Building Linear Programming Models

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The importance of Linearity

History of LP

Is the world Nonlinear or Linear?

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Koopmans, Dantzig, and Kantorovich. Koopmans and Kantorovich shared the Nobel Prize for economics in 1975. Dantzig's development of linear programming was fundamental to their work. Koopmans expressed regret that Dantzig was not named to share the honor.

Modeling for Decision-Making Processes

In **Operations Research**, the main focus is the study of MODELS and ALGORITHMS, which can support a *decision-making process*.

A decision-making process is divided into

1. Detection of the decision-making problem.
2. Fact analysis and data collection.
3. **Model building** as an abstraction of the real problem.
4. **Computation** of one or more solutions.
5. Analysis of the results obtained and their interpretation in relation to the real case.

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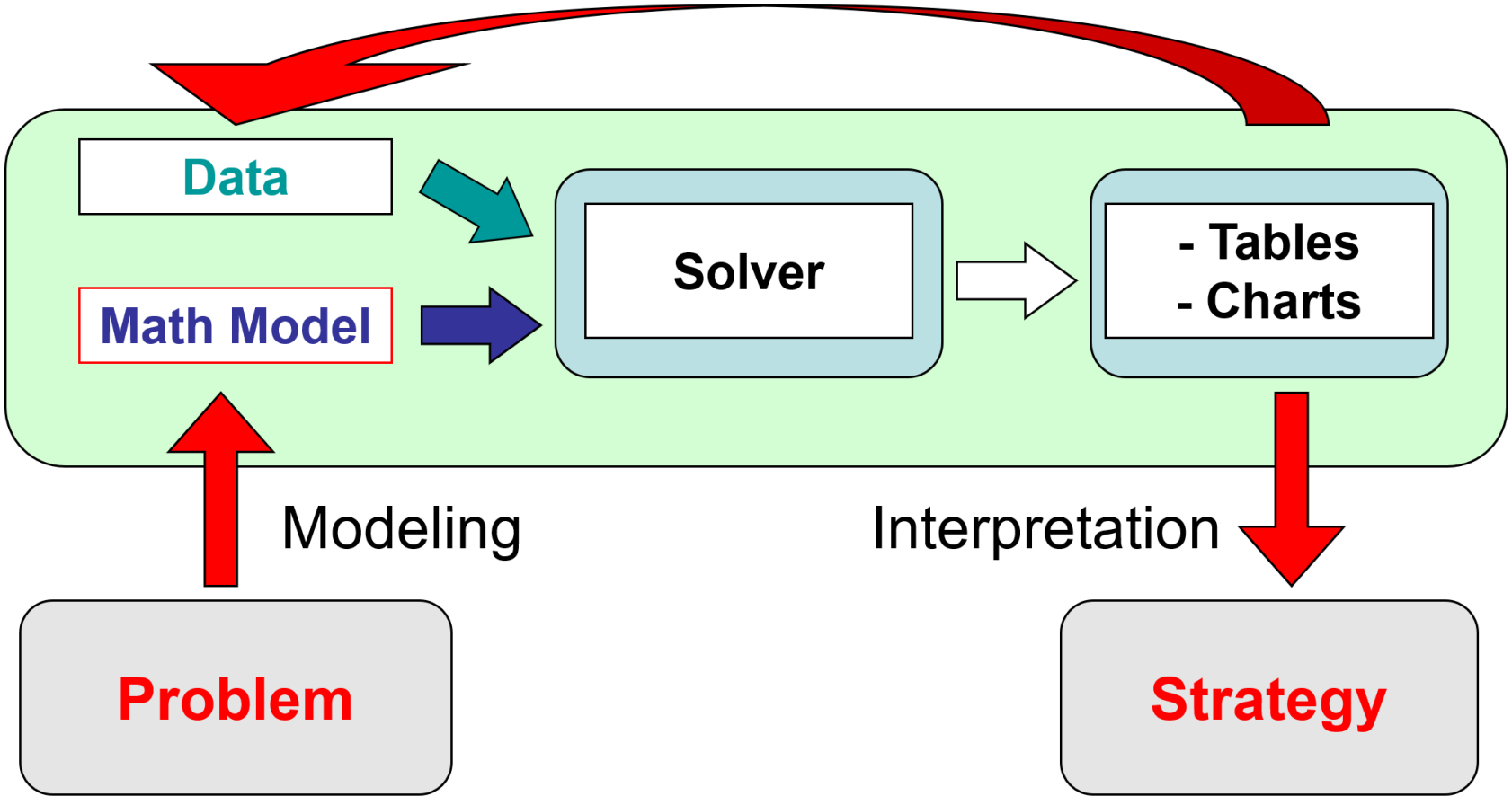
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Model Abstraction

The model is "just" an abstraction permitting to describe in mathematical terms the outstanding features of the problem we want to study and solve.

Modeling as closed-loop activity



Types of Decision-Making Models

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3. **Simulation Models:** The investigated problem is very complex and the data of the problem are very uncertain (stochastic) variables. The idea is to reproduce in the most accurate possible way the behavior of the system we want to examine, and then to perform scenario analyses.

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Model Validation: When we interpret the results of models, we must never forget that they are an *abstraction of reality*. It is strictly necessary to consider with the utmost attention the soundness of the model: it should represent with good accuracy the relevant features of the decision-making problem.

