \tag{2.8}$$

Moreover the potentials V_1 and V_2 are symmetric.

Remark 2.3. The constants C and C' given by Lemma 2.2 depend on $\mathbf{b}_1$, $\mathbf{b}_2$ or both in the case that, as V, we consider V_1, V_2, or $V_{1,2}$, respectively. In particular, as we are going to point out in the proof, the hypothesis $\mathbf{b}_1 \cdot \mathbf{b}_2 > 0$ is needed in order to ensure the positivity of the constant C' for $V = V_{1,2}$.

Proof. The symmetry of the potentials V_1 and V_2 follows from that of the elasticity tensor. For the proof of (i), fix $\mathbf{y}, \mathbf{z} \in \Omega$ with $\mathbf{y} \neq \mathbf{z}$. Choose $L \geq 2|\mathbf{y}-\mathbf{z}|$ such that $\Omega \subseteq B(\mathbf{z}, L)$, say $L = 2\,\mathrm{diam}\Omega$, and set $\tilde{\mathbf{x}} = \mathbf{x}-\mathbf{z}$ and $\tilde{\mathbf{y}} = \mathbf{y}-\mathbf{z}$. Then

$$
\begin{aligned}
|V(\mathbf{y},\mathbf{z})| &\leq C \int_\Omega \frac{1}{|\mathbf{x}-\mathbf{y}|\,|\mathbf{x}-\mathbf{z}|}\,\mathrm{d}\mathbf{x} \\
&\leq C \int_{B(\mathbf{z},L)} \frac{1}{|\mathbf{x}-\mathbf{y}|\,|\mathbf{x}-\mathbf{z}|}\,\mathrm{d}\mathbf{x} \\
&= C \int_{B(\mathbf{0},L)} \frac{1}{|\tilde{\mathbf{x}}|\,|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}|}\,\mathrm{d}\tilde{\mathbf{x}} \\
&= C \int_{B(\mathbf{0},2|\tilde{\mathbf{y}}|)} \frac{1}{|\tilde{\mathbf{x}}|\,|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}|}\,\mathrm{d}\tilde{\mathbf{x}} + C \int_{B(\mathbf{0},L)\setminus B(\mathbf{0},2|\tilde{\mathbf{y}}|)} \frac{1}{|\tilde{\mathbf{x}}|\,|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}|}\,\mathrm{d}\tilde{\mathbf{x}}
\end{aligned}
\tag{2.9}
$$

For the first integral, we split it as

$$
\begin{aligned}
\int_{B(\mathbf{0},2|\tilde{\mathbf{y}}|)} \frac{1}{|\tilde{\mathbf{x}}|\,|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}|}\,\mathrm{d}\tilde{\mathbf{x}} &= \int_{B(\mathbf{0},|\tilde{\mathbf{y}}|/2)} \frac{1}{|\tilde{\mathbf{x}}|\,|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}|}\,\mathrm{d}\tilde{\mathbf{x}} + \int_{B(\tilde{\mathbf{y}},|\tilde{\mathbf{y}}|/2)} \frac{1}{|\tilde{\mathbf{x}}|\,|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}|}\,\mathrm{d}\tilde{\mathbf{x}} \\
&\quad + \int_{B(\mathbf{0},2|\tilde{\mathbf{y}}|)\setminus\{B(\mathbf{0},|\tilde{\mathbf{y}}|/2)\cup B(\tilde{\mathbf{y}},|\tilde{\mathbf{y}}|/2)\}} \frac{1}{|\tilde{\mathbf{x}}|\,|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}|}\,\mathrm{d}\tilde{\mathbf{x}}.
\end{aligned}
$$

Note that, if $|\tilde{\mathbf{x}}| \leq |\tilde{\mathbf{y}}|/2$, then

$$
|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}| \geq ||\tilde{\mathbf{x}}|-|\tilde{\mathbf{y}}|| = |\tilde{\mathbf{y}}|-|\tilde{\mathbf{x}}| \geq |\tilde{\mathbf{y}}|-|\tilde{\mathbf{y}}|/2 = |\tilde{\mathbf{y}}|/2
$$

and thus

$$
\int_{B(\mathbf{0},|\tilde{\mathbf{y}}|/2)} \frac{1}{|\tilde{\mathbf{x}}|\,|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}|}\,\mathrm{d}\tilde{\mathbf{x}} \leq \frac{2}{|\tilde{\mathbf{y}}|} \int_{B(\mathbf{0},|\tilde{\mathbf{y}}|/2)} \frac{1}{|\tilde{\mathbf{x}}|}\,\mathrm{d}\tilde{\mathbf{x}} = 2\pi. \tag{2.10}
$$

Analogously, if $|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}| \leq |\tilde{\mathbf{y}}|/2$, then it is easy to see that $|\tilde{\mathbf{x}}| \geq |\tilde{\mathbf{y}}|/2$ and, using this, to show that

$$
\int_{B(\tilde{\mathbf{y}},|\tilde{\mathbf{y}}|/2)} \frac{1}{|\tilde{\mathbf{x}}|\,|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}|}\,\mathrm{d}\tilde{\mathbf{x}} \leq 2\pi. \tag{2.11}
$$

Finally,

$$
\int_{B(\mathbf{0},2|\tilde{\mathbf{y}}|)\setminus\{B(\mathbf{0},|\tilde{\mathbf{y}}|/2)\cup B(\tilde{\mathbf{y}},|\tilde{\mathbf{y}}|/2)\}} \frac{1}{|\tilde{\mathbf{x}}|\,|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}|}\,\mathrm{d}\tilde{\mathbf{x}} \leq \frac{4}{|\tilde{\mathbf{y}}|^2} \int_{B(\mathbf{0},2|\tilde{\mathbf{y}}|)} \mathrm{d}\tilde{\mathbf{x}} = 16\pi. \tag{2.12}
$$

Hence, combining (2.10)-(2.12), we obtain

$$
\int_{B(\mathbf{0},2|\tilde{\mathbf{y}}|)} \frac{1}{|\tilde{\mathbf{x}}|\,|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}|}\,\mathrm{d}\tilde{\mathbf{x}} \leq 20\pi.
$$

The second integral in the last line of (2.9) can be easily bounded using the reverse triangle inequality and polar coordinates as follows:

$$
\begin{aligned}
\int_{B(\mathbf{0},L)\setminus B(\mathbf{0},2|\tilde{\mathbf{y}}|)} \frac{1}{|\tilde{\mathbf{x}}|\,|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}|}\,\mathrm{d}\tilde{\mathbf{x}} &\leq \int_{B(\mathbf{0},L)\setminus B(\mathbf{0},2|\tilde{\mathbf{y}}|)} \frac{1}{|\tilde{\mathbf{x}}|\,(|\tilde{\mathbf{x}}|-|\tilde{\mathbf{y}}|)}\,\mathrm{d}\tilde{\mathbf{x}} = 2\pi \int_{2|\tilde{\mathbf{y}}|}^{L} \frac{1}{r-|\tilde{\mathbf{y}}|}\,\mathrm{d}r \\
&= 2\pi \log\left(\frac{L}{|\tilde{\mathbf{y}}|}-1\right) = 2\pi \log\left(\frac{L}{|\mathbf{y}-\mathbf{z}|}-1\right) \\
&\leq 2\pi \log \frac{L}{|\mathbf{y}-\mathbf{z}|}.
\end{aligned}
$$

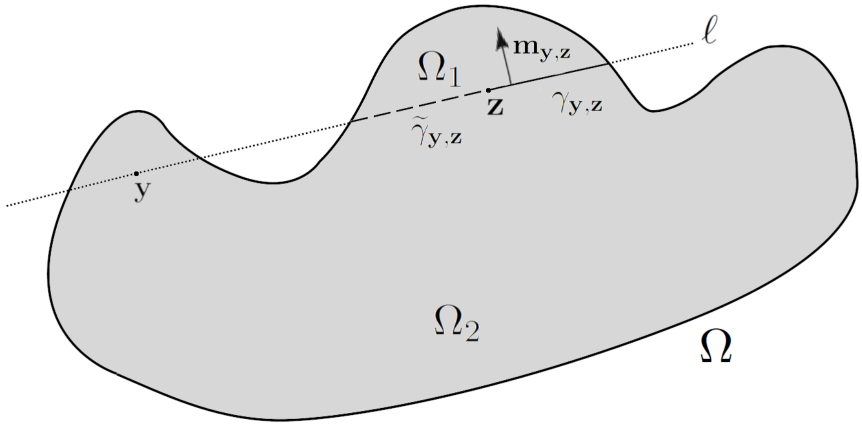


Figure 2.1: Connecting cut technique

Thus, we conclude that

$$|V(\mathbf{y},\mathbf{z})| \leq C\left(20\pi + 2\pi\log\frac{L}{|\mathbf{y}-\mathbf{z}|}\right) \leq C\left(1+\log\frac{L}{|\mathbf{y}-\mathbf{z}|}\right)$$

which is exactly (2.7).
We now prove (ii). For simplicity, we deal with the case $V = V_1$. Take an open set Ω' with $\Omega' \subset\subset \Omega$ and fix two distinct points $\mathbf{y},\mathbf{z} \in \Omega'$. Here we use an argument taken from [6]. Consider the line ℓ passing through $\mathbf{y}$ and $\mathbf{z}$ and denote by $\gamma_{\mathbf{y},\mathbf{z}}$ the segment on it connecting $\mathbf{z}$ to $\partial\Omega$; this can be parametrized via $s \mapsto \mathbf{z} + s\ (\mathbf{z}-\mathbf{y})/|\mathbf{z}-\mathbf{y}|$ where $0 \leq s \leq s_{\mathbf{y},\mathbf{z}}$, so that the unit normal is given by $\mathbf{m}_{\mathbf{y},\mathbf{z}} = (\mathbf{z}-\mathbf{y})^{\perp}/|\mathbf{z}-\mathbf{y}|$. Since $\Omega \setminus \gamma_{\mathbf{y},\mathbf{z}}$ is simply-connected and $\mathbf{K}_1(\cdot;\mathbf{z})$ is smooth in this set with vanishing curl, by the classical Poincaré Lemma there exists a function $\mathbf{v}_{\mathbf{y},\mathbf{z}} \in C^{\infty}(\Omega\setminus\gamma_{\mathbf{y},\mathbf{z}};\mathbb{R}^2)$ such that $\mathrm{D}\mathbf{v}_{\mathbf{y},\mathbf{z}} = \mathbf{K}_1(\cdot;\mathbf{z})$ with zero mean on B_0. It is easy to see that, except at the ending points, if $\mathbf{x} \in \gamma_{\mathbf{y},\mathbf{z}}$, then the one-sided limits

$$\mathbf{v}^{+}_{\mathbf{y},\mathbf{z}}(\mathbf{x}) = \lim_{\boldsymbol{\xi}\to\mathbf{x},\ \boldsymbol{\xi}\cdot\mathbf{m}_{\mathbf{y},\mathbf{z}}>0} \mathbf{v}_{\mathbf{y},\mathbf{z}}(\boldsymbol{\xi}), \qquad \mathbf{v}^{-}_{\mathbf{y},\mathbf{z}}(\mathbf{x}) = \lim_{\boldsymbol{\xi}\to\mathbf{x},\ \boldsymbol{\xi}\cdot\mathbf{m}_{\mathbf{y},\mathbf{z}}<0} \mathbf{v}_{\mathbf{y},\mathbf{z}}(\boldsymbol{\xi})$$

exists and are finite, and thus the jump $[\mathbf{v}_{\mathbf{y},\mathbf{z}}](x) = \mathbf{v}^{+}_{\mathbf{y},\mathbf{z}}(\mathbf{x}) - \mathbf{v}^{-}_{\mathbf{y},\mathbf{z}}(\mathbf{x})$ is well-defined. Moreover, if we compute the circulation of $\mathbf{K}_1(\cdot;\mathbf{z})$, for example, along $\partial B(\mathbf{z},r)$ with $r = |\mathbf{x}-\mathbf{z}|$ using a counter-clockwise parametrization $\boldsymbol{\alpha}\colon [0,1] \to \mathbb{R}^2$ with $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}(1) = \mathbf{x}$, then we obtain

$$\begin{aligned}
\mathbf{b}_1 &= \int_{\partial B(z,r)} \mathbf{K}_1(\boldsymbol{\xi};\mathbf{z})\mathbf{t}(\boldsymbol{\xi})\ \mathrm{d}\mathscr{H}^1(\boldsymbol{\xi}) = \int_{\partial B(\mathbf{z},r)} \mathrm{D}\mathbf{v}_{\mathbf{y},\mathbf{z}}(\boldsymbol{\xi})\mathbf{t}(\boldsymbol{\xi})\ \mathrm{d}\mathscr{H}^1(\boldsymbol{\xi}) = \\
&= \int_0^1 \mathrm{D}\mathbf{v}_{\mathbf{y},\mathbf{z}}(\boldsymbol{\alpha}(t))\boldsymbol{\alpha}'(t)\ \mathrm{d}t = \lim_{h\to0^+}\int_h^{1-h} \mathrm{D}\mathbf{v}_{\mathbf{y},\mathbf{z}}(\boldsymbol{\alpha}(t))\boldsymbol{\alpha}'(t)\ \mathrm{d}t = \\
&= \lim_{h\to0^+}\{\mathbf{v}_{\mathbf{y},\mathbf{z}}(\boldsymbol{\alpha}(1-h)) - \mathbf{v}_{\mathbf{y},\mathbf{z}}(\boldsymbol{\alpha}(h))\} = \mathbf{v}^{-}_{\mathbf{y},\mathbf{z}}(\mathbf{x}) - \mathbf{v}^{+}_{\mathbf{y},\mathbf{z}}(\mathbf{x}) = -[\mathbf{v}_{\mathbf{y},\mathbf{z}}](\mathbf{x}).
\end{aligned} \tag{2.13}$$

Thus we have that $[\mathbf{v}_{\mathbf{y},\mathbf{z}}] = -\mathbf{b}_1$. Clearly $\mathbf{v}_{\mathbf{y},\mathbf{z}} \in W^{1,1}_{\mathrm{loc}}(\Omega\setminus\gamma_{\mathbf{y},\mathbf{z}};\mathbb{R}^2)$ with $\mathrm{D}\mathbf{v}_{\mathbf{y},\mathbf{z}} \in L^1(\Omega;\mathbb{R}^{2\times2})$. Therefore we can conclude that $\mathbf{v}_{\mathbf{y},\mathbf{z}} \in L^1(\Omega;\mathbb{R}^2)$ (see Corollary at p.23 in [19] or Theorem 7.6 in [22]). This can be also proved directly using the following truncation argument. Set $\mathbf{v}_{\mathbf{y},\mathbf{z}} = (v_1, v_2)$ and consider $v_1^M = (v_1 \wedge M) \vee (-M)$ where $M > 0$. Clearly $v_1^M \in L^\infty(\Omega)$, moreover it is easy to see that $v_1^M \in W^{1,1}(\Omega \setminus \gamma_{\mathbf{y},\mathbf{z}})$ with $\mathrm{D}v_1^M = \mathrm{D}v_1 \chi_{\{|v_1|<M\}}$. Using the Poincaré-Wirtinger inequality, we get

$$\left\| v_1^M - \fint_{B_0} v_1^M \, \mathrm{d}\mathbf{x} \right\|_{L^1(\Omega)} \leq C ||\mathrm{D}v_1^M||_{L^1(\Omega;\mathbb{R}^2)} \leq C||\mathrm{D}v_1||_{L^1(\Omega;\mathbb{R}^2)}.$$

Since $v_1^M \to v_1$ a.e. and $\fint_B v_1^M \, \mathrm{d}\mathbf{x} \to \fint_B v_1 \, \mathrm{d}\mathbf{x} = 0$ as $M \to +\infty$, by the Fatou Lemma we conclude that

$$\begin{aligned}\int_\Omega |v_1| \, \mathrm{d}\mathbf{x} &= \int_\Omega \left| v_1 - \fint_{B_0} v_1 \, \mathrm{d}\boldsymbol{\xi} \right| \, \mathrm{d}\mathbf{x} \\ &\leq \liminf_{M\to+\infty} \int_\Omega \left| v_1^M - \fint_{B_0} v_1^M \, \mathrm{d}\boldsymbol{\xi} \right| \, \mathrm{d}\mathbf{x} \leq C \int_\Omega |\mathrm{D}v_1| \, \mathrm{d}\mathbf{x} < +\infty,\end{aligned}$$

that is, $v_1 \in L^1(\Omega)$. Applying the same argument to v_2, we prove that $\mathbf{v}_{\mathbf{y},\mathbf{z}} \in L^1(\Omega;\mathbb{R}^2)$, hence $\mathbf{v}_{\mathbf{y},\mathbf{z}} \in W^{1,1}(\Omega \setminus \gamma_{\mathbf{y},\mathbf{z}};\mathbb{R}^2)$.
Let $\widetilde{\gamma}_{\mathbf{y},\mathbf{z}}$ be the segment on ℓ connecting $\mathbf{z}$ to $\partial\Omega$ parametrized by $s \mapsto \mathbf{z} + s(\mathbf{z}-\mathbf{y})/|\mathbf{z}-\mathbf{y}|$ with $\tilde{s}_{\mathbf{y},\mathbf{z}} \leq s \leq 0$. Denote by Ω_1 and Ω_2 the two open sets in which Ω is partitioned by $\widetilde{\gamma}_{\mathbf{y},\mathbf{z}} \cup \gamma_{\mathbf{y},\mathbf{z}}$. Consider $\boldsymbol{\varphi} \in C_c^\infty(\Omega;\mathbb{R}^2)$. Integrating by parts, we have

$$\begin{aligned}\int_{\Omega_1} \mathbf{v}_{\mathbf{y},\mathbf{z}} \, \mathrm{div}\boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} &= -\int_{\Omega_1} \mathrm{D}\mathbf{v}_{\mathbf{y},\mathbf{z}} \, \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega_1} \mathbf{v}_{\mathbf{y},\mathbf{z}} \, (\boldsymbol{\varphi}\cdot\mathbf{n}) \, \mathrm{d}\mathscr{H}^1 = \\ &= -\int_{\Omega_1} \mathbf{K}_1(\mathbf{x};\mathbf{z}) \, \boldsymbol{\varphi}(\mathbf{x}) \, \mathrm{d}\mathbf{x} - \int_{\widetilde{\gamma}_{\mathbf{y},\mathbf{z}}} \mathbf{v}_{\mathbf{y},\mathbf{z}} \, (\boldsymbol{\varphi}\cdot\mathbf{m}_{\mathbf{y},\mathbf{z}}) \, \mathrm{d}\mathscr{H}^1 \\ &\quad - \int_{\gamma_{\mathbf{y},\mathbf{z}}} \mathbf{v}^+_{\mathbf{y},\mathbf{z}} \, (\boldsymbol{\varphi}\cdot\mathbf{m}_{\mathbf{y},\mathbf{z}}) \, \mathrm{d}\mathscr{H}^1\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}\int_{\Omega_2} \mathbf{v}_{\mathbf{y},\mathbf{z}} \, \mathrm{div}\boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} &= -\int_{\Omega_2} \mathrm{D}\mathbf{v}_{\mathbf{y},\mathbf{z}} \, \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega_2} \mathbf{v}_{\mathbf{y},\mathbf{z}} \, (\boldsymbol{\varphi}\cdot\mathbf{n}) \, \mathrm{d}\mathscr{H}^1 = \\ &= -\int_{\Omega_2} \mathbf{K}_1(\mathbf{x};\mathbf{z}) \, \boldsymbol{\varphi}(\mathbf{x}) \, \mathrm{d}\mathbf{x} + \int_{\widetilde{\gamma}_{\mathbf{y},\mathbf{z}}} \mathbf{v}_{\mathbf{y},\mathbf{z}} \, (\boldsymbol{\varphi}\cdot\mathbf{m}_{\mathbf{y},\mathbf{z}}) \, \mathrm{d}\mathscr{H}^1 \\ &\quad + \int_{\gamma_{\mathbf{y},\mathbf{z}}} \mathbf{v}^-_{\mathbf{y},\mathbf{z}} \, (\boldsymbol{\varphi}\cdot\mathbf{m}_{\mathbf{y},\mathbf{z}}) \, \mathrm{d}\mathscr{H}^1.\end{aligned} \tag{2.15}$$

Hence, summing (2.14) and (2.15) and recalling that $[\mathbf{v}_{\mathbf{y},\mathbf{z}}] = -\mathbf{b}_1$, we obtain

$$-\int_\Omega \mathbf{v}_{\mathbf{y},\mathbf{z}} \, \mathrm{div}\boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} = \int_\Omega \mathbf{K}_1(\mathbf{x};\mathbf{z}) \, \boldsymbol{\varphi}(\mathbf{x}) \, \mathrm{d}\mathbf{x} - \int_{\gamma_{\mathbf{y},\mathbf{z}}} \mathbf{b}_1(\boldsymbol{\varphi}\cdot\mathbf{m}_{\mathbf{y},\mathbf{z}}) \, \mathrm{d}\mathscr{H}^1$$

which says that $\mathrm{D}\mathbf{v}_{\mathbf{y},\mathbf{z}} = \mathbf{K}_1(\cdot;\mathbf{z})\mathscr{L}^2 \llcorner \Omega - \mathbf{b}_1 \otimes \mathbf{m}_{\mathbf{y},\mathbf{z}} \mathscr{H}^1 \llcorner \gamma_{\mathbf{y},\mathbf{z}} \in \mathcal{M}_\mathrm{b}(\Omega;\mathbb{R}^{2\times 2})$ in the sense of distributions. Thus $\mathbf{v}_{\mathbf{y},\mathbf{z}} \in BV(\Omega;\mathbb{R}^2)$.
We now claim that the family $\{\mathbf{v}_{\mathbf{y},\mathbf{z}} : \mathbf{y},\mathbf{z} \in \Omega'\}$ is bounded in $BV(\Omega;\mathbb{R}^2)$. Indeed, if we set $\delta = d(\Omega', \partial\Omega)$, we have

$$\begin{aligned}\int_\Omega |\mathbf{K}_1(\mathbf{x};\mathbf{z})| \, \mathrm{d}\mathbf{x} &\leq C \int_\Omega \frac{1}{|\mathbf{x}-\mathbf{z}|} \, \mathrm{d}\mathbf{x} \\ &= C \int_{B(\mathbf{z},\delta/2)} \frac{1}{|\mathbf{x}-\mathbf{z}|} \, \mathrm{d}\mathbf{x} + C \int_{\Omega \setminus B(\mathbf{z},\delta/2)} \frac{1}{|\mathbf{x}-\mathbf{z}|} \, \mathrm{d}\mathbf{x} \\ &\leq C\pi\delta + 2C/\delta \mathscr{L}^2(\Omega)\end{aligned}$$

and

$$\int_{\gamma_{\mathbf{y},\mathbf{z}}} |\mathbf{b}_1 \otimes \mathbf{m}_{\mathbf{y},\mathbf{z}}| \, \mathrm{d}\mathscr{H}^1 \leq |\mathbf{b}_1| \mathscr{H}^1(\gamma_{\mathbf{y},\mathbf{z}}) \leq |\mathbf{b}_1| \mathrm{diam}\, \Omega$$

so that $||\mathrm{D}\mathbf{v}_{\mathbf{y},\mathbf{z}}||_{\mathcal{M}_{\mathrm{b}}(\Omega;\mathbb{R}^{2\times 2})} \leq C(\Omega, \delta)$. Moreover, using the Poincaré-Wirtinger inequality (see Remark 3.50 in [1]), we see that

$$||\mathbf{v}_{\mathbf{y},\mathbf{z}}||_{L^1(\Omega;\mathbb{R}^2)} = \left\| \mathbf{v}_{\mathbf{y},\mathbf{z}} - \fint_{B_0} \mathbf{v}_{\mathbf{y},\mathbf{z}} \, \mathrm{d}\mathbf{x} \right\|_{L^1(\Omega;\mathbb{R}^2)} \leq C ||\mathrm{D}\mathbf{v}_{\mathbf{y},\mathbf{z}}||_{\mathcal{M}_{\mathrm{b}}(\Omega;\mathbb{R}^{2\times 2})} \leq C(\Omega, \delta),$$

and thus we conclude that $||\mathbf{v}_{\mathbf{y},\mathbf{z}}||_{BV(\Omega;\mathbb{R}^2)} \leq C(\Omega, \delta)$ for every $\mathbf{y}, \mathbf{z} \in \Omega'$, as claimed. From this bound and from the continuity of the trace operator on $BV(\Omega; \mathbb{R}^2)$, we also deduce that $||\mathbf{v}_{\mathbf{y},\mathbf{z}}||_{L^1(\partial\Omega;\mathbb{R}^2)} \leq C(\Omega, \delta)$ for every $\mathbf{y}, \mathbf{z} \in \Omega'$. We compute

$$\begin{aligned} V_1(\mathbf{y}, \mathbf{z}) &= \int_{\Omega \setminus \gamma_{\mathbf{y},\mathbf{z}}} \mathbb{C}\mathbf{K}_1(\mathbf{x}; \mathbf{y}) : \mathbf{K}_1(\mathbf{x}; \mathbf{z}) \, \mathrm{d}\mathbf{x} = \int_{\Omega \setminus \gamma_{\mathbf{y},\mathbf{z}}} \mathbb{C}\mathbf{K}_1(\mathbf{x}; \mathbf{y}) : \mathrm{D}\mathbf{v}_{\mathbf{y},\mathbf{z}}(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \\ &= \int_{\partial\Omega} \mathbf{v}_{\mathbf{y},\mathbf{z}}(\mathbf{x}) \cdot \mathbb{C}\mathbf{K}_1(\mathbf{x}; \mathbf{y}) \mathbf{n}(\mathbf{x}) \, \mathrm{d}\mathscr{H}^1(\mathbf{x}) \\ &\quad - \int_{\gamma_{\mathbf{y},\mathbf{z}}} [\mathbf{v}_{\mathbf{y},\mathbf{z}}](\mathbf{x}) \cdot \mathbb{C}\mathbf{K}_1(\mathbf{x}; \mathbf{y}) \mathbf{m}_{\mathbf{y},\mathbf{z}}(\mathbf{x}) \, \mathrm{d}\mathscr{H}^1(\mathbf{x}), \end{aligned} \tag{2.16}$$

where we have used that $\operatorname{div} \mathbb{C}\mathbf{K}_1(\cdot; \mathbf{y}) = 0$. For the first integral at the right hand side of (2.16), we note that $|\mathbf{x} - \mathbf{y}| > \delta$ for $\mathbf{x} \in \partial\Omega$ and $\mathbf{y} \in \Omega'$. This implies that

$$\left| \int_{\partial\Omega} \mathbf{v}_{\mathbf{y},\mathbf{z}}(\mathbf{x}) \cdot \mathbb{C}\mathbf{K}_1(\mathbf{x}; \mathbf{y}) \mathbf{n}(\mathbf{x}) \, \mathrm{d}\mathscr{H}^1(\mathbf{x}) \right| \leq \frac{C}{\delta} \int_{\partial\Omega} |\mathbf{v}_{\mathbf{y},\mathbf{z}}| \, \mathrm{d}\mathscr{H}^1 \leq C_1, \tag{2.17}$$

where $C_1 > 0$ depends only on Ω and δ. For the second integral at the right hand side of (2.16), we consider again polar coordinates (ϱ, ϑ) centered at $\mathbf{y}$ and we note that $\mathbf{e}_\varrho(\mathbf{x}) \cdot \mathbf{m}_{\mathbf{y},\mathbf{z}}(\mathbf{x}) = 0$ and $\mathbf{e}_\vartheta(\mathbf{x}) \cdot \mathbf{m}_{\mathbf{y},\mathbf{z}}(\mathbf{x}) = 1$ for $\mathbf{x} \in \gamma_{\mathbf{y},\mathbf{z}}$. Therefore we have that

$$\begin{aligned} \mathbb{C}\mathbf{K}_1(\mathbf{x}; \mathbf{y}) \, \mathbf{m}_{\mathbf{y},\mathbf{z}}(\mathbf{x}) &= \frac{\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)} \Big\{ (\mathbf{b}_1 \cdot \mathbf{e}_\varrho(\mathbf{x})) \mathbf{e}_\varrho(\mathbf{x}) + (\mathbf{b}_1 \cdot \mathbf{e}_\vartheta(\mathbf{x})) \mathbf{e}_\vartheta(\mathbf{x}) \Big\} \frac{1}{|\mathbf{x} - \mathbf{y}|} \\ &= \frac{\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)} \frac{\mathbf{b}_1}{|\mathbf{x} - \mathbf{y}|} \end{aligned}$$

for every $\mathbf{x} \in \gamma_{\mathbf{y},\mathbf{z}}$, and we obtain

$$\begin{aligned} - \int_{\gamma_{\mathbf{y},\mathbf{z}}} & [\mathbf{v}_{\mathbf{y},\mathbf{z}}](\mathbf{x}) \cdot \mathbb{C}\mathbf{K}_1(\mathbf{x}; \mathbf{y}) \mathbf{m}_{\mathbf{y},\mathbf{z}}(\mathbf{x}) \, \mathrm{d}\mathscr{H}^1(\mathbf{x}) \\ &= \frac{\mu(\lambda + \mu)|\mathbf{b}_1|^2}{\pi(\lambda + 2\mu)} \int_{\gamma_{\mathbf{y},\mathbf{z}}} \frac{1}{|\mathbf{x} - \mathbf{y}|} \, \mathrm{d}\mathscr{H}^1(\mathbf{x}) \\ &= C_2 \int_0^{s_{\mathbf{y},\mathbf{z}}} \frac{1}{|\mathbf{y} - \mathbf{z}| + s} \, \mathrm{d}s \\ &= C_2 \Big(\log(|\mathbf{y} - \mathbf{z}| + s_{\mathbf{y},\mathbf{z}}) - \log |\mathbf{y} - \mathbf{z}| \Big) \end{aligned} \tag{2.18}$$

where we set $C_2 = \mu(\lambda + \mu)|\mathbf{b}_1|^2 / (\pi(\lambda + 2\mu))$. Combining (2.16)-(2.18) with the inequality $|\mathbf{y} - \mathbf{z}| + s_{\mathbf{y},\mathbf{z}} \geq s_{\mathbf{y},\mathbf{z}} \geq d(\mathbf{z}, \partial\Omega) \geq \delta$, we can bound the interaction potential from below as follows:

$$\begin{aligned} V_1(\mathbf{y}, \mathbf{z}) &\geq -C_1 + C_2(\log(|\mathbf{y} - \mathbf{z}| + s_{\mathbf{y},\mathbf{z}}) - \log |\mathbf{y} - \mathbf{z}|) \geq \\ &\geq -C_1 + C_2 \log \delta - C_2 \log |\mathbf{y} - \mathbf{z}|. \end{aligned}$$

