The geometry of Linear Programming

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Definition 1 (Polyhedron)

A polyhedron is a set that can be described in the form

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\},$$

where $\boldsymbol{A}$ is an $m \times n$ matrix and $\boldsymbol{b}$ is a vector in $\mathbb{R}^m$.

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where $\boldsymbol{A}$ is an $m \times n$ matrix and $\boldsymbol{b}$ is a vector in $\mathbb{R}^m$. A set of the form

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\},$$

is also a polyhedron, and it is called a polyhedron in standard form.

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Definition 2 (Bounded sets)

A set $S \subset \mathbb{R}^n$ is bounded if there exists a constant K such that the absolute value of every component of every element of S is less than or equal to K: $||x||_\infty < K, \forall x \in S$.

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Definition 3 (Hyperplanes and halfspaces)

Let $\boldsymbol{a}$ be a nonzero vector in $\mathbb{R}^n$ and let b be a scalar.

1. The set $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}^\intercal \boldsymbol{x} = b\}$ is called a hyperplane.
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NOTE: The vector $\boldsymbol{a}$ is orthogonal to the hyperplane itself (to show).

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A set $S \subset \mathbb{R}^n$ is convex if for any $\boldsymbol{x}, \boldsymbol{y} \in S$, and any $\lambda \in [0, 1]$, we have $\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \in S$.

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Definition 5 (Convex combination and convex hull)

Let $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k$ be vectors in $\mathbb{R}^n$ and let $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_+$ s.t. $\sum_{i=1}^k \lambda_i = 1$.

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Theorem 6

1. *The intersection of convex sets is convex.*
2. *Every polyhedron is a convex set.*
3. *A convex combination of a finite number of elements of a convex set also belongs to that set.*
4. *The convex hull of a finite number of vectors is a convex set.*

Proof.

Definition 7 (Extreme point)

Let P be a polyhedron. A vector $\boldsymbol{x} \in P$ is an **extreme point** of P if we cannot find two vectors $\boldsymbol{y}, \boldsymbol{z} \in P$, both different from $\boldsymbol{x}$, and a scalar $\lambda \in [0, 1]$, such that $\boldsymbol{x} = \lambda \boldsymbol{y} + (1 - \lambda)\boldsymbol{z}$. (*draw figure*)

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Definition 8 (Vertex)

Let P be a polyhedron. A vector $\boldsymbol{x} \in P$ is a **vertex** of P if there exists some vector $\boldsymbol{c}$ such that $\boldsymbol{c}^\intercal \boldsymbol{x} < \boldsymbol{c}^\intercal \boldsymbol{y}$ for all $\boldsymbol{y}$ satisfying $\boldsymbol{y} \in P$ and $\boldsymbol{y} \neq \boldsymbol{x}$.
NOTE: ($\boldsymbol{x}$ is a vertex of P if and only if P is on one side of a hyperplane)

Consider a polyhedron $P \subset \mathbb{R}^n$ defined in terms of linear equality and inequality constraints:

$$\begin{aligned} \boldsymbol{a}_i^\mathsf{T} \boldsymbol{x} &\geq b_i, \quad i \in M_1 && (1) \\ \boldsymbol{a}_i^\mathsf{T} \boldsymbol{x} &\leq b_i, \quad i \in M_2 && (2) \\ \boldsymbol{a}_i^\mathsf{T} \boldsymbol{x} &= b_i, \quad i \in M_3 && (3) \end{aligned}$$

where M_1, M_2, M_3 are finite index sets.

Definition 9 (Active or binding constraints)

If a vector $\boldsymbol{x}^*$ satisfies $\boldsymbol{a}_i^\mathsf{T} \boldsymbol{x}^* = b_i$ for some $i \in M_1, M_2, M_3$, we say that the corresponding constraint is active or binding at $\boldsymbol{x}^*$.

Theorem 10

Le $\boldsymbol{x}^*$ be an element of $\mathbb{R}^n$ and let $I = \{i \mid \boldsymbol{a}_i^\mathsf{T} \boldsymbol{x} = b_i\}$ be the set of indices of constraints that are active at $\boldsymbol{x}^*$. Then, the following are equivalent:

1. *There exist n vectors in the set $\{\boldsymbol{a}_i \mid i \in I\}$, which are linearly independent.*
2. *The span of the vectors $\boldsymbol{a}_i, i \in I$, is all of $\mathbb{R}^n$, that is, every element of $\mathbb{R}^n$ can be expressed as a linear combination of the vectors $\boldsymbol{a}_i, i \in I$.*
3. *The system of equations $\boldsymbol{a}_i^\mathsf{T} \boldsymbol{x} = b_i, i \in I$, has a unique solution.*

(Sketch of Proof):

Definition 11 (Basic solution and basic feasible solution)

Consider a polyhedron P defined by linear equality and inequality constraints, and let x^* be an element of $\mathbb{R}^n$.

