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Invited review

A comparative analysis of several asymmetric traveling salesman problem formulations

Temel Öncan[a], İ. Kuban Altınel[b], Gilbert Laporte[c,*]

[a]Department of Industrial Engineering, Galatasaray University, Ortaköy, İstanbul 34357, Turkey
[b]Department of Industrial Engineering, Boğaziçi University, Bebek, İstanbul 34342, Turkey
[c]Canada Research Chair in Distribution Management, HEC Montréal, 3000, chemin de la Côte-Sainte-Catherine, Montréal, Canada, H3T 2A7

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Abstract

In this survey, a classification of 24 asymmetric traveling salesman problem (ATSP) formulations is presented. The strength of their LP relaxations is discussed and known relationships from the literature are reviewed. Some new relationships are also introduced, and computational results are reported.

Keywords: Integer linear programming; Asymmetric traveling salesman problem; Formulations; Projections

1. Introduction

The *asymmetric traveling salesman problem* (ATSP) is defined on a directed graph $G=(V, A)$, where $V=\{1, \dots, n\}$ is the vertex set, $A=\{(i, j) : i, j \in V\}$ is the arc set, and a non-symmetric cost matrix (c_{ij}) is defined on A. To simplify the notation, we associate variables to arcs (i, i) but we force these variables to be equal to 0 by setting $c_{ii}=\infty$. The ATSP consists of determining a least cost Hamiltonian circuit or tour over G. The problem is commonly interpreted as that of determining an optimal salesman's tour over n cities. The ATSP is NP-hard even if the costs are Euclidean [1].

As is the case with most combinatorial optimization problems, exact algorithms for the ATSP combine polyhedral results with enumeration. The efficiency of the enumeration depends on the strength of the LP (or linear) relaxation of a given formulation. It is possible to state that given two formulations, the one yielding the larger relaxation value is better. The strengths of LP relaxations, or equivalently the strengths of two formulations, can also be compared by using polyhedral information. Suppose two different formulations F_1 and F_2 are stated in the same space of variables $x \in \mathbb{R}^p$ and the objective is to minimize. Let $P(F_1)$ and $P(F_2)$ be the polyhedra associated with these formulations. If $P(F_1) \subset P(F_2)$, then F_1 is a better formulation than F_2 since the lower bound obtained by solving the LP relaxation of F_1 is at least equal to the one obtained by solving the LP relaxation of F_2.

Different formulations of a given problem can be frequently stated in terms of different sets of variables, as is the case with some ATSP formulations that use an extended set of variables with respect to another formulation. If F_3 is a formulation with this property, it is possible to project the extended polyhedron $P(F_3)$ of F_3 into the subspace of the

* Corresponding author.
E-mail addresses: ytoncan@gsu.edu.tr (T. Öncan), altinel@boun.edu.tr (İ.K. Altınel), gilbert@crt.umontreal.ca (G. Laporte).

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original variables without losing any integer solution, namely any of the tours, and compare the lower bound obtained over the projected polyhedron $TP(F_3)$. To be precise, given the polyhedron

$$P(F_3) = \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : Ax + By \leqslant b\},$$

where A, B and b have m rows, the *projection of* $P(F_3)$ *into the subspace of x variables*, or *into* $\mathbb{R}^p$, is

$$TP(F_3) = \{x \in \mathbb{R}^p : \text{ there exists } y \in \mathbb{R}^q \text{ such that } (x, y) \in P(F_3)\},$$

and if $TP(F_3) \subset P(F_1)$, then F_3 is a better formulation since its LP bound is larger than that of F_1 [2].

Several attempts have been made to compare a number of existing ATSP formulations, see, e.g., Wong [3], Padberg and Sung [4], Langevin et al. [5], Gouveia and Voss [6], Altınel et al. [7] and Orman and Williams [8]. The main motivation of this paper is to survey the existing ATSP formulations and to establish new relationships between them. The formulations are described in Section 2. Section 3 summarizes the known relationships between these formulations, while Section 4 introduces some new ones. Comparative computational results are presented in Section 5, and conclusions follow in Section 6.

2. Classical formulations and their subtour elimination constraints

Many ATSP formulations consist of an assignment problem with integrality and subtour elimination constraints. They use binary x_{ij} variables equal to 1 if and only if arc (i, j) belongs to the optimal solution. The basic model is as follows:

$$\text{(ATSP)} : \text{Minimize} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{1}$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \ldots, n, \tag{2}$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, \ldots, n, \tag{3}$$

$$0 \leqslant x_{ij} \leqslant 1, \quad i, j = 1, \ldots, n, \tag{4}$$

$$x_{ij} = 0, 1, \quad i, j = 1, \ldots, n, \tag{5}$$

$$\{(i, j) : x_{ij} = 1,\ i, j = 2, \ldots, n\} \text{ does not contain subtours.} \tag{6}$$

In this formulation, the assignment constraints (2) and (3) ensure that each vertex is incident to one outgoing arc and one incoming arc. Constraints (6) break subtours in the set $\{2, \ldots, n\}$. Together with (2) and (3) they also eliminate subtours containing vertex 1, and the subtour elimination constraints for $i = 1$ become redundant. To be precise with notation, note that the projected polyhedron $TP(F)$ of an ATSP formulation F is stated in the space of x_{ij} variables.

2.1. The Dantzig, Fulkerson and Johnson (DFJ) formulation

In their seminal work Dantzig et al. [9] have formulated the subtour elimination constraints as

$$\sum_{i,j \in S} x_{ij} \leqslant |S| - 1, \quad S \subseteq \{2, \ldots, n\}, \quad 2 \leqslant |S| \leqslant n - 1. \tag{7}$$

These inequalities are facet defining [10] and can be gradually added to the formulation through a branch-and-bound scheme, which results in an optimal solution. In practice, relatively few subtour elimination constraints are needed.

2.2. Circuit packing (CP) inequalities

The following inequalities are known as the *circuit packing* or simply *circuit* inequalities.

$$\sum_{(i,j) \in C} x_{ij} \leqslant |C| - 1 \quad \text{for all circuits } D_C = (V_C, C), \quad V_C \subseteq \{2, \ldots, n\}. \tag{8}$$

The formulation obtained by replacing inequalities (6) with circuit inequalities (8) is a valid ATSP formulation [10].

2.3. The Miller, Tucker and Zemlin (MTZ) formulation

The earliest known extended formulation of the ATSP is due to Miller et al. [11]. It was originally proposed for a *vehicle routing problem* (VRP) where the number of vertices of each route is limited. It uses u_i variables to define the order in which each vertex i is visited on a tour. The subtour elimination constraints of the MTZ formulation are stated as

$$u_i - u_j + (n-1)x_{ij} \leqslant n-2, \quad i, j = 2, \ldots, n, \tag{9}$$

$$1 \leqslant u_i \leqslant n-1, \quad i = 2, \ldots, n. \tag{10}$$

We should mention that the u_i variables are unrestricted in the original paper. Simple bounds (10) were introduced later on, and do not affect the LP bound obtained on P(MTZ).

2.4. The Desrochers and Laporte (DL) formulation

The compact polynomial representation of P(MTZ) extends naturally to a number of variants of the VRP [12]. Another advantage of the MTZ formulation is that the subtour elimination constraints (9) and (10) can be incorporated into other type of problem formulations together with stronger constraints. Motivated by these facts, Desrochers and Laporte [12] have lifted subtour elimination constraints (9) and (10) to obtain the stronger forms:

$$u_i - u_j + (n-1)x_{ij} + (n-3)x_{ji} \leqslant n-2, \quad i, j = 2, \ldots, n, \tag{11}$$

$$1 + (n-3)x_{i1} + \sum_{j=2}^{n} x_{ji} \leqslant u_i \leqslant n - 1 - (n-3)x_{1i} - \sum_{j=2}^{n} x_{ij}, \quad i = 2, \ldots, n. \tag{12}$$

They have shown that inequalities (11) are facet defining. The authors have also proved that (12) are valid inequalities, which have been shown to be facet defining by Driscoll [13].