If we set $C' = C_2/2$ and we choose $R' > 0$ such that $C_2 \log|\mathbf{y}-\mathbf{z}| \leq -2C_1 - C_2 + C_2 \log \delta$ for $0 < |y-z| < R'$, the previous inequality implies (2.8).
The proof of (ii) for $V = V_2$ and $V = V_{1,2}$ is completely analogous. Simply note that, in the second case, in (2.18) we are going to obtain $\mathbf{b}_1 \cdot \mathbf{b}_2$ in place of $|\mathbf{b}_1|^2$. Thus we see that the assumption $\mathbf{b}_1 \cdot \mathbf{b}_2 > 0$ is necessary in order to obtain a positive constant C_2.
It remains to show the continuity of the interaction potential. Again, for simplicity, we take $V = V_1$. Consider two points $\mathbf{y}, \mathbf{z} \in \Omega$ and two sequences $(\mathbf{y}_n), (\mathbf{z}_n) \subset \Omega$ such that $\mathbf{y}_n \to \mathbf{y}$ and $\mathbf{z}_n \to \mathbf{z}$ for $n \to \infty$. If $\mathbf{y} \neq \mathbf{z}$, then $\mathbf{y}_n \neq \mathbf{z}_n$ for $n \gg 1$. Moreover, by continuity, $\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_n) : \mathbf{K}_1(\mathbf{x};\mathbf{z}_n) \to \mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}) : \mathbf{K}_1(\mathbf{x};\mathbf{z})$ for a.e. $\mathbf{x} \in \Omega$, as $n \to \infty$, and thus we only need to find a domination in order to apply the Dominated Convergence Theorem. Take $0 < \delta < |\mathbf{y}-\mathbf{z}|/4$. Then, by (1.12), one has

$$\chi_{\Omega \setminus \{B(\mathbf{y}_n,\delta) \cup B(\mathbf{z}_n,\delta)\}} |\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_n) : \mathbf{K}_1(\mathbf{x};\mathbf{z}_n)| \leq \frac{C}{\delta^2}. \tag{2.19}$$

On the other hand, if $\mathbf{x} \in B(\mathbf{y}_n,\delta) \cup B(\mathbf{z}_n,\delta)$, then either $\mathbf{x} \in B(\mathbf{y}_n,\delta)$ or $\mathbf{x} \in B(\mathbf{z}_n,\delta)$. In the first case, for $n \gg 1$ we have

$$\begin{aligned} |\mathbf{x}-\mathbf{z}_n| &\geq |\mathbf{y}-\mathbf{z}| - |\mathbf{y}-\mathbf{y}_n| - |\mathbf{y}_n-\mathbf{x}| - |\mathbf{z}_n-\mathbf{z}| \\ &\geq \frac{|\mathbf{y}-\mathbf{z}|}{2} + 2\delta - \delta/2 - \delta - \delta/2 = \frac{|\mathbf{y}-\mathbf{z}|}{2} \end{aligned}$$

and thus

$$\frac{1}{|\mathbf{x}-\mathbf{z}_n|} \leq \frac{2}{|\mathbf{y}-\mathbf{z}|}$$

for $n \gg 1$. In the second case, we obtain in a similar way that

$$\frac{1}{|\mathbf{x}-\mathbf{y}_n|} \leq \frac{2}{|\mathbf{y}-\mathbf{z}|}$$

for $n \gg 1$. Thus, by (1.12)

$$\begin{aligned} &\chi_{B(\mathbf{y}_n,\delta)\cup B(\mathbf{z}_n,\delta)}(\mathbf{x}) |\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_n) : \mathbf{K}_1(\mathbf{x};\mathbf{z}_n)| \\ &\qquad \leq \left(\chi_{B(\mathbf{y}_n,\delta)}(\mathbf{x}) + \chi_{B(\mathbf{z}_n,\delta)}(\mathbf{x})\right) \frac{C}{|\mathbf{x}-\mathbf{y}_n|\,|\mathbf{x}-\mathbf{z}_n|} \\ &\qquad \leq \chi_{B(\mathbf{y}_n,\delta)}(\mathbf{x}) \frac{2C}{|\mathbf{y}-\mathbf{z}|\,|\mathbf{x}-\mathbf{y}_n|} + \chi_{B(\mathbf{z}_n,\delta)}(\mathbf{x}) \frac{2C}{|\mathbf{y}-\mathbf{z}|\,|\mathbf{x}-\mathbf{z}_n|} \\ &\qquad \leq \frac{2C}{|\mathbf{y}-\mathbf{z}|}\left(\frac{1}{|\mathbf{x}-\mathbf{y}_n|} + \frac{1}{|\mathbf{x}-\mathbf{z}_n|}\right) \end{aligned} \tag{2.20}$$

for every $\mathbf{x} \in \Omega$. Moreover, using translation operators, one can show that

$$\frac{1}{|\cdot - \mathbf{y}_n|} \to \frac{1}{|\cdot - \mathbf{y}|} \quad \text{in } L^1(\Omega), \qquad \frac{1}{|\cdot - \mathbf{z}_n|} \to \frac{1}{|\cdot - \mathbf{z}|} \quad \text{in } L^1(\Omega), \qquad \text{as } n \to \infty,$$

so that there exists a subsequence (n_k) and exist two functions $g_1, g_2 \in L^1(\Omega)$ such that for every k we have $\frac{1}{|\mathbf{x}-\mathbf{y}_{n_k}|} \leq g_1(\mathbf{x})$ and $\frac{1}{|\mathbf{x}-\mathbf{z}_{n_k}|} \leq g_2(\mathbf{x})$ for a.e. $\mathbf{x} \in \Omega$. Hence, taking into account (2.19) and (2.20), we deduce that

$$|\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_{n_k}) : \mathbf{K}_1(\mathbf{x};\mathbf{z}_{n_k})| \leq \frac{C}{\delta^2} + \frac{2C}{|\mathbf{y}-\mathbf{z}|}(g_1(\mathbf{x}) + g_2(\mathbf{x})),$$

which gives the desired domination. Therefore, we obtain that $V_1(\mathbf{y}_{n_k}, \mathbf{z}_{n_k}) \to V_1(\mathbf{y},\mathbf{z})$, as $k \to \infty$, and by the Urysohn property we conclude that $V_1(\mathbf{y}_n,\mathbf{z}_n) \to V_1(\mathbf{y},\mathbf{z})$, as $n \to \infty$.

Finally, we consider the case $\mathbf{y} = \mathbf{z}$. Take an open set $\Omega' \subset\subset \Omega$ with $\mathbf{y} \in \Omega'$, so that $\mathbf{y}_n, \mathbf{z}_n \in \Omega'$ for $n \gg 1$. If $\mathbf{y}_n = \mathbf{z}_n$ for $n \gg 1$, then $V_1(\mathbf{y}_n, \mathbf{z}_n) = +\infty$ for $n \gg 1$ and there is nothing to prove. Otherwise we can construct two subsequences $(\mathbf{y}_{n_k})$ and $(\mathbf{z}_{n_k})$ such that $\mathbf{y}_{n_k} \neq \mathbf{z}_{n_k}$ for every k. By (2.8) we have that $V_1(\mathbf{y}_{n_k}, \mathbf{z}_{n_k}) \geq C'(1 - \log|\mathbf{y}_{n_k} - \mathbf{z}_{n_k}|)$, at least for $k \gg 1$, and we conclude that $V_1(\mathbf{y}_{n_k}, \mathbf{z}_{n_k}) \to +\infty = V_1(\mathbf{y}, \mathbf{z})$, as $k \to \infty$. Again, it is sufficient to apply the Urysohn property to conclude. The proof of the continuity for V_2 and $V_{1,2}$ is exactly the same. □

Remark 2.4. From Lemma 2.2, it follows that all the interaction potentials are bounded from below on $\Omega' \times \Omega'$ for every $\Omega' \subset\subset \Omega$. To see this, consider $R' > 0$ as in (ii) and take $\mathbf{y}, \mathbf{z} \in \Omega'$. If $|\mathbf{y} - \mathbf{z}| \geq R'$, then by (2.7) we have

$$V(\mathbf{y}, \mathbf{z}) \geq -|V(\mathbf{y}, \mathbf{z})| \geq -C\left(1 - \log\frac{|\mathbf{y} - \mathbf{z}|}{L}\right) \geq -C + C\log\frac{R'}{L};$$

otherwise, if $|\mathbf{y} - \mathbf{z}| < R'$, then by (2.8) we get

$$V(\mathbf{y}, \mathbf{z}) \geq C'(1 - \log|\mathbf{y} - \mathbf{z}|) \geq C'(1 - \log R')$$

Therefore we have $V \geq -C$ on $\Omega' \times \Omega'$. In particular, if $\Omega_0 \subset\subset \Omega'$, then we easily deduce that the functional $\mathcal{F}$ is bounded from below.

Under the assumptions of this chapter, we have that every minimizer of the auxiliary functional given by Lemma 1.4 admits an extension to $H^1(\Omega; \mathbb{R}^2)$ which is uniformly bounded with respect to the pair of admissible empirical measures considered. This is the content of the following result.

Lemma 2.5. *For a given n, consider two admissible measures $\mu^1 = \sum_{i=1}^{N_n} \delta_{\mathbf{y}_i} \in X_n^1$ and $\mu^2 = \sum_{i=1}^{M_n} \delta_{\mathbf{z}_i} \in X_n^2$ such that $\mu^1 + \mu^2 \in X_n$. Then $\mathbf{u}^{\varepsilon_n}_{\mu^1+\mu^2} \in H^1(\Omega_{\varepsilon_n}(\mu^1 + \mu^2); \mathbb{R}^2)$ admits an extension $\widetilde{\mathbf{u}}^{\varepsilon_n}_{\mu^1+\mu^2} \in H^1(\Omega; \mathbb{R}^2)$ such that*

$$||\widetilde{\mathbf{u}}^{\varepsilon_n}_{\mu^1+\mu^2}||_{H^1(\Omega;\mathbb{R}^2)} \leq C \tag{2.21}$$

with a constant $C > 0$ independent of n, μ^1 and μ^2.

Proof. For simplicity, we set $\mathbf{u} = \mathbf{u}^{\varepsilon_n}_{\mu^1+\mu^2}$. Conditions (2.1) ensure that $r_n > 4\varepsilon_n$ for $n \gg 1$. Thus, by Theorem A.1 in the Appendix, there exists an extension $\widetilde{\mathbf{u}} \in H^1(\Omega; \mathbb{R}^2)$ satisfying

$$||\mathrm{E}\widetilde{\mathbf{u}}||_{L^2(\Omega;\mathbb{R}^{2\times 2})} \leq C||\mathrm{E}\mathbf{u}||_{L^2(\Omega_{\varepsilon_n}(\mu^1+\mu^2);\mathbb{R}^{2\times 2})} \tag{2.22}$$

with some constant $C > 0$ independend of n, μ^1 and μ^2. Since $\widetilde{\mathbf{u}} = \mathbf{u}$ on B_0, we can use the Korn inequality which, combined with (2.22), gives

$$||\widetilde{\mathbf{u}}||_{H^1(\Omega;\mathbb{R}^2)} \leq C||\mathrm{E}\mathbf{u}||_{L^2(\Omega_{\varepsilon_n}(\mu^1+\mu^2);\mathbb{R}^{2\times 2})}. \tag{2.23}$$

Hence, in order to prove (2.21), it is sufficient to show that

$$||\mathrm{E}\mathbf{u}||_{L^2(\Omega_{\varepsilon_n}(\mu^1+\mu^2);\mathbb{R}^{2\times 2})} \leq C \tag{2.24}$$

for some constant $C > 0$ independent of n, μ^1 and μ^2. Taking $\mathbf{v} = \mathbf{0}$ as a competitor in $H^1(\Omega; \mathbb{R}^2)$ we see that $I_{\varepsilon_n}(\mu^1 + \mu^2, \mathbf{u}) \leq I_{\varepsilon_n}(\mu^1 + \mu^2, \mathbf{v}) = 0$. From this, using (1.4), we obtain

$$
\begin{aligned}
&C||\mathrm{E}\mathbf{u}||^2_{L^2(\Omega_{\varepsilon_n}(\mu^1+\mu^2);\mathbb{R}^{2\times 2})} \\
&\quad\le \frac{1}{n}\sum_{i=1}^{N_n}||\mathbb{C}\mathbf{K}_1^{\varepsilon_n}(\cdot;\mathbf{y}_i)||_{L^2(\partial\Omega;\mathbb{R}^{2\times 2})}||\mathbf{u}||_{L^2(\partial\Omega;\mathbb{R}^2)} \\
&\quad+\frac{1}{n}\sum_{i=1}^{M_n}||\mathbb{C}\mathbf{K}_2^{\varepsilon_n}(\cdot;\mathbf{z}_i)||_{L^2(\partial\Omega;\mathbb{R}^{2\times 2})}||\mathbf{u}||_{L^2(\partial\Omega;\mathbb{R}^2)} \\
&\quad+\frac{1}{n}\sum_{i=1}^{N_n}\sum_{j\neq i}||\mathbb{C}\mathbf{K}_1^{\varepsilon_n}(\cdot;\mathbf{y}_i)||_{L^2(\partial B(\mathbf{y}_j,\varepsilon_n);\mathbb{R}^{2\times 2})}||\mathbf{u}||_{L^2(\partial B(\mathbf{y}_j,\varepsilon_n);\mathbb{R}^2)} \\
&\quad+\frac{1}{n}\sum_{i=1}^{N_n}\sum_{j=1}^{M_n}||\mathbb{C}\mathbf{K}_1^{\varepsilon_n}(\cdot;\mathbf{y}_i)||_{L^2(\partial B(\mathbf{z}_j,\varepsilon_n);\mathbb{R}^{2\times 2})}||\mathbf{u}||_{L^2(\partial B(\mathbf{z}_j,\varepsilon_n);\mathbb{R}^2)} \\
&\quad+\frac{1}{n}\sum_{i=1}^{M_n}\sum_{j=1}^{N_n}||\mathbb{C}\mathbf{K}_2^{\varepsilon_n}(\cdot;\mathbf{z}_i)||_{L^2(\partial B(\mathbf{y}_j,\varepsilon_n);\mathbb{R}^{2\times 2})}||\mathbf{u}||_{L^2(\partial B(\mathbf{y}_j,\varepsilon_n);\mathbb{R}^2)} \\
&\quad+\frac{1}{n}\sum_{i=1}^{M_n}\sum_{j\neq i}||\mathbb{C}\mathbf{K}_2^{\varepsilon_n}(\cdot;\mathbf{z}_i)||_{L^2(\partial B(\mathbf{z}_j,\varepsilon_n);\mathbb{R}^{2\times 2})}||\mathbf{u}||_{L^2(\partial B(\mathbf{z}_j,\varepsilon_n);\mathbb{R}^2)},
\end{aligned}
\tag{2.25}
$$

where we used that $\mathbb{C}\mathbf{K}_1^{\varepsilon_n}(\cdot;\mathbf{y}_j)\mathbf{n}=\mathbf{0}$ on $\partial B(\mathbf{y}_j,\varepsilon_n)$ for $j=1,\dots,N_n$ and $\mathbb{C}\mathbf{K}_2^{\varepsilon_n}(\cdot;\mathbf{z}_j)\mathbf{n}=\mathbf{0}$ on $\partial B(\mathbf{z}_j,\varepsilon_n)$ for $j=1,\dots,M_n$. Moreover, from (1.12), (1.33), and (2.1), we can deduce the following inequalities:

$$
\begin{aligned}
\sup_{\mathbf{x}\in\partial\Omega}|\mathbb{C}\mathbf{K}_1^{\varepsilon_n}(\mathbf{x};\mathbf{y}_i)| &\le C\sup_{\mathbf{x}\in\partial\Omega}\left(\frac{1}{|\mathbf{x}-\mathbf{y}_i|}+\frac{\varepsilon_n^2}{|\mathbf{x}-\mathbf{y}_i|^3}\right)\\
&\le C\left(\frac{1}{r_0}+\frac{\varepsilon_n^2}{r_0}\right)\le C,\quad \text{for } i=1,\dots,N_n,
\end{aligned}
\tag{2.26}
$$

$$
\begin{aligned}
\sup_{\mathbf{x}\in\partial B(\mathbf{y}_j,\varepsilon_n)}|\mathbb{C}\mathbf{K}_1^{\varepsilon_n}(\mathbf{x};\mathbf{y}_i)| &\le C\sup_{\mathbf{x}\in\partial B(\mathbf{y}_j,\varepsilon_n)}\left(\frac{1}{|\mathbf{x}-\mathbf{y}_i|}+\frac{\varepsilon_n^2}{|\mathbf{x}-\mathbf{y}_i|^3}\right)\\
&\le C\left(\frac{1}{r_n}+\frac{\varepsilon_n^2}{r_n^3}\right)\le\frac{C}{r_n},\quad \text{for } i=1,\dots,N_n \text{ and } j\neq i,
\end{aligned}
\tag{2.27}
$$

$$
\begin{aligned}
\sup_{\mathbf{x}\in\partial B(\mathbf{z}_j,\varepsilon_n)}|\mathbb{C}\mathbf{K}_1^{\varepsilon_n}(\mathbf{x};\mathbf{y}_i)| &\le C\sup_{\mathbf{x}\in\partial B(\mathbf{z}_j,\varepsilon_n)}\left(\frac{1}{|\mathbf{x}-\mathbf{y}_i|}+\frac{\varepsilon_n^2}{|\mathbf{x}-\mathbf{y}_i|^3}\right)\\
&\le C\left(\frac{1}{r_n}+\frac{\varepsilon_n^2}{r_n^3}\right)\le\frac{C}{r_n},\quad \text{for } i=1,\dots,N_n \text{ and } j=1,\dots,M_n
\end{aligned}
\tag{2.28}
$$

and analogously

$$
\sup_{\mathbf{x}\in\partial\Omega}|\mathbb{C}\mathbf{K}_2^{\varepsilon_n}(\mathbf{x};\mathbf{z}_i)|\le C,\quad \text{for } i=1,\dots,M_n,
\tag{2.29}
$$

$$
\sup_{\mathbf{x}\in\partial B(\mathbf{y}_j,\varepsilon_n)}|\mathbb{C}\mathbf{K}_2^{\varepsilon_n}(\mathbf{x};\mathbf{z}_i)|\le\frac{C}{r_n},\quad \text{for } i=1,\dots,M_n \text{ and } j=1,\dots,N_n,
\tag{2.30}
$$

$$
\sup_{\mathbf{x}\in\partial B(\mathbf{z}_j,\varepsilon_n)}|\mathbb{C}\mathbf{K}_1^{\varepsilon_n}(\mathbf{x};\mathbf{z}_i)|\le\frac{C}{r_n},\quad \text{for } i=1,\dots,M_n \text{ and } j\neq i.
\tag{2.31}
$$

Hence, combining (2.25) with the inequalities (2.26)-(2.31), we obtain

$$\begin{aligned}||\mathrm{E}\mathbf{u}||^2_{L^2(\Omega_{\varepsilon_n}(\mu^1+\mu^2);\mathbb{R}^{2\times 2})} &\le C||\mathbf{u}||_{L^2(\partial\Omega;\mathbb{R}^2)}\\ &+ C\frac{\sqrt{\varepsilon_n}}{r_n}\left\{\sum_{j=1}^{N_n}||\mathbf{u}||_{L^2(B(\mathbf{y}_j,\varepsilon_n);\mathbb{R}^2)} + \sum_{j=1}^{M_n}||\mathbf{u}||_{L^2(B(\mathbf{z}_j,\varepsilon_n);\mathbb{R}^2)}\right\}.\end{aligned} \tag{2.32}$$

The norm of of $\mathbf{u}$ on $\partial\Omega$ is easily controlled using the trace inequality and (2.23). Indeed we have

$$||\mathbf{u}||_{L^2(\partial\Omega;\mathbb{R}^2)} = ||\tilde{\mathbf{u}}||_{L^2(\partial\Omega;\mathbb{R}^2)} \le C||\tilde{\mathbf{u}}||_{H^1(\Omega;\mathbb{R}^2)} \le C||\mathrm{E}\mathbf{u}||_{L^2(\Omega_{\varepsilon_n}(\mu^1+\mu^2);\mathbb{R}^{2\times 2})}. \tag{2.33}$$

To control the norm of $\mathbf{u}$ on the boundaries $\partial B(\mathbf{y}_j,\varepsilon_n)$, for $j=1,\dots,N_n$, we proceed as follows. Denote by $\hat{\mathbf{u}}$ the vector field $\mathbf{u}$ as a function of the polar coordinates (ϱ,ϑ) centered at $\mathbf{y}_j$. For a.e. $\vartheta\in(0,2\pi)$, we have that $\hat{\mathbf{u}}(\cdot,\vartheta)\in AC([\varepsilon_n,r_n];\mathbb{R}^2)\cap H^1((\varepsilon_n,r_n);\mathbb{R}^2)$. Hence, for $\varepsilon_n\le r\le r_n/2$, we may write

$$\hat{\mathbf{u}}(\varepsilon_n,\vartheta) = \hat{\mathbf{u}}(r,\vartheta) - \int_{\varepsilon_n}^{r}\frac{\partial\hat{\mathbf{u}}}{\partial\varrho}\,\mathrm{d}\varrho$$

and, using Jensen inequality, we easily deduce

$$|\hat{\mathbf{u}}(\varepsilon_n,\vartheta)|^2 \le 2|\hat{\mathbf{u}}(r,\vartheta)|^2 + 2(r-\varepsilon_n)\int_{\varepsilon_n}^{r}\left|\frac{\partial\hat{\mathbf{u}}}{\partial\varrho}\right|^2\mathrm{d}\varrho \le 2|\hat{\mathbf{u}}(r,\vartheta)|^2 + 2r_n\int_{\varepsilon_n}^{r/2}\left|\frac{\partial\hat{\mathbf{u}}}{\partial\varrho}\right|^2\mathrm{d}\varrho.$$

Integrating respect to ϑ over $(0,2\pi)$ and multiplying by ε_n, we obtain

$$\int_0^{2\pi}\varepsilon_n|\hat{\mathbf{u}}(\varepsilon_n,\vartheta)|^2\,\mathrm{d}\vartheta \le 2\int_0^{2\pi}\varepsilon_n|\hat{\mathbf{u}}(r,\vartheta)|^2\,\mathrm{d}\vartheta + 2r_n\int_0^{2\pi}\int_{\varepsilon_n}^{r_n/2}\left|\frac{\partial\hat{\mathbf{u}}}{\partial\varrho}\right|^2\mathrm{d}\varrho\,\mathrm{d}\vartheta$$

from which, since $\varepsilon_n\le r\le r_n/2$, it follows that

$$\int_{\partial B(\mathbf{y}_j,\varepsilon_n)}|\mathbf{u}|^2\,\mathrm{d}\mathscr{H}^1 \le 2\int_0^{2\pi} r|\hat{\mathbf{u}}(r,\vartheta)|^2\,\mathrm{d}\vartheta + 2r_n\int_{B(\mathbf{y}_j,r_n/2)\setminus\overline{B}(\mathbf{y}_j,\varepsilon_n)}|\mathrm{D}\mathbf{u}|^2\,\mathrm{d}\mathbf{x}.$$

Finally, averaging with respect to r over $(\varepsilon_n,r_n/2)$, we have

$$\begin{aligned}\int_{\partial B(\mathbf{y}_j,\varepsilon_n)}|\mathbf{u}|^2\,\mathrm{d}\mathscr{H}^1 &\le \frac{C}{r_n}\int_{B(\mathbf{y}_j,r_n/2)\setminus\overline{B}(\mathbf{y}_j,\varepsilon_n)}|\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\\ &+ C\int_{B(\mathbf{y}_j,r_n/2)\setminus\overline{B}(\mathbf{y}_j,\varepsilon_n)}|\mathrm{D}\mathbf{u}|^2\,\mathrm{d}\mathbf{x},\end{aligned} \tag{2.34}$$

where we used (2.1). In a completely analogous way, it can be shown that

$$\begin{aligned}\int_{\partial B(\mathbf{z}_j,\varepsilon_n)}|\mathbf{u}|^2\,\mathrm{d}\mathscr{H}^1 &\le \frac{C}{r_n}\int_{B(\mathbf{z}_j,r_n/2)\setminus\overline{B}(\mathbf{z}_j,\varepsilon_n)}|\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\\ &+ C\int_{B(\mathbf{z}_j,r_n/2)\setminus\overline{B}(\mathbf{z}_j,\varepsilon_n)}|\mathrm{D}\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\end{aligned} \tag{2.35}$$

for $j=1,\dots,M_n$. Thus, using (2.23) and the trace inequality, from (2.34) and (2.35) we conclude

$$\begin{aligned}\sum_{j=1}^{N_n}||\mathbf{u}||_{L^2(B(\mathbf{y}_j,\varepsilon_n);\mathbb{R}^2)} + \sum_{j=1}^{M_n}||\mathbf{u}||_{L^2(B(\mathbf{z}_j,\varepsilon_n);\mathbb{R}^2)} &\le \frac{C}{\sqrt{r_n}}||\mathbf{u}||_{H^1(\Omega_{\varepsilon_n}(\mu^1+\mu^2);\mathbb{R}^2)}\\ &\le \frac{C}{\sqrt{r_n}}||\tilde{\mathbf{u}}||_{H^1(\Omega;\mathbb{R}^2)} \le \frac{C}{\sqrt{r_n}}||\mathrm{E}\mathbf{u}||_{L^2(\Omega_{\varepsilon_n}(\mu^1+\mu^2);\mathbb{R}^{2\times 2})}.\end{aligned} \tag{2.36}$$

Therefore, combining (2.32), (2.33) and (2.36) and using (2.1), we obtain

$$\begin{aligned}||\mathbf{Eu}||^2_{L^2(\Omega_{\varepsilon_n}(\mu^1+\mu^2);\mathbb{R}^{2\times 2})} &\leq C||\mathbf{Eu}||_{L^2(\Omega_{\varepsilon_n}(\mu^1+\mu^2);\mathbb{R}^{2\times 2})} + C\sqrt{\frac{\varepsilon_n}{r_n^3}}||\mathbf{Eu}||_{L^2(\Omega_{\varepsilon_n}(\mu^1+\mu^2);\mathbb{R}^{2\times 2})} \\ &\leq C||\mathbf{Eu}||_{L^2(\Omega_{\varepsilon_n}(\mu^1+\mu^2);\mathbb{R}^{2\times 2})},\end{aligned}$$

which gives (2.24). Note that the constant C obtained is independent of n, μ^1 and μ^2, as desired. □

In the next lemma we study existence and uniqueness for the minimization problem (2.5). The argument is analogous to that of Lemma 1.4. Here we look for a minimizer in the class

$$Y(\Omega) = \left\{ \mathbf{v} \in H^1(\Omega;\mathbb{R}^2) : \int_{B_0} \mathbf{v} = 0,\ \int_{B_0} (\mathrm{D}\mathbf{v} - \mathrm{D}\mathbf{v}^\top) = 0 \right\}. \tag{2.37}$$

Lemma 2.6. ***(Minimization of the limiting auxiliary functional)*** *For every pair of measures $\mu^1, \mu^2 \in \mathcal{X}(\Omega)$ there exists a unique minimizer $\mathbf{u}_{\mu^1,\mu^2} \in H^1(\Omega;\mathbb{R}^2)$ of the functional $I(\mu^1,\mu^2,\cdot)$ in the class $Y(\Omega)$.*

Proof. The functional $I(\mu^2,\mu^2,\cdot)$ is clearly weakly lower semicontinuous on $H^1(\Omega;\mathbb{R}^2)$. Moreover, it is also weakly coercive. Indeed, since $|\mathbf{x}-\mathbf{y}| \geq r_0$ for every $\mathbf{x} \in \partial\Omega$ and $\mathbf{y} \in \overline{\Omega}_0$, using (1.12) we have

$$\begin{aligned}&\left| \int_\Omega \int_{\partial\Omega} \mathbf{v}(\mathbf{x}) \cdot \mathbb{C}\mathbf{K}_i(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\ \mathrm{d}\mathscr{H}^1(\mathbf{x})\ \mathrm{d}\mu^i(\mathbf{y}) \right| \leq \\ &\leq \frac{C|\mathbf{b}_i|}{r_0}||\mathbf{v}||_{L^2(\partial\Omega;\mathbb{R}^2)}\ \mu^i(\Omega) \leq C||\mathbf{v}||_{L^2(\partial\Omega;\mathbb{R}^2)}\end{aligned}$$

for every $\mathbf{v} \in Y(\Omega)$ and $i = 1, 2$. Hence, using the Korn inequality, we obtain that

$$I(\mu^2,\mu^2,\mathbf{v}) \geq C_1||\mathbf{v}||^2_{H^1(\Omega;\mathbb{R}^2)} - C_2||\mathbf{v}||_{H^1(\Omega;\mathbb{R}^2)} \tag{2.38}$$

for every $\mathbf{v} \in Y(\Omega)$. Therefore, by the Direct Method, a minimizer exists. Now, if $\mathbf{u}_1, \mathbf{u}_2 \in Y(\Omega)$ are two minimizers, then they solve the following Euler-Lagrange equations:

$$\forall \mathbf{v} \in Y(\Omega), \quad \int_\Omega \mathbb{C}\mathrm{D}\mathbf{u}_1(\mathbf{x}) : \mathrm{D}\mathbf{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} + \sum_{i=1}^2 \int_\Omega \int_{\partial\Omega} \mathbf{v}(\mathbf{x}) \cdot \mathbb{C}\mathbf{K}_i(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\ \mathrm{d}\mathscr{H}^1(\mathbf{x})\ \mathrm{d}\mu^i(\mathbf{y}) = 0, \tag{2.39}$$

$$\forall \mathbf{v} \in Y(\Omega), \quad \int_\Omega \mathbb{C}\mathrm{D}\mathbf{u}_2(\mathbf{x}) : \mathrm{D}\mathbf{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} + \sum_{i=1}^2 \int_\Omega \int_{\partial\Omega} \mathbf{v}(\mathbf{x}) \cdot \mathbb{C}\mathbf{K}_i(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\ \mathrm{d}\mathscr{H}^1(\mathbf{x})\ \mathrm{d}\mu^i(\mathbf{y}) = 0. \tag{2.40}$$

Taking the difference between (2.39) and (2.40) and then choosing $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$, we get that $\int_\Omega \mathbb{C}\mathrm{D}(\mathbf{u}_1 - \mathbf{u}_2) : \mathrm{D}(\mathbf{u}_1 - \mathbf{u}_2) = 0$, which entails that $\mathrm{E}(\mathbf{u}_1 - \mathbf{u}_2) = 0$. This implies that $\mathbf{u}_1 - \mathbf{u}_2$ is an infinitesimal rigid-body motion in $Y(\Omega)$. By the definition of $Y(\Omega)$, we deduce that $\mathbf{u}_1 - \mathbf{u}_2 = 0$. □

Remark 2.7. In order to prove that $\mathcal{G}(\mu^1,\mu^2) = I(\mu^1,\mu^2,\mathbf{u}_{\mu^1,\mu^2})$, taking into account (2.5), we have to check that

$$\inf_{\mathbf{u}\in H^1(\Omega;\mathbb{R}^2)} I(\mu^1,\mu^2,\mathbf{u}) = \inf_{\mathbf{u}\in Y(\Omega)} I(\mu^1,\mu^2,\mathbf{u}).$$

To do this we can argue as in Remark 1.5. We note the invariance of $I(\mu^1,\mu^2,\cdot)$ with respect to infinitesimal rigid-body motions. Therefore, for every $\mathbf{u}\in H^1(\Omega;\mathbb{R}^2)$ we can choose a competitor of the form $\mathbf{v} = \mathbf{u}+\boldsymbol{\eta}$, with $\boldsymbol{\eta}$ an infinitesimal rigid-body motion, such that $\mathbf{v}\in Y(\Omega)$. Thus we have $I(\mu^1,\mu^2,\mathbf{v}) = I(\mu^1,\mu^2,\mathbf{u})$ and the claim follows.