1. The vector x^* is a basic solution if: (i) All equality constraints are active; (ii) Out of the constraints that are active at x^*, there are n of them that are linearly independent.

2. If x^* is a basic solution that satisfies all of the constraints, we say that it is a **Basic Feasible Solution (BFS)**. *(TO SHOW)*

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Theorem 12

Let P be a nonempty polyhedron and let $x^* \in P$. Then, the following are equivalent:

1. *x^* is a vertex.*
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Corollary 13

Given a finite number of linear inequality constraints, there can be only a finite number of basic or basic feasible solutions.

Proof. + Example unit cube.

Assumption: Given a polyhedron in standard form $P = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0\}$, we will assume next that the m rows of $\boldsymbol{A}$ are linearly independent.

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Theorem 14

Consider the constraints $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$ and assume that the $m \times n$ matrix $\boldsymbol{A}$ has linearly independent rows. A vector $\boldsymbol{x} \in \mathbb{R}^n$ is a basic solution if and only if we have $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, and there exist indices $B(1), \ldots, B(m)$ such that

1. *The columns $A_{B(1)}, \ldots, A_{B(m)}$ are linearly independent.*
2. *If $i \neq B(1), \ldots, B(m)$, then $x_i = 0$.*

Procedure for constructing basic solutions

1. Choose m linearly independent columns $A_{B(1)}, \ldots, A_{B(m)}$.
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$.
3. Solve the system of m equations $\boldsymbol{Ax} = \boldsymbol{b}$ for the unknowns $x_{B(1)}, \ldots, x_{B(m)}$.

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If we get a nonnegative vector $\boldsymbol{x} \geq 0$, we have a *basic feasible solution*, and the variables $x_{B(1)}, \ldots, x_{B(m)}$ are called basic variables. The remaining variables are nonbasic.

The columns $A_{B(1)}, \ldots, A_{B(m)}$ are called the basic columns, and, since they are linearly independent, they form a basis of $\mathbb{R}^m$.

Two bases are distinct if they are defined by two different set of basic indexes $B(1), \ldots, B(m)$.

By arranging the m basic columns next to each other, we obtain an $m \times m$ matrix $\boldsymbol{B}$, called a basis matrix.

We can hence solve ($\boldsymbol{B}$ is invertible)

$$\boldsymbol{B}\boldsymbol{x}_B = \boldsymbol{b} \qquad \Rightarrow \qquad \boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}.$$

Example: Basic solutions

Consider the constraints $\boldsymbol{Ax} = \boldsymbol{b}$ given as follows.

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}$$

QUESTION: Which columns can give a basic solution? A basic feasible solution?

Rank of a matrix A

Let A be a matrix of dimensions $m\times$.

1. The column space of A is the subspace of $\mathbb{R}^m$ spanned by the columns of A.
2. The row space of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.
3. The dimension of the column space is always equal to the dimension of the rows space, and this number is called the rank of A, and it is denoted by $\text{rank}(A)$. Clearly, $\text{rank}(A) \leq \min\{m, n\}$.
4. The matrix A is said to have full rank if $\text{rank}(A) = \min\{m, n\}$.

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Theorem 15

Let $P = \{x \mid Ax = b, x \geq 0\}$ be a nonempty polyhedron, where A is a matrix of dimension $m \times n$, with rows $a_{i_1}^T, \ldots, a_{i_m}^T$. Suppose that $\text{rank}(A) = k < m$ and that the rows $a_{i_1}^T, \ldots, a_{i_k}^T$ are linearly independent. Consider the polyhedron

$$Q = \{x \mid a_{i_1}^T x = b_{i_1}, \ldots, a_{i_k}^T x = b_{i_k}, x \geq 0\}.$$

Then,

$$P = Q.$$

Example: Rank assumption on A

Consider the polyhedron defined by the constraints

$$\begin{array}{rrrcl} 2x_1 + & x_2 & +x_3 & \leq & 2 \\ x_1 + & x_2 & & = & 1 \\ x_1 & & +x_3 & = & 1 \\ & x_1, x_2, x_3 \geq 0 & & & \end{array}$$

QUESTION: Can we found an equivalent polyhedron?