2.5. The Sherali and Driscoll (SD) formulation

Sherali and Driscoll [14] have also strengthened the relaxations of the MTZ formulation for the ATSP. The authors apply only a partial first level version of the *Reformulation–linearization technique* (RLT) [15] to a nonlinear reformulation of the MTZ subtour elimination constraints. The new variables y_{ij} assume a non-zero value representing the order of arc (i, j) on the tour. Their $\mathcal{O}(n^2)$ subtour elimination constraints are as follows:

$$\sum_{j=2}^{n} y_{ij} + (n-1)x_{i1} = u_i, \quad i = 2, \ldots, n, \tag{13}$$

$$\sum_{i=2}^{n} y_{ij} + 1 = u_j, \quad j = 2, \ldots, n, \tag{14}$$

$$x_{ij} \leqslant y_{ij} \leqslant (n-2)x_{ij}, \quad i, j = 2, \ldots, n, \tag{15}$$

$$u_j + (n-2)x_{ij} - (n-1)(1 - x_{ji}) \leqslant y_{ij} + y_{ji} \leqslant u_j - (1 - x_{ji}), \quad i, j = 2, \ldots, n, \tag{16}$$

$$1 + (1 - x_{1j}) + (n-3)x_{j1} \leqslant u_j \leqslant (n-1) - (n-3)x_{1j} - (1 - x_{j1}), \quad j = 2, \ldots, n. \tag{17}$$

2.6. The Gavish and Graves (GG) formulation

A large class of extended ATSP formulations are known as *commodity flow* formulations [5], where the additional variables represent commodity flows through the arcs and satisfy additional flow conservation constraints. These models belong to three classes: single commodity flow (SCF), two-commodity flow (TCF) and multi-commodity flow (MCF) formulations.

The earliest SCF formulation is due to Gavish and Graves [16]. The additional continuous non-negative variables g_{ij} describe the flow of a single commodity to vertex 1 from every other vertex. A set of $\mathcal{O}(n^2)$ subtour elimination constraints is given by

$$\sum_{j=1}^{n} g_{ji} - \sum_{j=2}^{n} g_{ij} = 1, \quad i = 2, \ldots, n, \tag{18}$$

$$0 \leqslant g_{ij} \leqslant (n-1)x_{ij}, \quad i = 1, \ldots, n; \quad j = 2, \ldots, n. \tag{19}$$

An interpretation similar to the one given for the additional variables u_i of the MTZ formulation can also be applied to variables g_{ij}. These can be viewed as the number of arcs included on the path from vertex 1 and arc (i, j) in the optimal tour [17].

As an example of TCF formulation, we can mention the formulation due to Finke et al. [18] (FCG). For the sake of brevity, we do not present FCG here since Langevin et al. [5] have shown that FCG and GG are equivalent.

2.7. Multi-commodity flow (MCF) formulations

Wong [3] was the first to formulate the ATSP as an MCF model using additional non-negative variables to describe the flow of $2(n-1)$ commodities between vertex 1 and the other vertices k, namely commodities k taking the value $w_{ij}^{(k,1)}$, $k = 2, \ldots, n$ and commodities l taking the value $w_{ij}^{(1,l)}$, $l = 2, \ldots, n$. The flow variable $w_{ij}^{(1,l)}$ is equal to one if and only if the commodity going from 1 to l flows on arc (i, j). The flow variable $w_{ij}^{(k,1)}$ is equal to one if and only if the commodity going from k to 1 flows on arc (i, j). His $\mathcal{O}(n^3)$ subtour elimination constraints are defined as follows:

$$\sum_{j=1}^{n} w_{ij}^{(1,l)} - \sum_{j=1}^{n} w_{ji}^{(1,l)} = 0, \quad i, l = 2, \ldots, n; \quad i \neq l, \tag{20}$$

$$\sum_{j=2}^{n} w_{1j}^{(1,l)} - \sum_{j=2}^{n} w_{j1}^{(1,l)} = 1, \quad l = 2, \ldots, n, \tag{21}$$

$$\sum_{j=1}^{n} w_{ij}^{(1,i)} - \sum_{j=1}^{n} w_{ji}^{(1,i)} = -1, \quad i = 2, \ldots, n, \tag{22}$$

$$\sum_{j=1}^{n} w_{ij}^{(k,1)} - \sum_{j=1}^{n} w_{ji}^{(k,1)} = 0, \quad i, k = 2, \ldots, n; \quad i \neq k, \tag{23}$$

$$\sum_{j=2}^{n} w_{1j}^{(k,1)} - \sum_{j=2}^{n} w_{j1}^{(k,1)} = -1, \quad k = 2, \ldots, n, \tag{24}$$

$$\sum_{j=1}^{n} w_{ij}^{(i,1)} - \sum_{j=1}^{n} w_{ji}^{(i,1)} = 1, \quad i = 2, \ldots, n, \tag{25}$$

$$0 \leqslant w_{ij}^{(1,l)} \leqslant x_{ij}, \quad i, j = 1, \ldots, n; \quad l = 2, \ldots, n, \tag{26}$$

$$0 \leqslant w_{ij}^{(k,1)} \leqslant x_{ij}, \quad i, j = 1, \ldots, n; \quad k = 2, \ldots, n. \tag{27}$$

Note that only the x_{ij} are subject to integrality since $w_{ij}^{(1,l)}$ and $w_{ij}^{(k,1)}$ variables are integer whenever the x_{ij} are, because the flow matrix for commodities k and l is totally unimodular. In fact the variable x_{ij} can be interpreted as the capacity bound on arc (i, j).

Modifications of the Wong MCF model (WONG) were proposed by Langevin [19] and Loulou [20]. The Langevin formulation (LANGEVIN) is a restriction of WONG. The Loulou formulation (LOULOU) is a restriction of both LANGEVIN and WONG. Hence, WONG, LOULOU and LANGEVIN are equivalent. For the sake of conciseness, we do not present LOULOU and LANGEVIN.

Another MCF formulation proposed by Claus [21] (CLAUS) uses only $(n-1)$ commodities. Langevin et al. [5] observed that CLAUS can be obtained from WONG by eliminating half of the flow variables and related constraints. Claus [21] uses only non-negative flow variables $w_{ij}^{(k,1)}$ and all the constraints in which they appear. Hence, the Claus subtour elimination constraints consists of (23)–(25) and (27).

2.8. *The Fox, Gavish and Graves (FGG) formulations*

The next formulations exploit a relationship between the ATSP and machine scheduling. Consider the following single machine scheduling problem where n jobs must be scheduled on a single machine. The machine is in the initial state which we denote by job 1. We assume that the machine will again be in that state after processing all of the remaining $n-1$ jobs. A set of $n-1$ jobs, denoted by $2, \ldots, n$, are to be performed on a single machine and a set up cost c_{ijk} is incurred when job j is processed immediately after job i in the kth position. The objective is to determine a cheapest sequence of the n jobs. This problem is known as the *Time Dependent* TSP (TDTSP), a generalization of the standard TSP where the cost of any given arc depends on its position in the tour.

Fox et al. [22] have proposed three time-dependent formulations. The one we present below has four set of constraints and will be denoted as FGG4 in the sequel. Let r_{ijk} be a variable equal to 1 if and only if vertex j is visited immediately after vertex i in the kth position. The FGG4 model is

$$\text{(FGG4) : Minimize} \quad \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk} r_{ijk} \tag{28}$$

$$\text{subject to} \quad \sum_{j=1}^{n}\sum_{k=1}^{n} r_{ijk} = 1, \quad i = 1, \ldots, n, \tag{29}$$

$$\sum_{i=1}^{n}\sum_{k=1}^{n} r_{ijk} = 1, \quad j = 1, \ldots, n, \tag{30}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ijk} = 1, \quad k = 1, \ldots, n, \tag{31}$$

$$\sum_{j=1}^{n}\sum_{k=2}^{n} k r_{ijk} - \sum_{j=1}^{n}\sum_{k=1}^{n} k r_{jik} = 1, \quad i = 2, \ldots, n, \tag{32}$$

$$r_{ijk} \in \{0, 1\}, \quad i, j, k = 1, \ldots, n. \tag{33}$$

Fox et al. [22] have noted that constraints (31) are not needed and may be dropped. Their second formulation, namely FGG3, consists of the objective function (28) and, constraints (29), (30), (32) and (33).