We now start approaching the proof of Theorem 2.1. In the next result we present some asymptotic estimates. In order to simplify the notation, we set $\mathbf{K}_i^n = \mathbf{K}_i^{\varepsilon_n}$ and, given $\mu_n^1,\mu_n^2\in\mathcal{X}(\Omega)$, we write $\Omega_n = \Omega_{\varepsilon_n}(\mu_n^1+\mu_n^2)$, $I_n = I_{\varepsilon_n}(\mu_n^1+\mu_n^2,\cdot)$, $Y_n(\Omega) = Y^{\varepsilon_n}_{\mu_n^1+\mu_n^2}(\Omega)$, and $\mathbf{u}_n = \mathbf{u}^{\varepsilon_n}_{\mu_n^1+\mu_n^2}$.

Lemma 2.8. ***(Preliminary estimates)*** *Consider two sequences (μ_n^1) and (μ_n^2) in $\mathcal{X}(\Omega)$ such that for every n we have*

$$\mu_n^1 = \frac{1}{n}\sum_{i=1}^{N_n}\delta_{y_i^n}\in X_n^1, \qquad \mu_n^2 = \frac{1}{n}\sum_{i=1}^{M_n}\delta_{z_i^n}\in X_n^2, \qquad \mu_n^1+\mu_n^2\in X_n.$$

Then, as $n\to\infty$, the following estimates hold:

(2.41)
$$\frac{1}{n^2}\sum_{i=1}^{N_n}\sum_{j\neq i}\int_{\Omega_n}\mathbb{C}\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_i^n):\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x} = \iint_{\Omega\times\Omega}V_1(\mathbf{y},\mathbf{z})\,\mathrm{d}(\mu_n^1\boxtimes\mu_n^1)(\mathbf{y},\mathbf{z}) + o(1),$$

(2.42)
$$\frac{1}{n^2}\sum_{i=1}^{M_n}\sum_{j\neq i}\int_{\Omega_n}\mathbb{C}\mathbf{K}_2^n(\mathbf{x};\mathbf{z}_i^n):\mathbf{K}_2^n(\mathbf{x};\mathbf{z}_j^n)\,\mathrm{d}\mathbf{x} = \iint_{\Omega\times\Omega}V_2(\mathbf{y},\mathbf{z})\,\mathrm{d}(\mu_n^2\boxtimes\mu_n^2)(\mathbf{y},\mathbf{z}) + o(1),$$

(2.43)
$$\frac{1}{n^2}\sum_{i=1}^{N_n}\sum_{j=1}^{M_n}\int_{\Omega_n}\mathbb{C}\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_i^n):\mathbf{K}_2^n(\mathbf{x};\mathbf{z}_j^n)\,\mathrm{d}\mathbf{x} = \iint_{\Omega\times\Omega}V_{1,2}(\mathbf{y},\mathbf{z})\,\mathrm{d}(\mu_n^1\otimes\mu_n^2)(\mathbf{y},\mathbf{z}) + o(1),$$

(2.44)
$$\frac{1}{n}\sum_{i=1}^{N_n}\int_{\partial\Omega_n}\mathbf{u}_n(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) = $$
$$= \frac{1}{n}\sum_{i=1}^{N_n}\int_{\partial\Omega}\mathbf{u}_n(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) + o(1),$$

(2.45)
$$\frac{1}{n}\sum_{i=1}^{M_n}\int_{\partial\Omega_n}\mathbf{u}_n(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_2^n(\mathbf{x};\mathbf{z}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) = $$
$$= \frac{1}{n}\sum_{i=1}^{M_n}\int_{\partial\Omega}\mathbf{u}_n(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_2(\mathbf{x};\mathbf{z}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) + o(1).$$

Moreover if $\mu_n^1\to\mu^1$ and $\mu_n^2\to\mu^2$ in $\mathcal{X}(\Omega)$, as $n\to\infty$, so that $\mu^1(\Omega) = \mathsf{m}$ and $\mu^2(\Omega) = 1-\mathsf{m}$, then

$$I_n(\mathbf{u}_n)\to I(\mu^1,\mu^2,\mathbf{u}_{\mu^1,\mu^2}), \qquad \textit{as } n\to\infty,$$

that is,

$$\mathcal{G}_n(\mu_n^1,\mu_n^2)\to\mathcal{G}(\mu^1,\mu^2),\qquad as\ n\to\infty. \tag{2.46}$$

Proof. We begin with the proof of (2.41).Using (1.3) and (1.32), we compute

$$\begin{aligned}\int_{\Omega_n}\mathbb{C}\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_i^n):\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}&=\int_{\Omega_n}\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n):\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}\\&+\varepsilon_n^2\int_{\Omega_n}\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n):\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}\\&+\varepsilon_n^2\int_{\Omega_n}\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n):\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\,\mathrm{d}\mathbf{x}\\&+\varepsilon_n^4\int_{\Omega_n}\mathbb{C}\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n):\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}.\end{aligned}\tag{2.47}$$

We focus on the last three integrals. Applying the Divergence Theorem and using the fact that $\operatorname{div}\mathbb{C}\mathbf{K}_1(\cdot;y_i^n)=0$ in Ω_n, we obtain

$$\begin{aligned}\varepsilon_n^2\int_{\Omega_n}\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n):\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}=\varepsilon_n^2\Bigg\{&\int_{\partial\Omega}\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n)\,\mathrm{d}\mathbf{x}\\&-\sum_{k=1}^{N_n}\int_{\partial B(\mathbf{y}_k^n,\varepsilon_n)}\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\&-\sum_{k=1}^{M_n}\int_{\partial B(\mathbf{z}_k^n,\varepsilon_n)}\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\Bigg\},\end{aligned}\tag{2.48}$$

$$\begin{aligned}\varepsilon_n^2\int_{\Omega_n}\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n):\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\,\mathrm{d}\mathbf{x}=\varepsilon_n^2\Bigg\{&\int_{\partial\Omega}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}\\&-\sum_{k=1}^{N_n}\int_{\partial B(\mathbf{y}_k^n,\varepsilon_n)}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\&-\sum_{k=1}^{M_n}\int_{\partial B(\mathbf{z}_k^n,\varepsilon_n)}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\Bigg\},\end{aligned}\tag{2.49}$$

$$\begin{aligned}\varepsilon_n^4\int_{\Omega_n}\mathbb{C}\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n):\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}=\varepsilon_n^4\Bigg\{&\int_{\partial\Omega}\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\cdot\mathbb{C}\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\,\mathrm{d}\mathbf{x}\\&-\sum_{k=1}^{N_n}\int_{\partial B(\mathbf{y}_k^n,\varepsilon_n)}\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\cdot\mathbb{C}\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\&-\sum_{k=1}^{M_n}\int_{\partial B(\mathbf{z}_k^n,\varepsilon_n)}\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\cdot\mathbb{C}\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\Bigg\}.\end{aligned}\tag{2.50}$$

Since $|\mathbf{x}-\mathbf{y}_i^n|\geq r_0$ for every $\mathbf{x}\in\partial\Omega$ and for $i=1,\dots,n$, using (1.12) and (1.33), we have that

$$\varepsilon_n^2\int_{\partial\Omega}|\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)|\,|\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n)|\,\mathrm{d}\mathbf{x}\leq\frac{C}{r_0^3}\varepsilon_n^2\to0,\quad\text{as }n\to\infty,\tag{2.51}$$

and similary

$$\varepsilon_n^4\int_{\partial\Omega}|\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)|\,|\mathbb{C}\mathrm{D}\mathbf{y}(\mathbf{x}-\mathbf{y}_i^n)|\,\mathrm{d}\mathbf{x}\leq\frac{C}{r_0^5}\varepsilon_n^4\to0,\quad\text{as }n\to\infty,\tag{2.52}$$

both for every $i,j=1,\dots,N_n$. Moreover, nothing that $|\mathbf{x}-\mathbf{y}_i^n|\geq r_n-\varepsilon_n$ for every $\mathbf{x}\in\partial B(\mathbf{z}_k,\varepsilon_n)$, for $k=1,\dots,M_n$ and $i=1,\dots,N_n$, we can use (1.12), (1.33) and (2.1), to show that

$$\begin{aligned}\varepsilon_n^2\sum_{k=1}^{M_n}\int_{\partial B(\mathbf{z}_k,\varepsilon_n)}|\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)|\,|\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)|\,\mathrm{d}\mathbf{x}&\leq C\frac{n\varepsilon_n^3}{(r_n-\varepsilon_n)^3}\\&\leq C\frac{n\varepsilon_n^3}{r_n^3}\to 0,\quad\text{as } n\to\infty,\end{aligned}\tag{2.53}$$

and

$$\begin{aligned}\varepsilon_n^4\sum_{k=1}^{M_n}\int_{\partial B(\mathbf{z}_k,\varepsilon_n)}|\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)|\,|\mathbb{C}D\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)|\,\mathrm{d}\mathbf{x}&\leq C\frac{n\varepsilon_n^5}{(r_n-\varepsilon_n)^5}\\&\leq C\frac{n\varepsilon_n^5}{r_n^5}\to 0,\quad\text{as } n\to\infty,\end{aligned}\tag{2.54}$$

both for $i,j=1,\dots,N_n$. Thus from (2.47)-(2.50) and (2.51)-(2.54), we deduce that

$$\begin{aligned}\int_{\Omega_n}\mathbb{C}\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_i^n):\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}&=\int_{\Omega_n}\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n):\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}\\&-\varepsilon_n^2\sum_{k=1}^{N_n}\int_{\partial B(\mathbf{y}_k^n,\varepsilon_n)}\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\&-\varepsilon_n^2\sum_{k=1}^{N_n}\int_{\partial B(\mathbf{y}_k^n,\varepsilon_n)}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\mathbf{n}(x)\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\&-\varepsilon_n^4\sum_{k=1}^{N_n}\int_{\partial B(\mathbf{y}_k^n,\varepsilon_n)}\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\cdot\mathbb{C}D\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})+o(1),\end{aligned}\tag{2.55}$$

as $n\to\infty$. For $k\neq i,j$ and $i=1,\dots,N_n$, we have $|\mathbf{x}-\mathbf{y}_i^n|\geq r_n-\varepsilon_n$ for every $\mathbf{x}\in\partial B(\mathbf{y}_k^n,\varepsilon_n)$. Hence, using again (1.12), (1.33) and (2.1), we compute

$$\begin{aligned}&\int_{\partial B(\mathbf{y}_k^n,\varepsilon_n)}|\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})|\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\&\quad+\int_{\partial B(\mathbf{y}_k^n,\varepsilon_n)}|\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\mathbf{n}(\mathbf{x})|\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\&\quad+\varepsilon_n^2\int_{\partial B(\mathbf{y}_k^n,\varepsilon_n)}|\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\cdot\mathbb{C}D\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})|\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\&\quad\leq C\varepsilon_n\left(\frac{1}{(r_n-\varepsilon_n)^3}+\frac{1}{(r_n-\varepsilon_n)^3}+\frac{\varepsilon_n^2}{(r_n-\varepsilon_n)^5}\right)\leq C\left(\frac{\varepsilon_n}{r_n^3}+\frac{\varepsilon_n^3}{r_n^5}\right).\end{aligned}\tag{2.56}$$

Similarly, we have

$$\begin{aligned}&\int_{\partial B(\mathbf{y}_i^n,\varepsilon_n)}|\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})|\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\&+\int_{\partial B(\mathbf{y}_i^n,\varepsilon_n)}|\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\mathbf{n}(\mathbf{x})|\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\&+\varepsilon_n^2\int_{\partial B(\mathbf{y}_i^n,\varepsilon_n)}|\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\cdot\mathbb{C}D\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})|\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\&\leq C\varepsilon_n\left(\frac{1}{\varepsilon_n(r_n-\varepsilon_n)^2}+\frac{1}{\varepsilon^2(r_n-\varepsilon_n)}+\frac{\varepsilon_n^2}{\varepsilon_n^3(r_n-\varepsilon_n)^2}\right)\leq C\left(\frac{1}{r_n^2}+\frac{1}{\varepsilon_n r_n}\right),\end{aligned}\tag{2.57}$$

and

$$\begin{aligned}
&\int_{\partial B(\mathbf{y}_j^n,\varepsilon_n)} |\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})|\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\
&+\int_{\partial B(\mathbf{y}_j^n,\varepsilon_n)} |\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\mathbf{n}(\mathbf{x})|\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\
&+\varepsilon_n^2\int_{\partial B(\mathbf{y}_j^n,\varepsilon_n)} |\mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\cdot\mathbb{C}\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})|\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\
&\le C\varepsilon_n\left(\frac{1}{\varepsilon_n^2(r_n-\varepsilon_n)}+\frac{1}{\varepsilon_n(r_n-\varepsilon_n)^2}+\frac{\varepsilon_n^2}{\varepsilon_n^2(r_n-\varepsilon_n)^3}\right)\le C\left(\frac{1}{\varepsilon_n r_n}+\frac{1}{r_n^2}+\frac{\varepsilon_n}{r_n^3}\right).
\end{aligned}\tag{2.58}$$

Therefore, from (2.56), (2.57) and (2.58), we obtain

$$\begin{aligned}
\varepsilon_n^2\sum_{k=1}^{N_n}&\left|\int_{\partial B(\mathbf{y}_j^n,\varepsilon_n)} \mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\right|\\
&+\varepsilon_n^2\sum_{k=1}^{N_n}\left|\int_{\partial B(\mathbf{y}_j^n,\varepsilon_n)} \mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\right|\\
&+\varepsilon_n^4\sum_{k=1}^{N_n}\left|\int_{\partial B(\mathbf{y}_j^n,\varepsilon_n)} \mathbf{w}(\mathbf{x}-\mathbf{y}_j^n)\cdot\mathbb{C}\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\right|\\
&\le C\varepsilon_n^2 n\left\{\left(\frac{\varepsilon_n}{r_n^3}+\frac{\varepsilon_n^3}{r_n^5}\right)+\left(\frac{1}{r_n^2}+\frac{1}{\varepsilon_n r_n}\right)+\left(\frac{1}{\varepsilon_n r_n}+\frac{1}{r_n^2}+\frac{\varepsilon_n}{r_n^3}\right)\right\}\\
&\le Cnr_n^2\frac{\varepsilon_n^2}{r_n^2}\left(\frac{\varepsilon_n}{r_n^3}+\frac{\varepsilon_n^3}{r_n^5}+\frac{1}{r_n^2}+\frac{1}{\varepsilon_n r_n}\right)\le Cnr_n^2\frac{\varepsilon_n}{r_n^3}\left(\frac{\varepsilon_n^2}{r_n^2}+\frac{\varepsilon_n^4}{r_n^4}+\frac{\varepsilon_n}{r_n}+1\right)
\end{aligned}\tag{2.59}$$

which, due to (2.1), vanishes as $n\to\infty$. Thus, looking back at (2.47), we proved that

$$\begin{aligned}
\frac{1}{n^2}\sum_{i=1}^{N_n}\sum_{j\neq i}\int_{\Omega_n}&\mathbb{C}\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_i^n):\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}\\
&=\frac{1}{n^2}\sum_{i=1}^{N_n}\sum_{j\neq i}\int_{\Omega_n}\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n):\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}+o(1),
\end{aligned}\tag{2.60}$$

as $n\to\infty$. If we prove that

$$\frac{1}{n^2}\sum_{i=1}^{N_n}\sum_{j\neq i}\int_{\Omega\setminus\Omega_n}\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n):\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}\to 0,\quad\text{as } n\to\infty,\tag{2.61}$$

then we can change the domain of integration in the right hand side of (2.60) from Ω_n to Ω and thus obtain (2.41). Since $\Omega\setminus\Omega_n=\bigcup_{k=1}^{N_n}\overline{B}(\mathbf{y}_k^n,\varepsilon_n)\cup\bigcup_{k=1}^{M_n}\overline{B}(\mathbf{z}_k^n,\varepsilon_n)$, we have

$$\begin{aligned}
\frac{1}{n^2}\sum_{i=1}^{N_n}\sum_{j\neq i}\int_{\Omega\setminus\Omega_n}&\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n):\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}\\
&=\frac{1}{n^2}\sum_{i=1}^{N_n}\sum_{j\neq i}\sum_{k=1}^{N_n}\int_{B(\mathbf{y}_k^n,\varepsilon_n)}\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n):\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}\\
&+\frac{1}{n^2}\sum_{i=1}^{N_n}\sum_{j\neq i}\sum_{k=1}^{M_n}\int_{B(\mathbf{z}_k^n,\varepsilon_n)}\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n):\mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}.
\end{aligned}\tag{2.62}$$

For $k \neq i, j$ we have

$$\left|\int_{B(\mathbf{y}_k^n,\varepsilon_n)} \mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n) : \mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}\right| \leq C\int_{B(\mathbf{y}_k^n,\varepsilon_n)} \frac{1}{|\mathbf{x}-\mathbf{y}_i^n|\,|\mathbf{x}-\mathbf{y}_j^n|}\,\mathrm{d}\mathbf{x} \leq \leq C\frac{\varepsilon_n^2}{(r_n-\varepsilon_n)^2} \leq C\frac{\varepsilon_n^2}{r_n^2}, \tag{2.63}$$

and analogously for the integrals on $B(\mathbf{z}_k^n,\varepsilon_n)$ with $k = 1,\ldots,M_n$. For $k = i$, using polar coordinates, we compute

$$\left|\int_{B(\mathbf{y}_i^n,\varepsilon_n)} \mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n) : \mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x}\right| \leq C\int_{B(\mathbf{y}_k^n,\varepsilon_n)} \frac{1}{|\mathbf{x}-\mathbf{y}_i^n|\,|\mathbf{x}-\mathbf{y}_j^n|}\,\mathrm{d}\mathbf{x} \leq \leq \frac{C}{r_n-\varepsilon_n}\int_{B(\mathbf{y}_i^n,\varepsilon_n)} \frac{1}{|\mathbf{x}-\mathbf{y}_i^n|}\,\mathrm{d}\mathbf{x} \leq C\frac{\varepsilon_n}{r_n}, \tag{2.64}$$

and analogously for $k = j$. Thus, from (2.62), using (2.63) and (2.64) we deduce that

$$\frac{1}{n^2}\sum_{i=1}^{N_n}\sum_{j\neq i}\int_{\Omega\setminus\Omega_n} \mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n) : \mathbf{K}_1(\mathbf{x};\mathbf{y}_j^n)\,\mathrm{d}\mathbf{x} \leq \frac{C}{n^2}\sum_{i=1}^{N_n}\sum_{j\neq i}\left(\frac{\varepsilon_n^2}{r_n^2}N_n + \frac{2\varepsilon_n}{r_n} + \frac{\varepsilon_n^2}{r_n^2}M_n\right) \leq C\left(\frac{\varepsilon_n^2}{r_n^2}n + \frac{\varepsilon_n}{r_n}\right)$$

which vanishes, as $n \to \infty$. Hence (2.61) holds and this concludes the proof of (2.41). The proof of (2.42) and (2.43) is completely analogous.
We now prove (2.44). Taking into account that $\mathbb{C}\mathbf{K}_1^n(\cdot;\mathbf{y}_i^n)\mathbf{n} = 0$ on $\partial B(\mathbf{y}_i^n,\varepsilon_n)$, we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{N_n}\int_{\partial\Omega_n} \mathbf{u}_n(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) &= \frac{1}{n}\sum_{i=1}^{N_n}\int_{\partial\Omega} \mathbf{u}_n(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) \\ &- \frac{1}{n}\sum_{i=1}^{N_n}\sum_{k\neq i}\int_{\partial B(y_k^n,\varepsilon_n)} \mathbf{u}_n(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) \\ &- \frac{1}{n}\sum_{i=1}^{N_n}\sum_{k=1}^{M_n}\int_{\partial B(z_k^n,\varepsilon_n)} \mathbf{u}_n(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}).\end{aligned} \tag{2.65}$$

Recalling (1.32), we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{N_n}\int_{\partial\Omega} \mathbf{u}_n(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) &= \frac{1}{n}\sum_{i=1}^{N_n}\int_{\partial\Omega} \mathbf{u}_n(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) \\ &+ \frac{\varepsilon_n^2}{n}\sum_{i=1}^{N_n}\int_{\partial\Omega} \mathbf{u}_n(\mathbf{x})\cdot\mathbb{C}\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}).\end{aligned} \tag{2.66}$$

Moreover,

$$\frac{\varepsilon_n^2}{n}\sum_{i=1}^{N_n}\left|\int_{\partial\Omega}\mathbf{u}_n(\mathbf{x})\cdot\mathbb{C}\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\right|$$
$$\leq \frac{\varepsilon_n^2}{n}\sum_{i=1}^{N_n}\int_{\partial\Omega}|\mathbf{u}_n(\mathbf{x})|\,|\mathbb{C}\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)|\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\leq$$
$$\leq \frac{\varepsilon_n^2}{n}\sum_{i=1}^{N_n}||\mathbf{u}_n||_{L^2(\partial\Omega;\mathbb{R}^2)}||\mathbb{C}\mathrm{D}\mathbf{w}(\cdot-\mathbf{y}_i^n)||_{L^2(\partial\Omega;\mathbb{R}^{2\times 2})}\leq C\varepsilon_n^2\frac{N_n}{n},$$

which vanishes, as $n\to\infty$. Note that here we used the hypothesis of confinement, combined with (1.33), to deduce that $|\mathbb{C}\mathrm{D}\mathbf{w}(\mathbf{x}-\mathbf{y}_i^n)|\leq Cr_0^{-3}$ for every $\mathbf{x}\in\partial\Omega$ and $i=1,\ldots,N_n$, and the uniform boundedness of the traces of the functions $\mathbf{u}_n$. The latter can be obtained using the extensions $\tilde{\mathbf{u}}_n$ given by Lemma 2.5 and the continuity of the trace operator on $H^1(\Omega;\mathbb{R}^2)$. From the same lemma, looking back at (2.36), we can also deduce the following estimate

$$\sum_{k=1}^{N_n}||\mathbf{u}_n||_{L^2(\partial B(y_k^n,\varepsilon_n);\mathbb{R}^2)}+\sum_{k=1}^{M_n}||\mathbf{u}_n||_{L^2(\partial B(z_k^n,\varepsilon_n);\mathbb{R}^2)}\leq\frac{C}{\sqrt{r_n}}. \tag{2.67}$$

Hence, since the first integral in (2.66) is, up to an infinitesimal term, the right hand side of (2.44), it is enough to show that the last two integrals at the right hand side of (2.65) vanish. Using that $|\mathbb{C}\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_i^n)|\leq C/r_n$ for $\mathbf{x}\in\partial B(\mathbf{y}_k^n,\varepsilon_n)$ and $i=1,\ldots,N_n$, and the estimate (2.67), we compute

$$\frac{1}{n}\sum_{i=1}^{N_n}\sum_{k\neq i}\left|\int_{\partial B(\mathbf{y}_k^n,\varepsilon_n)}\mathbf{u}_n(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\right|$$
$$\leq\frac{1}{n}\frac{C}{r_n}\sum_{i=1}^{N_n}\sum_{k\neq i}\int_{\partial B(\mathbf{y}_k^n,\varepsilon_n)}|\mathbf{u}_n(\mathbf{x})|\,\mathrm{d}\mathscr{H}^1(\mathbf{x})$$
$$\leq\frac{N_n}{n}\frac{C}{r_n}\sum_{k=1}^{N_n}\int_{\partial B(\mathbf{y}_k^n,\varepsilon_n)}|\mathbf{u}_n(x)|\,\mathrm{d}\mathscr{H}^1(x)$$
$$\leq\frac{N_n}{n}\frac{C}{r_n}\sqrt{\varepsilon_n}\sum_{k=1}^{N_n}||\mathbf{u}_n(x)||_{L^2(\partial B(y_k^n,\varepsilon_n))}\leq C\frac{N_n}{n}\sqrt{\frac{\varepsilon_n}{r_n^3}},$$

which vanishes, as $n\to\infty$. With analogous computations, one can estimate also the last integral at the right hand side of (2.65), hence (2.44) is proven.
Equation (2.45) can be proven exactly in the same way.
We move to the proof of the final statement of the lemma. By hypothesis we have that $\mu_n^1\to\mu^1$ and $\mu_n^2\to\mu^2$ in $\mathcal{X}(\Omega)$ as $n\to\infty$, then, by narrow convergence, we deduce that $\mu^1(\Omega)=\mathsf{m}$ and $\mu^2(\Omega)=1-\mathsf{m}$. We know from Lemma 2.5 that we can extend each $\mathbf{u}_n\in H^1(\Omega_n;\mathbb{R}^2)$ to $\tilde{\mathbf{u}}_n\in H^1(\Omega;\mathbb{R}^2)$ in such a way as to have $||\tilde{\mathbf{u}}_n||_{H^1(\Omega;\mathbb{R}^2)}\leq C$ for every n and for some $M>0$. Thus, by weak compactness, there exists a subsequence $(\tilde{\mathbf{u}}_{n_k})$ and $\mathbf{u}\in H^1(\Omega;\mathbb{R}^2)$ such that $\tilde{\mathbf{u}}_{n_k}\rightharpoonup\mathbf{u}$ in $H^1(\Omega;\mathbb{R}^2)$, as $k\to\infty$. . We claim that

$$\frac{1}{n_k}\sum_{i=1}^{N_{n_k}}\int_{\partial\Omega}\mathbf{u}_{n_k}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^{n_k})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\to\int_\Omega\int_{\partial\Omega}\mathbf{u}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\,\mathrm{d}\mu^1(\mathbf{y}) \tag{2.68}$$

and

$$\frac{1}{n_k}\sum_{i=1}^{M_{n_k}}\int_{\partial\Omega}\mathbf{u}_{n_k}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_2(\mathbf{x};\mathbf{y}_i^{n_k})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) \rightarrow \int_\Omega\int_{\partial\Omega}\mathbf{u}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_2(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\,\mathrm{d}\mu^2(\mathbf{y}), \tag{2.69}$$

as $k\to\infty$. For the first claim, we have

$$\begin{aligned}\frac{1}{n_k}\sum_{i=1}^{N_{n_k}}\int_{\partial\Omega}&\mathbf{u}_{n_k}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y}_i^{n_k})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\\ &=\int_\Omega\int_{\partial\Omega}(\tilde{\mathbf{u}}_{n_k}(\mathbf{x})-\mathbf{u}(\mathbf{x}))\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\,\mathrm{d}\mu^1_{n_k}(\mathbf{y})\\ &+\int_\Omega\int_{\partial\Omega}\mathbf{u}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\,\mathrm{d}\mu^1_{n_k}(\mathbf{y}).\end{aligned} \tag{2.70}$$

For every $\mathbf{y}\in\Omega_0$ we have

$$\begin{aligned}&\left|\int_{\partial\Omega}(\tilde{\mathbf{u}}_{n_k}(\mathbf{x})-\mathbf{u}(\mathbf{x}))\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\right|\\ &\le||\tilde{\mathbf{u}}_{n_k}-\mathbf{u}||_{L^2(\partial\Omega;\mathbb{R}^2)}\,||\mathbb{C}\mathbf{K}_1(\cdot;\mathbf{y})||_{L^2(\partial\Omega;\mathbb{R}^{2\times2})}\le\frac{C}{r_0}||\tilde{\mathbf{u}}_{n_k}-\mathbf{u}||_{L^2(\partial\Omega;\mathbb{R}^2)}.\end{aligned}$$

Since $||\tilde{\mathbf{u}}_{n_k}-\mathbf{u}||_{L^2(\partial\Omega;\mathbb{R}^2)}\to 0$, as $k\to\infty$, by the compactness of the trace operator on $H^1(\Omega;\mathbb{R}^2)$, we easily deduce that the first integral at the right hand side of (2.70) vanishes, as $k\to\infty$. For the second term, we note that the function

$$\mathbf{y}\mapsto\int_{\partial\Omega}\mathbf{u}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})$$

is continuous (this can be easily checked using the Dominated Convergence Theorem) and bounded on Ω_0, thus by narrow convergence we have

$$\begin{aligned}\int_\Omega\int_{\partial\Omega}&\mathbf{u}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\,\mathrm{d}\mu^1_{n_k}(\mathbf{y})\\ &\rightarrow\int_\Omega\int_{\partial\Omega}\mathbf{u}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\,\mathrm{d}\mu^1(\mathbf{y}),\end{aligned}$$

as $k\to\infty$, so that (2.68) is proven. The proof of (2.69) is completely analogous.
Owing to (2.68) and (2.69) we can characterize the weak limit $\mathbf{u}$. Since $\tilde{\mathbf{u}}_{n_k}\rightharpoonup\mathbf{u}$ in $H^1(\Omega;\mathbb{R}^2)$ and, by (2.1), $\mathscr{L}^2(\Omega\setminus\Omega_{n_k})\to 0$, as $k\to\infty$, we have that $\chi_{\Omega_{n_k}}\mathrm{D}\tilde{\mathbf{u}}_{n_k}\rightharpoonup\mathrm{D}\mathbf{u}$ in $L^2(\Omega;\mathbb{R}^{2\times2})$, as $k\to\infty$. Thus, by the lower semicontinuity of the elastic energy, we have

$$\begin{aligned}\int_\Omega\mathbb{C}\mathrm{D}\mathbf{u}:\mathrm{D}\mathbf{u}\,\mathrm{d}\mathbf{x}&\le\liminf_k\int_\Omega\mathbb{C}(\chi_{\Omega_{n_k}}\mathrm{D}\tilde{\mathbf{u}}_{n_k}):\mathrm{D}(\chi_{\Omega_{n_k}}\mathrm{D}\tilde{\mathbf{u}}_{n_k})\,\mathrm{d}\mathbf{x}\\ &=\liminf_k\int_\Omega\mathbb{C}\mathrm{D}\mathbf{u}_{n_k}:\mathrm{D}\mathbf{u}_{n_k}\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{2.71}$$