Definition 16 (Degeneracy)

A basic solution $x \in \mathbb{R}^n$ is said to be degenerate if more than n of the constraints are active at x.

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Example: Degeneracy

Consider the polyhedron P defined by the constraints

$$\begin{array}{rrrcr} 2x_1 + & x_2 + & x_3 & \leq & 8 \\ & x_2 + & 6x_3 & \leq & 12 \\ x_1 & & & \leq & 4 \\ & x_2 & & \leq & 6 \\ x_1, x_2, x_3 \geq 0 & & & & \end{array}$$

Definition 17 (Degeneracy in standard form polyhedra)

Consider the standard form polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ and let $\boldsymbol{x}$ be a basic solution. Let m be the number of rows of $\boldsymbol{A}$. The vector $\boldsymbol{x}$ is a degenerate basic solution if more than $(m - n)$ of the components of $\boldsymbol{x}$ are zero.

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Example: Degeneracy

Consider the polyhedron P defined by the constraints

$$\begin{aligned}
2x_1 + x_2 + x_3 + x_4 &= 8 \\
x_2 + 6x_3 + x_5 &= 12 \\
x_1 + x_6 &= 4 \\
x_2 + x_7 &= 6 \\
x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\geq 0
\end{aligned}$$

Definition 18 (Line contained in a polyhedra)

A polyhedron $P \subset \mathbb{R}^n$ contains a line if there exists a vector $\boldsymbol{x} \in P$ and a nonzero vector $\boldsymbol{d} \in \mathbb{R}^n$ such that $\boldsymbol{x} + \lambda \boldsymbol{d} \in P$ for all scalars λ.

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Theorem 19

Suppose that the polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}_i^\mathsf{T} \geq b_i, i = 1, \ldots, m\}$ is nonempty. Then, the following are equivalent:

- *The polyhedron P has at least one extreme point.*
- *The polyhedron P does not contain a line.*
- *There exist n vectors out of the family $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$, which are linearly independent.*

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NOTE 2: The positive orthant $\{\boldsymbol{x} \mid \boldsymbol{x} \geq \boldsymbol{0}\}$ does not contain a line.

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Corollary 20

Every nonempty bounded polyhedron and every nonempty polyhedron in a standard form has at least one basic feasible solution.

Theorem 21

Consider the linear programming problem of minimizing $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$ over a polyhedron P. Suppose that P has at least one extreme point and that there exists an optimal solution. Then, there exists an optimal solution which is an extreme point of P.

Theorem 21

Consider the linear programming problem of minimizing $\boldsymbol{c}^\top \boldsymbol{x}$ over a polyhedron P. Suppose that P has at least one extreme point and that there exists an optimal solution. Then, there exists an optimal solution which is an extreme point of P.

Corollary 22

Consider the linear programming problem of minimizing $\boldsymbol{c}^\top \boldsymbol{x}$ over a nonempty polyhedron P. Then, either the optimal cost is equal to $-\infty$ or there exists an optimal solution.

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Theorem 23

A nonempty and bounded polyhedron is the convex hull of its extreme points.

Theorem 24

Consider the linear programming problem of minimizing $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$ over a polyhedron P. Suppose the P has at least one extreme point. Then, either the optimal cost is equal to $-\infty$, or there exists an extreme point which is optimal.

(Sketch of Proof):

Solution of a Linear Programming problem

- If the feasible set is nonempty and bounded, there exists an optimal solution. In addition, there is an optimal solution which is an extreme point.
- If the feasible set is unbounded, there are the following possibilities:
  1. There exists an optimal solution which is an extreme point.
  2. There exists an optimal solution, but no optimal solution is an extreme point. (This can only happen if the feasible set has not extreme points; it never happens when the problem is standard form).
  3. The optimal cost is $-\infty$.

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- If the feasible set is unbounded, there are the following possibilities:
  1. There exists an optimal solution which is an extreme point.
  2. There exists an optimal solution, but no optimal solution is an extreme point. (This can only happen if the feasible set has not extreme points; it never happens when the problem is standard form).
  3. The optimal cost is $-\infty$.

Geometric and algebraic representations

1. Whether or not a point is an extreme point (equivalently, vertex, or basic feasible solution) is a geometric property.
2. Whether or not a point is a basic solution may depend on the way that a polyhedron is represented.
3. Whether or not a basic or basic feasible solution is degenerate may depend on the way that a polyhedron is represented.

References for these slides

- Chap. 2 in Bertsimas, Dimitris, and John N. Tsitsiklis. **Introduction to Linear Optimization**, 1997. Athena Scientific.