The third formulation by Fox et al. [22], denoted by FGG2, is more compact than the FGG4 formulation. It uses (34) as an aggregation of constraints (29)–(31):

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} r_{ijk} = n. \tag{34}$$

To ensure the validity of the FGG2 formulation, Fox et al. [22] assume that $r_{ij1} = r_{jin}$ for $i = 2, \ldots, n$, $r_{1ik} = 0$ for $k = 2, \ldots, n$ and $r_{i1k} = 0$ for $k = 1, \ldots, n-1$.

2.9. *The Gouveia and Pires (GP) formulations*

Gouveia and Pires [17] have first proposed generalized formulations of the MTZ. They have shown that these formulations are stronger than the MTZ formulation. Then, Gouveia and Pires [23] have devised even stronger formulations by generalizing CLAUS.

2.9.1. Generalizations of the MTZ formulation

In their first extension of the MTZ formulation Gouveia and Pires [17], have shown that the set of inequalities

$$x_{ij} + v_{ki} - v_{kj} \leqslant 1, \quad i, j, k = 2, \dots, n, \tag{35}$$

$$x_{ij} - v_{ij} \leqslant 0, \quad i, j = 2, \dots, n, \tag{36}$$

$$x_{ij} + v_{ji} \leqslant 1, \quad i, j = 2, \dots, n \tag{37}$$

eliminate subtours and, together with constraints (2)–(5), provide a valid ATSP formulation. Additional variables v_{ij} indicate whether vertex i is on the path from vertex 1 to vertex j. This first formulation of Gouveia and Pires [17] will be referred to as the GP1 formulation. Recall that variables u_i of the MTZ formulation can be interpreted as the number of intermediate vertices on the path from vertex 1 to vertex i of the optimal tour. These variables are related to the new variables v_{ij} as follows:

$$u_i = \sum_{k=2}^{n} v_{ki}, \quad i = 2, \dots, n. \tag{38}$$

Constraints (35) can be lifted in two different ways [17]

$$x_{ji} + x_{ij} + v_{ki} - v_{kj} \leqslant 1, \quad i, j, k = 2, \dots, n, \tag{39}$$

$$x_{kj} + x_{ik} + x_{ij} + v_{ki} - v_{kj} \leqslant 1, \quad i, j, k = 2, \dots, n. \tag{40}$$

New formulations, called GP2 and GP3, obtained by replacing (35) first with (39) and then (40), respectively, are valid for the ATSP. In their fourth formulation, which we refer to as GP4, Gouveia and Pires [17] consider constraints (39) and (40) together with constraints (2)–(5), (36) and (37).

2.9.2. Generalizations of the CLAUS formulation

Gouveia and Pires [23] have considered an equivalent version of CLAUS. Although both formulations are equivalent and yield the same LP bound, the order of indexing of commodity flow variables is different. They use f_{kij} instead of $w_{ij}^{(k,1)}$, which only affects their interpretation. Namely, $f_{kij} = 1$ indicates that arc (i, j) is on the path from vertex $k = 2, \dots, n$ to vertex 1. Their MCF formulation consists of the assignment constraints (2), (3), integrality restrictions (5), and the following subtour elimination constraints [23]:

$$\sum_{i=1}^{n} f_{kji} - \sum_{i=1}^{n} f_{kij} = 0, \quad j, k = 2, \dots, n; \quad j \neq k, \tag{41}$$

$$\sum_{i=1}^{n} f_{jji} = 1, \quad j = 2, \dots, n, \tag{42}$$

$$f_{kij} \leqslant x_{ij}, \quad i, j = 1, \dots, n; \quad k = 2, \dots, n, \tag{43}$$

$$f_{kij} \geqslant 0, \quad i, j = 1, \dots, n; \quad k = 2, \dots, n. \tag{44}$$

Gouveia and Pires [23] have stated that the variables v_{ij} are related to the new variables f_{kij} as follows:

$$\sum_{j=1}^{n} f_{kij} = v_{ki}, \quad i, k = 2, \dots, n. \tag{45}$$

It is also interesting to observe that the left-hand side of (45) counts the number of arcs leaving vertex i on the path from vertex k to vertex 1.

Gouveia and Pires [23] have generalized inequalities (39) and (40), respectively, as follows:

$$\sum_{p,q\in S} x_{pq} + v_{ki} - v_{kj} \leqslant |S| - 1, \quad i, j, k = 2, \ldots, n, \quad S \subseteq \{2, \ldots, n\}, \quad |S| \geqslant 2; \quad k \notin S; \quad i, j \in S, \tag{46}$$

$$v_{ki} + \sum_{m\in S}(x_{im} + x_{km}) + \sum_{m\in S}(x_{mj} + x_{mk}) + \sum_{p,q\in S} x_{pq} + x_{ik} + x_{kj} + x_{ij} \leqslant 1 + |S| + v_{kj},$$
$$i, j, k = 2, \ldots, n; \quad S \subset V\backslash\{1, i, j, k\}. \tag{47}$$

Note that for $|S| = 2$ constraints (46) reduce to (39) and, for $|S| = 0$ constraints (47) reduce to (40). Consequently, the GP5 formulation is obtained by replacing inequalities (40) of GP3 with inequalities (47) and GP6 is obtained by replacing inequalities (39) of GP2 formulation with inequalities (46) [23].

Finally, observing that inequalities (39) and the linear transformation (45) are not redundant for the polyhedron P(CLAUS), Gouveia and Pires [23] have proposed the GP7 formulation which consists of (2)–(5), (3) and (41)–(45). Gouveia and Pires [23] have further generalized the GP7 formulation by replacing constraints (39) of GP7 formulation with (46). Hence, the GP8 formulation consists of (2)–(5) and (41)–(46).

2.10. The Sarin, Sherali and Bhootra (SSB) formulations

Sarin et al. [24] have proposed the following $\mathcal{O}(n^3)$ path constraints for the ATSP. Let $d_{ij} = 1$ if and only if vertex i precedes (not necessarily immediately) vertex j in a tour, and let

$$d_{ij} \geqslant x_{ij}, \quad i, j = 2, \ldots, n, \tag{48}$$

$$d_{ij} + d_{ji} = 1, \quad i, j = 2, \ldots, n, \tag{49}$$

$$d_{ij} + d_{jk} + d_{ki} \leqslant 2, \quad i, j, k = 2, \ldots, n. \tag{50}$$

Hence, the SSB1 formulation consists of inequalities (2)–(5) and (48)–(50). Note that the auxiliary variables d_{ij} determine a precedence relationship between the vertices and play the same role as the v_{ij} variables of Gouveia and Pires [17]. That is to say, $v_{ij} = d_{ij}$.

Sarin et al. [24] have proposed two stronger formulations, namely SSB2 (L1ATSPxy therein) and SSB3 (SL1ATSPxy therein), which are polynomial size extensions of the SSB1 formulation by, respectively, replacing (50) with one of the following inequalities

$$(d_{ij} + x_{ji}) + d_{jk} + d_{ki} \leqslant 2, \quad i, j, k = 2, \ldots, n, \tag{51}$$

$$3(d_{ij} + d_{jk} + d_{ki}) + x_{ji} + x_{kj} + x_{ik} \leqslant 6, \quad i, j, k = 2, \ldots, n. \tag{52}$$

Inequalities (51) have also been discussed by Altınel et al. [7]. To further tighten the LP relaxations of the SSB2 and SSB3 formulations, Sarin et al. [24] have included the following constraints to both of the SSB2 and SSB3 formulations:

$$x_{1j} + x_{j1} \leqslant 1, \quad j = 2, \ldots, n. \tag{53}$$

2.11. The Sherali, Sarin and Tsai (SST) formulations

The latest ATSP formulation is due to Sherali et al. [25]. They have started with the following lifted path constraints:

$$d_{ij} \geqslant x_{1i}, \quad i, j = 2, \ldots, n, \tag{54}$$

$$d_{ji} \geqslant x_{i1}, \quad i, j = 2, \ldots, n, \tag{55}$$

$$-(1 - x_{ik}) \leqslant d_{ij} - d_{kj} \leqslant 1 - x_{ik}, \quad i, j, k = 2, \ldots, n \tag{56}$$

which they have combined with (48) and (49) to obtain the subtour elimination constraints of the SST1 (called ATSP0 therein) formulation.

