The conserved phase–field system with memory

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The standard conserved phase–field system in its simplest form reads

$$\partial_t(\vartheta + \ell\chi) - \Delta\vartheta = g \tag{1}$$

$$\partial_t\chi - \Delta\bigl(-\Delta\chi + \chi^3 - \chi - \ell\vartheta\bigr) = 0 \tag{2}$$

where ϑ and χ are the relative temperature and the order parameter respectively, ℓ represents the latent heat, and g is a source term. The first equation comes from the energy balance, provided that the heat flux $\mathbf{q}$ is given by the Fourier law

$$\mathbf{q}(x,t) = -\nabla\vartheta(x,t). \tag{3}$$

We replace (3) by the Gurtin–Pipkin law, namely

$$\mathbf{q}(x,t) = -\int_{-\infty}^{t} k(t-s)\,\nabla\vartheta(x,s)\,ds = \mathbf{q}_0(x,t) - (k * \nabla\vartheta)(x,s) \tag{4}$$

where k is a smooth memory kernel. In (4), the flux $\mathbf{q}_0$ and the convolution account for the integrals over $(-\infty,0)$ and $(0,t)$, respectively. Hence, $\mathbf{q}_0$ is known whenever ϑ is known up to $t=0$. Moreover, we generalize the free energy which leads to the Cahn–Hilliard equation (2) and allow the latent heat ℓ to depend on χ itself. Hence, we study the system

$$\partial_t(\vartheta + \lambda(\chi)) - \Delta(k * \vartheta) = g \tag{5}$$

$$\partial_t\chi - \Delta w = 0 \tag{6}$$

$$w = -\Delta\chi + \xi + \sigma'(\chi) - \lambda'(\chi)\,\vartheta \tag{7}$$

$$\xi \in \beta(\chi) \tag{8}$$

where g is assumed to be known, λ and σ are smooth real functions on $\mathbb{R}$, and β is a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$, which accounts also for constraints on χ if $D(\beta) \neq \mathbb{R}$. Finally, system (5–8) is complemented by appropriate initial and boundary conditions, namely

$$\vartheta\,|_{t=0} = \vartheta_0, \qquad \chi\,|_{t=0} = \chi_0 \tag{9}$$

$$\nabla(k * \vartheta)\cdot\mathbf{n} = \nabla\chi\cdot\mathbf{n} = \nabla w\cdot\mathbf{n} = 0 \tag{10}$$

where ϑ_0, χ_0 are given and $\mathbf{n}$ is the normal unit vector on the boundary.

We discuss the boundary value problem (5–10) from the mathematical viewpoint, looking for existence, uniqueness, continuous dependence, and regularity of solutions. The results we present have been obtaind in a joint work with P. Colli (Pavia), M. Grasselli (Milan), and G. Schimperna (Pavia). The main structure assumptions are: $k(0)$ is strictly positive and λ, σ' have linear growth at infinity.