Introduction to Linear Programming

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Introduction

In this lecture

Before giving a formal definition of **Linear Programming (LP)**, we present two related real-world problems, and we discuss how they can be modeled as LP problems.

Managing a Production Facility

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- Each product is composed of raw materials. The the set of raw materials used by the company is $I = \{1, \ldots, m\}$.
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- For each raw material in I, the facility has b_i units.
- The market unit value of each raw material is ρ_i.
- The facility produces a set of products $J = \{1, \ldots, n\}$.
- Each product $j \in J$ requires a_{ij} units of raw material $i \in I$.
- Each product $j \in J$ can be sold for σ_j euro per unit.

Production Manager as Optimist

The production manager every day has to decide the quantity x_j produced for each product $j \in J$, with the objective to maximize the company profits, while respecting the availability constraints on the raw materials.

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Net Revenue

The net revenue for selling a unit of product j is

$$c_j = \sigma_j - \sum_{i=1}^{m} \rho_i a_{ij} \tag{1}$$

where σ_j is the selling price, ρ_i is the unit cost of raw material, and a_{ij} is the amount of raw material j required (per unit of product) by product i.

Decision Variables

Each production quantity has to be non negative

$$x_j \geq 0, \quad j = 1, \ldots, n \tag{2}$$

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The total revenue for the company is

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Constraints

The availability of raw materials (resources) is limited

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m. \tag{4}$$

Resource Allocation Problem

To summarize, the production manager's job is to determine production values x_j, so as to maximize (3), subject to constraints (4) and (2).

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In mathematical notation, the problem is

$$\max \quad \sum_{j=1}^{n} c_j x_j \tag{5}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1, \ldots, m \tag{6}$$

$$x_j \geq 0 \qquad j = 1, \ldots, n \tag{7}$$

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To summarize, the production manager's job is to determine production values x_j, so as to maximize (3), subject to constraints (4) and (2).

By using the notation $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$, and $\boldsymbol{x} \in \mathbb{R}^n$, we can write in compact notation:

$$\begin{aligned} \max \quad & \boldsymbol{c}^\mathsf{T} \boldsymbol{x} && (8) \\ \text{s.t.} \quad & A\boldsymbol{x} \leq \boldsymbol{b} && (9) \\ & \boldsymbol{x} \geq 0, && (10) \end{aligned}$$

where all vectors are column vectors, and the inequalities apply component-wise (as in the previous slide).

Outline

1 Introduction

2 Production Manager

3 Comptroller

4 Linear Programming

5 Homework

Comptroller as Pessimist

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Lost Opportunity Cost

Let w_i denote the assigned unit value to raw material i, which are in this case the **decision variables**. The *lost opportunity cost* of having b_i units of raw material in depot is $b_i w_i$, which yields the overall cost

$$\sum_{i=1}^{m} b_i w_i$$

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QUESTION: Which are the constraints in this case?

Objective Function

The total *lost opportunity cost* of the company is

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The unit value w_i cannot be less than the market value ρ_i

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where the **decision variables** are

$$w_i \geq 0, \quad 0 = 1, \ldots, m \tag{14}$$

The Comptroller's Problem

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we get the following Linear Programming model

$$\min \quad \sum_{i=1}^{m} b_i \, y_i \tag{15}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} a_{ij} y_i \leq c_j \qquad j = 1, \ldots, n y_i \geq 0 \qquad i = 1, \ldots, m \tag{16}$$

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QUESTION: Are we missing some terms in this model?

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$$f(\boldsymbol{x}) = \boldsymbol{c}^\intercal \boldsymbol{x} = c_1 x_1 + \cdots + c_j x_j + \cdots + c_n x_n.$$

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- The values assigned to the decision variables must satisfies a given set of **constraints**, expressed in the form:

$$a_1 x_1 + \cdots + a_j x_j + \cdots + a_n x_n \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} b.$$

Constraint Conversion

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An equality constraint

$$a_1x_1 + \cdots + a_jx_j + \cdots + a_nx_n = b$$

can be converted to inequality by adding two inequalities:

$$a_1x_1 + \cdots + a_jx_j + \cdots + a_nx_n \leq b$$
$$a_1x_1 + \cdots + a_jx_j + \cdots + a_nx_n \geq b$$

