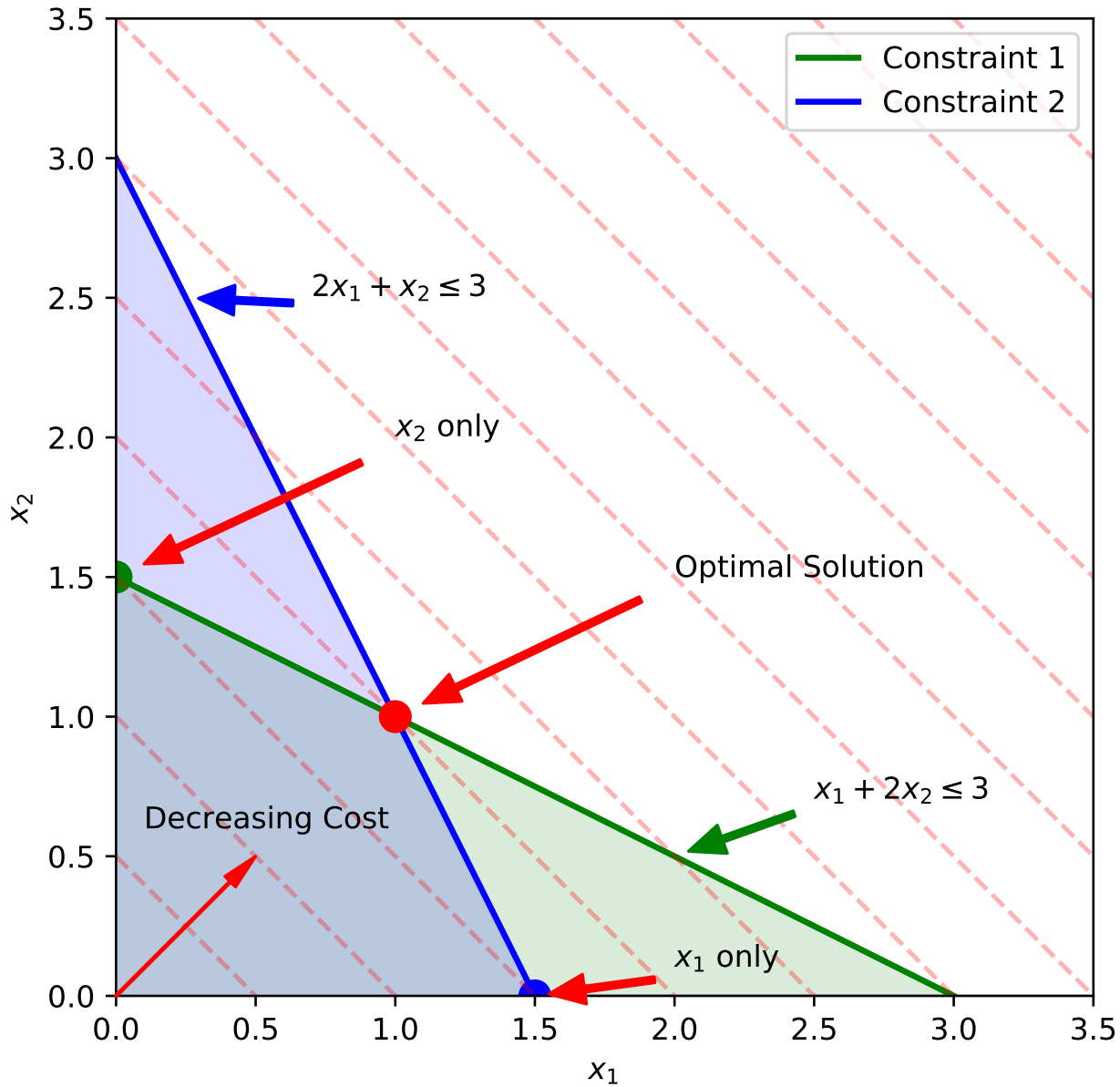
The Primal Simplex Algorithm

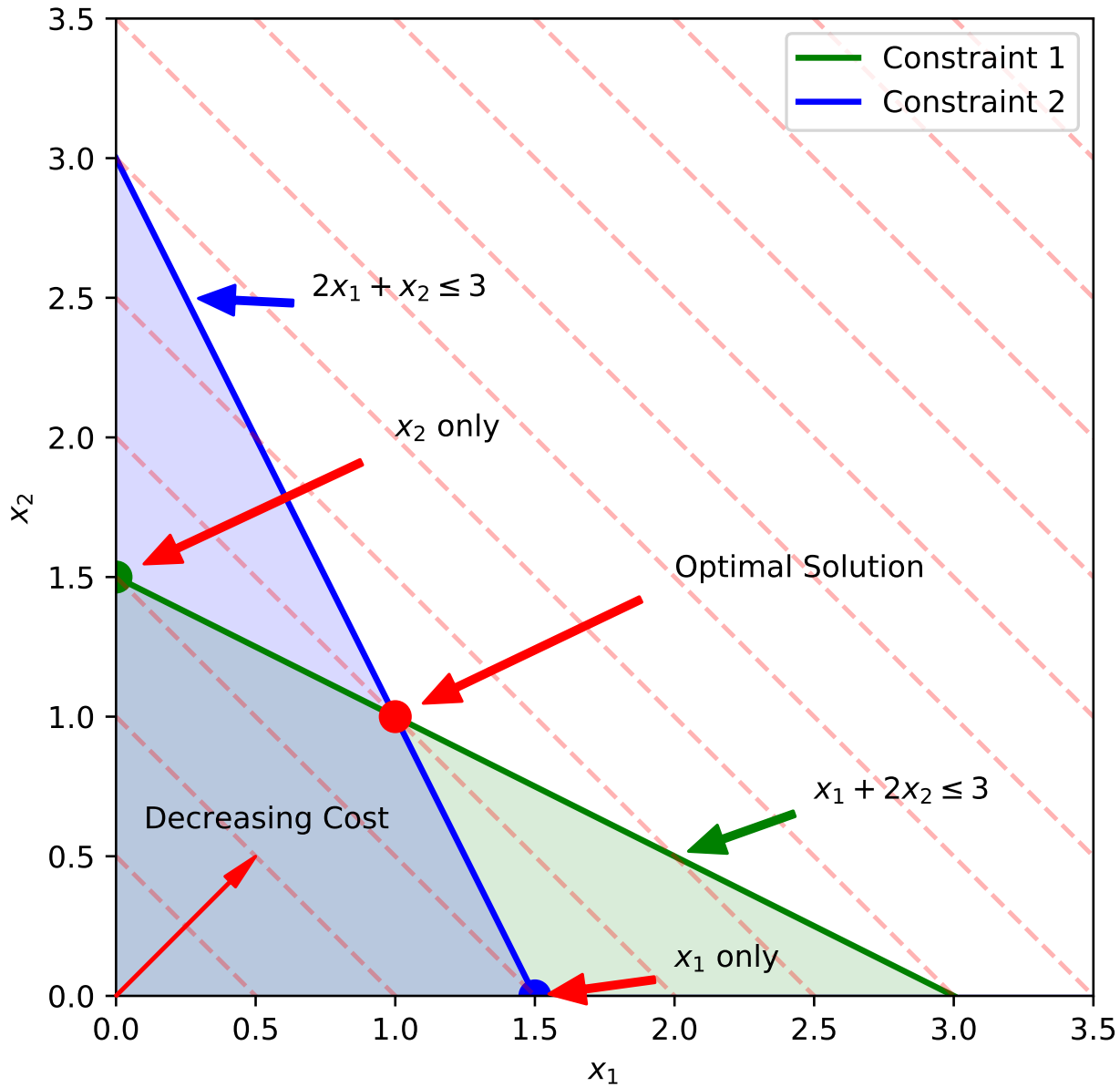
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Notation

In this lecture, we focus on Linear Programming problems written in standard form:

$$\begin{aligned} \min \quad & \boldsymbol{c}^\intercal \boldsymbol{x} && (1) \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} && (2) \\ & \boldsymbol{x} \geq 0. && (3) \end{aligned}$$

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Convex function over a convex set

In Linear Programming, local optimality implies global optimality, since we are minimizing a **convex function** over a **convex set**.

Definition 1 (Feasible direction)

Let x be an element of a polyhedron P. A vector $\boldsymbol{d} \in \mathbb{R}^n$ is said to be a feasible direction at $\boldsymbol{x}$, if there exists a positive scalar θ for which $\boldsymbol{x} + \theta \boldsymbol{d} \in P$.

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Exercise

Consider the LP problem:

$$\begin{array}{lrrrrcl} \min & c_1 x_1 + & c_2 x_2 + & c_3 x_3 + & c_4 x_4 & & \\ \text{s.t.} & x_1 + & x_2 + & x_3 + & x_4 & = & 2 \\ & 2x_1 + & & 3x_3 + & 4x_4 & = & 2 \\ & & x_1, x_2, x_3, x_4 \geq 0 & & & & \end{array}$$

The reduced cost of every basic variable is zero

SHOW.

