The theorems of Riemann-Roch and Abel

Lectures delivered at the 1987 ICTP College on Riemann Surfaces

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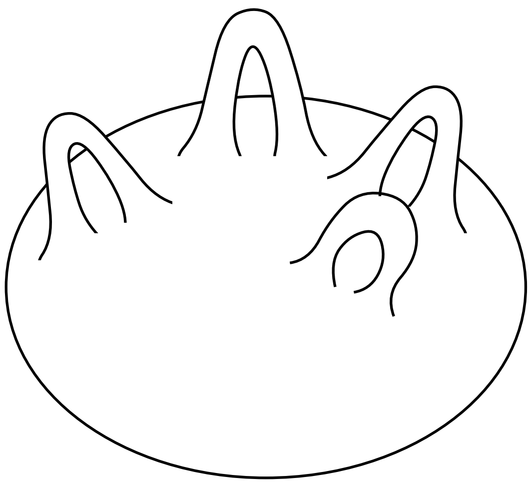
Abstract. The aim of these notes is to give simple proofs of the theorems of Abel and Riemann-Roch for compact Riemann surfaces, and to explain some of their most elementary geometric applications. The treatment of the subject is quite classical; in particular, no cohomological machinery is used, although some knowledge of it on the part of the reader would certainly help.

1. Abelian differentials. Let C be a compact connected Riemann surface (henceforth abbreviated RS). As is well known, topologically C is just a sphere with handles attached:



The number of handles is the <u>genus</u> of C: thus the above figure depicts a RS of genus 4.

An <u>abelian</u> <u>differential</u> on C is a global holomorphic (1,0)-form, i. e., a (1,0)-form α such that $\bar\partial\alpha = 0$; put otherwise, if we write, locally, $\alpha = a(z)dz$, a has to be holomorphic. Let's illustrate this with a few examples.

(1.1) Example: the Riemann sphere. This is obtained from the disjoint union of two copies of $\mathbb{C}$, with coordinates z and w, via the identification $w = 1/z$, for every nonzero z. If α is an abelian differential, we can write it, in the z coordinate, as $\alpha = a(z)dz$, where

$$a(z) = \sum_{i\geq 0} c_i z^i \, ;$$

in the w coordinate, then,

$$\alpha = \sum_{i\geq 0} c_i w^{-i} d(1/w) = -\sum_{i\geq 0} c_i w^{-i-2} dw .$$

Thus α is not holomorphic at $w = 0$ unless it vanishes identically. In other words, in this case the vector space of abelian differentials has dimension zero.

(1.2) Example: elliptic Riemann surfaces. An elliptic RS is the quotient $\mathbb{C}/\Lambda = C$, where Λ is a lattice in $\mathbb{C}$. If z is a linear coordinate in $\mathbb{C}$, the differential dz is translation-invariant, hence it descends to an abelian differential on C, to be denoted by the same symbol. Notice that dz does not vanish anywhere; thus, if α is another abelian differential, α/dz is a holomorphic function on C, hence a constant. This shows that, for an elliptic RS, the space of abelian differentials is one-dimensional; furthermore, a nonzero abelian differential does not vanish anywhere.

Before moving on to the next example, we make a general remark. Let φ, ψ be abelian differentials on a RS C: we claim that φ and ψ have the same number of zeros (an n-fold zero is counted as n zeros). Consider in fact the quotient φ/ψ: it is a meromorphic function on C, and applying the residue theorem to its logarithmic derivative shows that it has as many zeros as it has poles, if we count zeros and poles according to their multiplicity. Since the poles of φ/ψ arise from zeros of ψ, and its zeros from zeros of φ, this proves our claim.

(1.3) Example: hyperelliptic Riemann surfaces. Consider the locus Z in $\mathbb{C}^2$ defined by the equation

$$x^2 = F(y) \, ,$$

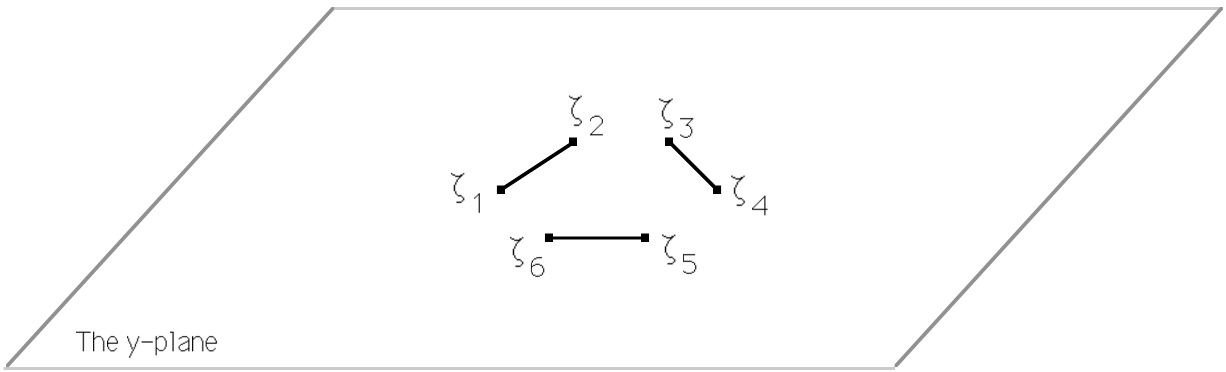
where F is a polynomial of even degree $2n$ with distinct roots. Thus

