Building Linear Programming Models

Part 2

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Homework 1





Given that there are 40 hours in total of production time available this week (to produce both bands and coils), the problem is to **decide** how many tons of bands and how many tons of coils should be produced to yield the **maximum** profit.

QUESTION 1: Formulate this problem as a LP problem.

QUESTION 2: Can you solve this problem by inspection?

Solution Homework 1

Data

r_B	production rate for bands in *tons/hour* (default 200)
r_C	production rate for coils in *tons/hour* (default 140)
p_B	profit for bands *euro/tons* (default 25)
p_C	profit for coils *euro/tons* (default 30)
d_B	max demand for bands in *tons* (default 6 000)
d_C	max demand for coils in *tons* (default 4 000)
T	number of weekly working hours (default 40 hours)

Decision Variables: Production Quantities

x_B	amount of production for bands in *tons*
x_C	amount of production for coils in *tons*

Solution Homework 1: Full Model

Decision Variables: Production Quantities

x_B amount of production for bands in *tons*
x_C amount of production for coils in *tons*

Objective Function: Maximize profit

$$\max \quad p_B x_B + p_C x_C$$

Constraints

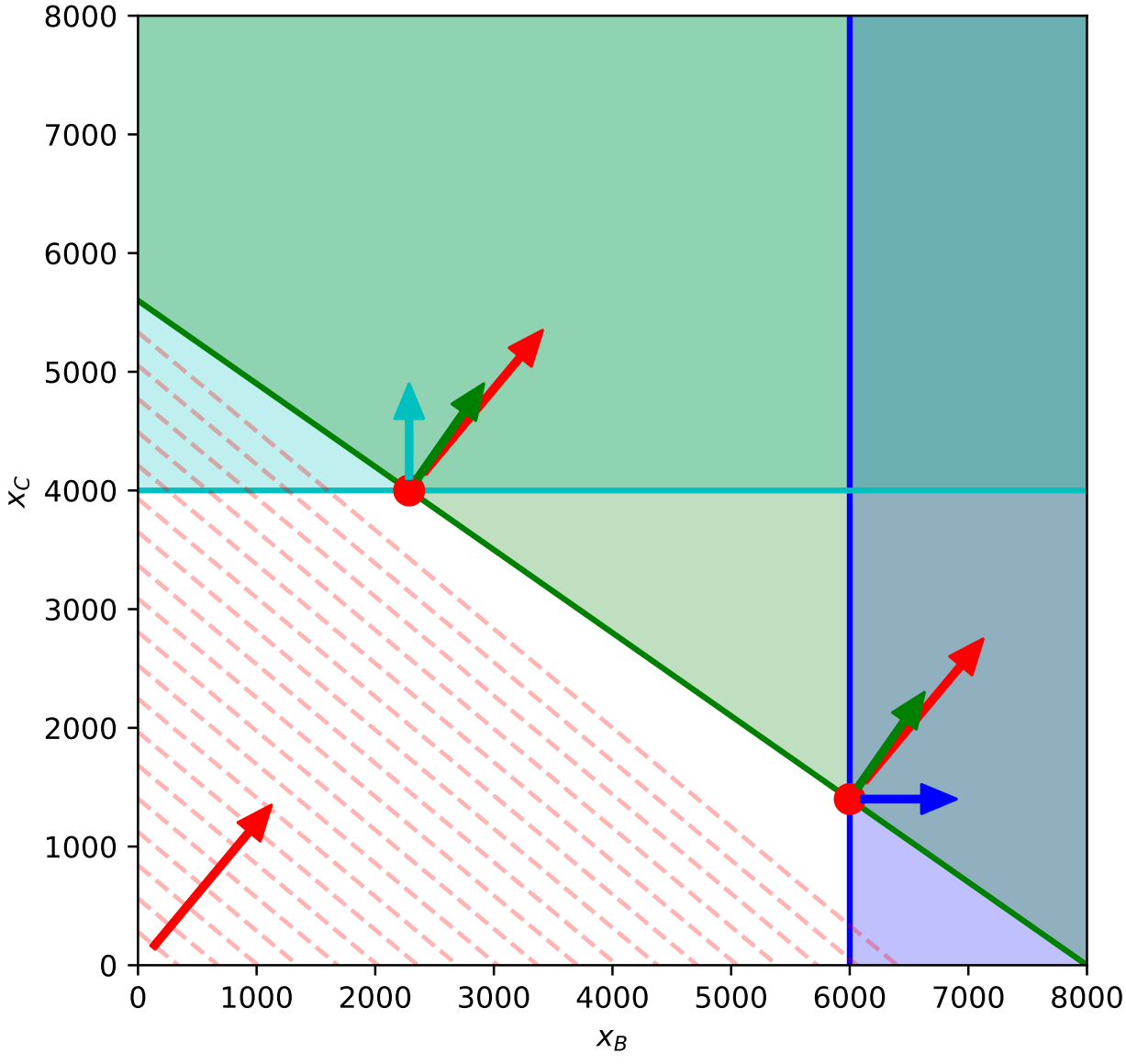
Maximum number of hours available per week