Table 1
Classification of the ATSP formulations

Category	Formulations	Reference	Variables	Constraints
Exponential sized	DFJ	Dantzig et al. [9]	$\mathcal{O}(n^2)$	$\mathcal{O}(2^n)$
	GP5, GP6, GP8	Gouveia and Pires [23]		
	CP	Grötschel and Padberg [10]		
Miller–Tucker–Zemlin based	MTZ	Miller et al. [11]	$\mathcal{O}(n^2)$	$\mathcal{O}(n^2)$
	DL	Desrochers and Laporte [12]		
	SD	Sherali and Driscoll [14]		
Single commodity flow	GG	Gavish and Graves [16]	$\mathcal{O}(n^2)$	$\mathcal{O}(n^2)$
Two-commodity flow	FCG	Finke et al. [18]	$\mathcal{O}(n^2)$	$\mathcal{O}(n)$
Multi commodity flow	WONG	Wong [3]	$\mathcal{O}(n^3)$	$\mathcal{O}(n^3)$
	CLAUS	Claus [21]		
	LANGEVIN	Langevin [19]		
	LOULOU	Loulou [20]		
	GP7	Gouveia and Pires [23]		
	SST2	Sherali et al. [25]		
Time dependent	FGG2, FGG3, FGG4	Fox et al. [22]	$\mathcal{O}(n^3)$	$\mathcal{O}(n)$
Precedence variable based	GP1, GP2, GP3, GP4	Gouveia and Pires [17]	$\mathcal{O}(n^2)$	$\mathcal{O}(n^3)$
	SSB1, SSB2, SSB3	Sarin et al. [24]		
	SST1	Sherali et al. [25]		

These authors have also applied a first-order RLT to the SST1 formulation and derived the following constraints:

$$0 \leqslant t_{ij}^k \leqslant x_{ik}, \quad i, j, k = 2, \ldots, n; \quad i, k \neq j, \tag{57}$$

$$\sum_{k=2; k \neq j}^{n} t_{ij}^k + x_{ij} = d_{ij}, \quad i, j = 2, \ldots, n, \tag{58}$$

$$x_{1k} + \sum_{i=2; i \neq j}^{n} t_{ij}^k = d_{kj}, \quad k, j = 2, \ldots, n, \tag{59}$$

which are shown to be tighter than the MCF subtour elimination constraints. Sherali et al. [25] have proposed several augmented formulations which are tighter than the SSB1 and SSB2. For the sake of conciseness, we will only present here SST2 (called ATSP6 therein) which yields the tightest bounds. The SST2 formulation is defined with inequalities (2)–(5), (49), (51), (54), (55) and (57)–(59).

The ATSP formulations presented in this section are summarized in Table 1.

3. Known relationships between ATSP formulations

We now summarize several known relationships between the ATSP formulations just described.

3.1. The MTZ formulation

In their analytical comparison of different ATSP formulations Padberg and Sung [4] have shown that the polyhedron obtained by projecting P(MTZ) into the subspace of x_{ij} variables, namely TP(MTZ), is the *weak circuit polytope* because of *weak circuit* inequalities which are a weaker version of the *circuit* inequalities (8). Hence they have shown that P(CP) is a proper subset of TP(MTZ) and, the MTZ formulation is weaker than both CP and DFJ formulations.

The weakness of the MTZ formulation was also demonstrated by Langevin et al. [5]. Desrochers and Laporte [12] have lifted some MTZ constraints, while Sherali and Driscoll [14] have proposed a reformulation of the MTZ constraints which imply the DL constraints.

Table 2
Relative percent deviations between LP relaxation bounds and optimal values

Instance	Optimum	SST1	SSB1	SSB3	SSB2	SST2	GP1	GP2	GP3	GP4	CLAUS	MTZ	FGG2	FGG4	GG	SD	DL
ftv33	1286	7.574	4.769	4.589	4.426	0.000	4.769	4.646	0.000	0.000	0.000	7.642	75.492	6.959	7.031	4.782	5.351
ftv35	1473	5.974	3.288	3.138	3.123	0.651	3.404	3.235	1.322	1.288	1.064	6.118	77.065	5.530	5.594	3.903	4.039
ftv38	1530	5.752	3.194	3.058	2.913	0.641	3.201	3.094	1.272	1.240	1.024	5.869	76.952	5.838	5.437	3.264	3.454
ftv44	1613	5.456	2.036	1.957	1.922	na	2.118	1.992	1.837	1.837	1.744	5.555	72.229	5.158	5.175	2.433	2.433
ftv47	1776	6.729	2.528	2.451	2.365	na	2.528	2.412	1.896	1.858	1.542	6.767	74.347	6.521	6.544	2.747	2.835
ftv55	1608	10.759	4.740	4.484	4.167	na	4.740	4.571	2.539	2.332	1.493	10.554	74.184	10.288	10.311	5.891	6.049
ftv64	1839	6.417	4.002	3.954	3.800	na	4.001	3.982	3.144	3.144	1.713	6.314	74.792	5.823	5.845	4.003	4.241
ftv70	1950	9.436	4.463	4.379	4.251	na	4.463	4.374	3.536	3.523	2.103	9.260	74.279	8.729	8.751	4.637	4.691
ft70	38 673	1.211	0.700	0.673	0.598	na	1.271	1.252	1.146	1.144	0.053	1.773	37.470	1.085	1.103	0.798	0.878
ft53	6905	14.096	12.548	12.437	12.210	na	12.548	12.438	11.078	10.692	0.000	14.042	72.992	12.376	12.454	11.392	12.934
Average		7.340	4.227	4.112	3.977	0.431	4.304	4.200	2.777	2.706	1.073	7.389	70.980	6.831	6.825	4.385	4.691

3.2. *The MCF formulations*

We will next present the relationships associated with the MCF formulations which are among the most used ATSP formulations because of their relative strength. To the best of our knowledge, there exist three results on the equivalence of MCF formulations with the DFJ formulation. The first result is by Wong [3] who developed the first MCF formulation and showed that it is equivalent to DFJ. The second result is due to Langevin et al. [5] who have shown the equivalence of a class of MCF formulations: DFJ, LANGEVIN, LOULOU, WONG and CLAUS. The third result is due to Padberg and Sung [4] who have derived the projection TP(CLAUS) of polyhedron P(CLAUS) into the subspace of x_{ij} variables and have shown that it is equivalent to P(DFJ).

3.3. *The SCF formulation*

Langevin et al. [5] have demonstrated that by definition of the flow variables of the SCF and MCF formulations, SCF is an aggregation of MCF. In fact the LP relaxation bound of the instances from Table 2 for the GG and CLAUS formulations show that TP(MCF) is a proper subset of TP(SCF).

We now consider the GG formulation. As a consequence of a result by Gouveia [26] for the *Capacitated Vehicle Routing Problem*, the polyhedron obtained by projecting P(GG) into the subspace of x_{ij} variables is the *weak clique polytope* because of *weak clique* inequalities. These inequalities are weaker versions of inequalities (7). This result shows that P(DFJ) is a proper subset of TP(GG) and therefore the GG formulation is weaker than the DFJ formulation. However, GG is stronger than MTZ since TP(GG) is a proper subset of TP(MTZ). This follows from the fact that *weak clique* inequalities include also constraints *weak circuit* inequalities. This result was shown by Wong [3] and by Padberg and Sung [4].