Optimization Problems

Given $f, g : \mathbb{R}^n \times \mathbb{Z}^p \to \mathbb{R}$, we are interested in solving the following Non-convex Global Optimization problem:

$$\begin{aligned} \min \quad & f(\boldsymbol{x}, \boldsymbol{y}) && && (1)\\ \text{s.t.} \quad & g_i(\boldsymbol{x}, \boldsymbol{y}) \leq 0 && i = 1, \ldots, m. && (2)\\ & \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{y} \in \mathbb{Z}^p. && && (3) \end{aligned}$$

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- If $p = 0$ and f, g_i are convex functions $\to$ Convex Optimization
- If $p = 0$ and f, g_i are linear functions $\to$ Linear Programming
- If f, g_i are linear functions $\to$ Integer Linear Programming
- If $n = 0$, $\boldsymbol{y} \in \{0, 1\}^p \to$ Combinatorial Optimization

Optimization Solvers

- Non-convex Constrained Optimization: Baron, Couenne
- Unconstrained Optimization: IPOPT, L-BFGS
- Convex Optimization: Bonmin, CVX, Mosek
- (Integer) Linear Programming
  - Commercial: CPLEX, GUROBI, FICO (Quadratic Pr.)
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Modeling Languages

Modeling languages permit to write easily optimization problems:

- PYOMO, PuIP, AMPL, GAMS, ...

Outline

1 Introduction

2 Classical Examples

3 LP Modeling


tella®
SERVIZIO CONSUMATORI
TEL.011/2876600
8013 5463
7150803
Valori nutrizionali medi | per 100 g | per porzione (15 g) | (%*) per 15 g
Energia | 2252 kJ/ 539 kcal | 336 kJ/ 80 kcal | 4
grassi | 30,9 g | 4,6 g | 7
di cui: acidi grassi saturi | 10,6 g | 1,6 g | 8
carboidrati | 57,5 g | 8,6 g | 3
di cui: zuccheri | 56,3 g | 8,4 g | 9
proteine | 6,3 g | 0,9 g | 2
sale | 0,107 g | 0,016 g | 0
* Assunzioni di riferimento di un adulto medio (8400 kJ/ 2000 kcal)

Diet Problem: Stigler, 1945, *Journal of Farm Economics*



Life: 1911 - 1981

THE COST OF SUBSISTENCE

GEORGE J. STIGLER
University of Minnesota

ELABORATE investigations have been made of the adequacy of diets at various income levels, and a considerable number of "low-cost," "moderate," and "expensive" diets have been recommended to consumers. Yet, so far as I know, no one has determined the minimum cost of obtaining the amounts of calories, protein, minerals, and vitamins which these studies accept as adequate or optimum. This will be done in the present paper, not only for its own interest but because it sheds much light on the meaning of conventional "low-cost" diets.

1982: Nobel Memorial Price in Economic Sciences

Model 1: Diet Problem

"The minimum cost of an adequate diet is obviously governed by the nutritive values and costs of the foods eligible for inclusion."

Daily allowances of nutrients for a moderately active man (weighing 154 pounds or 69.85 Kg). Source: National Research Council, Recommended Dietary Allowances, Reprint and Circular Series No. 115, January, 1943.

Nutrient	Allowance	
Calories	3 000	calories
Protein	70	grams
Calcium	0.8	grams
Iron	12	milligrams
Vitamin A	5 000	International Units
Thiamine (B1)	1.8	milligrams
Riboflavin (B2 or G)	2.7	milligrams
Niacin (Nicotinic Acid)	18	milligrams
Ascorbic Acid (C)	75	milligrams