$$\frac{x_B}{r_B} + \frac{x_C}{r_C} \leq T \tag{1}$$

Non-negative production and maximum demand for each product

$$0 \leq \quad x_B \quad \leq d_B, \tag{2}$$
$$0 \leq \quad x_C \quad \leq d_C \tag{3}$$

Solution Homework 1: Graphical Solution



Solution Homework 1: Python Script with Pyomo

Model 5: Data Fitting

Given m data points $(\boldsymbol{a}_i, b_i)$ with $i = 1, \ldots, m$, where $\boldsymbol{a}_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, we want to find a model that predicts b given the vector $\boldsymbol{a}$

$$b = f_{\boldsymbol{x}}(\boldsymbol{a}) \tag{4}$$

where $\boldsymbol{x}$ is a parameter vector.

The **prediction error** or **residual** at the i-th data point is

$$| \, b_i - f_{\boldsymbol{x}}(\boldsymbol{a}) \, | \tag{5}$$

Given a choice between different models $f_{\boldsymbol{x}}(\boldsymbol{a})$, we should choose a model that "*explains*" the available data as best as possible, that is, that minimizes the **residuals**.

A Linear Model is of the form

$$b = f_{\boldsymbol{x}}(\boldsymbol{a}) = \boldsymbol{a}^{\intercal}\boldsymbol{x} \tag{6}$$

Model 5: Data Fitting ⇒ HOMEWORK

Given a Linear Model of the form

$$b = f_{\boldsymbol{x}}(\boldsymbol{a}) = \boldsymbol{a}^{\intercal}\boldsymbol{x} \tag{7}$$

We have two options to minimize the **residuals**:

1. To minimize the largest residual:

$$\max_{i=1,\ldots,m} \mid b_i - \boldsymbol{a}^{\intercal}\boldsymbol{x} \mid \tag{8}$$

2. To minimize the average residual:

$$\frac{1}{m} \sum_{i=1,\ldots,m} \mid b_i - \boldsymbol{a}^{\intercal}\boldsymbol{x} \mid \tag{9}$$

HOMEWORK: Can you formulate the **Data Fitting** problem that minimizes these two functions using Linear Programming?

Solution Homework 2: Minimize the largest residual

The problem of minimizing the largest residual

$$\max_{i=1,\ldots,m} \mid b_i - \boldsymbol{a}^\mathsf{T}\boldsymbol{x} \mid$$

can be formalized with the following LP:

$$\begin{aligned}
\min \quad & z && && (10)\\
\text{s.t.} \quad & + b_i - \boldsymbol{a}_i^\mathsf{T}\boldsymbol{x} \leq z && i = 1,\ldots,m && (11)\\
& - b_i + \boldsymbol{a}_i^\mathsf{T}\boldsymbol{x} \leq z && i = 1,\ldots,m && (12)\\
& && && (13)
\end{aligned}$$

where the decision variables are the vector $\boldsymbol{x}$ and z.

Solution Homework 2: Minimize the average residual

The problem of minimizing the average residual

$$\frac{1}{m} \sum_{i=1,\ldots,m} \mid b_i - \boldsymbol{a}^\mathsf{T} \boldsymbol{x} \mid$$

can be formalized with the following LP:

$$\begin{aligned}
\min \quad & \frac{1}{m} \sum_{i=1}^{m} x_i && && (14)\\
\text{s.t.} \quad & + b_i - \boldsymbol{a}_i^\mathsf{T} \boldsymbol{x} \leq z_i && i = 1, \ldots, m && (15)\\
& - b_i + \boldsymbol{a}_i^\mathsf{T} \boldsymbol{x} \leq z_i && i = 1, \ldots, m && (16)\\
& && && (17)
\end{aligned}$$

where the decision variables are the vector $\boldsymbol{x}$ and z.