3.4. *The GP1, GP2, GP3 and GP4 formulations*

We now summarize the projection results by Gouveia and Pires [17] on the strengths of the GP1, GP2, GP3 and GP4 formulations. The authors have shown that the projection of the polyhedron P(GP1) into the subspace of original x_{ij} variables, which is the set of feasible solutions in the LP relaxation of GP1, is exactly the circuit polytope P(CP). Since P(CP) is a proper subset of TP(MTZ) this implies that GP1 is stronger than MTZ.

Gouveia and Pires [17] have also provided the projections TP(GP2) and TP(GP3) of the LP relaxations of GP2 and GP3 formulations into the subspace of the original x_{ij} variables. Both TP(GP2) and TP(GP3) are proper subsets of the circuit polytope P(CP), which is equivalent to TP(GP1). Hence this implies that the GP2 and GP3 formulations are stronger that the GP1 formulation. Unfortunately, the GP2 and GP3 formulations are dominated by the DFJ formulation because P(DFJ) is a proper subset of both TP(GP2) and TP(GP3) [17].

The authors have shown that the GP2 and GP3 formulations are incomparable. They have also stated that the GP4 formulation is stronger than the GP2 and GP3 formulations since it includes the constraints of both GP2 and GP3 formulations. Gouveia and Pires [17] have shown that a dominance relation between the GP4 and CLAUS formulations does not exist. Hence, the GP4 and DFJ formulations are incomparable. Finally, they have proved that the GG and GP1 formulations are not comparable [17].

3.5. The GP5, GP6, GP7 and GP8 formulations

Gouveia and Pires [23] have analyzed the strength of the GP5, GP6, GP7 and GP8 formulations. They have proved that the CLAUS formulation is stronger than the GP5 formulation, which is in its turn stronger than the GP3 formulation. The equivalence of the GP6 and DFJ formulations has later been established by Myung [27]. Recall that Padberg and Sung [4] have shown that TP(CLAUS), the projection of the polyhedron P(CLAUS) into the subspace of x_{ij} variables, is the polyhedron P(DFJ). As a consequence TP(GP7), the projection of the polyhedron P(GP7) into the subspace of x_{ij} variables, is contained in the set described by the inequalities (7), and therefore TP(GP7) is a proper subset of TP(CLAUS) and also P(DFJ). This implies that GP7 is stronger than CLAUS. Finally, Gouveia and Pires [23] have stated that the GP8 formulation is stronger than both the GP6 and GP7 formulations, since the constraints of GP8 imply the constraints of the GP6 and GP7 formulations.

3.6. The time-dependent TSP formulation

The time-dependent TSP formulations were analyzed and compared by Gouveia and Voss [6]. In this overview we have only considered the time-dependent formulation by Fox et al. [22]. Gouveia and Voss [6] have shown that P(FGG4) is a proper subset of P(FGG3). Furthermore, Gouveia and Voss [6] have proposed an SCF formulation based on the subtour elimination constraints of the GG formulation. Their SCF formulation was slightly stronger than the GG formulation. They have proved that their SCF formulation is equivalent to the FGG3 formulation which implies that the FGG3 formulation is stronger than the GG formulation. Another relationship regarding the FGG formulation is due to Padberg and Sung [4] who have derived the projection of FGG2 formulation into the subspace of x_{ij} and then shown that the FGG2 formulation is weaker than the DFJ formulation.

3.7. The SSB and SST formulations

Sarin et al. [24] have proposed three formulations: SSB1, SSB2 and SSB3. Two of them, SSB2 and SSB3, contain lifted inequalities (51) and (52), respectively, which are liftings of constraints (50). Therefore, both the SSB2 and SSB3 formulations are stronger than the SSB1 formulation.

Sherali et al. [25] have observed that the LP relaxation of the SST1 formulation is as strong as the GP1 formulation since the left-hand side of (56) reduces to that of constraint (35). On the other hand it is also possible to show that the right-hand side of (56) reduces to $d_{ij} + d_{jk} \geqslant d_{ik} \geqslant x_{ik}$ which is implied by (48)–(49). Therefore SSB1 is as strong as SST1. By considering the instances from Table 2 we can say that TP(SSB1) is a proper subset of TP(SST1) which is in turn a proper subset of TP(GP1). Finally, Sherali et al. [25] have shown that SST2 is stronger than SSB2 since all the inequalities of SSB2 formulation are implied by the constraints of SST2 formulation.

Fig. 1 illustrates the relative strengths of the LP relaxations of the ATSP formulations we have presented.

4. New relationships between ATSP formulations

We now develop new relationships between some of the formulations presented in Section 2. These results do not close the discussion on the comparison of ATSP formulations because there are still some missing relationships.

4.1. Time-dependent TSP formulation

We start with an observation on the time-dependent TSP formulation of Fox et al. [22].

Proposition 1. *The FGG3 formulation is stronger than the FGG2 formulation.*

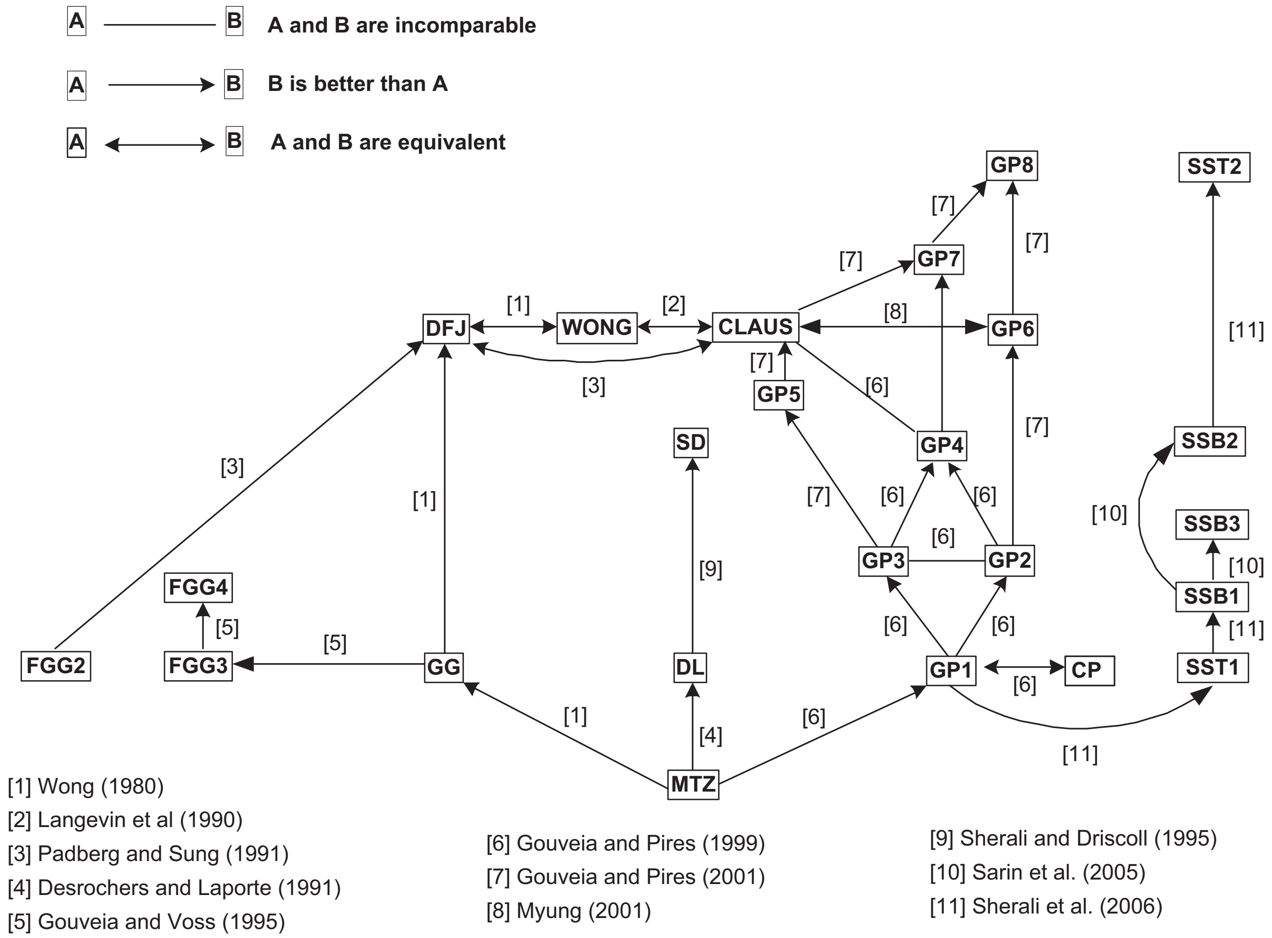

Fig. 1. Known relationships between 24 ATSP formulations.