TABLE A. NUTRITIVE VALUES OF COMMON FOODS PER DOLLAR OF EXPENDITURE, AUGUST 15, 1939

Commodity	Unit	Price Aug. 15, 1939 (cents)	Edible Weight per $1.00 (grams)	Calories (1,000)	Protein (grams)	Calcium (grams)	Iron (mg.)	Vitamin A (1,000 I.U.)	Thiamine (mg.)	Ribo-flavin (mg.)	Niacin (mg.)	Ascorbic Acid (mg.)
**1. Wheat Flour (Enriched)	10 lb.	36.0	12,600	44.7	1,411	2.0	365		55.4	33.3	441	
2. Macaroni	1 lb.	14.1	3,217	11.6	418	.7	54		3.2	1.9	68	
3. Wheat Cereal (Enriched)	28 oz.	24.2	3,280	11.8	377	14.4	175		14.4	8.8	114	
4. Corn Flakes	8 oz.	7.1	3,194	11.4	252	.1	56		13.5	2.3	68	
5. Corn Meal	1 lb.	4.6	9,861	36.0	897	1.7	99	30.9	17.4	7.9	106	
6. Hominy Grits	24 oz.	8.5	8,005	28.6	680	.8	80		10.6	1.6	110	
7. Rice	1 lb.	7.5	6,048	21.2	460	.6	41		2.0	4.8	60	
8. Rolled Oats	1 lb.	7.1	6,389	25.3	907	5.1	341		37.1	8.9	64	
9. White Bread (Enriched)	1 lb.	7.9	5,742	15.6	488	2.5	115		13.8	8.5	126	
10. Whole Wheat Bread	1 lb.	9.1	4,985	12.2	484	2.7	125		13.9	6.4	160	
11. Rye Bread	1 lb.	9.2	4,930	12.4	439	1.1	82		9.9	3.0	66	
12. Pound Cake	1 lb.	24.8	1,829	8.0	130	.4	31	18.9	2.8	3.0	17	
13. Soda Crackers	1 lb.	15.1	3,004	12.5	288	.5	50					
14. Milk	1 qt.	11.0	8,867	6.1	310	10.5	18	16.8	4.0	16.0	7	177
**15. Evaporated Milk (can)	14½ oz.	6.7	6,035	8.4	422	15.1	9	26.0	3.0	23.5	11	60
16. Butter	1 lb.	30.8	1,473	10.8	9	.2	3	44.2		.2	2	
*17. Oleomargarine	1 lb.	16.1	2,817	20.6	17	.6	6	55.8	.2			
18. Eggs	1 doz.	32.6	1,857	2.9	238	1.0	52	18.6	2.8	6.5	1	
**19. Cheese (Cheddar)	1 lb.	24.2	1,874	7.4	448	16.4	19	28.1	.8	10.3	4	
20. Cream	½ pt.	14.1	1,689	3.5	49	1.7	3	16.9	.6	2.5		17
21. Peanut Butter	1 lb.	17.9	2,534	15.7	661	1.0	48		9.6	8.1	471	
22. Mayonnaise	½ pt.	16.7	1,198	8.6	18	.2	8	2.7	.4	.5		
23. Crisco	1 lb.	20.3	2,234	20.1								
24. Lard	1 lb.	9.8	4,628	41.7				.2		.5	5	
25. Sirloin Steak	1 lb.	39.6	1,145*	2.9	166	.1	34	.2	2.1	2.9	69	
26. Round Steak	1 lb.	36.4	1,246*	2.2	214	.1	32	.4	2.5	2.4	87	
27. Rib Roast	1 lb.	29.2	1,553*	3.4	213	.1	33			2.0		
28. Chuck Roast	1 lb.	22.6	2,007*	3.6	309	.2	46	.4	1.0	4.0	120	
29. Plate	1 lb.	14.6	3,107*	8.5	404	.2	62		.9			
**30. Liver (Beef)	1 lb.	26.8	1,692*	2.2	333	.2	139	169.2	6.4	50.8	316	525
31. Leg of Lamb	1 lb.	27.6	1,643*	3.1	245	.1	20		2.8	3.9	86	
32. Lamb Chops (Rib)	1 lb.	36.6	1,239*	3.3	140	.1	15		1.7	2.7	54	
33. Pork Chops	1 lb.	30.7	1,477*	3.5	196	.2	30		17.4	2.7	60	
34. Pork Loin Roast	1 lb.	24.2	1,874*	4.4	249	.3	37		18.2	3.6	79	
35. Bacon	1 lb.	25.6	1,772*	10.4	152	.2	23		1.8	1.8	71	
36. Ham—smoked	1 lb.	27.4	1,655*	6.7	212	.2	31		9.9	3.3	50	
37. Salt Pork	1 lb.	16.0	2,835*	18.8	164	.1	26		1.4	1.8		
38. Roasting Chicken	1 lb.	30.3	1,497*	1.8	184	.1	30	.1	.9	1.8	68	46
39. Veal Cutlets	1 lb.	42.3	1,072*	1.7	156	.1	24		1.4	2.4	57	
40. Salmon, Pink (can)	16 oz.	13.0	3,489	5.8	705	6.8	45	3.5	1.0	4.9	209	
41. Apples	1 lb.	4.4	9,072	5.8	27	.5	36	7.3	3.6	2.7	5	544
42. Bananas	1 lb.	6.1	4,982	4.9	60	.4	30	17.4	2.5	3.5	28	498
43. Lemons	1 doz.	26.0	2,380	1.0	21	.5	14		.5		4	952
44. Oranges	1 doz.	30.9	4,439	2.2	40	1.1	18	11.1	3.6	1.3	10	1,998
*45. Green Beans	1 lb.	7.1	5,750	2.4	138	3.7	80	69.0	4.3	5.8	37	862
**46. Cabbage	1 lb.	3.7	8,949	2.6	125	4.0	36	7.2	9.0	4.5	26	5,369
47. Carrots	1 bunch	4.7	6,080	2.7	73	2.8	43	188.5	6.1	4.3	89	608
48. Celery	1 stalk	7.3	3,915	.9	51	3.0	23	.9	1.4	1.4	9	313
49. Lettuce	1 head	8.2	2,247	.4	27	1.1	22	112.4	1.8	3.4	11	449
*50. Onions	1 lb.	3.6	11,844	5.8	166	3.8	59	16.6	4.7	5.9	21	1,184