Remarks on basic Norms Minimization

The two previous examples are special cases of norm minimization:

$$||\boldsymbol{x}||_1 = \sum_i | z_i | \tag{18}$$

$$||\boldsymbol{x}||_\infty = \max_i | x_i | \tag{19}$$

Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$ be the input data in matrix and vector notation, and let $\boldsymbol{x} \in \mathbb{R}^m$ be the parameter vector.

In general, the **Linear Data Fitting** problem asks to minimize a distance:

$$d(\boldsymbol{Ax}, \boldsymbol{b}) = ||\boldsymbol{Ax} - \boldsymbol{b}||_p \tag{20}$$

When $p = 2$, we get a **Least Square** unconstrained minimization problem (easier to solve):

$$\min_{\boldsymbol{x} \in \mathbb{R}^m} ||\boldsymbol{Ax} - \boldsymbol{b}||_2 \quad \Rightarrow \quad \min_{\boldsymbol{x} \in \mathbb{R}^m} | \boldsymbol{a}_i x_i - b_i |^2 \tag{21}$$

Regularization vs. Constraint Optimization

In general, the **Linear Data Fitting** problem asks to minimize a distance:

$$\min_{\boldsymbol{x}\in\mathbb{R}^m} d(\boldsymbol{Ax}, \boldsymbol{b}) \quad \Rightarrow \quad \min_{\boldsymbol{x}\in\mathbb{R}^m} ||\boldsymbol{Ax} - \boldsymbol{b}||_p \tag{22}$$

A common requirement in **Data Fitting** is to look for **sparse solutions**, that are solution with a few positive variables $x_i > 0$. A common approach consists of adding a regularization parameter:

$$\min_{\boldsymbol{x}\in\mathbb{R}^m} ||\boldsymbol{Ax} - \boldsymbol{b}||_p + \lambda ||\boldsymbol{x}||_q \tag{23}$$

where $\lambda > 0$ is a problem dependent parameter.

QUESTION: Can we formulate a Data Fitting problem using Linear Programming and by imposing the constraint that at most k out of m variables x_i can be greater than zero?

Outline

1 Homework

2 Linear Objectives

3 Linear Constraints

4 Integer Models

Ratio Objectives

In some applications the following non-linear objectives arises

$$\max \frac{\sum_{i=1}^{n} a_i x_i}{\sum_{i=1}^{n} b_i x_i}$$

Also this objective can be turned into a linear function by the following transformations:

1. Replace the expression $\frac{1}{\sum_{i=1}^{n} b_i x_i}$ by a variable z
2. Represents the products $x_i z$ by a variable y_i. The objective now becomes:

$$\max \sum_{i=1}^{n} a_i y_i.$$

3. To satisfy the first relation, introduce the constraint: $\sum_{i=1}^{n} b_i y_i = 1$
4. Replace the following constraints: $\sum_{i=1}^{n} d_i x_i \gtreqless e \quad \Rightarrow \quad \sum_{i=1}^{n} d_i y_i \gtreqless ez$
5. To obtain the x_i values, divide the y_i values by z.

Multiple Objectives

The basic LP problem is a **single objective** problem.

However, basic multiple objective problems can be solved with two basic approaches:

- We can solve the same problem several times, every time with a new objective, while the previous objective is kept as a constraint. Moreover, objectives and constraints can often be interchanged.

  This is known as Lexicographic Optimization.

- Another way of tackling multiple objectives is to take a suitable linear combination of all the objective functions and optimize that.

- If the objectives are in conflict, care must be taken in turning objectives into constraints, since the problem can be easily become infeasible.

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Hard and Soft Constraints

A linear constraint such as

$$\sum_{i=1}^{n} a_i x_i \leq b$$

is a **hard constraint**: any solution which violates this solution is forbidden.
Sometimes, we would like to weaken the constraint, and we need a **soft constraint** which can be violated at a certain cost. A possibility is to rewrite the constraint as

$$\sum_{i=1}^{n} a_i x_i - u \leq b$$

where u is a variable with a positive cost coefficient for a minimization problem (negative cost for maximization). If the constraint is $\sum_{i=1}^{n} a_i x_i \geq b$, we can add a *slack* variable with appropriate weight.
With an equality constraint we have to consider both over- and under-budget violations:

$$\sum_{i=1}^{n} a_i x_i + u - v = b$$

where again the additional variables u and v have appropriate cost coefficients in the objective function.