Theorem 3

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1. *If $\bar{\boldsymbol{c}} \geq 0$, then $\boldsymbol{x}$ is optimal.*
2. *If $\boldsymbol{x}$ is optimal and nondegenerate, then $\bar{\boldsymbol{c}} \geq 0$.*

(Sketch of Proof)

Definition 4 (Optimal basis)

A basis matrix $\boldsymbol{B}$ is said to be optimal if:

1. $\boldsymbol{B}^{-1}\boldsymbol{b} \geq \mathbf{0}$, and
2. $\bar{\boldsymbol{c}}^{\intercal} = \bar{\boldsymbol{c}}^{\intercal} - \bar{\boldsymbol{c}}_B^{\intercal}\boldsymbol{B}^{-1}\boldsymbol{A} \geq \mathbf{0}^{\intercal}$.

Theorem 5

1. *The columns* $\boldsymbol{A}_{B(i)}, i \neq l$, *and* $\boldsymbol{A}_j$ *are linearly independent and, therefore,* $\bar{\boldsymbol{B}}$ *is a basis matrix.*
2. *The vector* $\boldsymbol{y} = \boldsymbol{x} + \theta^* \boldsymbol{d}$ *is a basic feasible solution associated with the basis matrix* $\bar{\boldsymbol{B}}$.

Definition 6 (Degeneracy)

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Definition 7 (Degeneracy in standard form polyhedra)

Consider the standard form polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ and let $\boldsymbol{x}$ be a basic solution. Let m be the number of rows of $\boldsymbol{A}$. The vector $\boldsymbol{x}$ is a degenerate basic solution if more than $(m - n)$ of the components of $\boldsymbol{x}$ are zero.

We denote by $\mathcal{B} := \{B(1), \ldots, B(m)\}$ the set of basic indices and $\mathcal{N} := \{1, \ldots, n\} \setminus \mathcal{B}$ the nonbasic indices.

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An iteration of the simplex method

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3. Compute $\boldsymbol{u} = -\boldsymbol{d}_B = \boldsymbol{B}^{-1} \boldsymbol{A}_j$. If no component if $\boldsymbol{u}$ is positive, we have $\theta^* = \infty$, the optimal cost is $-\infty$, and the algorithm terminates.

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5. Let ℓ be such that $\theta^* = \frac{x_{B(\ell)}}{u_\ell}$. Form a new basis by replacing $\boldsymbol{A}_{B(\ell)}$ with $\boldsymbol{A}_j$. If $\boldsymbol{y}$ is the new BFS, the values of the new basic variables are:

$$y_j = \theta^* \tag{4}$$

$$y_{B(i)} = x_{B(i)} - \theta^* u_i, \qquad \forall i \neq \ell. \tag{5}$$

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1. *We have an optimal basis $\boldsymbol{B}$ and an associated basic feasible solution which is optimal.*
2. *We have found a vector $\boldsymbol{d}$ satisfying $\boldsymbol{Ad} = \boldsymbol{0}, \boldsymbol{d} \geq \boldsymbol{0}$, and $\boldsymbol{c}^\intercal \boldsymbol{d} < 0$, and the optimal cost is $-\infty$.*

The simplex method for degenerate problems

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Cycling

A sequence of degenerate basis change might lead back to the initial basis, in which case the algorithm may loop indefinitely.

This undesirable phenomenon is called cycling.

Cycling can be avoided by using appropriate pivoting rules for selecting

- the entering variable/column (Step 2)
- the leaving variable/column (Step 5)

Pivoting rule

Fundamental rules for selecting the entering column/variable:

1. Choose a column $\boldsymbol{A}_j$, with $\bar{c}_j < 0$ whose reduced cost is the most negative (aka Dantzig's rule).

2. Choose a column with $\bar{c}_j < 0$ for which the corresponding cost decrease $\theta^* |\bar{c}_j|$ is largest. Note that we need to compute θ^* for each column with $\bar{c}_j$.

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Cycling

By the following the smallest subscript rule both for the entering and the leaving variables, we can avoid the cycling undesired phenomenon.

Exercise

Consider the LP problem:

$$
\begin{array}{lrrrcl}
\min & -10x_1- & 12x_2- & 12x_3 & & \\
\text{s.t.} & x_1+ & 2x_2+ & 2x_3 & \leq & 20 \\
& 2x_1+ & x_2+ & 2x_3 & \leq & 20 \\
& 2x_1+ & 2x_2+ & x_3 & \leq & 20 \\
& & x_1, x_2, x_3, x_4 & \geq 0 & &
\end{array}
$$

References for these slides

- Chap. 3 in Bertsimas, Dimitris, and John N. Tsitsiklis. **Introduction to Linear Optimization**, 1997. Athena Scientific.