Hence, by (2.44), (2.45), (2.68), (2.69) and (2.71), we obtain

$$
\begin{aligned}
\liminf_k I_{n_k}(\mathbf{u}_{n_k}) \geq{}& \liminf_k \int_{\Omega_{n_k}} W(\mathrm{D}\mathbf{u}_{n_k}(\mathbf{x}))\,\mathrm{d}\mathbf{x} \\
&+ \liminf_k \frac{1}{n_k}\sum_{i=1}^{N_{n_k}} \int_{\partial\Omega_{n_k}} \mathbf{u}_{n_k}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1^{n_k}(\mathbf{x};\mathbf{y}_i^{n_k})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) \\
&+ \liminf_k \frac{1}{n_k}\sum_{i=1}^{M_{n_k}} \int_{\partial\Omega_{n_k}} \mathbf{u}_{n_k}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_2^{n_k}(\mathbf{x};\mathbf{y}_i^{n_k})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) \\
\geq{}& \int_\Omega W(\mathrm{D}\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x} \\
&+ \int_\Omega\int_{\partial\Omega} \mathbf{u}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\,\mathrm{d}\mu^1(\mathbf{y}) \\
&+ \int_\Omega\int_{\partial\Omega} \mathbf{u}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_2(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\,\mathrm{d}\mu^2(\mathbf{y}) = I(\mu^1,\mu^2,\mathbf{u}).
\end{aligned} \tag{2.72}
$$

By minimality we have $I_n(\mathbf{u}_n) \leq I_n(\mathbf{v})$ for every $\mathbf{v}\in H^1(\Omega_n;\mathbb{R}^2)$. Thus, if we show that

$$
\forall \mathbf{w}\in H^1(\Omega;\mathbb{R}^2), \quad I_n(\mathbf{w}) \to I(\mu^1,\mu^2,\mathbf{w}), \quad \text{as } n\to\infty, \tag{2.73}
$$

then, using (2.72), we have

$$
\forall \mathbf{v}\in H^1(\Omega;\mathbb{R}^2), \quad I(\mu^1,\mu^2,\mathbf{u}) \leq \liminf_k I_{n_k}(\mathbf{u}_{n_k}) \leq \liminf_k I_{n_k}(\mathbf{v}) = I(\mu^1,\mu^2,\mathbf{v}),
$$

that is, $\mathbf{u}$ is a minimizer of $I(\mu^1,\mu^2,\cdot)$ on $H^1(\Omega;\mathbb{R}^2)$. For the proof of (2.73), the convergence of the elastic energies can be easily obtained using the Monotone Convergence Theorem. Hence it is sufficient to show that for $\mathbf{v}\in H^1(\Omega;\mathbb{R}^2)$ we have

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{N_n}\int_{\partial\Omega_n} \mathbf{v}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_i^n)\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) \\
&\qquad\to \int_\Omega\int_{\partial\Omega} \mathbf{v}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\,\mathrm{d}\mu^1(\mathbf{y}),
\end{aligned} \tag{2.74}
$$

as $n\to\infty$. This can be done splitting the integral as in (2.65) and using analogous arguments. Therefore $\mathbf{u}$ is a minimizer of $I(\mu^1,\mu^2,\cdot)$ on $H^1(\Omega;\mathbb{R}^2)$. Moreover, since $\mathbf{u}_n\in Y_n(\Omega)$ for every n, by weak convergence $\mathbf{u}\in Y(\Omega)$ and thus, by Lemma 2.6, $\mathbf{u} = \mathbf{u}_{\mu^1,\mu^2}$. Then, using the Urysohn property, we conclude that $\tilde{\mathbf{u}}_n \rightharpoonup \mathbf{u}_{\mu^1,\mu^2}$ in $H^1(\Omega;\mathbb{R}^2)$ and $I_n(\mathbf{u}_n)\to I(\mu^1,\mu^2,\mathbf{u}_{\mu^1,\mu^2})$ as $n\to\infty$. Futhermore, it follows that the convergences in (2.68) and (2.69) are true along every subsequence, hence, using (2.44) and (2.45), we deduce that

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{N_n}\int_{\partial\Omega_n} \mathbf{u}_n(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1^n(\mathbf{x};\mathbf{y}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) \\
&\qquad\to \int_\Omega\int_{\partial\Omega} \mathbf{u}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\,\mathrm{d}\mu^1(\mathbf{y})
\end{aligned} \tag{2.75}
$$

and

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{M_n}\int_{\partial\Omega_n} \mathbf{u}_n(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_2^n(\mathbf{x};\mathbf{z}_i^n)\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x}) \\
&\qquad\to \int_\Omega\int_{\partial\Omega} \mathbf{u}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_2(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\,\mathrm{d}\mu^2(\mathbf{y}),
\end{aligned} \tag{2.76}
$$

as $n\to\infty$. □

Remark 2.9. Looking at the proof of the previous lemma and paying attention to the arguments used, we can easily convince ourselves that if (μ_n^1) and (μ_n^2) are two sequences in $\mathcal{X}(\Omega)$ such that along subsequences indexed by (n_k) we have $\mu_{n_k}^1 \in X_{n_k}^1$, $\mu_{n_k}^2 \in X_{n_k}^2$ and $\mu_{n_k}^1 + \mu_{n_k}^2 \in X_{n_k}$ for every k, then estimates analogous to (2.41)-(2.45) hold for the corresponding subsequences. Moreover, if $\mu_n^1 \to \mu^1$ and $\mu_n^2 \to \mu^2$ in $\mathcal{X}(\Omega)$ as $n \to \infty$, then we can also conclude that $\mathcal{G}_{n_k}(\mu_{n_k}^1, \mu_{n_k}^2) \to \mathcal{G}(\mu^1, \mu^2)$, as $k \to \infty$.

2.3 The proof

We are now ready to present the proof of Theorem 2.1. By definition, we have to show that the *Liminf inequality* and the *Limsup inequality* hold.

Liminf inequality. *For every $\mu^1, \mu^2 \in \mathcal{X}(\Omega)$ and for every sequences (μ_n^1) and (μ_n^2) in $\mathcal{X}(\Omega)$ such that $\mu_n^1 \to \mu^1$ and $\mu_n^2 \to \mu^2$ in $\mathcal{X}(\Omega)$, as $n \to \infty$, we have*

$$\mathcal{E}(\mu^1, \mu^2) \le \liminf_n \mathcal{E}_n(\mu_n^1, \mu_n^2). \tag{2.77}$$

Proof. Note that if $\liminf_n \mathcal{E}_n(\mu_n^1, \mu_n^2) = +\infty$, then the inequality is trivially true, thus we assume that $\liminf_n \mathcal{E}_n(\mu_n^1, \mu_n^2) < +\infty$. In this case, we choose a subsequence along which the limit is equal to the liminf, that is, $\lim_k \mathcal{E}_{n_k}(\mu_{n_k}^1, \mu_{n_k}^2) = \liminf_n \mathcal{E}_n(\mu_n^1, \mu_n^2)$. Then, for $k \gg 1$, we have $\mathcal{E}_{n_k}(\mu_{n_k}^1, \mu_{n_k}^2) < +\infty$, which implies that $\mu_{n_k}^1 \in X_{n_k}^1$, $\mu_{n_k}^2 \in X_{n_k}^2$ and $\mu_{n_k}^1 + \mu_{n_k}^2 \in X_{n_k}$. Since, by Lemma 2.8 and Remark 2.9, we have $\liminf_n \mathcal{G}_n(\mu_n^1, \mu_n^2) = \lim_k \mathcal{G}_{n_k}(\mu_{n_k}^1, \mu_{n_k}^2) = \mathcal{G}(\mu^1, \mu^2)$, we only need to prove that

$$\mathcal{F}(\mu^1, \mu^2) \le \lim_k \mathcal{F}_{n_k}(\mu_{n_k}^1, \mu_{n_k}^2) = \liminf_n \mathcal{F}_n(\mu_n^1, \mu_n^2). \tag{2.78}$$

Again by Lemma 2.8 and Remark 2.9, we can use (2.41)-(2.43) to obtain that

$$\begin{aligned} \lim_k \mathcal{F}_{n_k}(\mu_{n_k}^1, \mu_{n_k}^2) \ge{} & \liminf_k \frac{1}{2} \iint_{\Omega\times\Omega} V_1 \, \mathrm{d}(\mu_{n_k}^1 \boxtimes \mu_{n_k}^1) \\ & + \liminf_k \frac{1}{2} \iint_{\Omega\times\Omega} V_2 \, \mathrm{d}(\mu_{n_k}^2 \boxtimes \mu_{n_k}^2) \\ & + \liminf_k \iint_{\Omega\times\Omega} V_{1,2} \, \mathrm{d}(\mu_{n_k}^1 \otimes \mu_{n_k}^2). \end{aligned} \tag{2.79}$$

For the first limit at the right hand side, we take $M > 0$ and we set $V_1^M = V_1 \wedge M$. Clearly V_1^M is continuous on $\Omega \times \Omega$. Moreover, by Remark 2.4, it is also bounded on $\overline{\Omega}_0 \times \overline{\Omega}_0$. Thus, for $\mu_{n_k}^1 = \frac{1}{n_k} \sum_{i=1}^{N_{n_k}} \delta_{\mathbf{y}_i^{n_k}}$, we have

$$\begin{aligned} \liminf_k \iint_{\Omega\times\Omega} V_1 \, \mathrm{d}(\mu_{n_k}^1 \boxtimes \mu_{n_k}^1) &\ge \lim_k \iint_{\Omega\times\Omega} V_1^M \, \mathrm{d}(\mu_{n_k}^1 \boxtimes \mu_{n_k}^1) \\ &= \lim_k \left\{ \frac{1}{n_k^2} \sum_{i,j=1}^{N_{n_k}} V_1^M(\mathbf{y}_i^{n_k}, \mathbf{y}_j^{n_k}) - \frac{1}{n_k^2} \sum_{i=1}^{N_{n_k}} V_1^M(\mathbf{y}_i^{n_k}, \mathbf{y}_i^{n_k}) \right\} \\ &= \lim_k \left\{ \iint_{\Omega\times\Omega} V_1^M \, \mathrm{d}(\mu_{n_k}^1 \otimes \mu_{n_k}^1) - \frac{M}{n_k} \frac{N_{n_k}}{n_k} \right\} \\ &= \iint_{\Omega\times\Omega} V_1^M \, \mathrm{d}(\mu^1 \otimes \mu^1), \end{aligned} \tag{2.80}$$

where we have used that $\mu^1_{n_k} \otimes \mu^1_{n_k} \overset{*}{\rightharpoonup} \mu^1 \otimes \mu^1$ in $(C_b(\Omega \times \Omega))'$, as $k \to \infty$. Note that $V_1^M \nearrow V_1$ pointwise in $\Omega \times \Omega$, as $M \to +\infty$, and $V_1^M \geq -C$ on $\overline{\Omega}_0 \times \overline{\Omega}_0$, hence we can apply the Monotone Convergence Theorem to obtain

$$\iint_{\Omega\times\Omega} V_1^M \,\mathrm{d}(\mu^1 \otimes \mu^1) \to \iint_{\Omega\times\Omega} V_1 \,\mathrm{d}(\mu^1 \otimes \mu^1), \quad \text{as } M \to +\infty.$$

From (2.80) we conclude that

$$\liminf_k \iint_{\Omega\times\Omega} V_1 \,\mathrm{d}(\mu^1_{n_k} \boxtimes \mu^1_{n_k}) \geq \iint_{\Omega\times\Omega} V_1 \,\mathrm{d}(\mu^1 \otimes \mu^1). \tag{2.81}$$

The other two integrals in (2.79) can be treated in an analogous way leading to

$$\lim_k \iint_{\Omega\times\Omega} V_2 \,\mathrm{d}(\mu^2_{n_k} \boxtimes \mu^2_{n_k}) \geq \iint_{\Omega\times\Omega} V_2 \,\mathrm{d}(\mu^2 \otimes \mu^2) \tag{2.82}$$

and

$$\lim_k \iint_{\Omega\times\Omega} V_{1,2} \,\mathrm{d}(\mu^1_{n_k} \otimes \mu^2_{n_k}) \geq \iint_{\Omega\times\Omega} V_{1,2} \,\mathrm{d}(\mu^1 \otimes \mu^2). \tag{2.83}$$

Hence, combining (2.79)-(2.83), we obtain (2.78). □

We now pass to the Limsup inequality. The proof given here is an adaptation of that presented in [21], where the case of a system of dislocations with a single Burgers vector was considered. Recall that, in Theorem 2.1, we assume (2.6). This hypothesis is going to be used here.

Limsup inequality. *For every $\mu^1, \mu^2 \in \mathcal{X}(\Omega)$ there exists two sequences (μ^1_n) and (μ^2_n) in $\mathcal{X}(\Omega)$ such that $\mu^1_n \to \mu^1$ and $\mu^2_n \to \mu^2$ in $\mathcal{X}(\Omega)$, as $n \to \infty$, and*

$$\mathcal{E}(\mu^1, \mu^2) \geq \limsup_n \mathcal{E}_n(\mu^1_n, \mu^2_n). \tag{2.84}$$

Proof. First of all, we can assume that $\mathcal{E}(\mu^1, \mu^2) < +\infty$, since otherwise the inequality is trivially satisfied. From $\mathcal{F}(\mu^1, \mu^2) < +\infty$ it easily follows that, defining the diagonal set $D = \{(x, x) : x \in \Omega\}$, we have $(\mu^1 \otimes \mu^1)(D) = (\mu^2 \otimes \mu^2)(D) = (\mu^1 \otimes \mu^2)(D) = 0$. Moreover, since $\mathcal{E}(\mu^1, \mu^2) < +\infty$, we know that $\mu^1(\Omega) = \mathsf{m}$ and $\mu^2(\Omega) = 1 - \mathsf{m}$. We follow the common strategy of first proving the limsup inequality for measures that belong to a particular class and then regain the general statement by an approximation procedure. Note that, once we construct two sequences of admissible measures converging to μ^1 and μ^2, respectively, then the convergence of $\mathcal{G}_n$ follows from Lemma 2.8. Therefore we only need to focus on the interaction term. It can be noticed that the interaction energy is not continuous with respect to narrow convergence. Moreover, given the singular character of the interaction potentials, the dislocations need to be suitably allocated while constructing the recovery sequences. The proof is divided into several steps.

Step 1 *(First approximation)* The construction that we are going to use applies to measures in $\mathcal{X}(\Omega)$ whose support is compactly contained in Ω_0. Since, in general, this is not the case for μ^1 and μ^2, we perform a first approximation by shrinking their supports. This will be done in such a way that the energy is controlled. Recalling (2.6), we consider the homothety ω_ϑ which is given by $\omega_\vartheta(\mathbf{x}) = (1 - \vartheta)\mathbf{x}_0 + \vartheta\mathbf{x}$, where $\mathbf{x}_0 \in \Omega_0$ and $0 < \vartheta < 1$. Then we define the measures $\mu^1_\vartheta = \omega^\sharp_\vartheta \mu^1$ and $\mu^2_\vartheta = \omega^\sharp_\vartheta \mu^2$, whose supports are contained in the closure of the set $\Omega_0^\vartheta = \omega_\vartheta(\Omega_0)$. Note that, by (2.6), we have $\Omega_0^\vartheta \subset\subset \Omega_0$. Moreover, we claim that

$$\mu_\vartheta^1 \to \mu^1 \text{ in } \mathcal{X}(\Omega), \qquad \mu_\vartheta^2 \to \mu^2 \text{ in } \mathcal{X}(\Omega), \quad \text{as } \vartheta \to 1^-, \tag{2.85}$$

and

$$\mathcal{F}(\mu_\vartheta^1, \mu_\vartheta^2) \to \mathcal{F}(\mu^1, \mu^2), \quad \text{as } \vartheta \to 1^-. \tag{2.86}$$

In order to prove (2.85), we take $v \in C_b(\Omega)$ and $i = 1, 2$. By definition of push-forward, we have that $\int_\Omega v(\mathbf{x})\,\mathrm{d}\mu_\vartheta^i(\mathbf{x}) = \int_\Omega v(\omega_\vartheta(\boldsymbol{\xi}))\,\mathrm{d}\mu^i(\boldsymbol{\xi})$. Clearly, $\omega_\vartheta(\boldsymbol{\xi}) \to \boldsymbol{\xi}$ for every $\boldsymbol{\xi} \in \Omega$, as $\vartheta \to 1^-$, thus, since v is bounded, we can apply the Dominated Convergence Theorem to obtain

$$\int_\Omega v(\mathbf{x})\,\mathrm{d}\mu_\vartheta^i(\mathbf{x}) \to \int_\Omega v(\mathbf{x})\,\mathrm{d}\mu^i(\mathbf{x}), \quad \text{as } \vartheta \to 1^-.$$

For the proof of (2.86), we start with the first term. We have

$$\iint_{\Omega\times\Omega} V_1(\mathbf{y},\mathbf{z})\,\mathrm{d}(\mu_\vartheta^1 \otimes \mu_\vartheta^1)(\mathbf{y},\mathbf{z}) = \iint_{\Omega\times\Omega} V_1(\omega_\vartheta(\boldsymbol{\xi}), \omega_\vartheta(\boldsymbol{\eta}))\,\mathrm{d}(\mu^1 \otimes \mu^1)(\boldsymbol{\xi},\boldsymbol{\eta}). \tag{2.87}$$

In order to apply the Dominated Convergence Theorem we need to find a domination for the integrand at the right hand side of (2.87). Using (2.7), we see that for $1/2 < \vartheta < 1$ and for $(\boldsymbol{\xi},\boldsymbol{\eta}) \in (\Omega \times \Omega) \setminus D$, i.e., $\mu^1 \otimes \mu^1$-a.e., we have

$$\begin{aligned} |V_1(\omega_\vartheta(\boldsymbol{\xi}), \omega_\vartheta(\boldsymbol{\eta}))| &\le C(1 + \log L - \log|\omega_\vartheta(\boldsymbol{\xi}) - \omega_\vartheta(\boldsymbol{\eta})|) \\ &\le C(1 + \log L - \log\vartheta - \log|\boldsymbol{\xi} - \boldsymbol{\eta}|) \\ &\le C(1 + \log L - \log|\boldsymbol{\xi} - \boldsymbol{\eta}|). \end{aligned}$$

Therefore it is enough to show that the function $(\mathbf{y},\mathbf{z}) \mapsto \log|\mathbf{y}-\mathbf{z}|$ is in $L^1(\Omega\times\Omega, \mu^1\otimes\mu^1)$. Consider an open set Ω' such that $\Omega_0 \subset\subset \Omega' \subset\subset \Omega$. By (2.8), there exist $R' > 0$ and $C' > 0$ such that for every $\mathbf{y},\mathbf{z} \in \Omega'$ with $0 < |\mathbf{y}-\mathbf{z}| < R'$ we have $V_1(\mathbf{y},\mathbf{z}) \ge C'(1 - \log|\mathbf{y}-\mathbf{z}|)$. Then

$$\begin{aligned} \int_\Omega \int_{B(\mathbf{z},R'\wedge 1)} |\log|\mathbf{y}-\mathbf{z}||\,\mathrm{d}\mu^1(\mathbf{y})\,\mathrm{d}\mu^1(\mathbf{z}) &= -\int_\Omega \int_{B(\mathbf{z},R'\wedge 1)} \log|\mathbf{y}-\mathbf{z}|\,\mathrm{d}\mu^1(\mathbf{y})\,\mathrm{d}\mu^1(\mathbf{z}) \\ &\le \frac{1}{C'} \int_\Omega \int_{B(\mathbf{z},R'\wedge 1)} V_1(\mathbf{y},\mathbf{z})\,\mathrm{d}\mu^1(\mathbf{y})\,\mathrm{d}\mu^1(\mathbf{z}) < +\infty. \end{aligned}$$

Clearly, this is sufficient to conclude that $\iint_{\Omega\times\Omega} |\log|\mathbf{y}-\mathbf{z}||\,\mathrm{d}(\mu^1\otimes\mu^1)(\mathbf{y},\mathbf{z}) < +\infty$, as we wanted. Hence we can pass to the limit in (2.87) and obtain

$$\iint_{\Omega\times\Omega} V_1(\mathbf{y},\mathbf{z})\,\mathrm{d}(\mu_\vartheta^1 \otimes \mu_\vartheta^1)(\mathbf{y},\mathbf{z}) \to \iint_{\Omega\times\Omega} V_1(\mathbf{y},\mathbf{z})\,\mathrm{d}(\mu^1 \otimes \mu^1)(\mathbf{y},\mathbf{z}), \quad \text{as } \vartheta \to 1^-.$$

The terms with V_2 and $V_{1,2}$ can be treated analogously.

Step 2 *(Second approximation)* We now present the geometric construction used to approximate μ_ϑ^1 and μ_ϑ^2. Here the fact that their supports are compactly contained in Ω_0 will be used. Assume that $0 < \vartheta < 1$ is fixed. We introduce a parameter $h > 0$ and we consider the family of half-open squares of side $4h$ given by $\widetilde{\mathcal{Q}}^h = \bigcup_{p,q\in\mathbb{Z}} \big(4h(p,q) + [0,4h) \times [0,4h)\big)$. We denote by $\{\widetilde{Q}_\ell^h : \ell = 1, \dots, \Lambda(\vartheta,h)\}$ the set of squares in $\widetilde{\mathcal{Q}}^h$ that intersect Ω_0^ϑ (see Figure 2.2), so that, for h so small that $d(\Omega_0^\vartheta, \Omega_0) > 6h > 4\sqrt{2}h$, we

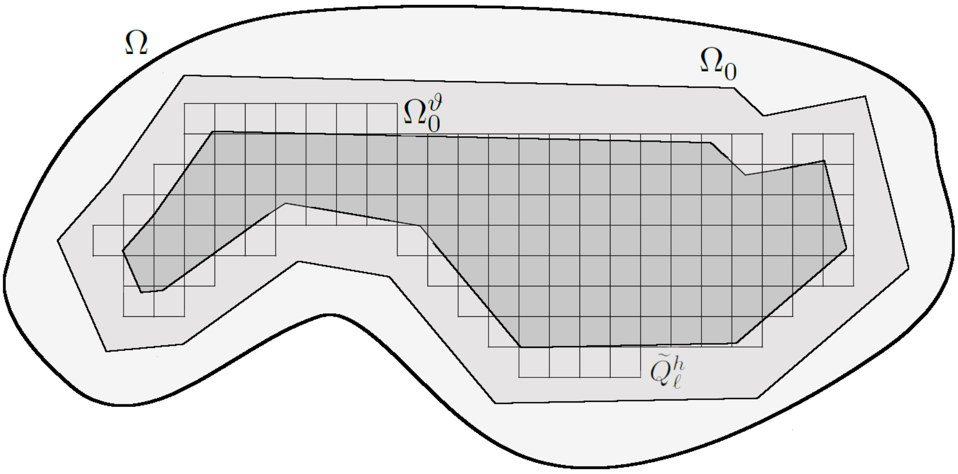


Figure 2.2: Ω_0^ϑ covered by the squares $\widetilde{Q}_\ell^h$.

have $\Omega_0^\vartheta \subset \bigcup_{\ell=1}^{\Lambda(\vartheta,h)} \widetilde{Q}_\ell^h \subset\subset \Omega_0$. Inside each $\widetilde{Q}_\ell^h$ we consider two smaller disjoint squares of side h given by $Q_\ell^h = 4h(p,q) + [0,h) \times [0,h)$ and $\widehat{Q}_\ell^h = 6h(p,q) + [0,h) \times [0,h)$, if $\widetilde{Q}_\ell^h = 4h(p,q) + [0,4h) \times [0,4h)$ (see Figure 2.3). The approximating measures are defined as measures supported in the union of the squares Q_ℓ^h and $\widehat{Q}_\ell^h$ obtained by distributing in each of those squares the masses $\mu_\vartheta^1(\widetilde{Q}_\ell^h)$ and $\mu_\vartheta^2(\widetilde{Q}_\ell^h)$ of the larger squares respectively in a uniform way. More precisely, we set

$$(2.88)\qquad \mu_{\vartheta,h}^1 = \sum_{\ell=1}^{\Lambda(\vartheta,h)} \frac{\mu_\vartheta^1(\widetilde{Q}_\ell^h)}{h^2} \mathscr{L}^2 \llcorner Q_\ell^h\,, \qquad \mu_{\vartheta,h}^2 = \sum_{\ell=1}^{\Lambda(\vartheta,h)} \frac{\mu_\vartheta^2(\widetilde{Q}_\ell^h)}{h^2} \mathscr{L}^2 \llcorner \widehat{Q}_\ell^h.$$

Clearly $\mathrm{supp}\mu_{\vartheta,h}^1 \subset \bigcup_{\ell=1}^{\Lambda(\vartheta,h)} \overline{Q}_\ell^h$ and $\mathrm{supp}\mu_{\vartheta,h}^2 \subset \bigcup_{\ell=1}^{\Lambda(\vartheta,h)} \overline{\widehat{Q}_\ell^h}$; moreover we easily see that $\mu_{\vartheta,h}^1(\widetilde{Q}_i^h) = \mu_\vartheta^1(\widetilde{Q}_i^h)$ and $\mu_{\vartheta,h}^2(\widetilde{Q}_i^h) = \mu_\vartheta^2(\widetilde{Q}_i^h)$. We claim that

$$(2.89)\qquad \mu_{\vartheta,h}^1 \to \mu_\vartheta^1 \text{ in } \mathcal{X}(\Omega), \qquad \mu_{\vartheta,h}^2 \to \mu_\vartheta^2 \text{ in } \mathcal{X}(\Omega), \quad \text{as } h \to 0^+.$$

To see this, we take $v \in C_b(\Omega)$. For every $\eta > 0$, by uniform continuity on $\overline{\Omega}_0$, there exists $\delta > 0$ such that $|v(\mathbf{x}) - v(\mathbf{y})| < \eta$ for any two points $\mathbf{x}, \mathbf{y} \in \overline{\Omega}_0$ with $|\mathbf{x} - \mathbf{y}| < \delta$. Then, using that $\mu_{\vartheta,h}^1(\widetilde{Q}_i^h) = \mu_\vartheta^1(\widetilde{Q}_i^h)$ and the Fubini Theorem, we compute

$$\begin{aligned}
\left| \int_\Omega v(\mathbf{x})\, \mathrm{d}\mu_{\vartheta,h}^1(\mathbf{x}) - \int_\Omega v(\mathbf{x})\, \mathrm{d}\mu_\vartheta^1(\mathbf{x}) \right| &\le \sum_{\ell=1}^{\Lambda(\vartheta,h)} \left| \int_{\widetilde{Q}_\ell^h} v(\mathbf{x})\, \mathrm{d}\mu_{\vartheta,h}^1(\mathbf{x}) - \int_{\widetilde{Q}_\ell^h} v(\mathbf{x})\, \mathrm{d}\mu_\vartheta^1(\mathbf{x}) \right| \\
&\le \sum_{\ell=1}^{\Lambda(\vartheta,h)} \left| \frac{1}{h^2} \int_{\widetilde{Q}_\ell^h} \int_{Q_\ell^h} v(\mathbf{x})\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mu_\vartheta^1(\mathbf{y}) - \frac{1}{h^2} \int_{Q_\ell^h} \int_{\widetilde{Q}_\ell^h} v(\mathbf{y})\, \mathrm{d}\mu_\vartheta^1(\mathbf{y})\, \mathrm{d}\mathbf{x} \right| \\
&\le \sum_{\ell=1}^{\Lambda(\vartheta,h)} \frac{1}{h^2} \int_{\widetilde{Q}_\ell^h} \int_{Q_\ell^h} |v(\mathbf{x}) - v(\mathbf{y})|\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mu_\vartheta^1(\mathbf{y}).
\end{aligned}$$

Therefore, for $4\sqrt{2}h < \delta$, we have

$$\left| \int_\Omega v\, \mathrm{d}\mu_{\vartheta,h}^1 - \int_\Omega v\, \mathrm{d}\mu_\vartheta^1 \right| \le \eta \sum_{\ell=1}^{\Lambda(\vartheta,h)} \mu_\vartheta^1(\widetilde{Q}_\ell^h) \le \eta$$

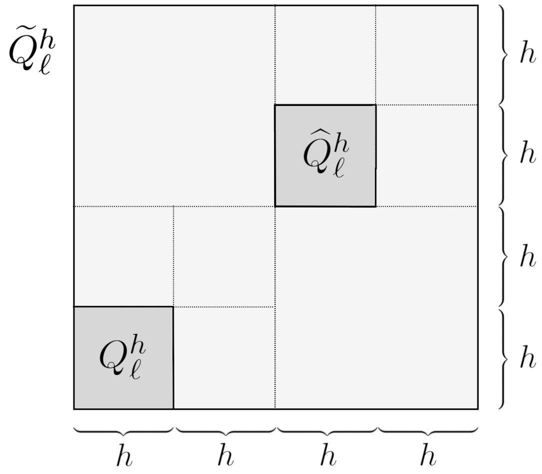


Figure 2.3: The square $\widetilde{Q}_\ell^h$ and the smaller squares Q_ℓ^h and $\widehat{Q}_\ell^h$.

which proves (2.89) for $\mu^1_{\vartheta,h}$. The same computation applies to $\mu^2_{\vartheta,h}$. For what concerns the energies we claim that

$$\mathcal{F}(\mu^1_\vartheta, \mu^2_\vartheta) \geq \limsup_{h\to 0^+} \mathcal{F}(\mu^1_{\vartheta,h}, \mu^2_{\vartheta,h}), \tag{2.90}$$

or, more precisely,

$$\iint_{\Omega\times\Omega} V_1 \,\mathrm{d}(\mu^1_\vartheta \otimes \mu^1_\vartheta) \geq \limsup_{h\to 0^+} \iint_{\Omega\times\Omega} V_1 \,\mathrm{d}(\mu^1_{\vartheta,h} \otimes \mu^1_{\vartheta,h}), \tag{2.91}$$

$$\iint_{\Omega\times\Omega} V_2 \,\mathrm{d}(\mu^2_\vartheta \otimes \mu^2_\vartheta) \geq \limsup_{h\to 0^+} \iint_{\Omega\times\Omega} V_2 \,\mathrm{d}(\mu^2_{\vartheta,h} \otimes \mu^2_{\vartheta,h}), \tag{2.92}$$

$$\iint_{\Omega\times\Omega} V_{1,2} \,\mathrm{d}(\mu^1_\vartheta \otimes \mu^2_\vartheta) \geq \limsup_{h\to 0^+} \iint_{\Omega\times\Omega} V_{1,2} \,\mathrm{d}(\mu^1_{\vartheta,h} \otimes \mu^2_{\vartheta,h}). \tag{2.93}$$

We begin with (2.91). As in the proof of the liminf inequality, we set $V_1^M = V_1 \wedge M$, where $M > 0$. We write

$$\begin{aligned}\iint_{\Omega\times\Omega} V_1 \,\mathrm{d}(\mu^1_{\vartheta,h} \otimes \mu^1_{\vartheta,h}) = &\iint_{\Omega\times\Omega} V_1^M \,\mathrm{d}(\mu^1_{\vartheta,h} \otimes \mu^1_{\vartheta,h}) \\ &+ \iint_{\Omega\times\Omega} (V_1 - V_1^M) \,\mathrm{d}(\mu^1_{\vartheta,h} \otimes \mu^1_{\vartheta,h}).\end{aligned} \tag{2.94}$$