Chance Constraints

It is possible specify that a certain constraint must be satisfied with a given probability, e.g., we may wish to be 95% confident that a constraint holds:

$$P\left[\sum_{i=1}^{n} a_i x_i \leq b\right] \geq \beta$$

where β is a probability.

A rather crude option, but that is used in practice, is to use a *deterministic equivalent* constraint

$$\sum_{i=1}^{n} a_i x_i \leq d$$

where d is larger than b, and such that the the solution of this constrain implies the chance constraint.

NOTE: Dealing efficiently with chance constraints is a research topic.

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Indicator Variables

Let $x \in \mathbb{R}_+$ and $z \in \{0, 1\}$ be two variables. Consider the logical implications:

$$x > 0 \Rightarrow z = 1$$

(if x is positive, then z is equal to one).

We can represent this condition with a linear constraint and a large enough coefficient M (M is an upper bound for x):

$$x - Mz \leq 0$$

Consider the logical implications:

$$z = 1 \Rightarrow x \geq m$$

This condition can be imposed by

$$x - mz \geq 0$$

Example: Fixed Charge problems.

Indicator Variables

Let $x \in \mathbb{R}_+$ and $z \in \{0, 1\}$. Consider the logical implications:

$$z = 1 \rightarrow \sum_{i=1}^{n} a_i x_i \leq b$$

can be represented by the constraint: $\sum_{i=1}^{n} a_i x_i \leq b + M(1 - z)$
where M is the smallest upper bound on $\sum_{i=1}^{n} a_i x_i - b$.
The logical relation

$$z = 0 \rightarrow \sum_{i=1}^{n} a_i x_i > b$$

is more involved, since in LP we cannot use *strict* inequalities. We need to introduce a tolerance ϵ and consider $\sum_{i=1}^{n} a_i x_i \geq b + \epsilon$. Overall, we get

$$\sum_{i=1}^{n} a_i x_i - (m - \epsilon)z \geq b + \epsilon$$

where m is a lower bound on $\sum_{i=1}^{n} a_i x_i - b$.

Homework 3

Use a $0-1$ variable z to indicate whether or not the following constraint is satisfied:

$$2x_1 + 3x_2 \leq 1.$$

Binary Relations

Let $z_1, z_2 \in \{0, 1\}$. Consider the logical implications:

$$z_1 \vee z_2, \quad z_1 \Rightarrow z_2, \quad z_1 \Leftrightarrow z_2$$

can be represented by the linear constraints:

$$z_1 + z_2 \geq 1 \tag{24}$$
$$z_1 - z_2 \leq 0 \tag{25}$$
$$z_1 - z_2 = 0 \tag{26}$$

Product of $0-1$ variables

If the product of two $0-1$ variables $z_1 z_2$ appears in a model, it can *linearized* by:

- Replace the product by a new $0-1$ variable z_3
- Impose the logical condition

$$z_3 \Leftrightarrow z_1 \wedge z_2$$

by means of the extra constraints

$$z_3 \leq z_1 \tag{27}$$
$$z_3 \leq z_2 \tag{28}$$
$$z_3 \geq z_1 + z_2 - 1 \tag{29}$$

Product of a $0-1$ and a real variable (bilinear term)

If $x \in \mathbb{R}$ and $z \in \{0,1\}$, and the product xz appear in a model, which is called a *bilinear term*, it can *linearized* by:

- Replace the product by a new variable y
- Impose the logical conditions

$$z = 0 \quad \Rightarrow \quad y = 0 \tag{30}$$
$$z = 1 \quad \Rightarrow \quad y = x \tag{31}$$

by means of the extra constraints

$$y \leq Mz \tag{32}$$
$$y \leq x \tag{33}$$
$$y \geq x + M(z-1) \tag{34}$$

where M is an upper bound of the domain of x.

References for Mathematical Modeling

- Chap. 3 and 9 in Williams, H.P., 2013. Model building in mathematical programming. John Wiley & Sons.
- Chap. 1 in Vanderbei, R.J., 2015. Linear programming. Heidelberg: Springer.
- Chap. 1 in Bertsimas, Dimitris, and John N. Tsitsiklis. Introduction to linear optimization, 1997. Athena Scientific.