Proof. Since the FGG2 formulation contains aggregated assignment constraint (34), which can be obtained by summing up constraints (29)–(31), every solution feasible for the LP relaxation of the FGG3 is also feasible for the LP relaxation of the FGG2. Indeed all the instances from Table 2 show that P(FGG3) is a proper subset of P(FGG2). □

4.2. The GG, SD and DL formulations

We now compare SD with GG and show that it is not possible to compare GG with DL.

Proposition 2. *The SD formulation is stronger than the GG formulation.*

Proof. Consider constraints (13)–(15) of the SD formulation and observe that the right-hand sides of (13) and (14) are in fact equal. As a result we can write

$$\sum_{j=2}^{n} y_{ij} + (n-1)x_{i1} = \sum_{j=2}^{n} y_{ji} + 1, \quad i = 2, \ldots, n. \tag{60}$$

Then by adding this set of inequalities for $S \subseteq V = \{1, \ldots, n\}$ we obtain

$$\sum_{\substack{i \in S \setminus \{1\} \\ j \in \overline{S} \setminus \{1\}}} y_{ij} + (n-1) \sum_{i \in S \setminus \{1\}} x_{i1} = \sum_{\substack{i \in \overline{S} \setminus \{1\} \\ j \in S \setminus \{1\}}} y_{ij} + |S|, \tag{61}$$

where $\overline{S} = V\backslash S$. On the other hand the aggregation of constraints (15) over S gives

$$(n-2)\sum_{\substack{i\in S\backslash\{1\}\\ j\in \overline{S}\backslash\{1\}}} x_{ij} \geqslant \sum_{\substack{i\in S\backslash\{1\}\\ j\in \overline{S}\backslash\{1\}}} y_{ij} \tag{62}$$

and

$$\sum_{\substack{i\in \overline{S}\backslash\{1\}\\ j\in S\backslash\{1\}}} y_{ij} \geqslant \sum_{\substack{i\in \overline{S}\backslash\{1\}\\ j\in S\backslash\{1\}}} x_{ij}. \tag{63}$$

Hence equality (61) and inequalities (62) and (63) imply

$$(n-2)\sum_{\substack{i\in S\backslash\{1\}\\ j\in \overline{S}\backslash\{1\}}} x_{ij} + (n-1)\sum_{i\in S\backslash\{1\}} x_{i1} \geqslant \sum_{\substack{i\in \overline{S}\backslash\{1\}\\ j\in S\backslash\{1\}}} x_{ij} + |S|, \tag{64}$$

which results in

$$(n-2)D(S\backslash\{1\}) + (n-1)\sum_{i\in S\backslash\{1\}} x_{i1} \geqslant (n-1)\sum_{\substack{i\in \overline{S}\backslash\{1\}\\ j\in S\backslash\{1\}}} x_{ij} + |S| \tag{65}$$

after adding $(n-2)\sum_{\substack{i\in \overline{S}\backslash\{1\}\\ j\in S\backslash\{1\}}} x_{ij}$ to both sides of (64). Here $D(S\backslash\{1\}) = \sum_{\substack{i\in S\backslash\{1\}\\ j\in \overline{S}\backslash\{1\}}} x_{ij} + \sum_{\substack{i\in \overline{S}\backslash\{1\}\\ j\in S\backslash\{1\}}} x_{ij}$ is the number of arcs having exactly one endpoint (head or tail but not both) in the subset $S\backslash\{1\}$. Inequality (65) can be equivalently rewritten as

$$(n-2)D(S\backslash\{1\}) + (n-1)\sum_{\substack{i\in S\backslash\{1\}\\ j\in \overline{S}\backslash\{1\}}} x_{ij} \geqslant (n-1)D(S\backslash\{1\}) + |S| \tag{66}$$

after adding $(n-1)\sum_{\substack{i\in S\backslash\{1\}\\ j\in \overline{S}\backslash\{1\}}} x_{ij}$ to its right- and left-hand sides, from which

$$(n-1)\sum_{\substack{i\in S\backslash\{1\}\\ j\in \overline{S}\backslash\{1\}}} x_{ij} \geqslant D(S\backslash\{1\}) + |S| \tag{67}$$

follows. Since

$$\sum_{i,j\in S} x_{ij} + \sum_{\substack{i\in S\\ j\in \overline{S}}} x_{ij} = |S| \tag{68}$$

for all subsets S of V, we can write

$$\sum_{i,j\in S\backslash\{1\}} x_{ij} + \sum_{\substack{i\in S\backslash\{1\}\\ j\in \overline{S}}} x_{ij} \leqslant |S|, \quad S \subseteq V \tag{69}$$

or equivalently

$$\sum_{i,j\in S\backslash\{1\}} x_{ij} \leqslant |S| - \sum_{\substack{i\in S\backslash\{1\}\\ j\in \overline{S}}} x_{ij}, \quad S \subseteq V. \tag{70}$$

Now, using (70) we can rewrite (67) as

$$(n-1)\left(|S| - \sum_{i,j\in S\backslash\{1\}} x_{ij}\right) \geqslant D(S\backslash\{1\}) + |S| \tag{71}$$

the weak clique inequalities

$$|S| - \frac{|S|}{n-1} \geqslant \sum_{i,j\in S} x_{ij}, \quad S \subseteq \{2,\ldots,n\}, 2 \leqslant |S| \leqslant n-1 \tag{72}$$

since $D(S\backslash\{1\})$ is non-negative. A direct consequence of this discussion is that the projection of a polyhedron that includes P(SD) is included in TP(GG) and consequently TP(SD) is in TP(GG). The SD formulation is as strong as GG. In addition it is always possible to find an ATSP instance for which the LP bound of the SD formulation is strictly larger than that of the GG formulation on the same instance. One such example is `ftv33` (Table 2). Therefore TP(SD) is a proper subset of TP(GG) and SD is stronger than GG. □

Proposition 3. *The DL and GG formulations are incomparable*.

Proof. Consider instances `ft53` and `ft70` in Table 2. The percent gaps from the optimum of the LP bounds corresponding to DL and GG formulations are, respectively, 12.934 and 12.454 for `ft53`. However, they are equal to 0.878 and 1.103 for `ft70`. □

4.3. The GP1, GP2, GP3, GP4, DL and SD formulations

Another interesting relationship exists between the GP1 and DL formulations. The fact that (11) is one of the subtour elimination constraints of the DL formulation, and that it is derived from constraints (36), (37), (39) and the linear transformation (38) may lead one to think that the GP1 formulation is stronger than the DL formulation. However, this is not the case.

Proposition 4. *The GP*1, *GP*2, *GP*3 *and GP*4 *formulations are not comparable to the DL formulation*.

Proof. Observe that for `ftv70` (Table 2) the LP bound of GP1 is strictly larger than the LP bound of DL. However, for `ft70` the LP bound of DL is strictly larger than that of GP4. □

A similar relationship exists between GP1, GP2, GP3, GP4 and SD.

Proposition 5. *The GP*1, *GP*2, *GP*3 *and GP*4 *formulations are not comparable to the SD formulation*.