*51. Potatoes	15 lb.	34.0	16,810	14.3	336	1.8	118	6.7	29.4	7.1	198	2,522
**52. Spinach	1 lb.	8.1	4,592	1.1	106	—	138	918.4	5.7	13.8	33	2,755
**53. Sweet Potatoes	1 lb.	5.1	7,649	9.6	138	2.7	54	290.7	8.4	5.4	83	1,912
54. Peaches (can)	No. 2½	16.8	4,894	3.7	20	.4	10	21.5	.5	1.0	31	196
55. Pears (can)	No. 2½	20.4	4,030	3.0	8	.3	8	.8	.8	.8	5	81
56. Pineapple (can)	No. 2½	21.3	3,993	2.4	16	.4	8	2.0	2.8	.8	7	399
57. Asparagus (can)	No. 2	27.7	1,945	.4	33	.3	12	16.3	1.4	2.1	17	272
58. Green Beans (can)	No. 2	10.0	5,386	1.0	54	2.0	65	53.9	1.6	4.3	32	431
59. Pork and Beans (can)	16 oz.	7.1	6,389	7.5	364	4.0	134	3.5	8.3	7.7	56	
60. Corn (can)	No. 2	10.4	5,452	5.2	136	.2	16	12.0	1.6	2.7	42	218
61. Peas (can)	No. 2	13.8	4,109	2.3	136	.6	45	84.9	4.9	2.5	37	370
62. Tomatoes (can)	No. 2	8.6	6,263	1.3	63	.7	38	53.2	3.4	2.5	36	1,253
63. Tomato Soup (can)	10½ oz.	7.6	3,917	1.6	71	.6	43	57.9	3.5	2.4	67	862
*64. Peaches, Dried	1 lb.	15.7	2,889	8.5	87	1.7	173	86.8	1.2	4.3	55	57
*65. Prunes, Dried	1 lb.	9.0	4,284	12.8	99	2.5	154	85.7	3.9	4.3	65	257
66. Raisins, Dried	15 oz.	9.4	4,524	13.5	104	2.5	136	4.5	6.3	1.4	24	136
67. Peas, Dried	1 lb.	7.9	5,742	20.0	1,367	4.2	345	2.9	28.7	18.4	162	
**68. Lima Beans, Dried	1 lb.	8.9	5,097	17.4	1,055	3.7	459	5.1	26.9	38.2	93	
**69. Navy Beans, Dried	1 lb.	5.9	7,688	26.9	1,691	11.4	792		38.4	24.6	217	
70. Coffee	1 lb.	22.4	2,025	—	—	—	—		4.0	5.1	50	
71. Tea	¼ lb.	17.4	652	—	—	—	—			2.3	42	
72. Cocoa	8 oz.	8.6	2,637	8.7	237	3.0	72		2.0	11.9	40	
73. Chocolate	8 oz.	16.2	1,400	8.0	77	1.3	39		.9	3.4	14	
74. Sugar	10 lb.	51.7	8,773	34.9	—	—	—					
75. Corn Sirup	24 oz.	13.7	4,966	14.7	—	.5	74				5	
76. Molasses	18 oz.	13.6	3,752	9.0	—	10.3	244		1.9	7.5	146	
77. Strawberry Preserves	1 lb.	20.5	2,213	6.4	11	.4	7	.2	.2	.4	3	

* Quantities including inedible portions.

Model 1: Diet Problem as Linear Programming

- $I = \{1, \ldots, n\}$ is the index set of foods
- $J = \{1, \ldots, m\}$ is the index set of nutrients

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- $J = \{1, \ldots, m\}$ is the index set of nutrients
- $A \in \mathbb{R}^{m \times n}$ is the food-to-nutrients matrix, where a_{ij} gives the amount of the i-th nutrient contained in the j-th food:

$$A = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{\text{foods}} \text{nutrients}$$

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- $\boldsymbol{b} \in \mathbb{R}^m$ is the vector of minimal allowance for each nutrient
- $\boldsymbol{c} \in \mathbb{R}^n$ is the vector of unitary costs for each food (as given in Table A, previous slide)

Model 1: Solving the Diet Problem with Linear Programming

$$\begin{aligned} \min \quad & \boldsymbol{c}^{\intercal}\boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

Model 1: Solving the Diet Problem with Linear Programming

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QUESTION: Do you think this model is valid? Any pitfall?

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But this is not the end of the diet tale. In the early 1950s, I moved to Santa Monica to work for the RAND Corporation. My doctor advised me to go on a diet to lose weight. I decided I would model my diet problem as a linear program and let the computer decide my diet. Some revisions of the earlier model, of course, would be necessary in order to give me a greater variety of foods to choose from; the calorie intake had to be reduced to under 1,500 calories per day; and the objective function had to be changed (I wasn't interested in saving money). I said to myself: "The trouble with a diet is that one's always hungry. What I need to do is maximize the feeling of feeling full." After giving much thought

Model 1: Solving the Diet Problem with Linear Programming

$$\begin{aligned} \min \quad & \boldsymbol{c}^\intercal \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

QUESTION: Do you think this model is valid? Any pitfall?

But this is not the end of the diet tale. In the early 1950s, I moved to Santa Monica to work for the RAND Corporation. My doctor advised me to go on a diet to lose weight. I decided I would model my diet problem as a linear program and let the computer decide my diet. Some revisions of the earlier model, of course, would be necessary in order to give me a greater variety of foods to choose from; the calorie intake had to be reduced to under 1,500 calories per day; and the objective function had to be changed (I wasn't interested in saving money). I said to myself: "The trouble with a diet is that one's always hungry. What I need to do is maximize the feeling of feeling full." After giving much thought

One day I said to Anne, my wife, "Today is Der Tag, whatever the 701 says that's what I want you to feed me each day starting with supper tonight."

Around 5:00 PM, Anne called, "Nu, it's five and you haven't called. What should I be cooking?" I replied that she didn't really want to know. I then read off the amounts of foods in the optimal diet. Her reaction: "The diet is a bit weird but conceivable. Is that it?"

"Not exactly," I replied, "AND 500 gallons of vinegar." She thought it funny and laughed.