Since V_1^M is continuous and bounded on $\overline{\Omega}_0 \times \overline{\Omega}_0$, using (2.89) we obtain that

$$\iint_{\Omega\times\Omega} V_1^M \,\mathrm{d}(\mu^1_{\vartheta,h} \otimes \mu^1_{\vartheta,h}) \to \iint_{\Omega\times\Omega} V_1^M \,\mathrm{d}(\mu^1_\vartheta \otimes \mu^1_\vartheta), \quad \text{as } h \to 0^+.$$

Moreover, we trivially have

$$\iint_{\Omega\times\Omega} V_1^M \,\mathrm{d}(\mu^1_\vartheta \otimes \mu^1_\vartheta) \leq \iint_{\Omega\times\Omega} V_1 \,\mathrm{d}(\mu^1_\vartheta \otimes \mu^1_\vartheta)$$

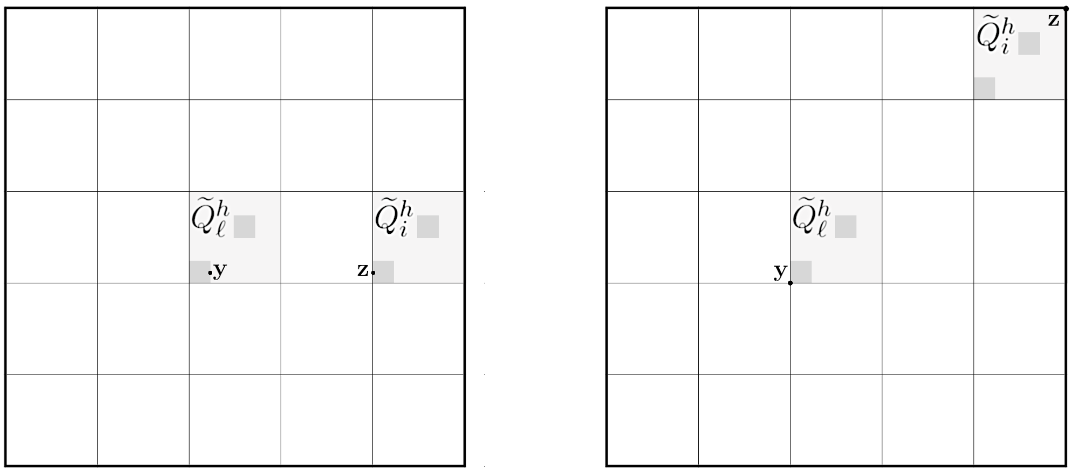


Figure 2.4: The "best" and the "worst" case in (2.98) and (2.99), for $p = 2$.

and we note that the right hand side is the desired upper bound. Hence, to conclude, we have to show that the second integral at the right hand side of (2.94) vanishes, as $h \to 0^+$. We recall that, by (2.7), for every $\mathbf{y}, \mathbf{z} \in \Omega$ with $\mathbf{y} \neq \mathbf{z}$ we have $|V_1(\mathbf{y},\mathbf{z})| \leq C(1+\log L - \log|\mathbf{y}-\mathbf{z}|)$. From this bound it follows that, if $V_1(\mathbf{y},\mathbf{z}) > M$, then $|\mathbf{y}-\mathbf{z}| < R_M$ where $R_M = \exp(1 + \log L - M/C)$. Note that $R_M \to 0^+$ as $M \to +\infty$. Moreover, if we consider an open set Ω' such that $\Omega_0 \subset\subset \Omega' \subset\subset \Omega$, by (2.8) there exist $C' > 0$ and $R' > 0$ such that $V_1(\mathbf{y},\mathbf{z}) \geq C'(1 - \log|\mathbf{y}-\mathbf{z}|)$ for any $\mathbf{y},\mathbf{z} \in \Omega'$ with $0 < |\mathbf{y}-\mathbf{z}| < R'$. In particular, when $|\mathbf{y}-\mathbf{z}| < R' \wedge e$, we have $V_1(\mathbf{y},\mathbf{z}) > 0$. Thus, we choose $M \gg 1$ such that $2R_M < R' \wedge e$ and $V_1(\cdot,\mathbf{z}) > 0$ on $B(\mathbf{z}, 2R_M)$ for every $\mathbf{z} \in \Omega'$. Hence, if we split the second integral in the right hand side of (2.94) as

$$\iint_{\Omega\times\Omega} (V_1 - V_1^M)\, \mathrm{d}(\mu^1_{\vartheta,h} \otimes \mu^1_{\vartheta,h})(\mathbf{y},\mathbf{z}) = \int_\Omega \int_{B(\mathbf{z},R_M)} (V_1 - V_1^M)\, \mathrm{d}(\mu^1_{\vartheta,h} \otimes \mu^1_{\vartheta,h})(\mathbf{y},\mathbf{z}) + \int_\Omega \int_{\Omega\setminus B(\mathbf{z},R_M)} (V_1 - V_1^M)\, \mathrm{d}(\mu^1_{\vartheta,h} \otimes \mu^1_{\vartheta,h})(\mathbf{y},\mathbf{z}),$$

then the second integral is equal to zero, since $|\mathbf{y}-\mathbf{z}| \geq R_M$ implies $V_1(\mathbf{y},\mathbf{z}) \leq M$, while the first can be bounded by $\int_\Omega \int_{B(\mathbf{z},R_M)} V_1(\mathbf{y},\mathbf{z})\, \mathrm{d}\mu^1_{\vartheta,h}(\mathbf{y})\, \mathrm{d}\mu^1_{\vartheta,h}(\mathbf{z})$, using the positivity of $V_1^M(\cdot,\mathbf{z})$. Therefore, in order to prove (2.91), it is sufficient to show that

$$\lim_{M\to+\infty} \limsup_{h\to 0^+} \int_\Omega \int_{B(\mathbf{z},R_M)} V_1(\mathbf{y},\mathbf{z})\, \mathrm{d}\mu^1_{\vartheta,h}(\mathbf{y})\, \mathrm{d}\mu^1_{\vartheta,h}(\mathbf{z}) = 0. \tag{2.95}$$

Using (2.88), we write

$$\begin{aligned} &\int_\Omega \int_{B(\mathbf{z},R_M)} V_1(\mathbf{y},\mathbf{z})\, \mathrm{d}\mu^1_{\vartheta,h}(\mathbf{y})\, \mathrm{d}\mu^1_{\vartheta,h}(\mathbf{z}) = \\ &\qquad = \sum_{\ell=1}^{\Lambda(\vartheta,h)} \sum_{i=1}^{\Lambda(\vartheta,h)} \frac{\mu^1_\vartheta(\widetilde{Q}^h_\ell)\, \mu^1_\vartheta(\widetilde{Q}^h_i)}{h^4} \int_{Q^h_\ell} \int_{Q^h_i \cap B(\mathbf{z},R_M)} V_1(\mathbf{y},\mathbf{z})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{z}. \end{aligned} \tag{2.96}$$

We set $P_{h,M} = \lfloor R_M/4h \rfloor + 1$. For $p = 1, \dots, P_{h,M}$ and $\ell = 1, \dots, \Lambda(\vartheta, h)$, we define

$$\mathcal{I}(\ell,p) = \{1 \le i \le \Lambda(\vartheta,h) : Q_i^h = Q_\ell^h + 4h(q,r),\ q,r \in \mathbb{Z},\ |q| \vee |r| = p\}$$

which is the set of indices of the squares that can be reached, starting from Q_ℓ^h, with p vertical (or horizontal) "jumps" of length $4h$ and at most p horizontal (or vertical) "jumps" of the same length. In this way, for every $\mathbf{z} \in Q_\ell^h$, we have $B(\mathbf{z}, R_M) \subset Q_\ell^h \cup \bigcup_{p=1}^{P_{h,M}} \bigcup_{i \in \mathcal{I}(\ell,p)} Q_i^h$ and the right hand side of (2.96) can be written as

$$\begin{aligned}
&\sum_{\ell=1}^{\Lambda(\vartheta,h)} \sum_{i=1}^{\Lambda(\vartheta,h)} \frac{\mu_\vartheta^1(\widetilde{Q}_\ell^h)\, \mu_\vartheta^1(\widetilde{Q}_i^h)}{h^4} \int_{Q_\ell^h} \int_{Q_i^h \cap B(\mathbf{z},R_M)} V_1(\mathbf{y},\mathbf{z})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{z} \\
&\qquad = \sum_{\ell=1}^{\Lambda(\vartheta,h)} \sum_{p=1}^{P_{h,M}} \sum_{i \in \mathcal{I}(\ell,p)} \frac{\mu_\vartheta^1(\widetilde{Q}_\ell^h)\, \mu_\vartheta^1(\widetilde{Q}_i^h)}{h^4} \int_{Q_\ell^h} \int_{Q_i^h \cap B(\mathbf{z},R_M)} V_1(\mathbf{y},\mathbf{z})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{z} \\
&\qquad + \sum_{\ell=1}^{\Lambda(\vartheta,h)} \frac{\mu_{\vartheta,h}^1(\widetilde{Q}_\ell^h)^2}{h^4} \int_{Q_\ell^h} \int_{Q_\ell^h \cap B(\mathbf{z},R_M)} V_1(\mathbf{y},\mathbf{z})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{z}.
\end{aligned} \tag{2.97}$$

Note that, given $\ell = 1, \dots, \Lambda(\vartheta,h)$, $p = 1, \dots, P_{h,M}$, $i \in \mathcal{I}(\ell,p)$, and given an open set Ω' with $\Omega_0 \subset\subset \Omega'$, the following hold (see Figure 2.4):

$$\begin{gathered}
\text{if } \mathbf{y} \in Q_\ell^h \text{ and } \mathbf{z} \in Q_i^h, \text{ then } |\mathbf{y}-\mathbf{z}| \ge (4p-1)h \text{ so that, by (2.7), we have} \\
|V_1(\mathbf{y},\mathbf{z})| \le C\Big(1 + \log L - \log((4p-1)h)\Big);
\end{gathered} \tag{2.98}$$

$$\begin{gathered}
\text{if } \mathbf{y} \in \widetilde{Q}_\ell^h \text{ and } \mathbf{z} \in \widetilde{Q}_i^h, \text{ then } |\mathbf{y}-\mathbf{z}| \le 4\sqrt{2}(p+1)h \le \sqrt{2}R_M + 8\sqrt{2}h, \text{ so that} \\
|\mathbf{y}-\mathbf{z}| < R' \text{ for } h < R'/(16\sqrt{2}) \text{ and } M \gg 1 \text{ such that } R_M < R'/(2\sqrt{2}) \text{ and,} \\
\text{by (2.8), we have } V_1(\mathbf{y},\mathbf{z}) \ge C'\Big(1 - \log\big(4\sqrt{2}(p+1)h\big)\Big).
\end{gathered} \tag{2.99}$$

Using these two facts, we can bound the integrals in the first term at the right hand side of (2.97) as follows:

$$\begin{aligned}
&\frac{\mu_\vartheta^1(\widetilde{Q}_\ell^h)\, \mu_\vartheta^1(\widetilde{Q}_i^h)}{h^4} \int_{Q_\ell^h} \int_{Q_i^h \cap B(\mathbf{z},R_M)} V_1(\mathbf{y},\mathbf{z})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{z} \\
&\le C\mu_\vartheta^1(\widetilde{Q}_\ell^h)\, \mu_\vartheta^1(\widetilde{Q}_i^h)\Big(1 + \log L - \log((4p-1)h)\Big) \\
&\le C\mu_\vartheta^1(\widetilde{Q}_\ell^h)\, \mu_\vartheta^1(\widetilde{Q}_i^h)\left(1 - \log(4\sqrt{2}(p+1)h) - \log\frac{(4p-1)h}{4\sqrt{2}(p+1)h} + \log L\right) \\
&\le C\mu_\vartheta^1(\widetilde{Q}_\ell^h)\, \mu_\vartheta^1(\widetilde{Q}_i^h)\left(\log\frac{4\sqrt{2}(p+1)L}{4p-1} + 1 - \log(4\sqrt{2}(p+1)h)\right) \\
&\le C\mu_\vartheta^1(\widetilde{Q}_\ell^h)\, \mu_\vartheta^1(\widetilde{Q}_i^h) + C\int_{\widetilde{Q}_\ell^h} \int_{\widetilde{Q}_i^h} V_1(\mathbf{y},\mathbf{z})\, \mathrm{d}\mu_\vartheta^1(\mathbf{y})\, \mathrm{d}\mu_\vartheta^1(\mathbf{z})
\end{aligned} \tag{2.100}$$

for $\ell = 1, \dots, \Lambda(\vartheta,h)$, $p = 1, \dots, P_{h,M}$ and $i \in \mathcal{I}(\ell,p)$. Note that here we used that

$$\frac{4\sqrt{2}(p+1)hL}{4p-1} \le \frac{8\sqrt{2}}{3}$$

for $p = 1, \dots, P_{h,M}$, since this quantity is decreasing for $p \ge 1$. For the integrals in the second term at the right hand side of (2.97), we choose $h \ll 1$, so that $2h < R_M$ and then

$Q_\ell^h \subset B(\mathbf{z}, \sqrt{2}h) \subset B(\mathbf{z}, 2h) \subset B(\mathbf{z}, R_M)$. Hence, using (2.7) and (2.99), and integrating by parts in polar coordinates, we have

$$
\begin{aligned}
&\frac{\mu^1_{\vartheta,h}(\widetilde{Q}_\ell^h)^2}{h^4}\int_{Q_\ell^h}\int_{Q_\ell^h\cap B(\mathbf{z},R_M)} V_1(\mathbf{y},\mathbf{z})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{z}\\
&\quad= \frac{\mu^1_{\vartheta,h}(\widetilde{Q}_\ell^h)^2}{h^4}\int_{Q_\ell^h}\int_{Q_\ell^h} V_1(\mathbf{y},\mathbf{z})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{z}\\
&\quad\le \frac{\mu^1_{\vartheta,h}(\widetilde{Q}_\ell^h)^2}{h^4}\int_{Q_\ell^h}\int_{B(\mathbf{z},\sqrt{2}h)} V_1(\mathbf{y},\mathbf{z})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{z}\\
&\quad\le C\frac{\mu^1_{\vartheta,h}(\widetilde{Q}_\ell^h)^2}{h^4}\int_{Q_\ell^h}\int_{B(\mathbf{z},\sqrt{2}h)} (1+\log L-\log|\mathbf{y}-\mathbf{z}|)\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{z}\\
&\quad\le C\mu^1_{\vartheta,h}(\widetilde{Q}_\ell^h)^2\left\{1-\frac{1}{h^2}\int_0^{\sqrt{2}h}\varrho\log\varrho\,\mathrm{d}\varrho\right\}\\
&\quad= C\mu^1_{\vartheta,h}(\widetilde{Q}_\ell^h)^2\left\{1-\frac{1}{h^2}\left[\frac{\varrho^2}{2}\log\varrho\right]_0^{\sqrt{2}h}-\int_0^{\sqrt{2}h}\frac{\varrho}{2}\,\mathrm{d}\varrho\right\}\\
&\quad= C\mu^1_{\vartheta,h}(\widetilde{Q}_\ell^h)^2 + C\mu^1_{\vartheta,h}(\widetilde{Q}_\ell^h)^2\left(1-\log(\sqrt{2}h)\right)\\
&\quad\le C\mu^1_{\vartheta,h}(\widetilde{Q}_\ell^h)^2 + C\mu^1_{\vartheta,h}(\widetilde{Q}_\ell^h)^2\left(1-\log(4\sqrt{2}h)\right)\\
&\quad\le C\mu^1_{\vartheta,h}(\widetilde{Q}_\ell^h)^2 + C\iint_{\widetilde{Q}_\ell^h\times\widetilde{Q}_\ell^h} V_1\,\mathrm{d}(\mu^1_\vartheta\otimes\mu^1_\vartheta)
\end{aligned}
\tag{2.101}
$$

for $\ell = 1, \dots, \Lambda(\vartheta, h)$. Note that, in the second line from below, we have used that (2.99) also holds for $p = 0$, that is, for every $\mathbf{y}, \mathbf{z} \in \widetilde{Q}_\ell^h$ we have $|\mathbf{y}-\mathbf{z}| \le 4\sqrt{2}h$, so that, by (2.8), $V_1(\mathbf{y}, \mathbf{z}) \ge C'(1 - \log(4\sqrt{2}h))$.
Hence, combining (2.97), (2.100) and (2.101), we deduce that

$$
\begin{aligned}
&\sum_{\ell=1}^{\Lambda(\vartheta,h)}\sum_{i=1}^{\Lambda(\vartheta,h)}\frac{\mu^1_\vartheta(\widetilde{Q}_\ell^h)\,\mu^1_\vartheta(\widetilde{Q}_i^h)}{h^4}\int_{Q_\ell^h}\int_{Q_i^h\cap B(\mathbf{z},R_M)} V_1(\mathbf{y},\mathbf{z})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{z}\\
&\quad\le \sum_{\ell=1}^{\Lambda(\vartheta,h)}\sum_{p=1}^{P_{h,M}}\sum_{i\in\mathcal{I}(\ell,p)}\left\{C\mu^1_\vartheta(\widetilde{Q}_\ell^h)\mu^1_\vartheta(\widetilde{Q}_i^h)+C\int_{\widetilde{Q}_\ell^h}\int_{\widetilde{Q}_i^h} V_1(\mathbf{y},\mathbf{z})\,\mathrm{d}\mu^1_\vartheta(\mathbf{y})\,\mathrm{d}\mu^1_\vartheta(\mathbf{z})\right\}\\
&\quad+\sum_{\ell=1}^{\Lambda(\vartheta,h)}\left\{C\mu^1_\vartheta(\widetilde{Q}_\ell^h)^2+C\int_{\widetilde{Q}_\ell^h}\int_{\widetilde{Q}_\ell^h} V_1(\mathbf{y},\mathbf{z})\,\mathrm{d}\mu^1_\vartheta(\mathbf{y})\,\mathrm{d}\mu^1_\vartheta(\mathbf{z})\right\}.
\end{aligned}
\tag{2.102}
$$

Note that, for a given $\mathbf{z} \in \widetilde{Q}_\ell^h$, we have $\widetilde{Q}_\ell^h \cup \bigcup_{p=1}^{P_{h,m}} \bigcup_{i\in\mathcal{I}(\ell,p)} \widetilde{Q}_i^h \subset B(\mathbf{z}, 2R_M)$ for $h \ll 1$. Indeed, for every $\mathbf{y} \in \widetilde{Q}_i^h$ with $i \in \mathcal{I}(\ell, p)$ and $p = 1, \dots, P_{h,M}$, we have

$$
|\mathbf{y}-\mathbf{z}| \le 4\sqrt{2}(P_{h,M}+1)h \le \sqrt{2}R_M + 8\sqrt{2}h < 2R_M, \quad \text{for } h < \frac{1}{4}\left(\frac{1}{\sqrt{2}}-\frac{1}{2}\right)R_M.
$$

Therefore, from (2.102), using the positivity of $V_1(\cdot, \mathbf{z})$ on $B(\mathbf{z}, 2R_M)$, we obtain

$$
\begin{aligned}
&\limsup_{h\to0^+}\sum_{\ell=1}^{\Lambda(\vartheta,h)}\sum_{i=1}^{\Lambda(\vartheta,h)}\frac{\mu^1_\vartheta(\widetilde{Q}_\ell^h)\,\mu^1_\vartheta(\widetilde{Q}_i^h)}{h^4}\int_{Q_\ell^h}\int_{Q_i^h\cap B(\mathbf{z},R_M)} V_1(\mathbf{y},\mathbf{z})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{z}\\
&\quad\le C\int_\Omega \mu^1_\vartheta(B(\mathbf{z},2R_M))\,\mathrm{d}\mu^1_\vartheta(\mathbf{z}) + C\int_\Omega\int_{B(\mathbf{z},2R_M)} V_1(\mathbf{y},\mathbf{z})\,\mathrm{d}\mu^1_\vartheta(\mathbf{y})\,\mathrm{d}\mu^1_\vartheta(\mathbf{z}).
\end{aligned}
\tag{2.103}
$$

Finally, using the Dominated Convergence Theorem and the fact that the integral of V_1 with respect to $\mu^1_\vartheta \otimes \mu^1_\vartheta$ is bounded, it is immediate to see that the last two integrals go to zero, as $M \to +\infty$. This proves (2.95) and, in turn, (2.91).

The proof of (2.92) is the same, while that of (2.93) is slightly simpler. By (2.7), we have that there exists $\widetilde{R}_M > 0$ such that $V_{1,2}(\mathbf{y},\mathbf{z}) > M$ implies $|\mathbf{y}-\mathbf{z}| < \widetilde{R}_M$ for every $\mathbf{y},\mathbf{z} \in \Omega$ with $\mathbf{y} \neq \mathbf{z}$. Thus we define similarly the quantity $\widetilde{P}_{h,M} = \lfloor \widetilde{R}_M/4h \rfloor + 1$ and the set of indices

$$\widetilde{\mathcal{I}}(\ell,p) = \{1 \le i \le \Lambda(\vartheta,h) : Q^h_i = \widehat{Q}^h_\ell + 2h(q,r),\ q,r \in \mathbb{Z},\ |q| \vee |r| = p\}$$

for $\ell = 1,\dots,\Lambda(\vartheta,h)$ and $p = 1,\dots,\widetilde{P}_{h,M}$. First, we split the integral at the right hand side of (2.93) as in (2.94) and, with the same argument used there, we see that it is sufficient to prove that

$$\lim_{M\to+\infty} \limsup_{h\to 0^+} \int_\Omega \int_{B(\mathbf{z},\widetilde{R}_M)} V_{1,2}(\mathbf{y},\mathbf{z})\, \mathrm{d}\mu^1_{\vartheta,h}(\mathbf{y})\, \mathrm{d}\mu^2_{\vartheta,h}(\mathbf{z}) = 0. \tag{2.104}$$

We have

$$\begin{aligned} &\int_{B(\mathbf{z},\widetilde{R}_M)} V_{1,2}(\mathbf{y},\mathbf{z})\, \mathrm{d}\mu^1_{\vartheta,h}(\mathbf{y})\, \mathrm{d}\mu^2_{\vartheta,h}(\mathbf{z}) \\ &\qquad = \sum_{\ell=1}^{\Lambda(\vartheta,h)} \sum_{p=0}^{\widetilde{P}_{h,M}} \sum_{i\in\widetilde{\mathcal{I}}(\ell,p)} \frac{\mu^1_\vartheta(\widetilde{Q}^h_i)\, \mu^2_\vartheta(\widetilde{Q}^h_\ell)}{h^4} \int_{\widehat{Q}^h_\ell} \int_{Q^h_i \cap B(z,\widetilde{R}_M)} V_{1,2}(\mathbf{y},\mathbf{z})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{z}. \end{aligned} \tag{2.105}$$

Note that, if $\mathbf{y} \in Q^h_i$ and $\mathbf{z} \in \widehat{Q}^h_\ell$, then we always have $|\mathbf{y}-\mathbf{z}| \ge \sqrt{2}h$; so, there is no need to distinguish the two cases as in (2.97). By (2.7), we have $V_{1,2}(\mathbf{y},\mathbf{z}) \le C(1+\log L - \log(\sqrt{2}h))$ for $\mathbf{y} \in Q^h_i$ and $\mathbf{z} \in \widehat{Q}^h_\ell$. Using this bound, we can argue as in (2.100) to conclude that

$$\begin{aligned} &\frac{\mu^1_\vartheta(\widetilde{Q}^h_i)\, \mu^2_\vartheta(\widetilde{Q}^h_\ell)}{h^4} \int_{\widehat{Q}^h_\ell} \int_{Q^h_i \cap B(z,\widetilde{R}_M)} V_{1,2}(\mathbf{y},\mathbf{z})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{z} \\ &\qquad \le C\mu^1_\vartheta(\widetilde{Q}^h_i)\, \mu^2_\vartheta(\widetilde{Q}^h_\ell) + C \int_{\widetilde{Q}^h_\ell} \int_{\widetilde{Q}^h_i} V_{1,2}(\mathbf{y},\mathbf{z})\, \mathrm{d}\mu^1_\vartheta(\mathbf{y})\, \mathrm{d}\mu^2_\vartheta(\mathbf{z}). \end{aligned} \tag{2.106}$$

Hence, if we sum up all the terms in (2.105) and we argue as in (2.102), we obtain

$$\begin{aligned} &\limsup_{h\to 0^+} \sum_{\ell=1}^{\Lambda(\vartheta,h)} \sum_{p=0}^{\widetilde{P}_{h,M}} \sum_{i\in\widetilde{\mathcal{I}}(\ell,p)} \frac{\mu^1_\vartheta(\widetilde{Q}^h_i)\, \mu^2_\vartheta(\widetilde{Q}^h_\ell)}{h^4} \int_{\widehat{Q}^h_\ell} \int_{Q^h_i \cap B(z,\widetilde{R}_M)} V_{1,2}(\mathbf{y},\mathbf{z})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{z} \\ &\qquad \le C \int_\Omega \mu^1_\vartheta(B(\mathbf{z},2\widetilde{R}_M))\, \mathrm{d}\mu^2_\vartheta(\mathbf{z}) + C \int_\Omega \int_{B(\mathbf{z},2\widetilde{R}_M)} V_{1,2}(\mathbf{y},\mathbf{z})\, \mathrm{d}\mu^1_\vartheta(\mathbf{y})\, \mathrm{d}\mu^2_\vartheta(\mathbf{z}), \end{aligned}$$

which tends to zero, as $M \to +\infty$. This proves (2.104) and, in turn, (2.90).

Step 3 *(Construction of the approximating measures)* We are now able to provide an approximation for the original measures μ^1 and μ^2 by means of a diagonal argument. Consider a sequence (ϑ_k) with $0 < \vartheta_k < 1$ such that $\vartheta_k \to 1^-$, as $k \to \infty$. By Step 1 we know that $\mu^1_{\vartheta_k} \to \mu^1$ and $\mu^2_{\vartheta_k} \to \mu^2$ in $\mathcal{X}(\Omega)$, as $k \to \infty$, and that $\mathcal{F}(\mu^1_{\vartheta_k}, \mu^2_{\vartheta_k}) \to \mathcal{F}(\mu^1,\mu^2)$, as $k \to \infty$. Moreover, by Step 2, for every k we have that

$$\mu^1_{\vartheta_k,h} \to \mu^1_{\vartheta_k} \quad \text{in } \mathcal{X}(\Omega), \qquad \mu^2_{\vartheta_k,h} \to \mu^2_{\vartheta_k} \quad \text{in } \mathcal{X}(\Omega), \quad \text{as } h \to 0^+, \tag{2.107}$$

and

$$\limsup_{h\to 0^+} \mathcal{F}(\mu^1_{\vartheta_k,h},\mu^2_{\vartheta_k,h}) \le \mathcal{F}(\mu^1_{\vartheta_k},\mu^2_{\vartheta_k}). \tag{2.108}$$

We denote by d a distance that induces the topology of $\mathcal{X}(\Omega)$. By (2.107) for every k we can find $h^1_k, h^2_k > 0$ such that for every $h \le h^1_k$ we have $d(\mu^1_{\vartheta_k,h},\mu^1_{\vartheta_k}) < 1/k$ and for every $h \le h^2_k$ we have $d(\mu^2_{\vartheta_k,h},\mu^2_{\vartheta_k}) < 1/k$. Moreover, by (2.108), there exists $h^0_k > 0$ such that, for any $h \le h^0_k$, we have $\mathcal{F}(\mu^1_{\vartheta_k,h},\mu^2_{\vartheta_k,h}) \le \mathcal{F}(\mu^1_{\vartheta_k},\mu^2_{\vartheta_k}) + 1/k$. Hence, for every k, we set $h_k = \min\{h^1_k, h^2_k, h^0_k, 1/k\}$, so that $h_k \to 0^+$, as $k\to\infty$. Then we define $\mu^1_k = \mu^1_{\vartheta_k,h_k}$ and $\mu^2_k = \mu^2_{\vartheta_k,h_k}$, more explicitly

$$\mu^1_k = \sum_{\ell=1}^{\Lambda_k} \frac{\mu^1_{\vartheta_k}(\widetilde{Q}^{h_k}_\ell)}{h_k^2}\mathscr{L}^2 \llcorner Q^{h_k}_\ell, \qquad \mu^2_k = \sum_{\ell=1}^{\Lambda_k} \frac{\mu^2_{\vartheta_k}(\widetilde{Q}^{h_k}_\ell)}{h_k^2}\mathscr{L}^2 \llcorner \widehat{Q}^{h_k}_\ell, \tag{2.109}$$

where we set $\Lambda_k = \Lambda(\vartheta_k, h_k)$. We claim that

$$\mu^1_k \to \mu^1 \quad \text{in } \mathcal{X}(\Omega), \qquad \mu^2_k \to \mu^2 \quad \text{in } \mathcal{X}(\Omega), \qquad \text{as } k\to\infty, \tag{2.110}$$

and

$$\limsup_{k} \mathcal{F}(\mu^1_k,\mu^2_k) \le \mathcal{F}(\mu^1,\mu^2). \tag{2.111}$$

To see (2.110), it is sufficient to use the triangle inequality. Indeed, for every k, we have $d(\mu^i_k,\mu^i) \le d(\mu^i_{\vartheta_k,h_k},\mu^i_{\vartheta_k}) + d(\mu^i_{\vartheta_k},\mu^i) < 1/k + d(\mu^i_{\vartheta_k},\mu^i)$ and the right hand side goes to zero, as $k\to\infty$. Equation (2.111) is also immediate since, for every k, we have $\mathcal{F}(\mu^1_k,\mu^2_k) = \mathcal{F}(\mu^1_{\vartheta_k,h_k},\mu^2_{\vartheta_k,h_k}) \le \mathcal{F}(\mu^1_{\vartheta_k},\mu^2_{\vartheta_k}) + 1/k$ and thus,

$$\limsup_k \mathcal{F}(\mu^1_k,\mu^2_k) \le \lim_k \mathcal{F}(\mu^1_{\vartheta_k},\mu^2_{\vartheta_k}) = \mathcal{F}(\mu^1,\mu^2).$$

Step 4 *(Construction of the recovery sequence)* We now construct the recovery sequences for the approximating measures μ^1_k and μ^2_k. These sequences have to be admissible in the sense that they have to contain the right number of dislocations that, in turn, need to satisfy the well-separation hypothesis. We fix k and for each positive integer n, representing the total number of defects, we look for two measures of the form $\frac{1}{n}\sum_{i=1}^{N_n}\delta_{\mathbf{y}^i_n}$ and $\frac{1}{n}\sum_{i=1}^{M_n}\delta_{\mathbf{z}^i_n}$. The idea is to allocate almost $\mu^1_k(\widetilde{Q}^{h_k}_\ell)N_n$ dislocations in each square $Q^{h_k}_\ell$, so that the measure $\frac{1}{n}\sum_{i=1}^{N_n}\delta_{\mathbf{y}^i_n}$ assigns to $Q^{h_k}_\ell$ almost the same mass as μ^1_k. More precisely, we recall that $\mu^1_k(\widetilde{Q}^{h_k}_\ell) = \mu^1_{\vartheta_k}(\widetilde{Q}^{h_k}_\ell)$ and $\mu^2_k(\widetilde{Q}^{h_k}_\ell) = \mu^2_{\vartheta_k}(\widetilde{Q}^{h_k}_\ell)$ and we set

$$\alpha^n_{k,\ell} = \left\lfloor \sqrt{\frac{N_n}{\mathsf{m}}\,\mu^1_{\vartheta_k}(\widetilde{Q}^{h_k}_\ell)} \right\rfloor, \qquad \beta^n_{k,\ell} = \left\lfloor \sqrt{\frac{M_n}{1-\mathsf{m}}\,\mu^2_{\vartheta_k}(\widetilde{Q}^{h_k}_\ell)} \right\rfloor. \tag{2.112}$$

These are clearly two non negative integers, and satisfy

$$\begin{gathered} \alpha^n_{k,\ell} \to \infty, \quad \text{as } n\to\infty, \quad \text{if } \mu^1_k(\widetilde{Q}^{h_k}_\ell) > 0, \\ \beta^n_{k,\ell} \to \infty, \quad \text{as } n\to\infty, \quad \text{if } \mu^2_k(\widetilde{Q}^{h_k}_\ell) > 0, \\ \frac{(\alpha^n_{k,\ell})^2}{n} \to \mu^1_{\vartheta_k}(\widetilde{Q}^{h_k}_\ell), \quad \frac{(\beta^n_{k,\ell})^2}{n} \to \mu^2_{\vartheta_k}(\widetilde{Q}^{h_k}_\ell), \quad \text{as } n\to\infty. \end{gathered} \tag{2.113}$$

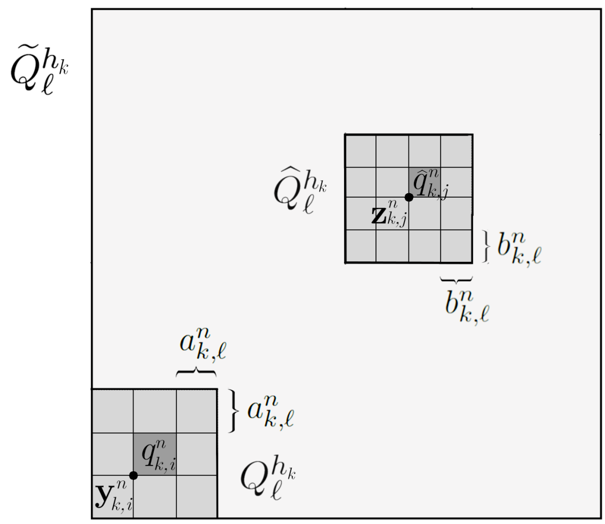


Figure 2.5: The smaller squares $q_{k,i}^n$ and $\hat{q}_{k,j}^n$ (here $\alpha_{k,\ell}^n = 3$ and $\beta_{k,\ell}^n = 4$).