Proof. For instance `ftv55`, the LP bound of GP1 is strictly larger than that of SD. However, for `ft70`, the LP bound of SD is strictly larger than that of GP4. □

4.4. The SSB formulations

We now comment on the relative strengths of the SSB1, SSB2 and SSB3 formulations. Our first remark is on the relation between the SSB1 and SSB2. Note that SSB2 is potentially stronger than SSB1 since inequalities (51) are lifted versions of the inequalities (50). Our second observation is related with the inequalities (48) of SSB2 which are in fact redundant. To see this, define $XD = \{(\mathbf{x},\mathbf{d}) : x_{ij} \leqslant d_{ij}, d_{ij} + d_{ji} = 1, i, j = 2,\ldots,n\}$ and also define the polyhedron $TP(XD) = \{\mathbf{x} \in \mathbb{R}^{n(n-1)} : x_{ij} + x_{ji} \leqslant 1, i, j = 2,\ldots,n\}$.

Proposition 6. *The projection of XD into the subspace of the x_{ij} variables is TP*(*XD*).

Proof. Let $(\overline{x},\overline{d})$ be a vector in *XD*. Summing up the inequalities $\overline{x}_{ij} \leqslant \overline{d}_{ij}$ and $\overline{x}_{ji} \leqslant \overline{d}_{ji}$ and using the identities $\overline{d}_{ij} + \overline{d}_{ji} = 1$ we obtain $\overline{x}_{ij} + \overline{x}_{ji} \leqslant 1$. Hence, $XD \subseteq TP(XD)$. Now select x from the polyhedron *TP*(*XD*). Then $x_{ij} \leqslant 1 - x_{ji}$ follows. In addition setting $d_{ij} = x_{ij}$ and $d_{ji} = 1 - x_{ij} \geqslant x_{ji}$ provides a feasible assignment, implying that there exists a vector d for every $x \in TP(XD)$ such that $(x,d) \in XD$. Therefore $TP(XD) \subseteq XD$. □

Now, adding inequalities $d_{ij} + d_{jk} + d_{ki} + x_{ji} \leqslant 2$ and $d_{ji} + d_{ik} + d_{kj} + x_{ij} \leqslant 2$ implies $x_{ij} + x_{ji} \leqslant 1$, namely *TP*(*XD*), since $d_{ij} + d_{ji} = 1$, $d_{kj} + d_{jk} = 1$ and $d_{ki} + d_{ik} = 1$. Hence, inequalities (51) together with inequalities (49) imply inequalities (48). We have thus proved the following proposition.

Proposition 7. *Inequalities* (48) *are redundant in SSB*2.

The next two results show that the SSB2 dominates both of the GP2 and SSB3 formulations.

Proposition 8. *TP*(*SSB*2) *is a proper subset of TP*(*SBB*3).

Proof. Adding inequalities (51) for (i, j), (j, k) and (k, i), i.e.,

$$d_{ij} + d_{jk} + d_{ki} + x_{ji} \leqslant 2,$$

$$d_{jk} + d_{ki} + d_{ij} + x_{kj} \leqslant 2,$$

$$d_{ki} + d_{ij} + d_{jk} + x_{ik} \leqslant 2$$

yields

$$3(d_{ki} + d_{ij} + d_{jk}) + x_{ji} + x_{kj} + x_{ik} \leqslant 6,$$

which is exactly (52). This implies that SSB2 is also potentially stronger than SSB3. Furthermore, Table 2 contains instances, such as `ftv47`, for which the LP bound SSB2 is strictly larger than that of SBB3. □

Proposition 9. *TP*(*SSB*2) *is a proper subset of TP*(*GP*2).

Proof. Consider a circuit $D_C = (V_C, C)$. Adding the subtour elimination constraints

$$x_{ij} + d_{ki} + d_{ij} \leqslant d_{kj} + 1, \quad (i, j) \in C \text{ with } i, j, k = 2, \ldots, n; \quad k \notin C$$

and

$$x_{ji} + d_{pj} + d_{ji} \leqslant d_{pi} + 1, \quad (i, j) \in C \text{ with } i, j, p = 2, \ldots, n; \quad i, j \neq p; \quad p \in C,$$

and using circularity constraints $d_{ij} + d_{ji} = 1$ we obtain inequalities

$$\sum_{(i,j)\in C} x_{ij} + \sum_{\substack{(i,j)\in C \\ i,j\neq p}} x_{ji} \leqslant |C| - 1, \quad p \in V_C, \quad V_C \subseteq \{2, \ldots, n\}. \tag{73}$$

On the other hand, adding the following constraints and using $d_{ij} + d_{ji} = 1$,

$$x_{ij} + d_{ki} + d_{ij} \leqslant d_{kj} + 1 \quad \text{for } i, j, k = 2, \ldots, n,$$

$$x_{ji} + d_{kj} + d_{ji} \leqslant d_{ki} + 1 \quad \text{for } i, j, k = 2, \ldots, n$$

we obtain

$$x_{ij} + x_{ji} \leqslant 1 \quad \text{for } i, j = 2, \ldots, n. \tag{74}$$

Constraints (73) and (74) are known to be the inequalities describing the projection of the subtour elimination constraints of GP2 formulation into the subspace of x_{ij} variables [17]. Therefore TP(SSB2) is a subset of TP(GP2). Moreover, Table 2 contains ATSP instances, such as `ftv33`, for which the LP bound of SSB2 is strictly larger than that of GP2. □

Our final remark is on the incomparability of the SSB2 and GP4 formulations.

Proposition 10. *The SSB2 and GP*4 *formulations are incomparable*.

Proof. To see this it suffices to consider instances `ft70` and `ft53`. For one of them GP4 give tighter LP bound but for the other SSB2 yields a tighter LP bound. □

4.5. The SST formulations

We will now prove that SST2 is the strongest formulation proposed by Sherali et al. [25].

Proposition 11. *The constraints* (57)–(59) *imply the lifted inequalities* (40).

Proof. By definition of the flow variables f_{kij} used in the constraints (41)–(44) [23] and by the definition of the flow variables t_{ik}^{j} used in the tightened MCF constraints [25], we have the identity

$$t_{ik}^{j} = f_{kij}. \tag{75}$$

Recall that f_{kij} is equal to one if and only if arc (i, j) is on the path from node k to node 1 and t_{ik}^{j} is equal to one if the commodity from 1 to k flows on arc (i, j). On the other hand, like the transformation (45) by Gouveia and Pires [23], namely $\sum_{j=1}^{n} f_{kij} = d_{ki}$ for $i, k = 2, \ldots, n$, we have the following transformation:

$$\sum_{j=1}^{n} t_{ik}^{j} = d_{ki}, \quad k, j = 2, \ldots, n. \tag{76}$$

Gouveia and Pires [23] have shown that it is possible to obtain constraints (40) from constraints (43), (44), and a weaker version of (41). Following similar steps and by using (75) and (76), we obtain

$$x_{ij} + x_{ik} + x_{kj} + d_{ki} \leqslant 1 + d_{kj}, \quad i, j, k = 2, \ldots, n, \tag{77}$$

which are exactly the lifted inequalities (40) proposed by Gouveia and Pires [17] since $d_{ij} = v_{ij}$. □

Therefore, Proposition 11 states that the strongest formulation proposed by Sherali et al. [25] is SST2 since none of their formulations contains any constraints stronger than (57)–(59).