Model 1: Solving the Diet Problem with Linear Programming

$$\begin{aligned} \min \quad & \boldsymbol{c}^\intercal \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

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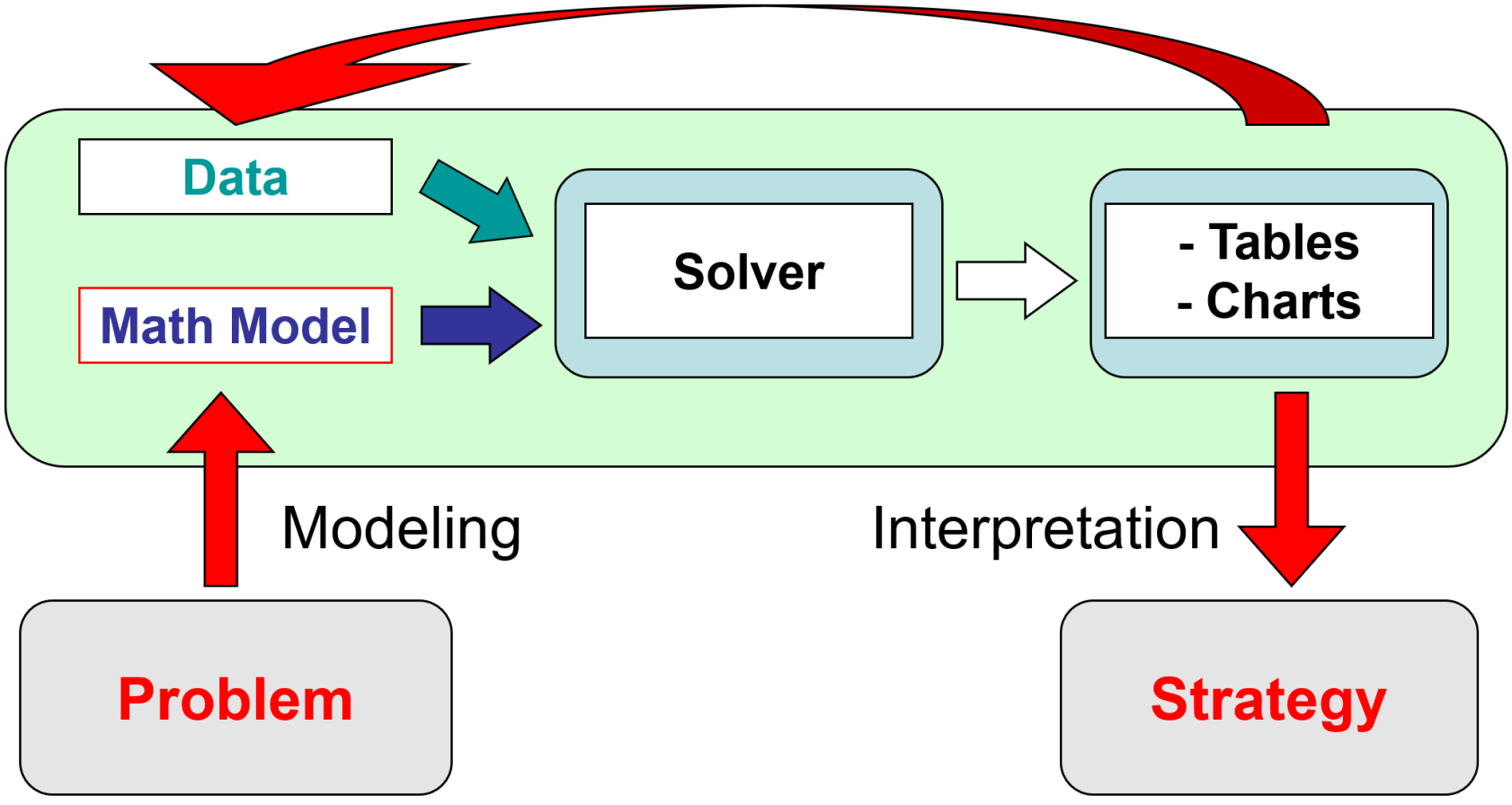
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At this point Anne got tired of the whole game. Speaking firmly so that I would know who was boss, she said,
"I have been studying the various menus the computer has been generating. There are some good ideas there that I can use. I'll put you on MY diet."
She did and I lost 22 pounds.

Modeling as closed-loop activity



Model 1: Solving the Diet Problem with Linear Programming

$$\begin{aligned} \min \quad & \boldsymbol{c}^\intercal \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{b} \leq A\boldsymbol{x} \leq \boldsymbol{d} \\ & 0 \leq \boldsymbol{x} \leq \boldsymbol{u} \end{aligned}$$

where

- $\boldsymbol{d} \in \mathbb{R}^m$ is a vector of maximum allowance for each nutrient
- $\boldsymbol{u} \in \mathbb{R}^n$ is a vector of maximum quantity for each food

Online demo at:
https://neos-guide.org/content/diet-problem-solver

[1] G.B. Dantzig. *The Diet Problem*. Interface, vol. 20(4), pp. 43–47, 1990.

[2] G.J. Stigler. *The Cost of Subsistence*. Journal of Farm Economics, vol. 27(2), pp. 303–314, 1945.