$$F(y) = \prod_{1 \leq i \leq 2n} (y - \zeta_i) \ , \qquad \zeta_i \neq \zeta_j \text{ if } i \neq j \ .$$

Notice, first of all, that Z is smooth. In fact, the partial of $x^2 - F(y)$ with respect to x vanishes on Z only at the roots of F, where the partial with respect to y does not vanish. Thus, by the implicit function theorem, we may take y as a local coordinate on Z away from the roots of F, and x as a local coordinate at the roots of F. To fix ideas, we shall work with

$$F(y) = y^6 - 1 \ ,$$

but our discussion will be valid in general. Partition the roots of F in pairs and join the two elements of each pair with a segment (a path, in general).



The function $(y - \zeta_i)(y - \zeta_{i+1})$ has two single-valued square roots in the complement of the segment $\zeta_i \zeta_{i+1}$: call one of these f_i, and set

$$f = f_1 \cdot f_3 \cdot f_5 \ .$$

Now cut the y-plane along the segments $\zeta_1 \zeta_2$, $\zeta_3 \zeta_4$, $\zeta_5 \zeta_6$. The part of Z lying over the remaining portion of the y-plane is the disjoint union of two copies of the slit y-plane, namely of the two sheets with equations

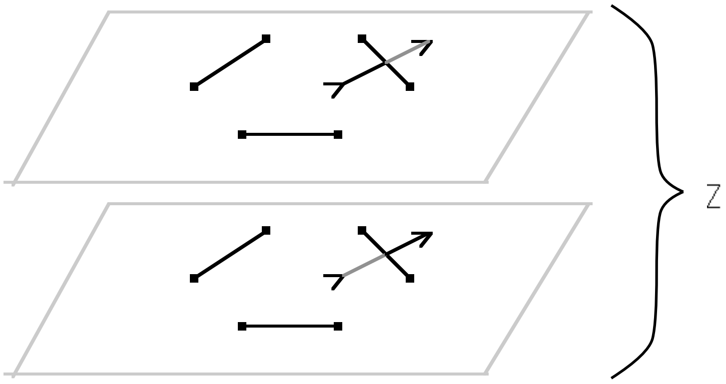
$$x = f(y) \ , \quad x = -f(y) \ .$$

As the complex plane can be compactified by adding one point at infinity, yielding the Riemann sphere, so Z can be compactified to a RS C by adding two points at infinity. More exactly, one can glue in two small disks $\{w \,|\, |w| < \varepsilon\}$ and $\{t \,|\, |t| < \varepsilon\}$ by identifying

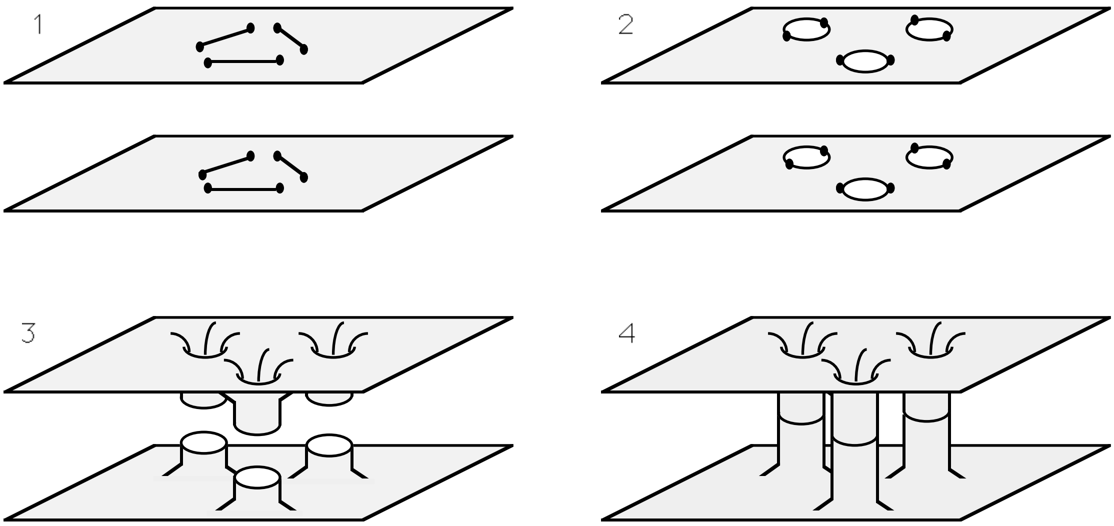
$$w \longmapsto (x,y) \qquad \text{where} \qquad x = f(1/w) \ , \ \ y = 1/w \ ,$$

$$t \longmapsto (x,y) \qquad \text{where} \qquad x = -f(1/t) \ , \ \ y = 1/t \ .$$