Proposition 12. *The SST2 formulation is stronger than the SSB2 formulation.*

Proof. It is clear that SST2 is stronger than SSB2 since the former includes all the constraints of the latter, as well as some additional non-redundant constraints. Hence, the LP relaxation of SST2 is tighter than that of SSB2. To see that TP(SST2) is a proper subset of TP(SSB2) it suffices to consider the instance `ftv33` of Table 2, for which the LP bound of SST2 is strictly larger than that of SSB2. □

Proposition 13. *The SST2 formulation is stronger than the GP7 formulation.*

Proof. The thesis follows from the results by Sherali et al. [25] who have observed that constraints (57)–(59) are stronger than the MCF constraints (41)–(44), and constraints (51) are liftings of constraints (35). To show that SST2 is stronger than GP7 it suffices to provide a feasible $(x_{ij}, v_{ij}, z_{kij})$ vector of P(GP7) such that there does not exist a vector (d_{ji}, z_{kij}) for which $(x_{ij}, d_{ji}, z_{kij})$ is included in P(SST2). Consider the following solution for a five-vertex instance $x_{13} = \frac{2}{3}$, $x_{15} = \frac{1}{3}$, $x_{21} = \frac{2}{3}$, $x_{25} = \frac{1}{3}$, $x_{34} = \frac{2}{3}$, $x_{35} = \frac{1}{3}$, $x_{41} = \frac{1}{3}$, $x_{42} = \frac{2}{3}$, $x_{52} = \frac{1}{3}$, $x_{53} = \frac{1}{3}$, $x_{54} = \frac{1}{3}$. This solution is feasible for GP7, but infeasible for SST2. Hence there exists at least one instance for which TP(SST2) is a proper subset of TP(GP7). □

5. Computational experiments

To complete our comparative study, we report some results on the empirical quality of the LP bounds obtained with some of the formulations presented in this paper. We have tested these formulations on standard ATSP instances obtained from TSPLIB Reinelt [28]. In Table 2 we present relative deviations from the optimal tour lengths, computed as

$$100 \times \left(\frac{z_{IP}^{*} - z_{LP}^{*}}{z_{IP}^{*}} \right),$$

where z^*_{IP} is the length of an optimal tour and z^*_{LP} is the lower bound obtained by solving the LP relaxation of the models. All computations were performed on a Pentium IV PC with a 3 GHz CPU.

As can be observed from Table 2, the SST2 and CLAUS formulations provide the best bounds. This is not surprising since SST2 is theoretically stronger than CLAUS, which is theoretically as strong as DFJ. The SSB2, SSB3, GP3 and GP4 formulations are the second best, while the performance of FGG4 is the worst. Table 3 provides the CPU times required for the computation of the lower bounds with the barrier solver of CPLEX 9.0. We use the barrier solver since the instances are quite large and the use of interior point methods can decrease the solution time considerably. However, barrier requires larger memory space which can cause difficulties for large linear programs as it is the case with the LP relaxation of SST2. This explains why we could not solve the LP relaxation of `ftv44` and of larger instances.

Table 3
CPU times (in seconds) with barrier option

Instance	SST1	SSB1	SSB3	SSB2	SST2	GP1	GP2	GP3	GP4	CLAUS	MTZ	FGG2	FGG4	GG	SD	DL
`ftv33`	21.9	5.1	7.3	7.8	668.0	15.7	17.3	57.4	29.9	17.2	2.3	0.9	2.4	0.2	0.8	2.8
`ftv35`	27.6	7.9	10.0	10.8	1357.0	21.9	34.2	71.0	71.9	18.9	3.9	1.0	8.9	0.2	0.9	2.6
`ftv38`	44.9	13.7	15.8	22.5	1741.0	89.9	66.0	74.9	91.6	29.3	4.7	1.3	4.6	0.2	1.2	5.8
`ftv44`	103.7	22.7	27.9	34.0	na	131.0	101.9	218.0	148.0	56.6	10.0	2.9	7.2	0.3	1.6	12.4
`ftv47`	140.4	31.9	37.3	42.9	na	171.8	199.0	194.7	184.8	232.0	20.3	2.9	7.7	0.3	2.1	20.6
`ftv55`	314.2	87.7	92.4	8.3	na	242.0	263.0	283.8	507.0	346.0	36.4	5.9	16.9	0.4	3.5	40.2
`ftv64`	876.0	201.2	234.5	254.1	na	823.7	910.1	779.7	2806.7	602.1	52.8	8.9	29.1	0.5	5.7	75.0
`ftv70`	1384.9	310.3	408.4	493.0	na	1030.4	1193.6	1013.9	3707.3	917.6	112.2	12.8	41.7	0.6	14.5	152.9
`ft70`	671.0	410.5	433.7	447.0	na	780.7	1337.1	1144.8	2986.2	832.8	110.4	26.9	32.4	0.6	7.5	149.9
`ft53`	902.0	193.6	194.0	167.0	na	186.0	200.0	349.2	419.0	382.0	26.9	5.5	12.9	0.3	3.2	33.1
Average	448.7	128.5	146.1	148.7	1255.3	349.32	432.22	418.75	1095.25	343.46	37.99	6.9	16.4	0.4	4.1	49.5

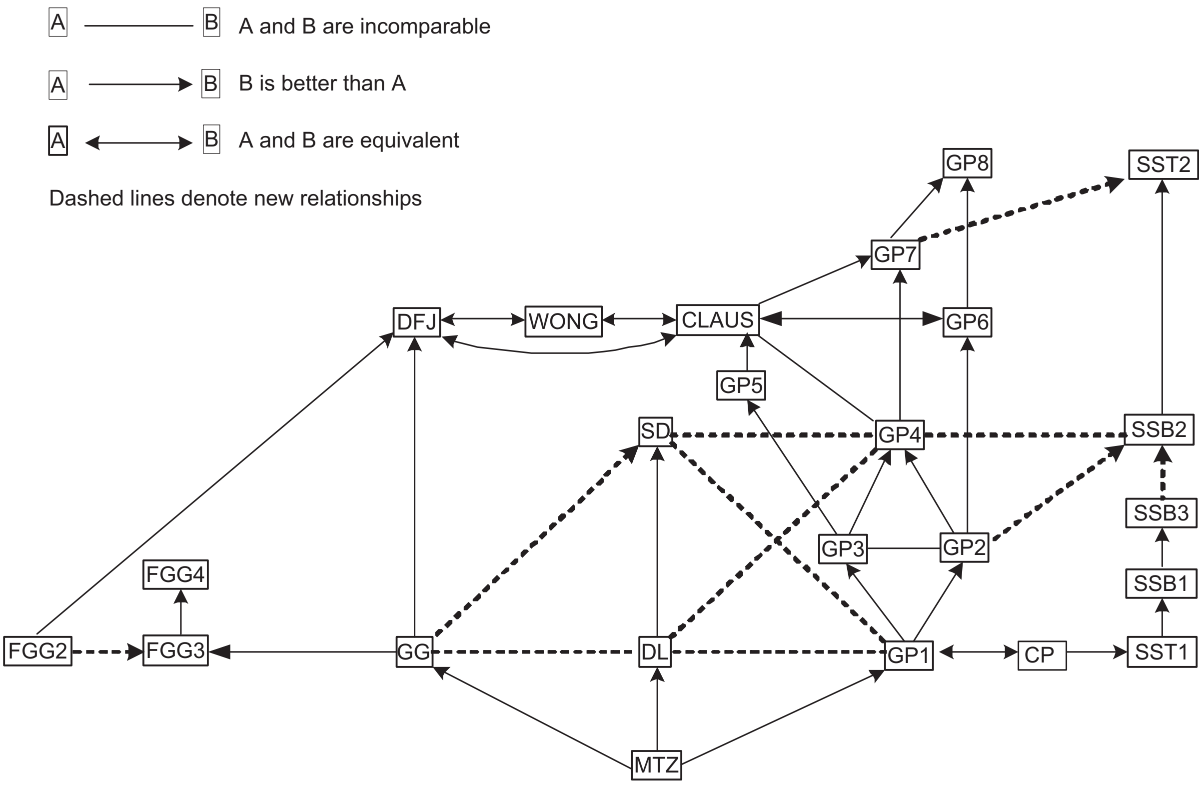

Fig. 2. Relative strength of the 24 ATSP formulations.

6. Conclusions

This comparative review and analysis clearly illustrates the wealth of available models for the ATSP. We have summarized the known relationships between these models and we have proved some new ones. Our study is supported by computational results on a subset of the TSPLIB instances. Fig. 2 depicts the relationships between 24 ATSP formulations, but also shows that some gaps exist. Clearly, there is scope for further research in this fascinating area.

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