Tolstoj's Transportation Problem

	Arkhangelsk	Yaroslavl'	Murom	Balakhonikha	Dzerzhinsk	Kishert'	Sverdlovsk	Artemovsk	Iledzhk	Dekonskaya	*Demand*
Agryz				709	1064	693					2
Aleksandrov					397			1180			4
Almaznaya								81		65	1.5
Alchevskaya								106		114	4
Baku								1554		1563	10
Barybino								985		968	2
Berendeevo		135			430						10
Bilimbai						200	59				1
Bobrinskaya								655		663	10
Bologoe		389						1398			1
Verkhov'e								678		661	1
Volovo								757		740	3
Vologda	634					1236					2
Voskresensk				427				1022		1005	1
V.Volochek		434						1353		1343	5
Galich	815	224				1056					0.5
Goroblagodatskaya						434	196				0.5
Zhlobin								882		890	8
Zverevo								227		235	5
Ivanovo					259						6
Inza				380	735					1272	2
Kagan								2445	2379		0.5
Kasimov			0								1
Kinel'				752		1208			454	1447	2
Kovylkino				355						1213	2
Kyshtym						421	159				3
Leningrad	1237	709						1667		1675	55
Likino			223		328						15
Liski								443		426	1
Lyuberdzhy			268		411					1074	1
Magnitogorskaya						932	678		818		1
Mauk						398	136				5
Moskva			288	378	405			1030		1022	141
Navashino			12	78							2

Tolstoj's Transportation Problem

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Navashino			12	78							2
Nizhegol'								333		316	1
Nerekhta		50			349						5
Nechaevskaya			92								0.5
N.-Novgorod					32						25
Omsk						1159	904		1746		5
Orenburg									76		1.5
Penza				411				1040	883	1023	7
Perm'	1749					121					1
Petrozavodsk	1394										1
Poltoradzhk								1739	3085	1748	4
Pskov								1497		1505	10
Rostov/Don								287		296	20
Rostov/Yarosl		56			454						2
Rtishchevo								880		863	1
Savelovo		325						1206		1196	5
Samara				711					495	1406	7
San-Donato						416	157				1
Saratov								1072		1055	15
Sasovo				504				1096		1079	1
Slavyanoserbsk								119		115	1.1
Sonkovo		193						1337			0.5
Stalingrad								624		607	15.4
St.Russa		558						1507		1515	5
Tambov								783		766	4
Tashkent								3051	1775		3
Tula								840		848	8
Tyumen'						584	329				6
Khar'kov								251		259	60
Chelyabinsk						511	257		949		2
Chishmy				1123		773			889		0.5
Shchigry								566		549	4
Yudino				403	757	999					0.5
Yama								44		52	5
Yasinovataya								85		93	6
Supply	5	11.5	8.5	12	100	12	15	314	10	55	543

Data:

Tolstoj's Transportation Problem

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Data:

- $I = \{1, \ldots, m\}$ set of destinations

Tolstoj's Transportation Problem

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- $J = \{1, \ldots, n\}$ set of sources

Tolstoj's Transportation Problem

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Data:

- $I = \{1, \ldots, m\}$ set of destinations
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- $C \in \mathbb{R}^{m \times n}$ with c_{ij} distance in km from i to j

Tolstoj's Transportation Problem

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- $J = \{1, \ldots, n\}$ set of sources
- $C \in \mathbb{R}^{m \times n}$ with c_{ij} distance in km from i to j
- $\boldsymbol{d} \in \mathbb{R}^m$ demands in tons

Tolstoj's Transportation Problem

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Baku								1554		1563	10
Barybino								985		968	2
Berendeevo		135			430						10
Bilimbai						200	59				1
Bobrinskaya								655		663	10
Bologoe		389						1398			1
Verkhov'e								678		661	1
Volovo								757		740	3
Vologda	634					1236					2
Voskresensk				427				1022		1005	1
V.Volochek		434						1353		1343	5
Galich	815	224				1056					0.5
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Zverevo								227		235	5
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Inza				380	735					1272	2
Kagan								2445	2379		0.5
Kasimov			0								1
Kinel'				752		1208			454	1447	2
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Moskva			288	378	405			1030		1022	141
Navashino			12	78							2
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Nechaevskaya			92								0.5
N.-Novgorod					32						25
Omsk						1159	904		1746		5
Orenburg									76		1.5
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Khar'kov								251		259	60
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Chishmy				1123		773			889		0.5
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Decision Variables:

$x_{ij} \geq 0$ quantity of material to i from j

Objective Function:

$$\min \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$$

Constraints:

$$\sum_{j \in J} x_{ij} \geq d_i, \forall i \in I$$
$$\sum_{i \in I} x_{ij} \leq s_j, \forall j \in J$$

Model 3: Multiperiod Planning of Electric Power Capacity

Problem Description:

A state wants to plan its electricity capacity for the next T years. The state has a forecast of d_t megawatts, presumed accurate, of the demand for electricity during year $t = 1, \ldots, T$. The existing capacity, which is in oil-fired plants, that will not be retired and will be available during year t, is e_t. There are two alternatives for expanding electric capacity: coal fired or nuclear power plants. There is a capital cost of c_t per megawatt of coal-fired capacity that becomes operational at the beginning of year t. The corresponding capital cost for nuclear power plants is n_t. For various political and safety reasons, it has been decided that no more than 20% of the total capacity should ever be nuclear. Coal plants last for 20 years, while nuclear plants last for 15 years.

A least cost capacity expansion plan is desired.