The first two claims in (2.113) are trivial. For the third, we recall that $t-1 < \lfloor t \rfloor \le t$ for any real number t. Hence $\sqrt{\frac{N_n}{\mathsf{m}}\,\mu^1_{\vartheta_k}(\widetilde{Q}_\ell^{h_k})} - 1 < \alpha_{k,\ell}^n \le \sqrt{\frac{N_n}{\mathsf{m}}\,\mu^1_{\vartheta_k}(\widetilde{Q}_\ell^{h_k})}$, so that

$$\frac{N_n}{n}\,\frac{\mu^1_{\vartheta_k}(\widetilde{Q}_\ell^{h_k})}{\mathsf{m}} - \frac{2}{n}\sqrt{\frac{N_n}{\mathsf{m}}\,\mu^1_{\vartheta_k}(\widetilde{Q}_\ell^{h_k})} + \frac{1}{n} < \frac{(\alpha_{k,\ell}^n)^2}{n} \le \frac{N_n}{n}\,\frac{\mu^1_{\vartheta_k}(\widetilde{Q}_\ell^{h_k})}{\mathsf{m}}$$

and passing to the limit, as $n \to \infty$, we get the claim. The fourth claim can be proved analogously. For every $\ell = 1, \ldots, \Lambda_k$ such that $\mu_k^1(\widetilde{Q}_\ell^{h_k}) > 0$ we set $a_{k,\ell}^n = h_k/\alpha_{k,\ell}^n$ and, for every $\ell = 1, \ldots, \Lambda_k$ such that $\mu_k^2(\widetilde{Q}_\ell^{h_k}) > 0$, we set $b_{k,\ell}^n = h_k/\beta_{k,\ell}^n$. We allocate the defects as follows. In every square $Q_\ell^{h_k}$ with $\mu_k^1(\widetilde{Q}_\ell^{h_k}) > 0$ we consider a square grid of side $a_{k,\ell}^n$. More precisely, if $Q_\ell^{h_k} = 4h_k(p,q) + [0,h_k) \times [0,h_k)$ with $p, q \in \mathbb{Z}$, we consider the family of points $\{4h(p,q) + a_{k,\ell}^n(r,s) : r, s = 0, \ldots, \alpha_{k,\ell}^n - 1\}$, that we denote by $\{\mathbf{y}_{k,i}^n : i \in I_{k,\ell}^n\}$, and the squares $q_{k,i}^n = \mathbf{y}_{k,i}^n + [0, a_{k,\ell}^n) \times [0, a_{k,\ell}^n)$ for $i \in I_{k,\ell}^n$. Thus $\mathbf{y}_{k,i}^n \in q_{k,i}^n$ and $Q_\ell^{h_k} = \bigcup_{i \in I_{k,\ell}^n} q_{k,i}^n$. Note that, for every $i, j \in I_{k,\ell}^n$ with $i \neq j$, by (2.1) we have that

$$|\mathbf{y}_{k,i}^n - \mathbf{y}_{k,j}^n| \ge a_{k,\ell}^n = \frac{h_k}{\alpha_{k,\ell}^n} \ge \frac{h_k}{\sqrt{\frac{N_n}{\mathsf{m}}\,\mu^1_{\vartheta_k}(\widetilde{Q}_\ell^{h_k})}} \ge \frac{h_k\sqrt{\mathsf{m}}}{\sqrt{n}} > r_n \tag{2.114}$$

for $n \gg 1$. Analogously, in every square $\widehat{Q}_\ell^{h_k}$ such that $\mu_k^2(\widetilde{Q}_\ell^{h_k}) > 0$, we consider a square grid of side $b_{k,\ell}^n$ and the corresponding nodes $\mathbf{z}_{k,j}^n$ and squares $\hat{q}_{k,j}^n$, where $j \in J_{k,\ell}^n$ and $J_{k,\ell}^n$ is a set of $(\beta_{k,\ell}^n)^2$ indices, so that $\mathbf{z}_{k,j}^n \in \hat{q}_{k,j}^n$ and $\widehat{Q}_\ell^{h_k} = \bigcup_{j \in J_{k,\ell}^n} q_{k,j}^n$ (see Figure 2.5). As in (2.114), by (2.1) we obtain that, for any $i, j \in J_{k,\ell}^n$ with $i \neq j$, we have

$$|\mathbf{z}_{k,i}^n - \mathbf{z}_{k,j}^n| > r_n \tag{2.115}$$

for $n \gg 1$. Moreover, if $i \in I_{k,\ell}^n$ and $j \in J_{k,m}^n$, for $n \gg 1$ we also have

$$|\mathbf{y}_{k,i}^n - \mathbf{z}_{k,j}^n| \ge \sqrt{2}h_k > r_n. \tag{2.116}$$

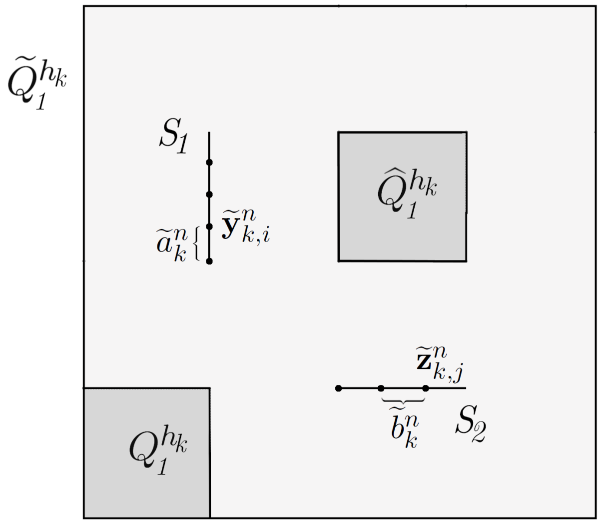


Figure 2.6: The segments S_1 and S_2 in $\widetilde{Q}_1^{h_k}$ (here $\widetilde{\mathsf{N}}_{n,k} = 4$ and $\widetilde{\mathsf{M}}_{n,k} = 3$).

The number of defects that we have placed in the first family of squares is equal to

$$\begin{aligned} \mathsf{N}_{n,k} &= \sum_{\ell=1}^{\Lambda_k} (\alpha_{k,\ell}^n)^2 \geq \sum_{\ell=1}^{\Lambda_k} \left(\sqrt{\frac{N_n}{\mathsf{m}} \, \mu_{\vartheta_k}^1 (\widetilde{Q}_\ell^{h_k})} - 1 \right)^2 = \\ &= N_n - 2\sqrt{\frac{N_n}{\mathsf{m}}} \sum_{\ell=1}^{\Lambda_k} \sqrt{\mu_{\vartheta_k}^1 (\widetilde{Q}_\ell^{h_k})} + \Lambda_k \geq N_n - 2\sqrt{\frac{n}{\mathsf{m}}} \Lambda_k. \end{aligned} \tag{2.117}$$

Thus, in order to construct an admissible measure corresponding to the Burgers vector $\mathbf{b}_1$, we still have to place a number of defects equal to

$$\widetilde{\mathsf{N}}_{n,k} = N_n - \mathsf{N}_{n,k} \leq 2\sqrt{\frac{n}{\mathsf{m}}} \Lambda_k. \tag{2.118}$$

To do this, we consider the cube $\widetilde{Q}_1^{h_k} = 4h_k(p_1, q_1) + [0, 4h_k) \times [0, 4h_k)$ where $p_1, q_1 \in \mathbb{Z}$. On the segment $S_1 = \{4h_k p_1 + h_k\} \times (4h_k q_1 + 2h_k + [0, h_k))$ we place the points $\widetilde{\mathbf{y}}_{k,i}^n$ with $i = 1, \dots, \widetilde{\mathsf{N}}_{n,k}$ in an equispaced way, that is at distance $\widetilde{a}_k^n = h_k / \widetilde{\mathsf{N}}_{n,k}$, starting from $\widetilde{\mathbf{y}}_{k,1}^n = (4h_k p_1 + h_k, 4h_k q_1 + 2h_k)$ (see Figure 2.6). Taking into account (2.118) and using (2.1), for any $i, j \in \{1, \dots, \widetilde{\mathsf{N}}_{n,k}\}$ with $i \neq j$, we have

$$|\widetilde{\mathbf{y}}_{k,i}^n - \widetilde{\mathbf{y}}_{k,j}^n| \geq \widetilde{a}_k^n = \frac{h_k}{\widetilde{\mathsf{N}}_{n,k}} \geq \frac{h_k \sqrt{\mathsf{m}}}{2\sqrt{n}\Lambda_k} > r_n \tag{2.119}$$

for $n \gg 1$. Clearly, if $i \in I_{k,\ell}^n$ and $j = 1, \dots, \widetilde{\mathsf{N}}_{n,k}$, then, for $n \gg 1$, we have

$$|\mathbf{y}_{k,i}^n - \widetilde{\mathbf{y}}_{k,j}^n| \geq h_k > r_n. \tag{2.120}$$

Analogously, the number of points placed in the second family of squares is

$$\mathsf{M}_{n,k} = \sum_{\ell=1}^{\Lambda_k} (\beta_{k,\ell}^n)^2 \geq M_n - 2\sqrt{\frac{n}{1-\mathsf{m}}} \Lambda_k,$$

hence we still have to place a number of defects equal to

$$\widetilde{\mathsf{M}}_{n,k} = M_n - \mathsf{M}_{n,k} \le 2\sqrt{\frac{n}{\mathsf{m}}}\Lambda_k. \tag{2.121}$$

Thus we place, along the segment $S_2 = (4h_k p_1 + 2h_k + [0, h_k)) \times \{4h_k q_1 + h_k\}$ in $\widetilde{Q}_1^{h_k}$, the points $\tilde{\mathbf{z}}_{k,j}^n$ with $j = 1, \dots, \widetilde{\mathsf{M}}_{n,k}$ at a distance $\widetilde{b}_k^n = h_k/\widetilde{\mathsf{M}}_{n,k}$, starting from $\tilde{\mathbf{z}}_{k,1}^n = (4h_k p_1 + 2h_k, 4h_k q_1 + h_k)$ (see Figure 2.6). As in (2.119), for any $i, j \in \{1, \dots, \widetilde{\mathsf{M}}_{n,k}\}$ with $i \neq j$, we have

$$|\tilde{\mathbf{z}}_{k,i}^n - \tilde{\mathbf{z}}_{k,j}^n| > r_n \tag{2.122}$$

and, as in (2.120), if $i \in J_{k,\ell}^n$ and $j = 1, \dots, \widetilde{\mathsf{M}}_{n,k}$, then we have

$$|\mathbf{z}_{k,i}^n - \tilde{\mathbf{z}}_{k,j}^n| > r_n \tag{2.123}$$

both for $n \gg 1$. Moreover, we trivially see that the distance between any point $\mathbf{y}_{k,i}^n$ and any point $\tilde{\mathbf{z}}_{k,j}^n$ and also between any point $\mathbf{z}_{k,i}^n$ and any point $\tilde{\mathbf{y}}_{k,j}^n$ is greater than or equal to r_n, for $n \gg 1$.
We define the empirical measures

$$\mu_{k,n}^1 = \frac{1}{n}\sum_{\ell=1}^{\Lambda_k}\sum_{i \in I_{k,\ell}^n} \delta_{\mathbf{y}_{k,i}^n} + \frac{1}{n}\sum_{i=1}^{\widetilde{\mathsf{N}}_{n,k}} \delta_{\tilde{\mathbf{y}}_{k,i}^n} \tag{2.124}$$

and

$$\mu_{k,n}^2 = \frac{1}{n}\sum_{\ell=1}^{\Lambda_k}\sum_{i \in J_{k,\ell}^n} \delta_{\mathbf{z}_{k,i}^n} + \frac{1}{n}\sum_{i=1}^{\widetilde{\mathsf{M}}_{n,k}} \delta_{\tilde{\mathbf{z}}_{k,i}^n} \tag{2.125}$$

where we set $I_{k,\ell}^n = \emptyset$ for ℓ such that $\mu_k^1(\widetilde{Q}_\ell^{h_k}) = 0$, and $J_{k,\ell}^n = \emptyset$ for ℓ such that $\mu_k^2(\widetilde{Q}_\ell^{h_k}) = 0$. It is clear that $\mu_{k,n}^1 \in X_n^1$ and $\mu_{k,n}^2 \in X_n^2$. Moreover, due to (2.114)-(2.116), (2.119), (2.120), (2.122) and (2.123), we have that $\mu_{k,n}^1 + \mu_{k,n}^2 \in X_n$. Therefore $(\mu_{k,n}^1)_n \subset \mathcal{X}(\Omega)$ and $(\mu_{k,n}^2)_n \subset \mathcal{X}(\Omega)$ are two sequences of admissible measures. Moreover the following hold:

$$\mu_{k,n}^1 \to \mu_k^1 \text{ in } \mathcal{X}(\Omega), \qquad \mu_{k,n}^2 \to \mu_k^2 \text{ in } \mathcal{X}(\Omega), \qquad \text{as } n \to \infty, \tag{2.126}$$

and

$$\limsup_n \mathcal{F}_n(\mu_{k,n}^1, \mu_{k,n}^2) \le \mathcal{F}(\mu_k^1, \mu_k^2), \tag{2.127}$$

that is $((\mu_{k,n}^1, \mu_{k,n}^2))_n \subset \mathcal{X}(\Omega) \times \mathcal{X}(\Omega)$ is a recovery sequence for $(\mu_k^1, \mu_k^2) \in \mathcal{X}(\Omega) \times \mathcal{X}(\Omega)$.
We begin with the proof of (2.126). Denote $\nu_{k,n}^1 = \frac{1}{n}\sum_{\ell=1}^{\Lambda_k}\sum_{i \in I_{k,\ell}^n} \delta_{\mathbf{y}_{k,i}^n}$. Take $v \in C_b(\Omega)$. Let $\eta > 0$ and let $\delta > 0$ be the modulus of uniform continuity of v in $\overline{\Omega}_0$ corresponding to η. We have

$$\left|\int_\Omega v \,\mathrm{d}\mu_{k,n}^1 - \int_\Omega v \,\mathrm{d}\mu_k^1\right| \le \left|\int_\Omega v \,\mathrm{d}\nu_{k,n}^1 - \int_\Omega v \,\mathrm{d}\mu_k^1\right| + \frac{1}{n}\sum_{i=1}^{\widetilde{\mathsf{N}}_{n,k}} |v(\tilde{\mathbf{y}}_{k,i}^n)|. \tag{2.128}$$

For the second term at the right hand side, we have

$$\frac{1}{n}\sum_{i=1}^{\widetilde{\mathsf{N}}_{n,k}} |v(\tilde{\mathbf{y}}^n_{k,i})| \leq \frac{\widetilde{\mathsf{N}}_{n,k}}{n}||v||_{L^\infty(\Omega)}$$

that goes to zero, as $n \to \infty$, by (2.118). As for the first term at the right hand side of (2.128) we have

$$\begin{aligned}
&\left|\int_{Q_\ell^{h_k}} v\,\mathrm{d}\nu^1_{k,n} - \int_{Q_\ell^{h_k}} v\,\mathrm{d}\mu^1_k\right| \leq \sum_{i\in I^n_{k,\ell}} \left|\int_{q^n_{k,i}} v\,\mathrm{d}\nu^1_{k,n} - \int_{q^n_{k,i}} v\,\mathrm{d}\mu^1_k\right| \\
&\leq \sum_{i\in I^n_{k,\ell}} \left|\frac{1}{n}\,v(\mathbf{y}^n_{k,i}) - \frac{\mu^1_{\vartheta_k}(\widetilde{Q}_\ell^{h_k})}{h_k^2}\int_{q^n_{k,i}} v(\mathbf{x})\,\mathrm{d}\mathbf{x}\right| \\
&= \sum_{i\in I^n_{k,\ell}} \left|\frac{1}{n(a^n_{k,\ell})^2}\int_{q^n_{k,i}} v(\mathbf{y}^n_{k,i})\,\mathrm{d}\mathbf{x} - \frac{\mu^1_{\vartheta_k}(\widetilde{Q}_\ell^{h_k})}{n(a^n_{k,\ell})^2}\,\frac{n(a^n_{k,\ell})^2}{h_k^2}\int_{q^n_{k,i}} v(\mathbf{x})\,\mathrm{d}\mathbf{x}\right| \qquad (2.129)\\
&\leq \sum_{i\in I^n_{k,\ell}} \frac{\mu^1_{\vartheta_k}(\widetilde{Q}_\ell^{h_k})}{h_k^2}\int_{q^n_{k,i}} |v(\mathbf{y}^n_{k,i}) - v(\mathbf{x})|\,\mathrm{d}\mathbf{x} \\
&+ \sum_{i\in I^n_{k,\ell}} \frac{1}{n(a^n_{k,\ell})^2}\left|1 - \mu^1_{\vartheta_k}(\widetilde{Q}_\ell^{h_k})\,\frac{n(a^n_{k,\ell})^2}{h_k^2}\right|\int_{q^n_{k,i}} |v(\mathbf{y}^n_{k,i})|\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

For $n \gg 1$, so that $\sqrt{2}a^n_{k,\ell} < \delta$, recalling that $\sharp I^n_{k,\ell} = (\alpha^n_{k,\ell})^2$ and that $a^n_{k,\ell} = h_k/\alpha^n_{k,\ell}$, for the first term at the right hand side of (2.129) we have

$$\sum_{i\in I^n_{k,\ell}} \frac{\mu^1_{\vartheta_k}(\widetilde{Q}_\ell^{h_k})}{h_k^2}\int_{q^n_{k,i}} |v(\mathbf{y}^n_{k,i}) - v(\mathbf{x})|\,\mathrm{d}\mathbf{x} < \eta\,\mu^1_{\vartheta_k}(\widetilde{Q}_\ell^{h_k}) \qquad (2.130)$$

while, for the last term in (2.129), we have

$$\begin{aligned}
&\sum_{i\in I^n_{k,\ell}} \frac{1}{n(a^n_{k,\ell})^2}\left|1 - \mu^1_{\vartheta_k}(\widetilde{Q}_\ell^{h_k})\,\frac{n(a^n_{k,\ell})^2}{h_k^2}\right|\int_{q^n_{k,i}} |v(\mathbf{y}^n_{k,i})|\,\mathrm{d}\mathbf{x} \qquad (2.131)\\
&\qquad\leq \frac{(\alpha^n_{k,\ell})^2}{n}\left|1 - \mu^1_{\vartheta_k}(\widetilde{Q}_\ell^{h_k})\,\frac{n(a^n_{k,\ell})^2}{h_k^2}\right| ||v||_{L^\infty(\Omega)}.
\end{aligned}$$

Summing over $\ell = 1, \dots, \Lambda_k$ in (2.130) and (2.131) and then passing to the limit, as $n \to \infty$, we easily see, using (2.113), that the first term of the right hand side of (2.129) goes to zero. This concludes the proof of (2.126) for $\mu^1_{k,n}$. Analogous arguments apply to $\mu^2_{k,n}$. Note that, in proving (2.126), we also have shown that

$$\mu^1_{k,n} - \nu^1_{k,n} \to 0 \quad \text{in } \mathcal{M}_{\mathrm{b}}(\Omega), \quad \text{as } n \to \infty$$

and

$$\nu^1_{k,n} \to \mu^1_k \quad \text{in } \mathcal{X}(\Omega), \quad \text{as } n \to \infty. \qquad (2.132)$$

Moreover, if we set $\nu^2_{k,n} = \frac{1}{n}\sum_{\ell=1}^{\Lambda_k}\sum_{i\in J^n_{k,\ell}} \delta_{\mathbf{z}^n_{k,i}}$, then we can prove in the same way that

$$\mu^2_{k,n} - \nu^2_{k,n} \to 0 \quad \text{in } \mathcal{M}_{\mathrm{b}}(\Omega), \quad \text{as } n \to \infty$$

and

$$\nu^2_{k,n} \to \mu^2_k \quad \text{in } \mathcal{X}(\Omega), \quad \text{as } n \to \infty. \tag{2.133}$$

We now move to the proof of (2.127). We are going to prove the following:

$$\limsup_n \iint_{\Omega\times\Omega} V_1 \, \mathrm{d}(\mu^1_{k,n} \boxtimes \mu^1_{k,n}) \le \iint_{\Omega\times\Omega} V_1 \, \mathrm{d}(\mu^1_k \otimes \mu^1_k), \tag{2.134}$$

$$\limsup_n \iint_{\Omega\times\Omega} V_2 \, \mathrm{d}(\mu^2_{k,n} \boxtimes \mu^2_{k,n}) \le \iint_{\Omega\times\Omega} V_2 \, \mathrm{d}(\mu^2_k \otimes \mu^2_k), \tag{2.135}$$

$$\limsup_n \iint_{\Omega\times\Omega} V_{1,2} \, \mathrm{d}(\mu^1_{k,n} \otimes \mu^2_{k,n}) \le \iint_{\Omega\times\Omega} V_{1,2} \, \mathrm{d}(\mu^1_k \otimes \mu^2_k). \tag{2.136}$$

Equation (2.127) will follow immediately by summing up (2.134)-(2.136). We start with the proof of (2.134). Using the symmetry of the interaction potential, we can write

$$\begin{aligned}
\iint_{\Omega\times\Omega} V_1 \, \mathrm{d}(\mu^1_{k,n} \boxtimes \mu^1_{k,n}) &= \frac{1}{n^2} \sum_{\ell=1}^{\Lambda_k} \sum_{i \in I^n_{k,\ell}} \sum_{j \neq i} V_1(\mathbf{y}^n_{k,i}, \mathbf{y}^n_{k,j}) \\
&+ \frac{1}{n^2} \sum_{\ell=1}^{\Lambda_k} \sum_{m \neq \ell} \sum_{i \in I^n_{k,\ell}} \sum_{j \in I^n_{k,m}} V_1(\mathbf{y}^n_{k,i}, \mathbf{y}^n_{k,j}) \\
&+ \frac{2}{n^2} \sum_{\ell=1}^{\Lambda_k} \sum_{i \in I^n_{k,\ell}} \sum_{j=1}^{\widetilde{\mathsf{N}}_{n,k}} V_1(\mathbf{y}^n_{k,i}, \tilde{\mathbf{y}}^n_{k,j}) \\
&+ \frac{1}{n^2} \sum_{i=1}^{\widetilde{\mathsf{N}}_{n,k}} \sum_{j \neq i} V_1(\tilde{\mathbf{y}}^n_{k,i}, \tilde{\mathbf{y}}^n_{k,j}).
\end{aligned} \tag{2.137}$$

The last three terms are easy to estimate. Indeed for the second term, we have that

$$\frac{1}{n^2} \sum_{\ell=1}^{\Lambda_k} \sum_{m \neq \ell} \sum_{i \in I^n_{k,\ell}} \sum_{j \in I^n_{k,m}} V_1(\mathbf{y}^n_{k,i}, \mathbf{y}^n_{k,j}) = \sum_{\ell=1}^{\Lambda_k} \sum_{m \neq \ell} \iint_{Q^{h_k}_\ell \times Q^{h_k}_m} V_1 \, \mathrm{d}(\nu^1_{k,n} \otimes \nu^1_{k,n}). \tag{2.138}$$

For $\ell \neq m$, using (2.7) we see that the potential V_1 is bounded on $Q^{h_k}_\ell \times Q^{h_k}_m$. Thus, by (2.132), we have that, as $n \to \infty$,

$$\sum_{\ell=1}^{\Lambda_k} \sum_{m \neq \ell} \iint_{Q^{h_k}_\ell \times Q^{h_k}_m} V_1 \, \mathrm{d}(\nu^1_{k,n} \otimes \nu^1_{k,n}) \to \sum_{\ell=1}^{\Lambda_k} \sum_{m \neq \ell} \iint_{Q^{h_k}_\ell \times Q^{h_k}_m} V_1 \, \mathrm{d}(\mu^1_k \otimes \mu^1_k). \tag{2.139}$$

For the third term in (2.137), using (2.7), (2.120), and that $\sharp(I^n_{k,\ell}) \le \mathsf{N}_{n,k}$, we obtain

$$\frac{2}{n^2} \sum_{\ell=1}^{\Lambda_k} \sum_{i \in I^n_{k,\ell}} \sum_{j=1}^{\widetilde{\mathsf{N}}_{n,k}} V_1(\mathbf{y}^n_{k,i}, \tilde{\mathbf{y}}^n_{k,j}) \le C \frac{\Lambda_k}{n^2} \, \mathsf{N}_{n,k} \, \widetilde{\mathsf{N}}_{n,k} \left(1 + \log L - \log \frac{a^n_{k,1}}{2} \right).$$

Thus, using (2.118) and the two estimates $\mathsf{N}_{n,k} \le n$ and $a^n_{k,1} \ge C/\sqrt{n}$, we can bound the right hand side in the previous line from above by $C(1/\sqrt{n} + 1/\sqrt{n} \log \sqrt{n})$, which goes to zero, as $n \to \infty$. Similarly, for the fourth term in (2.137), using (2.119) and the fact that $\tilde{a}_k^n \geq C/\sqrt{n}$, we can see that

$$\frac{1}{n^2} \sum_{i=1}^{\widetilde{\mathsf{N}}_{n,k}} \sum_{j \neq i} V_1(\tilde{\mathbf{y}}_{k,i}^n, \tilde{\mathbf{y}}_{k,j}^n) \leq \frac{C}{n^2} (\widetilde{\mathsf{N}}_{n,k})^2 \left(1 + \log L - \log \tilde{a}_k^n\right) \leq \frac{C}{n} + \frac{C}{n} \log \sqrt{n}$$

and the right hand side goes to zero, as $n \to \infty$.
In order to prove (2.134), it remains to show that

$$\limsup_n \frac{1}{n^2} \sum_{i \in I_{k,\ell}^n} \sum_{j \neq i} V_1(\mathbf{y}_{k,i}^n, \mathbf{y}_{k,j}^n) \leq \iint_{Q_\ell^{h_k} \times Q_\ell^{h_k}} V_1 \, \mathrm{d}(\mu_k^1 \otimes \mu_k^1) \tag{2.140}$$

for $\ell = 1, \dots, \Lambda_k$. To do this, we argue as in Step 2. We set $V_1^M = V_1 \wedge M$ where $M > 0$. Then we write

$$\begin{aligned} \frac{1}{n^2} \sum_{i \in I_{k,\ell}^n} \sum_{j \neq i} V_1(\mathbf{y}_{k,i}^n, \mathbf{y}_{k,j}^n) &= \frac{1}{n^2} \sum_{i \in I_{k,\ell}^n} \sum_{j \neq i} V_1^M(\mathbf{y}_{k,i}^n, \mathbf{y}_{k,j}^n) \\ &\quad + \frac{1}{n^2} \sum_{i \in I_{k,\ell}^n} \sum_{j \neq i} \left(V_1(\mathbf{y}_{k,i}^n, \mathbf{y}_{k,j}^n) - V_1^M(\mathbf{y}_{k,i}^n, \mathbf{y}_{k,j}^n) \right). \end{aligned} \tag{2.141}$$

Since V_1^M is continuous and bounded on $Q_\ell^{h_k} \times Q_\ell^{h_k}$ and by (2.132), we have

$$\begin{aligned} \frac{1}{n^2} \sum_{i \in I_{k,\ell}^n} \sum_{j \neq i} V_1^M(\mathbf{y}_{k,i}^n, \mathbf{y}_{k,j}^n) &= \iint_{Q_\ell^{h_k} \times Q_\ell^{h_k}} V_1^M \, \mathrm{d}(\nu_{k,n}^1 \boxtimes \nu_{k,n}^1) \\ &\to \iint_{Q_\ell^{h_k} \times Q_\ell^{h_k}} V_1^M \, \mathrm{d}(\mu_k^1 \otimes \mu_k^1) \leq \iint_{Q_\ell^{h_k} \times Q_\ell^{h_k}} V_1 \, \mathrm{d}(\mu_k^1 \otimes \mu_k^1), \end{aligned} \tag{2.142}$$

as $n \to \infty$. Hence, by (2.141) and (2.142), we only need to prove that

$$\lim_{M \to +\infty} \limsup_n \frac{1}{n^2} \sum_{i \in I_{k,\ell}^n} \sum_{j \neq i} \left(V_1(\mathbf{y}_{k,i}^n, \mathbf{y}_{k,j}^n) - V_1^M(\mathbf{y}_{k,i}^n, \mathbf{y}_{k,j}^n) \right) = 0. \tag{2.143}$$