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Given that we can convert any (linear) constraints, we can restrict our attention to Linear Programming problems formulated as

$$\begin{aligned} \max \quad & c_1 x_1 + \cdots + c_n x_n \\ \text{s.t.} \quad & a_{11} x_1 + \cdots + a_{1n} x_n \leq b_1 \\ & a_{21} x_1 + \cdots + a_{2n} x_n \leq b_2 \\ & \qquad \vdots \\ & a_{m1} x_1 + \cdots + a_{mn} x_n \leq b_m \\ & x_1, x_2, \ldots, x_n \geq 0. \end{aligned}$$

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- ✓ A solution that satisfies all the constraints is called **FEASIBLE**.
- ✓ A solution the attains a maximum value is called **OPTIMAL**.

Example 1

Consider the following LP problem:

$$\begin{aligned} \max \quad & 5x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 2 \\ & -2x_1 - 2x_2 \leq -9 \\ & x_1, x_2 \geq 0 \end{aligned}$$

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If a problem has no a **feasible** solution, then the problem itself is called **INFEASIBLE**.

Example 2

Consider the following LP problem:

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If a problem has an arbitrarily large positive value, then the problem itself is called **UNBOUNDED**.

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Consider the following LP problem:

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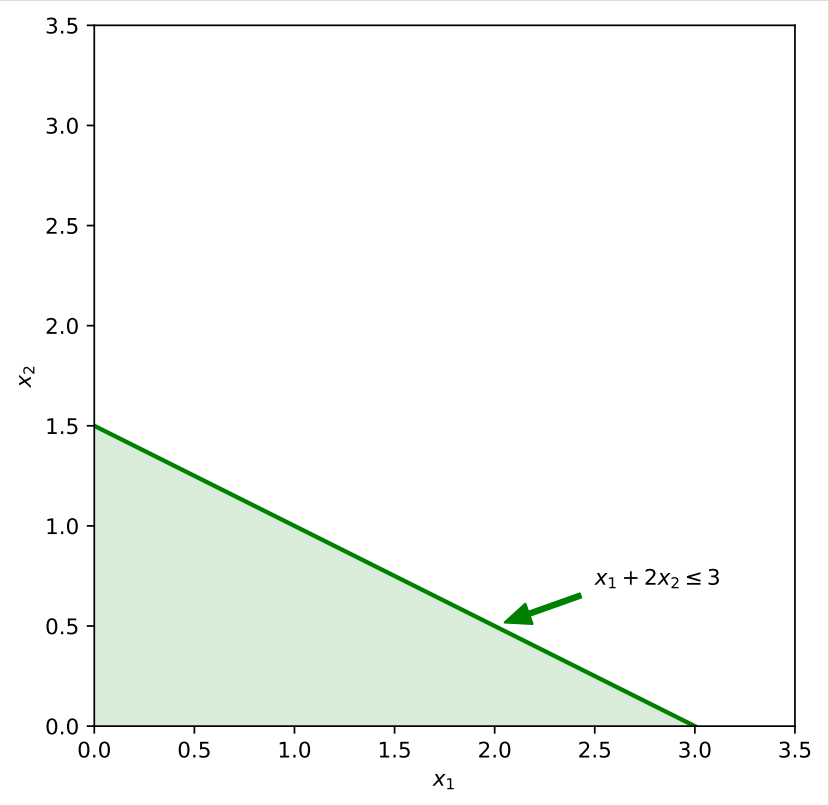
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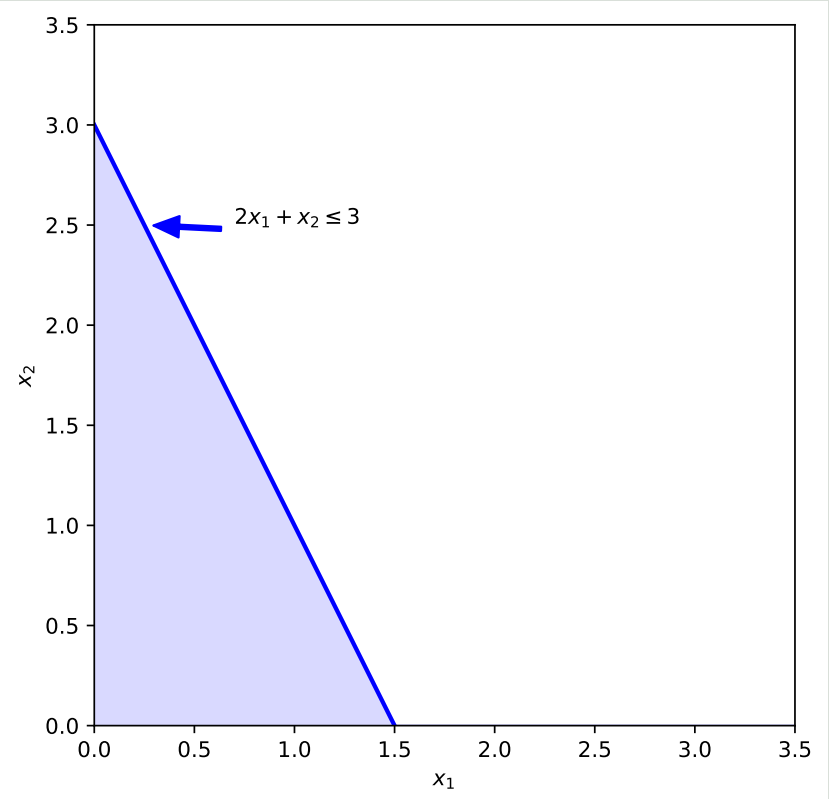
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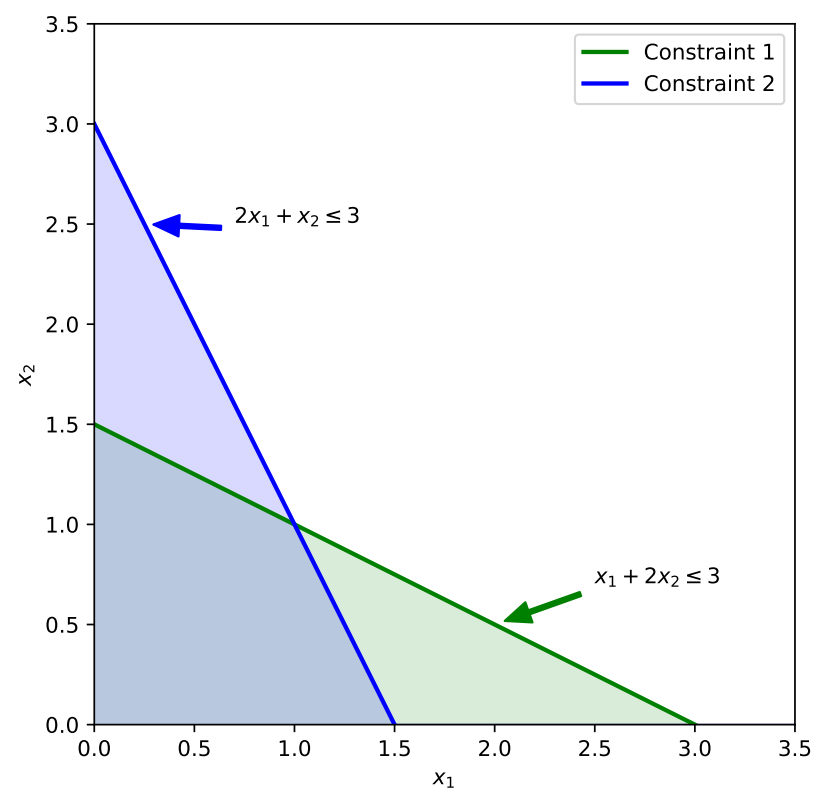
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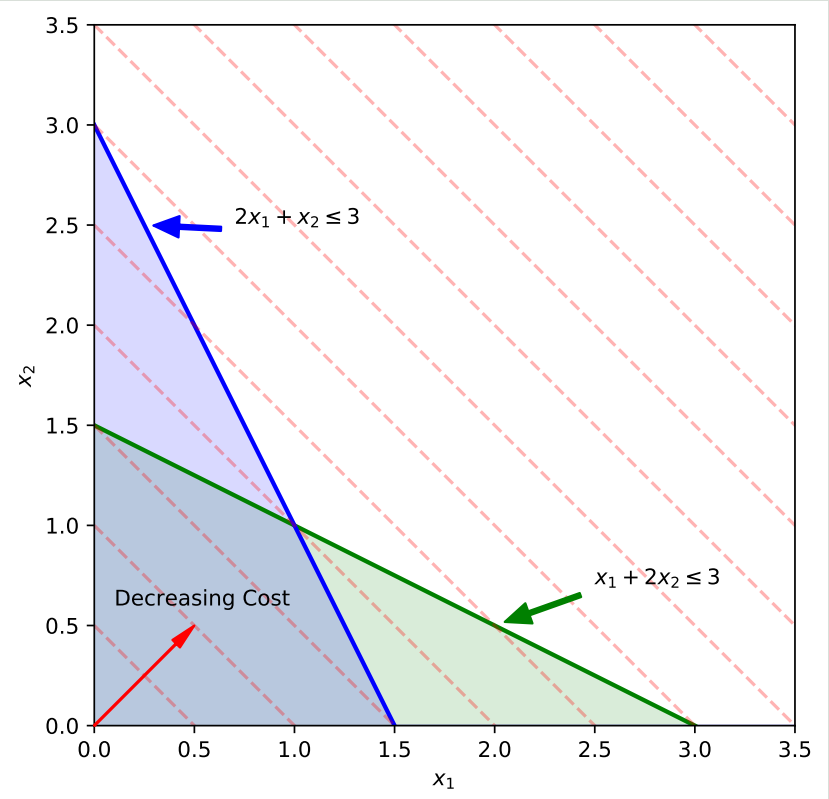
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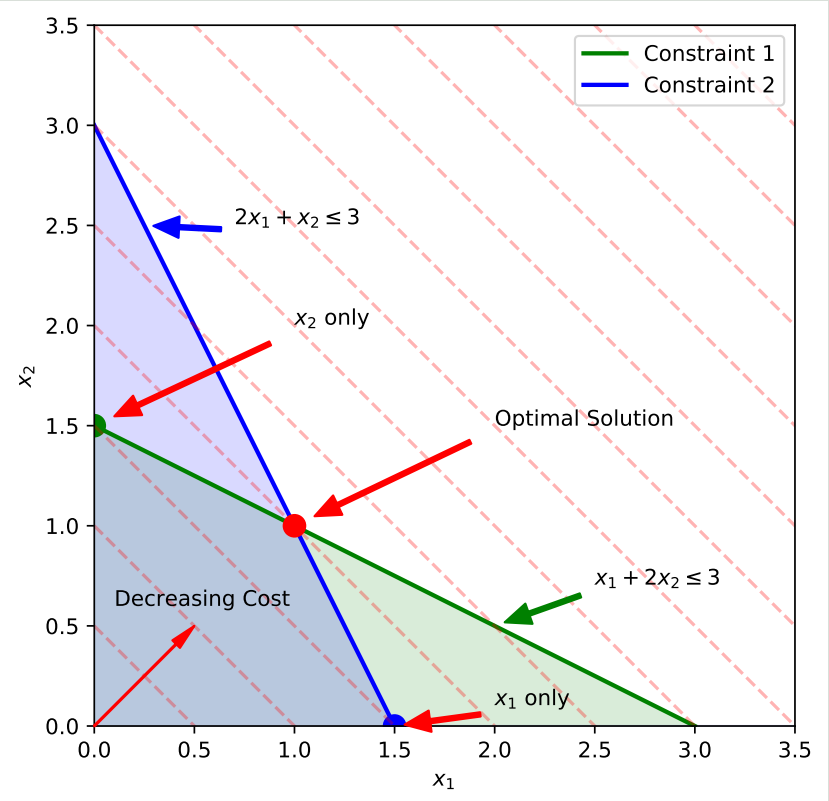
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Homework 1





A steel company must decide how to allocate next week's time on a rolling mill (see, `https://www.youtube.com/watch?v=AuuP8L-WppI`), which is a machine that takes unfinished slabs of steel as input and can produce either of two semi-finished products: bands and coils. The mill's two products come off the rolling line at different rates: Bands 200 tons/hour and Coils 140 tons/hour.

They also produce different profits: Bands 25 euro/ton and Coils 30 euro/ton. Based on the currently booked orders, the following upper bounds are placed on the amount of each product to produce: Bands 6 000 tons and Coils 4 000 tons.

Homework 1





Given that there are 40 hours in total of production time available this week (to produce both bands and coils), the problem is to **decide** how many tons of bands and how many tons of coils should be produced to yield the **maximum** profit.

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QUESTION 2: Can you solve this problem by inspection?