Data

T	number of years on the planning horizon
d_t	forecast of demand capacity at begging of period t
e_t	current available capacity at begging of period t
c_t	cost of expansion of coal capacity at begging of period t
n_t	cost of expansion of nuclear capacity at begging of period t
r	max ratio (percentage) of nuclear capacity over total capacity
T_c, T_n	duration in years of coal and nuclear plants

Data

T	number of years on the planning horizon
d_t	forecast of demand capacity at begging of period t
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r	max ratio (percentage) of nuclear capacity over total capacity
T_c, T_n	duration in years of coal and nuclear plants

Decision Variables: Expansion Production Plan

x_t	amount of coal capacity started at begging of period t
y_t	amount of nuclear capacity started at begging of period t

Data

T	number of years on the planning horizon
d_t	forecast of demand capacity at begging of period t
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n_t	cost of expansion of nuclear capacity at begging of period t
r	max ratio (percentage) of nuclear capacity over total capacity
T_c, T_n	duration in years of coal and nuclear plants

Decision Variables: Expansion Production Plan

x_t	amount of coal capacity started at begging of period t
y_t	amount of nuclear capacity started at begging of period t

Objective Function: Minimize cost of the expansion plan

$$\min \quad \sum_{t=1}^{T} (c_t x_t + n_t y_t)$$

Constraints

Total capacity of each year, given the expansion plan:

$$w_t = \sum_{s=\max\{1,t-T_c\}}^{t} x_s, \quad t = 1, \ldots, T \tag{4}$$

$$z_t = \sum_{s=\max\{1,t-T_n\}}^{t} y_s, \quad t = 1, \ldots, T \tag{5}$$

Constraints

Total capacity of each year, given the expansion plan:

$$w_t = \sum_{s=\max\{1,t-T_c\}}^{t} x_s, \quad t = 1, \ldots, T \tag{4}$$

$$z_t = \sum_{s=\max\{1,t-T_n\}}^{t} y_s, \quad t = 1, \ldots, T \tag{5}$$

Demand forecast in megawatts:

$$w_t + z_t + e_t \geq d_t, \quad t = 1, \ldots, T \tag{6}$$

Constraints

Total capacity of each year, given the expansion plan:

$$w_t = \sum_{s=\max\{1,t-T_c\}}^{t} x_s, \quad t = 1, \ldots, T \tag{4}$$

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Demand forecast in megawatts:

$$w_t + z_t + e_t \geq d_t, \quad t = 1, \ldots, T \tag{6}$$

Max yearly percentage of nuclear capacity:

$$\frac{z_t}{w_t + z_t + e_t} \leq r$$

Constraints

Total capacity of each year, given the expansion plan:

$$w_t = \sum_{s=\max\{1,t-T_c\}}^{t} x_s, \quad t = 1, \ldots, T \tag{4}$$

$$z_t = \sum_{s=\max\{1,t-T_n\}}^{t} y_s, \quad t = 1, \ldots, T \tag{5}$$

Demand forecast in megawatts:

$$w_t + z_t + e_t \geq d_t, \quad t = 1, \ldots, T \tag{6}$$

Max yearly percentage of nuclear capacity:

$$\frac{z_t}{w_t + z_t + e_t} \leq r = 0.2$$

Constraints

Total capacity of each year, given the expansion plan:

$$w_t = \sum_{s=\max\{1,t-T_c\}}^{t} x_s, \quad t = 1, \ldots, T \tag{4}$$

$$z_t = \sum_{s=\max\{1,t-T_n\}}^{t} y_s, \quad t = 1, \ldots, T \tag{5}$$

Demand forecast in megawatts:

$$w_t + z_t + e_t \geq d_t, \quad t = 1, \ldots, T \tag{6}$$

Max yearly percentage of nuclear capacity:

$$\frac{z_t}{w_t + z_t + e_t} \leq r = 0.2 \quad \Rightarrow \quad 0.8z_t - 0.2w_t \leq 0.2e_t, \quad t = 1, \ldots, T \tag{7}$$

Model 3: Full Linear Programming Model (Multiperiod Planning)

$$\min \quad \sum_{t=1}^{T}(c_t x_t + n_t y_t) \tag{8}$$

$$\text{s.t.} \quad w_t = \sum_{s=\max\{1,t-T_c\}}^{t} x_s, \qquad t = 1,\ldots,T \tag{9}$$

$$z_t = \sum_{s=\max\{1,t-T_n\}}^{t} y_s, \qquad t = 1,\ldots,T \tag{10}$$

$$w_t + z_t + e_t \geq d_t, \qquad t = 1,\ldots,T \tag{11}$$

$$0.8z_t - 0.2w_t \leq 0.2e_t, \qquad t = 1,\ldots,T \tag{12}$$

$$x_t, y_t \geq 0, \qquad t = 1,\ldots,T. \tag{13}$$



Model 4: A Scheduling Problem

Problem Description:
A hospital wants to make a weekly night shift (12pm-8am) schedule for its nurses. The demand for nurses for the night shift on day j is an integer d_j, $j = 1, \ldots, 7$. Every nurse works 5 days in a row on the night shift.

The problem is to find the minimal number of nurses the hospital needs to hire.

QUESTION: What do you think is a good choice as **decision variables** for this problem?

Data

7	days of scheduling
5	consecutive days of work for each nurse
d_j	demand of nurses for j-th day

Data

7	days of scheduling
5	consecutive days of work for each nurse
d_j	demand of nurses for j-th day

Decision Variables: Nurses starting at day j

x_j, $j = 1, \ldots, 7$: number of nurses that start their duty at the j-th day. If a nurse starts working at day 5, then she works on days $(5, 6, 7, 1, 2)$.

Data

7	days of scheduling
5	consecutive days of work for each nurse
d_j	demand of nurses for j-th day

Decision Variables: Nurses starting at day j

x_j, $j = 1, \ldots, 7$: number of nurses that start their duty at the j-th day. If a nurse starts working at day 5, then she works on days $(5, 6, 7, 1, 2)$.