This is going to require some work. Recall that, as we have seen in Step 2, for any $\mathbf{y}, \mathbf{z} \in \Omega$, the inequality $V_1(\mathbf{y}, \mathbf{z}) > M$ implies $|\mathbf{y} - \mathbf{z}| < R_M$. Moreover, we can choose $M \gg 1$ to ensure that $V_1(\cdot, \mathbf{z}) > 0$ on $B(\mathbf{z}, R_M)$ for every $\mathbf{z} \in \Omega_0$. Hence, we compute

$$\begin{aligned} &\frac{1}{n^2} \sum_{i \in I_{k,\ell}^n} \sum_{j \neq i} \left(V_1(\mathbf{y}_{k,i}^n, \mathbf{y}_{k,j}^n) - V_1^M(\mathbf{y}_{k,i}^n, \mathbf{y}_{k,j}^n) \right) \\ &\quad = \frac{1}{n^2} \sum_{i \in I_{k,\ell}^n} \sum_{\substack{j \neq i: \\ |\mathbf{y}_{k,i}^n - \mathbf{y}_{k,j}^n| < R_M}} \left(V_1(\mathbf{y}_{k,i}^n, \mathbf{y}_{k,j}^n) - V_1^M(\mathbf{y}_{k,i}^n, \mathbf{y}_{k,j}^n) \right) \\ &\quad \leq \frac{1}{n^2} \sum_{i \in I_{k,\ell}^n} \sum_{\substack{j \neq i: \\ |\mathbf{y}_{k,i}^n - \mathbf{y}_{k,j}^n| < R_M}} V_1(\mathbf{y}_{k,i}^n, \mathbf{y}_{k,j}^n). \end{aligned} \tag{2.144}$$

For every $\ell = \dots, \Lambda_k$ with $\mu_k^1(\widetilde{Q}_\ell^{h_k}) > 0$, we set $P_{k,\ell,M}^n = \left\lfloor R_M / a_{k,\ell}^n \right\rfloor + 1$. Note that, since $a_{k,\ell}^n \to 0$ as $n \to \infty$, we have $P_{k,\ell,M}^n \geq 2$ for $n \gg 1$. For $p = 2, \dots, P_{k,\ell,M}^n$, we define the set of indices

$$\mathcal{J}_{k,\ell}^n(i,p) = \{ j \in I_{k,\ell}^n : (p-1) a_{k,\ell}^n \leq |\mathbf{y}_{k,i}^n - \mathbf{y}_{k,j}^n| < p \, a_{k,\ell}^n \}.$$

These are the indices of the points $\mathbf{y}^n_{k,j}$ that lie in the annulus centered at $\mathbf{y}^n_{k,i}$ with internal and external radius given by $(p-1)a^n_{k,\ell}$ and $p\,a^n_{k,\ell}$, respectively. These points are contained in the set given by the difference between the square centered at $\mathbf{y}^n_{k,i}$ with side $2pa^n_{k,\ell}$ and the square centered at the same point with radius $(2p-4)a^n_{k,\ell}$. Thus we can easily estimate that $\sharp(\mathcal{J}^n_{k,\ell}(i,p)) \le 8(2p-1)$. Looking back at the last line of (2.144), using again (2.7) and the fact that $\sharp(I^n_{k,\ell}) = (\alpha^n_{k,\ell})^2$, we compute

$$\frac{1}{n^2}\sum_{i\in I^n_{k,\ell}}\sum_{\substack{j\neq i:\\ |\mathbf{y}^n_{k,i}-\mathbf{y}^n_{k,j}|<R_M}} V_1(\mathbf{y}^n_{k,i},\mathbf{y}^n_{k,j}) = \frac{1}{n^2}\sum_{i\in I^n_{k,\ell}}\sum_{p=2}^{P^n_{k,\ell,M}}\sum_{j\in\mathcal{J}^n_{k,\ell}(i,p)} V_1(\mathbf{y}^n_{k,i},\mathbf{y}^n_{k,j})$$
$$\le C\left(\frac{\alpha^n_{k,\ell}}{n}\right)^2\sum_{p=2}^{P^n_{k,\ell,M}}(2p-1)\Big(1+\log L-\log((p-1)a^n_{k,\ell})\Big)$$
$$\le C\left(\frac{\alpha^n_{k,\ell}}{n}\right)^2\sum_{p=2}^{P^n_{k,\ell,M}}\left(2p-1-2(p-1)\log\frac{(p-1)a^n_{k,\ell}}{L}-\log\frac{(p-1)a^n_{k,\ell}}{L}\right).$$

Note that, since $-t\log t\le C$ for any $t>0$, we have $-(p-1)\log((p-1)a^n_{k,\ell}/L)\le CL/a^n_{k,\ell}$ for every $p=2,\dots,P^n_{k,\ell,M}$. Using this property, the previous expression can be bounded by

$$C\left(\frac{\alpha^n_{k,\ell}}{n}\right)^2\sum_{p=2}^{P^n_{k,\ell,M}}\left(2p-1+\frac{C}{a^n_{k,\ell}}\right)\le C\left(\frac{\alpha^n_{k,\ell}}{n}\right)^2 P^n_{k,\ell,M}\left(2P^n_{k,M}-1+\frac{C}{a^n_{k,\ell}}\right)$$
$$\le C\left(\frac{\alpha^n_{k,\ell}}{n}\right)^2\left(\frac{R_M}{a^n_{k,\ell}}+1\right)\left(2\frac{R_M}{a^n_{k,\ell}}+1+\frac{C}{a^n_{k,\ell}}\right)$$
$$\le C\left(\frac{\alpha^n_{k,\ell}}{n}\right)^2\left\{\left(\frac{R_M}{a^n_{k,\ell}}\right)^2+\frac{R_M}{a^n_{k,\ell}}+\frac{R_M}{(a^n_{k,\ell})^2}+\frac{C}{a^n_{k,\ell}}+1\right\}$$
$$=C\left\{\left(\frac{R_M}{h_k}\right)^2\frac{(\alpha^n_{k,\ell})^4}{n^2}+\frac{R_M}{h_k}\frac{(\alpha^n_{k,\ell})^3}{n^2}+\frac{R_M}{h_k^2}\frac{(\alpha^n_{k,\ell})^4}{n^2}+\frac{C}{h_k}\frac{(\alpha^n_{k,\ell})^3}{n^2}+\left(\frac{\alpha^n_{k,\ell}}{n}\right)^2\right\}$$

where, in the last line, we applied the substitution $a^n_{k,\ell}=h_k/\alpha^n_{k,\ell}$. Thus, using (2.113), we obtain

$$\limsup_n\, C\left(\frac{\alpha^n_{k,\ell}}{n}\right)^2\sum_{p=2}^{P^n_{k,\ell,M}}\left(2p-1+\frac{C}{a^n_{k,\ell}}\right)\le C\left\{(R_M/h_k)^2+R_M/h_k^2\right\}\mu^1_{\vartheta_k}(\widetilde{Q}^{h_k}_\ell)^2$$

and the expression at the right hand side goes to zero, as $M\to+\infty$, since $R_M\to 0^+$, as $M\to+\infty$. Therefore we showed that

$$\lim_{M\to+\infty}\limsup_n\frac{1}{n^2}\sum_{i\in I^n_{k,\ell}}\sum_{\substack{j\neq i:\\ |\mathbf{y}^n_{k,i}-\mathbf{y}^n_{k,j}|<R_M}} V_1(\mathbf{y}^n_{k,i},\mathbf{y}^n_{k,j})=0$$

and this, by (2.144), proves (2.143) and, in turn, (2.134).
The claim (2.135) can be proved in the same way.

The proof of (2.136) is simpler. Indeed, we have

$$\begin{aligned}\iint_{\Omega\times\Omega} V_{1,2}\,\mathrm{d}(\mu^1_{k,n}\boxtimes\mu^1_{k,n}) =& \frac{1}{n^2}\sum_{\ell,m=1}^{\Lambda_k}\sum_{i\in I^n_{k,\ell}}\sum_{j\in J^n_{k,m}} V_{1,2}(\mathbf{y}^n_{k,i},\mathbf{z}^n_{k,j})\\ &+\frac{1}{n^2}\sum_{\ell=1}^{\Lambda_k}\sum_{i\in I^n_{k,\ell}}\sum_{j=1}^{\widetilde{\mathsf{M}}_{n,k}} V_{1,2}(\mathbf{y}^n_{k,i},\tilde{\mathbf{z}}^n_{k,j})\\ &+\frac{1}{n^2}\sum_{\ell=1}^{\Lambda_k}\sum_{j\in J^n_{k,\ell}}\sum_{i=1}^{\widetilde{\mathsf{N}}_{n,k}} V_{1,2}(\tilde{\mathbf{y}}^n_{k,i},\mathbf{z}^n_{k,j})\\ &+\frac{1}{n^2}\sum_{i=1}^{\widetilde{\mathsf{N}}_{n,k}}\sum_{j=1}^{\widetilde{\mathsf{M}}_{n,k}} V_{1,2}(\tilde{\mathbf{y}}^n_{k,i},\tilde{\mathbf{z}}^n_{k,j}).\end{aligned}\tag{2.145}$$

For the first term at the right hand side, we have

$$\frac{1}{n^2}\sum_{\ell,m=1}^{\Lambda_k}\sum_{i\in I^n_{k,\ell}}\sum_{j\in J^n_{k,m}} V_{1,2}(\mathbf{y}^n_{k,i},\mathbf{z}^n_{k,j}) = \sum_{\ell,m=1}^{\Lambda_k}\iint_{Q^{h_k}_\ell\times\widehat{Q}^{h_k}_m} V_{1,2}\,\mathrm{d}(\nu^1_{k,n}\otimes\nu^2_{k,n}).\tag{2.146}$$

By (2.132) and (2.133), we have that $\nu^1_{k,n}\otimes\nu^2_{k,n}\overset{*}{\rightharpoonup}\mu^1_k\otimes\mu^2_k$ in $\Big(C_b(\Omega\times\Omega)\Big)'$, as $n\to\infty$. Since $V_{1,2}$ is bounded on $Q^{h_k}_\ell\times\widehat{Q}^{h_k}_m$, we deduce that

$$\begin{aligned}&\sum_{\ell,m=1}^{\Lambda_k}\iint_{Q^{h_k}_\ell\times\widehat{Q}^{h_k}_m} V_{1,2}\,\mathrm{d}(\nu^1_{k,n}\otimes\nu^2_{k,n})\\ &\qquad\to\sum_{\ell,m=1}^{\Lambda_k}\iint_{Q^{h_k}_\ell\times\widehat{Q}^{h_k}_m} V_{1,2}\,\mathrm{d}(\mu^1_k\otimes\mu^2_k),\quad\text{as } n\to\infty.\end{aligned}\tag{2.147}$$

Hence, to prove (2.136), it is sufficient to prove that the last three terms in the right hand side of (2.145) vanish, as $n\to\infty$. Note that, if $\mathbf{y}\in Q^{h_k}_\ell$ and $\mathbf{z}\in\widehat{Q}^{h_k}_m$, then $|\mathbf{y}-\mathbf{z}|\ge\sqrt{2}h_k$ so that, by (2.7), we have $|V_{1,2}(\mathbf{y},\mathbf{z})|\le C(1+\log L-\log(\sqrt{2}h_k))$. Hence, using that $\mathsf{N}_{n,k}\le n$ and $\mathsf{M}_{n,k}\le n$, and recalling (2.118) and (2.121), we compute

$$\frac{1}{n^2}\sum_{\ell=1}^{\Lambda_k}\sum_{i\in I^n_{k,\ell}}\sum_{j=1}^{\widetilde{\mathsf{M}}_{n,k}} V_{1,2}(\mathbf{y}^n_{k,i},\tilde{\mathbf{z}}^n_{k,j})\le\frac{C}{n^2}\,\mathsf{N}_{n,k}\,\widetilde{\mathsf{M}}_{n,k}\Big(1+\log L-\log(\sqrt{2}h_k)\Big)\le\frac{C}{\sqrt{n}},\tag{2.148}$$

$$\frac{1}{n^2}\sum_{\ell=1}^{\Lambda_k}\sum_{j\in J^n_{k,\ell}}\sum_{i=1}^{\widetilde{\mathsf{N}}_{n,k}} V_{1,2}(\tilde{\mathbf{y}}^n_{k,i},\mathbf{z}^n_{k,j})\le\frac{C}{n^2}\,\mathsf{M}_{n,k}\,\widetilde{\mathsf{N}}_{n,k}\Big(1+\log L-\log(\sqrt{2}h_k)\Big)\le\frac{C}{\sqrt{n}},\tag{2.149}$$

$$\frac{1}{n^2}\sum_{i=1}^{\widetilde{\mathsf{N}}_{n,k}}\sum_{j=1}^{\widetilde{\mathsf{M}}_{n,k}} V_{1,2}(\tilde{\mathbf{y}}^n_{k,i},\tilde{\mathbf{z}}^n_{k,j})\le\frac{C}{n^2}\,\widetilde{\mathsf{N}}_{n,k}\,\widetilde{\mathsf{M}}_{n,k}\Big(1+\log L-\log(\sqrt{2}h_k)\Big)\le\frac{C}{n}.\tag{2.150}$$

Since the right hand sides of (2.148), (2.149) and (2.150) clearly go to zero, as $n\to\infty$, we have that the last three terms in (2.145) also go to zero, as desired. This concludes the proof of (2.136) and of (2.127).

Step 5 *(Conclusion)* Now that we have constructed a recovery sequence for each pair of approximating measures (μ_k^1, μ_k^2), we need to regain the limsup inequality for the original measures (μ^1, μ^2). To do this, we recall that, since the space $\mathcal{X}(\Omega)$ is metrizable, the Γ-limsup functional is defined as

(2.151)
$$\Gamma\text{-}\limsup_n \mathcal{F}_n(\mu^1, \mu^2) = \inf \left\{ \limsup_n \mathcal{F}_n(\nu_n^1, \nu_n^2) : (\nu_n^1, \nu_n^2) \to (\mu^1, \mu^2) \text{ in } \mathcal{X}(\Omega) \times \mathcal{X}(\Omega) \right\},$$

where the infimum is taken over over all sequences $\left((\nu_n^1, \nu_n^2)\right)_n \subset \mathcal{X}(\Omega) \times \mathcal{X}(\Omega)$ converging to $(\mu^1 m \mu^2)$. Moreover (see Remark 1.26 in [3] or Proposition 8.1 in [8]), there exists a sequence $\left((\mu_n^1, \mu_n^2)\right)_n \subset \mathcal{X}(\Omega) \times \mathcal{X}(\Omega)$ such that

$$\mu_n^1 \to \mu^1 \quad \text{in } \mathcal{X}(\Omega), \qquad \mu_n^2 \to \mu^2 \quad \text{in } \mathcal{X}(\Omega), \qquad \text{as } n \to \infty, \tag{2.152}$$

and

$$\Gamma\text{-}\limsup_n \mathcal{F}_n(\mu^1, \mu^2) = \limsup_n \mathcal{F}_n(\mu_n^1, \mu_n^2). \tag{2.153}$$

The functional $\Gamma\text{-}\limsup_n \mathcal{F}_n$ is lower semicontinuous on $\mathcal{X}(\Omega) \times \mathcal{X}(\Omega)$ (see Proposition 1.28 in [3] or Proposition 6.8 in [8]). Hence, we have

$$\begin{aligned}
\limsup_n \mathcal{F}_n(\mu_n^1, \mu_n^2) &= \Gamma\text{-}\limsup_n \mathcal{F}_n(\mu^1, \mu^2) \\
&\leq \liminf_k \left(\Gamma\text{-}\limsup_n \mathcal{F}_n(\mu_k^1, \mu_k^2) \right) \\
&\leq \liminf_k \left(\limsup_n \mathcal{F}_n(\mu_{k,n}^1, \mu_{k,n}^2) \right) \\
&\leq \liminf_k \mathcal{F}(\mu_k^1, \mu_k^2) \\
&\leq \limsup_k \mathcal{F}(\mu_k^1, \mu_k^2) \leq \mathcal{F}(\mu^1, \mu^2),
\end{aligned}$$

where we have used (2.110), (2.126) and (2.151), (2.127) and, finally, (2.111) in the last line. Thus, we proved that

$$\limsup_n \mathcal{F}_n(\mu_n^1, \mu_n^2) \leq \mathcal{F}(\mu^1, \mu^2). \tag{2.154}$$

Since $\mathcal{F}(\mu^1, \mu^2) < +\infty$, from (2.154) it follows that $\mathcal{F}_n(\mu_n^1, \mu_n^2) < +\infty$ for $n \gg 1$. Therefore, $\mu_n^1 \in X_n^1$, $\mu_n^2 \in X_n^2$ and $\mu_n^1 + \mu_n^2 \in X_n$, for $n \gg 1$, and we can use Lemma 2.8 to conclude that

$$\mathcal{G}_n(\mu_n^1, \mu_n^2) \to \mathcal{G}(\mu^1, \mu^2), \quad \text{as } n \to \infty. \tag{2.155}$$

Hence, combining (2.154) and (2.155), we obtain

$$\begin{aligned}
\limsup_n \mathcal{E}_n(\mu_n^1, \mu_n^2) &= \limsup_n \mathcal{F}_n(\mu_n^1, \mu_n^2) + \limsup_n \mathcal{G}_n(\mu_n^1, \mu_n^2) \\
&\leq \mathcal{F}(\mu^1, \mu^2) + \mathcal{G}(\mu^1, \mu^2) = \mathcal{E}(\mu^1, \mu^2),
\end{aligned}$$

which concludes the proof of the Limsup inequality. □

Remark 2.10. Since the space $\mathcal{X}(\Omega) \times \mathcal{X}(\Omega)$ is compact, by the Fundamental Theorem of Γ-convergence (see Theorem 1.21 in [3] or Theorem 7.8 in [8]) and by Theorem 2.1, we deduce the existence of a minimizer of the functional $\mathcal{E}$. That is, there exists two measures $\widetilde{\mu}^1, \widetilde{\mu}^2 \in \mathcal{X}(\Omega)$ such that

$$\mathcal{E}(\widetilde{\mu}^1, \widetilde{\mu}^2) = \min \Big\{ \mathcal{E}(\mu^1, \mu^2) : (\mu^1, \mu^2) \in \mathcal{X}(\Omega) \times \mathcal{X}(\Omega) \Big\}.$$

Moreover, for every sequence $((\widetilde{\mu}^1_n, \widetilde{\mu}^2_n))_n \subset \mathcal{X}(\Omega) \times \mathcal{X}(\Omega)$ such that $(\widetilde{\mu}^1_n, \widetilde{\mu}^2_n)$ is a minimizer of $\mathcal{E}_n$ for every n, there exists a subsequence converging to a minimum point of $\mathcal{E}$ and we have

$$\lim_n \mathcal{E}_n(\widetilde{\mu}^1_n, \widetilde{\mu}^2_n) = \min \mathcal{E}.$$

We conclude this chapter with a characterization of the class of measures where the Γ-limit $\mathcal{E}$ is finite.

Theorem 2.11. ***(Characterization of measures with finite energy)*** *Consider two measures $\mu^1, \mu^2 \in \mathcal{X}(\Omega)$. Then $\mathcal{E}(\mu^1, \mu^2) < +\infty$ if and only if $\mu^1(\Omega) = \mathsf{m}$, $\mu^2(\Omega) = 1 - \mathsf{m}$ and $\mu^1, \mu^2 \in H^{-1}(\Omega)$.*

Proof. For every $\mu^1, \mu^2 \in \mathcal{X}(\Omega)$, we set

$$\boldsymbol{\beta}_{\mu^1,\mu^2}(\mathbf{x}) = \mathrm{D}\mathbf{u}_{\mu^1,\mu^2}(\mathbf{x}) + \int_\Omega \mathbf{K}_1(\mathbf{x};\mathbf{y})\,\mathrm{d}\mu^1(\mathbf{y}) + \int_\Omega \mathbf{K}_2(\mathbf{x};\mathbf{y})\,\mathrm{d}\mu^2(\mathbf{y}), \quad \mathbf{x} \in \Omega, \tag{2.156}$$

where $\mathbf{u}_{\mu^1,\mu^2}$ is the function given by Lemma 2.6. We have $\boldsymbol{\beta}_{\mu^1,\mu^2} \in L^1(\Omega; \mathbb{R}^{2\times 2})$. Indeed, by Fubini Theorem and (1.12), for $i = 1, 2$, we compute

$$\int_\Omega \int_\Omega |\mathbf{K}_i(\mathbf{x};\mathbf{y})|\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mu^i(\mathbf{y}) \le C|\mathbf{b}_i| \int_\Omega \int_{B(\mathbf{y},R)} \frac{1}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mu^i(\mathbf{y}) \le C|\mathbf{b}_i|R,$$

where $0 < \operatorname{diam}\Omega < R$. We claim that $\boldsymbol{\beta}_{\mu^1,\mu^2}$ satisfies

$$\operatorname{curl} \boldsymbol{\beta}_{\mu^1,\mu^2} = \mathbf{b}_1\mu^1 + \mathbf{b}_2\mu^2 \tag{2.157}$$

in the sense of distributions and that it is a weak solution of the following Neumann problem

$$\begin{cases} \operatorname{div} \mathbb{C}\boldsymbol{\beta} = \mathbf{0} & \text{in } \Omega, \\ \mathbb{C}\boldsymbol{\beta}\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{2.158}$$

We begin with the proof of (2.157). Recall from (1.15) that, for $i = 1, 2$, the field $\mathbf{K}_i(\cdot;\mathbf{y})$ is a distributional solution of the system

$$\begin{cases} \operatorname{div} \mathbb{C}\mathbf{K}_i(\cdot;\mathbf{y}) = \mathbf{0}, \\ \operatorname{curl} \mathbf{K}_{\mathbf{b}}(\cdot;\mathbf{y}) = \mathbf{b}_i\,\delta_{\mathbf{y}} \end{cases} \quad \text{in } \mathbb{R}^2. \tag{2.159}$$

Therefore, for every $\varphi \in C^\infty_c(\Omega)$, by Fubini Theorem and the second equation in (2.159), we have

$$\begin{aligned} \int_\Omega \Big(\int_\Omega \mathbf{K}_i(\mathbf{x};\mathbf{y})\,\mathrm{d}\mu^i(\mathbf{y}) \Big) \mathrm{D}\varphi(\mathbf{x})^\perp\,\mathrm{d}\mathbf{x} &= \int_\Omega \int_\Omega \mathbf{K}_i(\mathbf{x};\mathbf{y}) \mathrm{D}\varphi(\mathbf{x})^\perp\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mu^i(\mathbf{y}) \\ &= \int_\Omega \mathbf{b}_i \varphi(\mathbf{y})\,\mathrm{d}\mu^i(\mathbf{y}) \end{aligned}$$

that is,

$$\operatorname{curl}\Big(\int_\Omega \mathbf{K}_i(\cdot;\mathbf{y})\,\mathrm{d}\mu^i \Big) = \mathbf{b}_i\mu^i$$

in the sense of distributions. Hence, (2.157) easily follows.
In order to prove that $\boldsymbol{\beta}_{\mu^1,\mu^2}$ is a weak solution of (2.158), we have to show that for every $\boldsymbol{\varphi} \in C^1(\overline{\Omega};\mathbb{R}^2)$, we have

$$\int_\Omega \mathbb{C}\boldsymbol{\beta}_{\mu^1,\mu^2} : \mathrm{D}\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} = 0. \tag{2.160}$$

Consider an extension $\widetilde{\boldsymbol{\varphi}} \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ of $\boldsymbol{\varphi}$ with $\operatorname{supp}\widetilde{\boldsymbol{\varphi}} \subset \Omega'$ for some open set Ω' with $\Omega \subset\subset \Omega'$. For $i = 1, 2$, we compute

$$\begin{aligned}\int_\Omega \mathbb{C}\Big(\int_\Omega \mathbf{K}_i(\mathbf{x};\mathbf{y})\,\mathrm{d}\mu^i(\mathbf{y})\Big) : \mathrm{D}\boldsymbol{\varphi}(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_\Omega\int_\Omega \mathbb{C}\mathbf{K}_i(\mathbf{x};\mathbf{y}) : \mathrm{D}\boldsymbol{\varphi}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mu^i(\mathbf{y})\\ &= \int_\Omega\int_{\Omega'} \mathbb{C}\mathbf{K}_i(\mathbf{x};\mathbf{y}) : \mathrm{D}\widetilde{\boldsymbol{\varphi}}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mu^i(\mathbf{y})\\ &\quad - \int_\Omega\int_{\Omega'\setminus\Omega} \mathbb{C}\mathbf{K}_i(\mathbf{x};\mathbf{y}) : \mathrm{D}\widetilde{\boldsymbol{\varphi}}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mu^i(\mathbf{y}).\end{aligned}$$

By the first equation in (2.159), the first integral at the right hand side is equal to zero. For the second one, integrating by parts and recalling that $\operatorname{div}\mathbb{C}\mathbf{K}_i(\mathbf{x};\mathbf{y}) = \mathbf{0}$ for every $\mathbf{x} \neq \mathbf{y}$, we obtain

$$\int_\Omega\int_{\Omega'\setminus\Omega} \mathbb{C}\mathbf{K}_i(\mathbf{x};\mathbf{y}) : \mathrm{D}\widetilde{\boldsymbol{\varphi}}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mu^i(\mathbf{y}) = -\int_\Omega\int_{\partial\Omega} \boldsymbol{\varphi}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_i(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\,\mathrm{d}\mu^i(\mathbf{y}).$$

Hence, we deduce

$$\begin{aligned}&\int_\Omega \mathbb{C}\Big(\int_\Omega \mathbf{K}_i(\mathbf{x};\mathbf{y})\,\mathrm{d}\mu^i(\mathbf{y})\Big) : \mathrm{D}\boldsymbol{\varphi}(\mathbf{x})\,\mathrm{d}\mathbf{x}\\ &\qquad = \int_\Omega\int_{\partial\Omega} \boldsymbol{\varphi}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_i(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\,\mathrm{d}\mu^i(\mathbf{y}).\end{aligned} \tag{2.161}$$

On the other hand, by the Euler-Lagrange equations satisfied by $\mathbf{u}_{\mu^1,\mu^2}$, we have

$$\begin{aligned}\int_\Omega \mathbb{C}\mathrm{D}\mathbf{u}_{\mu^1,\mu^2} : \mathrm{D}\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} = &-\int_\Omega\int_{\partial\Omega} \boldsymbol{\varphi}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_1(\mathbf{x};\mathbf{y})\mathbf{n}(x)\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\,\mathrm{d}\mu^1(\mathbf{y})\\ &-\int_\Omega\int_{\partial\Omega} \boldsymbol{\varphi}(\mathbf{x})\cdot\mathbb{C}\mathbf{K}_2(\mathbf{x};\mathbf{y})\mathbf{n}(x)\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\,\mathrm{d}\mu^2(\mathbf{y}).\end{aligned} \tag{2.162}$$

Therefore, combining (2.161) and (2.162), we obtain (2.160).
Suppose now that $\boldsymbol{\beta} \in L^1(\Omega;\mathbb{R}^{2\times 2})$ is another weak solution of (2.158) with

$$\operatorname{curl}\boldsymbol{\beta} = \mathbf{b}_1\mu^1 + \mathbf{b}_2\mu^2$$

in the sense of distributions. Then, by the weak Poincaré Lemma, $\boldsymbol{\beta}_{\mu^1,\mu^2} - \boldsymbol{\beta} = \mathrm{D}\mathbf{v}$ for some $\mathbf{v} \in W^{1,1}(\Omega;\mathbb{R}^2)$ that satisfies

$$\forall\boldsymbol{\varphi} \in C^1(\overline{\Omega};\mathbb{R}^2),\quad \int_\Omega \mathbb{C}\mathrm{D}\mathbf{v} : \mathrm{D}\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} = 0. \tag{2.163}$$