Objective Function: Minimize the number of nurses

$$\min \quad \sum_{j=1}^{7} x_j$$

Constraints

The constraints are better explained in expanded form:

$$
\begin{aligned}
x_1 \qquad\qquad + x_4 + x_5 + x_6 + x_7 &\geq d_1 && (14)\\
x_1 + x_2 \qquad\qquad + x_5 + x_6 + x_7 &\geq d_2 && (15)\\
x_1 + x_2 + x_3 \qquad\qquad + x_6 + x_7 &\geq d_3 && (16)\\
x_1 + x_2 + x_3 + x_4 \qquad\qquad + x_7 &\geq d_4 && (17)\\
x_1 + x_2 + x_3 + x_4 + x_5 &\geq d_5 && (18)\\
+x_2 + x_3 + x_4 + x_5 + x_6 &\geq d_6 && (19)\\
+x_3 + x_4 + x_5 + x_6 + x_7 &\geq d_7 && (20)
\end{aligned}
$$

In this formulation, every column represents the consecutive working days of the nurses.

Model 4: Staff Scheduling

$$
\begin{aligned}
\min \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 && (21)\\
\text{s.t.} \quad & x_1 + x_4 + x_5 + x_6 + x_7 \geq d_1 && (22)\\
& x_1 + x_2 + x_5 + x_6 + x_7 \geq d_2 && (23)\\
& x_1 + x_2 + x_3 + x_6 + x_7 \geq d_3 && (24)\\
& x_1 + x_2 + x_3 + x_4 + x_7 \geq d_4 && (25)\\
& x_1 + x_2 + x_3 + x_4 + x_5 \geq d_5 && (26)\\
& + x_2 + x_3 + x_4 + x_5 + x_6 \geq d_6 && (27)\\
& + x_3 + x_4 + x_5 + x_6 + x_7 \geq d_7 && (28)\\
& x_j \geq 0, && (29)
\end{aligned}
$$

QUESTION: Are we missing anything?

Model 4: Staff Scheduling

$$
\begin{aligned}
\min \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 && (21)\\
\text{s.t.} \quad & x_1 + x_4 + x_5 + x_6 + x_7 \geq d_1 && (22)\\
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& x_j \geq 0,\ x_j \in \mathbb{Z} && (29)
\end{aligned}
$$

QUESTION: Are we missing anything?

Outline

1 Introduction

2 Classical Examples

3 LP Modeling

Objective Function

Definition 1

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- A function f of the form

$$f(\boldsymbol{x}) = a_0 + \sum_{i=1}^{n} a_i x_i$$

with $a_i \in \mathbb{R}$, is an **affine** function, and is both convex and concave.

Piecewise Linear Convex Functions

Let $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m \in \mathbb{R}^n$, $d_1, \ldots, d_m \in \mathbb{R}$, and $f : \mathbb{R}^n \to R$ defined by

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□

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We can hence consider the following generalization of LP:

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which can be rewritten as

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & z \geq \boldsymbol{c}_i^\mathsf{T} \boldsymbol{x} + d_i, && i = 1, \ldots, m \\ & A\boldsymbol{x} \geq \boldsymbol{b} \end{aligned}$$

Problems with Absolute Values 1/2

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$$\begin{aligned} \min \quad & \sum_{i=1}^{n} c_i \mid x_i \mid \\ \text{s.t.} \quad & A\boldsymbol{x} \geq \boldsymbol{b} \end{aligned}$$

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Approach 2 (given $c_i > 0$): we introduce new variables $x_i^+, x_i^- \geq 0$, and we set $x_i = x_i^+ - x_i^-$. Then, we replace every occurrence of $\mid x_i \mid$ with $x_i^+ + x_i^-$.

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} c_i (x_i^+ + x_i^-) \\ \text{s.t.} \quad & A\boldsymbol{x}^+ - A\boldsymbol{x}^- \geq \boldsymbol{b} \\ & x_i^+, x_i^- \geq 0 \end{aligned}$$

Model 5: Data Fitting

Given m data points $(\boldsymbol{a}_i, b_i)$ with $i = 1, \ldots, m$, where $\boldsymbol{a}_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, we want to find a model that predicts b given the vector $\boldsymbol{a}$

$$b = f_{\boldsymbol{x}}(\boldsymbol{a}) \tag{34}$$

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The **prediction error** or **residual** at the i-th data point is

$$| \, b_i - f_{\boldsymbol{x}}(\boldsymbol{a}_i) \, | \tag{35}$$

Given a choice between different models $f_{\boldsymbol{x}}(\boldsymbol{a})$, we should choose a model that "*explains*" the available data as best as possible, that is, that minimizes the **residuals**.

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A Linear Model is of the form

$$b = f_{\boldsymbol{x}}(\boldsymbol{a}) = \boldsymbol{a}^{\intercal} \boldsymbol{x} \tag{36}$$

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HOMEWORK: Can you formulate the **Data Fitting** problem that minimizes these two functions using Linear Programming?

Model 6: Pattern Classification

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3. We want to design a **(binary) classifier** that given a new unseen object automatically infers its class by elaborating the vector $\boldsymbol{a}_i$.
4. A **linear classifier** is defined by a n-dimensional vector and a scalar x_{n+1}, such that, given a new object h with feature vector $\boldsymbol{a}_h$ we have:

$$\text{class}(h) = \begin{cases} C_1 & \text{if } \boldsymbol{a}_h^\intercal \boldsymbol{x} \geq x_{n+1} \\ C_2 & \text{if } \boldsymbol{a}_h^\intercal \boldsymbol{x} < x_{n+1} \end{cases}$$

A linear classifier makes decisions on the basis of a linear combination of the feature vector.

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Note: we "train" a linear classifier by finding a feasible solution to a Linear Programming problem