We consider a sequence of standard mollifiers (ρ_k) and we define the regularized functions $\mathbf{v}_k = \mathbf{v} * \rho_k \in C^\infty(\overline{\Omega};\mathbb{R}^2)$. These functions satisfy the same equation, namely

$$\forall\boldsymbol{\varphi} \in C^1(\overline{\Omega};\mathbb{R}^2),\quad \int_\Omega \mathbb{C}\mathrm{D}\mathbf{v}_k : \mathrm{D}\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} = 0 \tag{2.164}$$

for every k. Indeed, for every k and for every $\boldsymbol{\varphi} \in C^1(\overline{\Omega}; \mathbb{R}^2)$, we have

$$\int_\Omega \mathbb{C}\mathrm{D}\mathbf{v}_k : \mathrm{D}\boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} = \int_\Omega \mathbb{C}(\mathrm{D}\mathbf{v} * \rho_k) : \mathrm{D}\boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} = \int_\Omega \mathbb{C}\mathrm{D}\mathbf{v} : (\mathrm{D}\boldsymbol{\varphi} * \rho_k) \, \mathrm{d}\mathbf{x} = 0,$$

where we used (2.163), the symmetry of ρ_k and that $\boldsymbol{\varphi} * \rho_k \in C^\infty(\overline{\Omega}; \mathbb{R}^2)$. Hence, choosing $\boldsymbol{\varphi} = \mathbf{v}_k$ in (2.164), we deduce by (1.4) that $\mathrm{E}\mathbf{v}_k = \mathbf{0}$. Then $\mathrm{D}\mathbf{v}_k = \mathbf{A}_k$ for some constant matrix $\mathbf{A}_k \in \mathrm{Skew}(2)$, and, since $\mathrm{D}\mathbf{v}_k \to \mathrm{D}\mathbf{v}$ in $L^1(\Omega; \mathbb{R}^{2\times 2})$, as $k \to \infty$, we conclude that $\mathrm{D}\mathbf{v} = \mathbf{A}$ for some constant matrix $\mathbf{A} \in \mathrm{Skew}(2)$. Therefore, $\boldsymbol{\beta}_{\mu^1,\mu^2}$ and $\boldsymbol{\beta}$ differ for a constant skew-symmetric matrix.
Now suppose that $\mu^1(\Omega) = \mathsf{m}$, $\mu^2(\Omega) = 1 - \mathsf{m}$, and $\mu^1, \mu^2 \in H^{-1}(\Omega)$. We consider the minimization problem

$$\min_{\boldsymbol{\beta} \in \mathcal{A}(\mu^1,\mu^2;\mathbf{b}_1,\mathbf{b}_2)} E(\boldsymbol{\beta}), \tag{2.165}$$

where

$$E(\boldsymbol{\beta}) = \int_\Omega W(\boldsymbol{\beta}) \, \mathrm{d}\mathbf{x}$$

and

$$\mathcal{A}(\mu^1,\, \mu^2; \mathbf{b}_1, \mathbf{b}_2) = \Big\{ \boldsymbol{\beta} \in L^2(\Omega; \mathbb{R}^{2\times 2}) : \ \mathrm{curl}\, \boldsymbol{\beta} = \mathbf{b}_1 \mu^1 + \mathbf{b}_2 \mu^2 \Big\},$$

where the condition $\mathrm{curl}\, \boldsymbol{\beta} = \mathbf{b}_1\mu^1 + \mathbf{b}_2\mu^2$ should be intended in the sense of distributions. Note that the class $\mathcal{A}(\mu^1, \mu^2; \mathbf{b}_1, \mathbf{b}_2)$ is not empty. Indeed the field $\boldsymbol{\beta} = \mathrm{D}\mathbf{u}^\perp$ where $\mathbf{u} \in H^1_0(\Omega; \mathbb{R}^2)$ is the unique weak solution of the Dirichlet problem

$$\begin{cases} \Delta \mathbf{u} = \mathbf{b}_1\mu^1 + \mathbf{b}_2\mu^2 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega \end{cases}$$

is an element of that class. The functional E and the class $\mathcal{A}(\mu^1, \mu^2; \mathbf{b}_1, \mathbf{b}_2)$ are both convex, hence we have weak lower semicontinuity. Moreover, E is weakly coercive. Hence, by the Direct Method, we deduce the existence of a solution $\widetilde{\boldsymbol{\beta}} \in L^2(\Omega; \mathbb{R}^{2\times 2})$ of the minimization problem (2.165). Computing the Euler-Lagrange equations of the functional E, we obtain that $\widetilde{\boldsymbol{\beta}}$ is a solution of (2.158). Therefore we deduce that $\boldsymbol{\beta}_{\mu^1,\mu^2}$ and $\widetilde{\boldsymbol{\beta}}$ differ for a constant skew-symmetric matrix and, in turn, that $\boldsymbol{\beta}_{\mu^1,\mu^2} \in L^2(\Omega; \mathbb{R}^{2\times 2})$. Thus $\boldsymbol{\beta}_{\mu^1,\mu^2} \in \mathcal{A}(\mu^1, \mu^2; \mathbf{b}_1, \mathbf{b}_2)$ and, using Fubini Theorem and integration by parts, we compute

$$\begin{aligned} E(\boldsymbol{\beta}_{\mu^1,\mu^2}) = \int_\Omega W(\boldsymbol{\beta}_{\mu^1,\mu^2}) \, \mathrm{d}\mathbf{x} &= \int_\Omega W(\mathrm{D}\mathbf{u}_{\mu^1,\mu^2}) \, \mathrm{d}\mathbf{x} \\ &+ \int_\Omega \int_\Omega \mathbb{C}\mathbf{K}_1(\mathbf{x},\mathbf{y}) : \mathrm{D}\mathbf{u}_{\mu^1,\mu^2}(\mathbf{x}) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mu^1(\mathbf{y}) \\ &+ \int_\Omega \int_\Omega \mathbb{C}\mathbf{K}_2(\mathbf{x},\mathbf{y}) : \mathrm{D}\mathbf{u}_{\mu^1,\mu^2}(\mathbf{x}) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mu^2(\mathbf{y}) \\ &+ \frac{1}{2} \iint_{\Omega\times\Omega} V_1 \, \mathrm{d}(\mu^1 \otimes \mu^1) + \frac{1}{2} \iint_{\Omega\times\Omega} V_2 \, \mathrm{d}(\mu^2 \otimes \mu^2) \\ &+ \iint_{\Omega\times\Omega} V_{1,2} \, \mathrm{d}(\mu^1 \otimes \mu^2) = \mathcal{E}(\mu^1, \mu^2). \end{aligned}$$

Therefore, we conclude that $\mathcal{E}(\mu^1, \mu^2) < +\infty$.
Conversely, suppose that $\mathcal{E}(\mu^1, \mu^2) < +\infty$. Thus we have $\mu^1(\Omega) = \mathsf{m}$ and $\mu^2(\Omega) = 1 - \mathsf{m}$. By Step 3 in the proof of the Limsup inequality, there exist two sequences $(\mu^1_k), (\mu^2_k) \subset$

$\mathcal{X}(\Omega)$ such that $\mu_k^1 \to \mu^1$ and $\mu_k^2 \to \mu^2$ in $\mathcal{X}(\Omega)$, as $k \to \infty$, and $\limsup_k \mathcal{F}(\mu_k^1, \mu_k^2) \leq \mathcal{F}(\mu^1, \mu^2)$. Then, by the continuity of the functional $\mathcal{G}$, we easily deduce $\limsup_k \mathcal{E}(\mu_k^1, \mu_k^2) \leq \mathcal{E}(\mu^1, \mu^2)$. In analogy with (2.156), we define $\boldsymbol{\beta}_k \in L^1(\Omega; \mathbb{R}^{2\times 2})$ as

$$\boldsymbol{\beta}_k(\mathbf{x}) = \mathrm{D}\mathbf{u}_k(\mathbf{x}) + \int_\Omega \mathbf{K}_1(\mathbf{x}; \mathbf{y})\, \mathrm{d}\mu_k^1(\mathbf{y}) + \int_\Omega \mathbf{K}_2(\mathbf{x}; \mathbf{y})\, \mathrm{d}\mu_k^2(\mathbf{y}), \quad \mathbf{x} \in \Omega \tag{2.166}$$

where, for simplicity, we set $\mathbf{u}_k = \mathbf{u}_{\mu_k^1, \mu_k^2}$. Recalling (2.109), we see that $\mu_k^1, \mu_k^2 \in H^{-1}(\Omega)$. Hence, using the previous argument, we can conclude that $\boldsymbol{\beta}_k \in L^2(\Omega; \mathbb{R}^{2\times 2})$ and, by Fubini Theorem, we can check that $E(\boldsymbol{\beta}_k) = \mathcal{E}(\mu_k^1, \mu_k^2)$ for every k. Therefore, by (1.4),

$$C||\mathrm{sym}\boldsymbol{\beta}||^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} \leq E(\boldsymbol{\beta}_k) = \mathcal{E}(\mu_k^1, \mu_k^2) \leq \mathcal{E}(\mu^1, \mu^2)$$

for $k \gg 1$, so that there exist a subsequence $(\mathrm{sym}\boldsymbol{\beta}_{k_\ell})$ and a field $\boldsymbol{\xi} \in L^2(\Omega; \mathbb{R}^{2\times 2})$ such that

$$\mathrm{sym}\boldsymbol{\beta}_{k_\ell} \rightharpoonup \boldsymbol{\xi} \quad \text{in } L^2(\Omega; \mathbb{R}^{2\times 2}), \text{ as } \ell \to \infty. \tag{2.167}$$

Consider the sequence $(\mathbf{u}_{k_\ell}) \subset H^1(\Omega; \mathbb{R}^2)$. For every ℓ, taking $\mathbf{u} = \mathbf{0}$ as a competitor and using (2.38), we have

$$0 \geq I(\mu_{k_\ell}^1, \mu_{k_\ell}^2, \mathbf{u}_{k_\ell}) \geq C_1||\mathbf{u}_{k_\ell}||^2_{H^1(\Omega;\mathbb{R}^2)} - C_2||\mathbf{u}_{k_\ell}||_{H^1(\Omega;\mathbb{R}^2)},$$

from which we deduce that $||\mathbf{u}_{k_\ell}||_{H^1(\Omega;\mathbb{R}^2)} \leq C_2/C_1$ for every ℓ. Hence, there exist a subsequence $(\mathbf{u}_{k_{\ell_m}})$ and a function $\mathbf{u} \in H^1(\Omega; \mathbb{R}^2)$ such that $\mathbf{u}_{k_{\ell_m}} \rightharpoonup \mathbf{u}$ in $H^1(\Omega; \mathbb{R}^2)$, as $m \to \infty$. We claim that $\mathbf{u} = \mathbf{u}_{\mu^1,\mu^2}$. Indeed, by the lower semicontinuity of the elastic energy we have

$$\int_\Omega W(\mathrm{D}\mathbf{u})\, \mathrm{d}\mathbf{x} \leq \liminf_m \int_\Omega W(\mathrm{D}\mathbf{u}_{k_{\ell_m}})\, \mathrm{d}\mathbf{x}. \tag{2.168}$$

Moreover, for $i = 1, 2$ we have

$$\begin{aligned} &\int_\Omega \int_{\partial\Omega} \mathbf{u}_{k_{\ell_m}}(\mathbf{x}) \cdot \mathbb{C}\mathbf{K}_i(\mathbf{x}; \mathbf{y})\mathbf{n}(\mathbf{x})\, \mathrm{d}\mathscr{H}^1(\mathbf{x})\, \mathrm{d}\mu^i_{k_{\ell_m}}(\mathbf{y}) \\ &\quad = \int_\Omega \int_{\partial\Omega} \mathbf{u}(\mathbf{x}) \cdot \mathbb{C}\mathbf{K}_i(\mathbf{x}; \mathbf{y})\mathbf{n}(\mathbf{x})\, \mathrm{d}\mathscr{H}^1(\mathbf{x})\, \mathrm{d}\mu^i_{k_{\ell_m}}(\mathbf{y}) \\ &\quad + \int_\Omega \int_{\partial\Omega} (\mathbf{u}_{k_{\ell_m}}(\mathbf{x}) - \mathbf{u}(\mathbf{x})) \cdot \mathbb{C}\mathbf{K}_i(\mathbf{x}; \mathbf{y})\mathbf{n}(\mathbf{x})\, \mathrm{d}\mathscr{H}^1(\mathbf{x})\, \mathrm{d}\mu^i_{k_{\ell_m}}(\mathbf{y}). \end{aligned} \tag{2.169}$$

For the first integral at the right hand side of (2.169), since the function

$$\mathbf{y} \mapsto \int_{\partial\Omega} \mathbf{u}(\mathbf{x}) \cdot \mathbb{C}\mathbf{K}_i(\mathbf{x}; \mathbf{y})\mathbf{n}(\mathbf{x})\, \mathrm{d}\mathscr{H}^1(\mathbf{x})$$

is continuous and bounded, by narrow convergence we have

$$\begin{aligned} &\int_\Omega \int_{\partial\Omega} \mathbf{u}(\mathbf{x}) \cdot \mathbb{C}\mathbf{K}_i(\mathbf{x}; \mathbf{y})\mathbf{n}(\mathbf{x})\, \mathrm{d}\mathscr{H}^1(\mathbf{x})\, \mathrm{d}\mu^i_{k_{\ell_m}}(\mathbf{y}) \\ &\quad \to \int_\Omega \int_{\partial\Omega} \mathbf{u}(\mathbf{x}) \cdot \mathbb{C}\mathbf{K}_i(\mathbf{x}; \mathbf{y})\mathbf{n}(\mathbf{x})\, \mathrm{d}\mathscr{H}^1(\mathbf{x})\, \mathrm{d}\mu^i(\mathbf{y}), \quad \text{as } m \to \infty. \end{aligned} \tag{2.170}$$

For the second integral at the right hand side of (2.169), using (1.12), we compute

$$\begin{aligned}\left|\int_\Omega\int_{\partial\Omega}(\mathbf{u}_{k_{\ell_m}}(\mathbf{x})-\mathbf{u}(\mathbf{x}))\cdot\mathbb{C}\mathbf{K}_i(\mathbf{x};\mathbf{y})\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathscr{H}^1(\mathbf{x})\,\mathrm{d}\mu^i_{k_{\ell_m}}(\mathbf{y})\right|\\ \le\frac{C}{r_0}\|\mathbf{u}-\mathbf{u}_{k_{\ell_m}}\|_{L^2(\partial\Omega)}\end{aligned}\tag{2.171}$$

where the right hand side goes to zero, as $m\to\infty$, by the compactness of the trace operator. Thus, combining (2.170) and (2.171), we obtain that

$$I(\mu^1,\mu^2,\mathbf{u})\le\liminf_m I(\mu^1_{k_{\ell_m}},\mu^2_{k_{\ell_m}},\mathbf{u}_{k_{\ell_m}}).\tag{2.172}$$

Note that, by (2.170), we have that

$$I(\mu^1_{k_{\ell_m}},\mu^2_{k_{\ell_m}},\mathbf{v})\to I(\mu^1,\mu^2,\mathbf{v}),\quad\text{as } m\to\infty$$

for every $\mathbf{v}\in H^1(\Omega;\mathbb{R}^2)$. Therefore, using (2.172), we can conclude

$$\begin{aligned}I(\mu^1,\mu^2,\mathbf{u})&\le\liminf_m I(\mu^1_{k_{\ell_m}},\mu^2_{k_{\ell_m}},\mathbf{u}_{k_{\ell_m}})\\&\le I(\mu^1_{k_{\ell_m}},\mu^2_{k_{\ell_m}},\mathbf{u}_{\mu^1,\mu^2})=I(\mu^1,\mu^2,\mathbf{u}_{\mu^1,\mu^2})\end{aligned}$$

which entails $\mathbf{u}=\mathbf{u}_{\mu^1,\mu^2}$, as claimed. Therefore $\mathrm{D}\mathbf{u}_{k_{\ell_m}}\rightharpoonup\mathrm{D}\mathbf{u}_{\mu^1,\mu^2}$ in $L^2(\Omega;\mathbb{R}^{2\times2})$, as $m\to\infty$. Moreover, for $i=1,2$, we have that

$$\int_\Omega\mathbf{K}_i(\cdot;\mathbf{y})\,\mathrm{d}\mu^i_{k_{\ell_m}}(\mathbf{y})\overset{*}{\rightharpoonup}\int_\Omega\mathbf{K}_i(\cdot;\mathbf{y})\,\mathrm{d}\mu^i(\mathbf{y})$$

in the sense of distributions, as $m\to\infty$. Indeed, for every $\varphi\in C^\infty_c(\Omega)$, the function

$$\mathbf{y}\mapsto\int_\Omega\mathbf{K}_i(\mathbf{x};\mathbf{y})\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

is continuous and bounded. Thus, by Fubini Theorem and narrow convergence, we obtain

$$\begin{aligned}\int_\Omega\Big(\int_\Omega\mathbf{K}_i(\mathbf{x};\mathbf{y})\,\mathrm{d}\mu^i_{k_{\ell_m}}(\mathbf{y})\Big)\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}&=\int_\Omega\int_\Omega\mathbf{K}_i(\mathbf{x};\mathbf{y})\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mu^i_{k_{\ell_m}}(\mathbf{y})\\ \to\int_\Omega\int_\Omega\mathbf{K}_i(\mathbf{x};\mathbf{y})\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mu^i(\mathbf{y})&=\int_\Omega\Big(\int_\Omega\mathbf{K}_i(\mathbf{x};\mathbf{y})\,\mathrm{d}\mu^i(\mathbf{y})\Big)\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x},\end{aligned}$$

as $m\to\infty$. Therefore, $\boldsymbol{\beta}_k\overset{*}{\rightharpoonup}\boldsymbol{\beta}_{\mu^1,\mu^2}$ in the sense of distributions and, by (2.167), we deduce $\mathrm{sym}\boldsymbol{\beta}_{\mu^1,\mu^2}=\boldsymbol{\xi}$, so that $\mathrm{sym}\boldsymbol{\beta}_{\mu^1,\mu^2}\in L^2(\Omega;\mathbb{R}^{2\times2})$. Finally, by the generalized Korn inequality (see Theorem A.2 and Remark A.3 in the Appendix), we obtain that $\boldsymbol{\beta}_{\mu^1,\mu^2}\in L^2(\Omega;\mathbb{R}^{2\times2})$ and, in turn, $\mu^1,\mu^2\in H^{-1}(\Omega)$. □

Appendix

In this appendix we present two results. The first is an extension theorem for Sobolev functions defined on perforated domains and is taken from [21]. The proof is based on Lemma 4.1 in [23], which asserts the following: for every $\mathbf{v} \in H^1(B(\mathbf{0},2) \setminus \overline{B}(\mathbf{0},1);\mathbb{R}^2)$ there exists an extension $\widetilde{\mathbf{v}} \in H^1(B(\mathbf{0},2);\mathbb{R}^2)$ of $\mathbf{v}$ satisfying

$$||\mathrm{E}\widetilde{\mathbf{v}}||_{L^2(B(\mathbf{0},2);\mathbb{R}^{2\times 2})} \leq C_0 ||\mathrm{E}\mathbf{v}||_{L^2(B(\mathbf{0},2)\setminus B(\mathbf{0},1);\mathbb{R}^{2\times 2})} \tag{A.1}$$

with some constant $C_0 > 0$ independent of $\mathbf{v}$.

Theorem A.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Given $\mathbf{x}_1, \dots, \mathbf{x}_n \in \Omega$ and given $\delta > 0$ such that $d(\mathbf{x}_i, \partial\Omega) > 2\delta$ and $|\mathbf{x}_i - \mathbf{x}_j| > 4\delta$ for every $i \neq j$, define $\Omega_\delta = \Omega \setminus (\bigcup_{i=1}^n \overline{B}(\mathbf{x}_i, \delta))$. Then for every $\mathbf{u} \in H^1(\Omega_\delta;\mathbb{R}^2)$ there exist an extension $\widetilde{\mathbf{u}} \in H^1(\Omega;\mathbb{R}^2)$ of $\mathbf{u}$ satisfying*

$$||\mathrm{E}\widetilde{\mathbf{u}}||_{L^2(\Omega;\mathbb{R}^{2\times 2})} \leq C ||\mathrm{E}\mathbf{u}||_{L^2(\Omega_\delta;\mathbb{R}^{2\times 2})}$$

with some constant $C > 0$ independent of $\mathbf{u}$, of n, of the points $\mathbf{x}_i$, and of δ.

Proof. Take any $\mathbf{u} \in H^1(\Omega_\delta;\mathbb{R}^2)$ and denote by $\mathbf{u}_i$ its restriction to $B(\mathbf{x}_i, 2\delta) \setminus \overline{B}(\mathbf{x}_i, \delta)$. For every $i = 1, \dots, n$, we consider the affine map $\mathbf{g}_i$ on $\mathbb{R}^2$ given by $\mathbf{g}_i(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_i)/\delta$. We define $\mathbf{v}_i = \mathbf{u}_i \circ \mathbf{g}$, so that $\mathbf{v}_i \in H^1(B(\mathbf{0},2) \setminus \overline{B}(\mathbf{0},1);\mathbb{R}^2)$. Thus, by the result recalled previously, there exists an extension $\widetilde{\mathbf{v}}_i \in H^1(B(\mathbf{0},2);\mathbb{R}^2)$ of $\mathbf{v}_i$ satisfying (A.1). Then, the function $\widetilde{\mathbf{u}}_i = \widetilde{\mathbf{v}}_i \circ \mathbf{g}_i^{-1}$ is in $H^1(B(\mathbf{x}_i, 2\delta);\mathbb{R}^2)$ and gives an extension of $\mathbf{u}_i$. Moreover, an easy scaling argument shows that

$$||\mathrm{E}\widetilde{\mathbf{u}}_i||_{L^2(B(\mathbf{x}_i,2\delta);\mathbb{R}^{2\times 2})} \leq C_0 ||\mathrm{E}\mathbf{u}_i||_{L^2(B(\mathbf{x}_i,2\delta)\setminus\overline{B}(\mathbf{x}_i,\delta);\mathbb{R}^{2\times 2})}.$$

Hence, if we define

$$\widetilde{\mathbf{u}} = \begin{cases} \widetilde{\mathbf{u}}_i & \text{in } \overline{B}(\mathbf{x}_i,\delta), \text{ for } i = 1, \dots, n, \\ \mathbf{u} & \text{in } \Omega_\delta, \end{cases}$$

then we obtain a function in $H^1(\Omega;\mathbb{R}^2)$ that is an extension of $\mathbf{u}$ and satisfies

$$\begin{aligned} ||\mathrm{E}\widetilde{\mathbf{u}}||^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} &= ||\mathrm{E}\mathbf{u}||^2_{L^2(\Omega_\delta;\mathbb{R}^{2\times 2})} + \sum_{i=1}^n ||\mathrm{E}\widetilde{\mathbf{u}}_i||^2_{L^2(B(\mathbf{x}_i,\delta);\mathbb{R}^{2\times 2})} \\ &\leq (1 + C_0^2) ||\mathrm{E}\mathbf{u}||^2_{L^2(\Omega_\delta;\mathbb{R}^{2\times 2})}, \end{aligned}$$

where we remark that C_0 is the same constant as in (A.1). □

The second result is a generalized version of the Korn inequality which is taken from [12]. Consider a field $\boldsymbol{\beta} \in L^2(\Omega;\mathbb{R}^{2\times 2})$ with $\operatorname{curl} \boldsymbol{\beta} = \mathbf{0}$ in the sense of distributions. By the

Weak Poincaré Lemma, there exists $\mathbf{u} \in H^1(\Omega;\mathbb{R}^2)$ such that $\boldsymbol{\beta} = \mathrm{D}\mathbf{u}$. Hence, if skew$\boldsymbol{\beta}$ has zero mean on some ball $B \subset \Omega$, then by the classical Korn inequality we obtain

$$\int_\Omega |\boldsymbol{\beta}|^2 \,\mathrm{d}\mathbf{x} = \int_\Omega |\mathrm{D}\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} \leq C \int_\Omega |\mathrm{E}\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} = C \int_\Omega |\mathrm{sym}\,\boldsymbol{\beta}|^2 \,\mathrm{d}\mathbf{x} \tag{A.2}$$

for some constant $C > 0$ depending only on Ω. The next result shows that, in the plane, a similar estimate can be obtained also for fields whose curl is a measure of bounded variation, up to an error depending on the total variation of the measure.
The proof is based on the following result (see Theorem 3.1 and Remark 3.3 in [4]): if $\mathbf{f} \in L^1(\Omega;\mathbb{R}^2)$ is a field with $\mathrm{div}\,\mathbf{f} \in H^{-2}(\Omega)$, then $\mathbf{f} \in H^{-1}(\Omega;\mathbb{R}^2)$ and there exists a constant $C > 0$, independent of $\mathbf{f}$, such that

$$||\mathbf{f}||_{H^{-1}(\Omega;\mathbb{R}^2)} \leq C\Big(||\mathrm{div}\,\mathbf{f}||_{H^{-2}(\Omega)} + ||\mathbf{f}||_{L^1(\Omega;\mathbb{R}^2)}\Big). \tag{A.3}$$

Note that here and henceforth, the divergence and the curl operators are always intended in the distributional sense. By density, this result can be extended to measures of bounded variation. That is, for every $\mu \in \mathcal{M}_\mathrm{b}(\Omega;\mathbb{R}^2)$ such that $\mathrm{div}\,\mu \in H^{-2}(\Omega)$, we have that $\mu \in H^{-1}(\Omega;\mathbb{R}^2)$ and

$$||\mu||_{H^{-1}(\Omega;\mathbb{R}^2)} \leq C\Big(||\mathrm{div}\,\mu||_{H^{-2}(\Omega)} + |\mu|(\Omega)\Big). \tag{A.4}$$

Theorem A.2. ***(Generalized Korn inequality)*** *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and consider a ball $B \subset\subset \Omega$. There exists a constant $C > 0$ depending only on Ω such that, for every $\boldsymbol{\beta} \in L^2(\Omega;\mathbb{R}^{2\times 2})$ with*

$$\mathrm{curl}\,\boldsymbol{\beta} = \mu \in \mathcal{M}_\mathrm{b}(\Omega;\mathbb{R}^2), \qquad \int_B (\boldsymbol{\beta} - \boldsymbol{\beta}^\top)\,\mathrm{d}\mathbf{x} = \mathbf{0},$$

we have

$$\int_\Omega |\boldsymbol{\beta}|^2 \,\mathrm{d}\mathbf{x} \leq C \left(\int_\Omega |\mathrm{sym}\,\boldsymbol{\beta}|^2 \,\mathrm{d}\mathbf{x} + \big(|\mu|(\Omega)\big)^2 \right).$$

Proof. Set $\mu = (\mu_1, \mu_2)$ and $\boldsymbol{\beta} = (\beta_{ij})_{i,j=1,2}$. Consider $q = (\beta_{12} - \beta_{21})/2$, that is, the entry of position $(1,2)$ in the matrix skew $\boldsymbol{\beta}$. Since $\mathrm{curl}\,\boldsymbol{\beta} = \mu$, we can write

$$\begin{cases} \partial_1 q = \mu_1 + g_1, \\ \partial_2 q = \mu_2 + g_2, \end{cases}$$

where $g_1 = \partial_2\beta_{11} - \partial_1((\beta_{12}+\beta_{21})/2)$ and $g_2 = -\partial_1\beta_{22} + \partial_2((\beta_{12}+\beta_{21})/2)$. Note that $g_1, g_2 \in H^{-1}(\Omega)$ and that they are linear combinations of distributional derivatives of the entries of sym$\boldsymbol{\beta}$. Set $\mathbf{g} = (g_1, g_2)$. Trivially $\mathrm{curl}\,\mathrm{D}q = 0$. From this we deduce that $\mathrm{curl}\,\mu = -\,\mathrm{curl}\,\mathbf{g}$, or, equivalently, $\mathrm{div}\,\mu^\perp = -\,\mathrm{div}\,\mathbf{g}^\perp$. Note that $\mathrm{div}\,\mathbf{g}^\perp \in H^{-2}(\Omega)$. Therefore, by the result recalled in A.4, we obtain that $\mu^\perp \in H^{-1}(\Omega;\mathbb{R}^2)$, hence $\mu \in H^{-1}(\Omega;\mathbb{R}^2)$. Moreover, we have

$$\begin{aligned} ||\mu||_{H^{-1}(\Omega;\mathbb{R}^2)} &= ||\mu^\perp||_{H^{-1}(\Omega;\mathbb{R}^2)} \\ &\leq C\Big(||\mathrm{div}\,\mu^\perp||_{H^{-2}(\Omega)} + |\mu^\perp|(\Omega)\Big) \\ &= C\Big(||\mathrm{div}\,\mathbf{g}^\perp||_{H^{-2}(\Omega)} + |\mu|(\Omega)\Big) \\ &\leq C\Big(||\mathrm{sym}\boldsymbol{\beta}||_{L^2(\Omega;\mathbb{R}^{2\times 2})} + |\mu|(\Omega)\Big). \end{aligned} \tag{A.5}$$

Consider now the unique weak solution $\mathbf{u} \in H^1_0(\Omega;\mathbb{R}^2)$ of the following Dirichlet problem:

$$\begin{cases} -\Delta\mathbf{u} = \mu^{\perp} & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega. \end{cases}$$

Thus we have

$$||\mathrm{D}\mathbf{u}||_{L^2(\Omega;\mathbb{R}^{2\times 2})} \leq ||\mathbf{u}||_{H^1(\Omega;\mathbb{R}^2)} \leq C||\mu^{\perp}||_{H^{-1}(\Omega;\mathbb{R}^2)} = C||\mu||_{H^{-1}(\Omega;\mathbb{R}^2)} \tag{A.6}$$

where the constant $C > 0$ depends only on Ω. Set $\boldsymbol{\xi} = \mathrm{D}\mathbf{u}^{\perp}$. This field satisfies $\operatorname{curl}\boldsymbol{\xi} = \mu$. Moreover, by (A.5) and (A.6), we obtain

$$\begin{aligned} \int_{\Omega} |\boldsymbol{\xi}|^2 \,\mathrm{d}\mathbf{x} = \int_{\Omega} |\mathrm{D}\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} &\leq C||\mu||^2_{H^{-1}(\Omega;\mathbb{R}^2)} \\ &\leq C\Big(||\mathrm{sym}\boldsymbol{\beta}||^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + (|\mu|(\Omega))^2\Big). \end{aligned} \tag{A.7}$$

Define $\widetilde{\boldsymbol{\xi}} = \boldsymbol{\xi} - \mathbf{A}$, where we set $\mathbf{A} = 1/2 \fint_B (\boldsymbol{\xi} - \boldsymbol{\xi}^{\top}) \,\mathrm{d}\mathbf{x}$. Thus $\operatorname{curl}(\boldsymbol{\beta} - \widetilde{\boldsymbol{\xi}}) = \mathbf{0}$ and $\fint_B ((\boldsymbol{\beta} - \widetilde{\boldsymbol{\xi}}) - (\boldsymbol{\beta} - \widetilde{\boldsymbol{\xi}})^{\top}) = \mathbf{0}$. Therefore, we can apply the classical Korn inequality (A.2) and use (A.7) to conclude

$$\begin{aligned} \int_{\Omega} |\boldsymbol{\beta}|^2 \,\mathrm{d}\mathbf{x} &\leq 2\Big(\int_{\Omega} |\boldsymbol{\beta} - \widetilde{\boldsymbol{\xi}}|^2 \,\mathrm{d}\mathbf{x} + \int_{\Omega} |\widetilde{\boldsymbol{\xi}}|^2 \,\mathrm{d}\mathbf{x}\Big) \\ &\leq C\left(\int_{\Omega} |\mathrm{sym}\boldsymbol{\beta} - \mathrm{sym}\boldsymbol{\xi}|^2 \,\mathrm{d}\mathbf{x} + \int_{\Omega} |\boldsymbol{\xi}|^2 \,\mathrm{d}\mathbf{x}\right) \\ &\leq C\left(\int_{\Omega} |\mathrm{sym}\boldsymbol{\beta}|^2 \,\mathrm{d}\mathbf{x} + \int_{\Omega} |\boldsymbol{\xi}|^2 \,\mathrm{d}\mathbf{x}\right) \\ &\leq C\left(\int_{\Omega} |\mathrm{sym}\boldsymbol{\beta}|^2 \,\mathrm{d}\mathbf{x} + (|\mu|(\Omega))^2\right). \end{aligned}$$

□

Remark A.3. Let $\boldsymbol{\beta} \in L^2(\Omega;\mathbb{R}^{2\times 2})$ with $\operatorname{curl}\boldsymbol{\beta} = \mu$ be such that $\mathrm{skew}\boldsymbol{\beta}$ has not zero mean on B. We can consider the field $\boldsymbol{\beta} - \mathbf{A}$ where $\mathbf{A} = \frac{1}{2}\fint_B (\boldsymbol{\beta} - \boldsymbol{\beta}^{\top}) \,\mathrm{d}\mathbf{x}$. Its skew-symmetric part clearly satisfies the zero mean condition on B. Moreover $\operatorname{curl}(\boldsymbol{\beta} - \mathbf{A}) = \mu$. Therefore, applying the generalized Korn inequality, we obtain

$$\int_{\Omega} |\boldsymbol{\beta} - \mathbf{A}|^2 \,\mathrm{d}\mathbf{x} \leq C\left(\int_{\Omega} |\mathrm{sym}\,\boldsymbol{\beta}|^2 \,\mathrm{d}\mathbf{x} + (|\mu|(\Omega))^2\right).$$

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