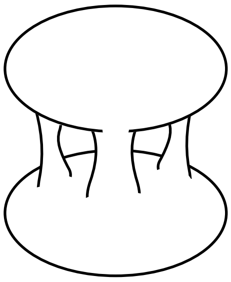
To get C from our two copies of the (completed) slit y-plane we have to glue them according to the following prescription: we attach one side of each slit on the "lower" sheet to the opposite side of the corresponding slit on the "upper" sheet. This is illustrated in the picture below: after glueing one gets an unbroken solid arrow and an unbroken dotted arrow.



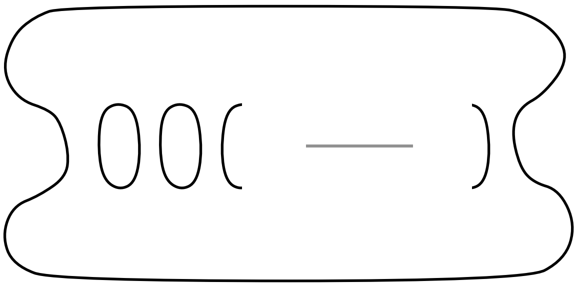
To picture what C looks like topologically it is convenient to perform an inversion with respect to the origin ($y \longmapsto 1/\bar{y}$) in the "upper" slit y-plane. This has the effect of reversing orientation, in particular reversing the "upper" arrows in the picture above. The glueing process can then be visualized, in successive steps, as follows:



The resulting RS therefore looks like this:



In other words, C has genus equal to 2. For general F, this same construction yields a RS of genus $g = n-1$:



Now we shall explicitly write down all abelian differentials on C. Set

$$\varphi = \frac{dx}{\frac{\partial F}{\partial y}} \ .$$

In our special case

$$\varphi = \frac{dx}{6y^5} = \frac{dy}{2x} \ .$$

By looking at the first expression near the 6^{th} roots of unity, where y does not vanish and x is a local coordinate, and at the second expression away from the 6^{th} roots of unity, where x does not vanish and y is a local coordinate, we see that φ is holomorphic and never vanishing on Z. In the w coordinate, instead, we have that

$$\varphi = \frac{w^3}{2\sqrt{1-w^6}} d\frac{1}{w} = -\frac{w}{2\sqrt{1-w^6}} dw \ ,$$

so φ has a simple zero at $w=0$; the same happens for $t=0$. For general F, similar considerations would show that φ is holomorphic, does not vanish on Z, and has zeros of multiplicity $n-2$ at the two points at infinity. We then have at least g independent abelian differentials on C, namely

$$\varphi\ ,\ \ y\varphi\ ,\ \ y^2\varphi\ ,\ \ \cdot\ \cdot\ \cdot\ ,\ \ y^{g-1}\varphi\ .$$

We claim that this is all, i. e., that any abelian differential on C is a linear combination of these. The argument runs as follows. Let h be the involution on C sending (x,y) to $(-x,y)$. Any abelian differential α can be written as the sum of its h-invariant and anti-invariant parts, namely of

$$\beta = \frac{\alpha + h(\alpha)}{2} \quad , \qquad \gamma = \frac{\alpha - h(\alpha)}{2} \ .$$

The invariant part, β, descends to a holomorphic differential on the y-plane, at least away from the roots of F. If p is one of these roots, β can be written, near p, in the local coordinate x, as $\beta = a(x)\cdot dx$, where

$$a(x) = \sum_{i\geq 0} c_i x^i \ .$$

Since β is invariant under the involution $x \longmapsto -x$, all the coefficients with even index must vanish, so that we can write, for some holomorphic function b,

$$\beta = 2b(x^2)\cdot x\,dx = b(y)\cdot dy\ .$$

So β descends to a holomorphic differential on <u>all</u> of the (completed) y-plane; since there are no nonzero holomorphic differentials on the Riemann sphere, β must be zero. Now we know that every abelian differential α on C is anti-invariant. The quotient α/φ is then an invariant meromorphic function on C which is holomorphic on Z, because φ does not vanish there. Therefore

$$\alpha/\varphi = P(y)$$

for some <u>polynomial</u> P. The degree of P cannot exceed $g-1=n-2$ since φ has $(n-2)$-fold zeros at infinity.

In conclusion, we have seen that, at least for the examples considered so far, the genus can be equivalently defined as:

i) the number of handles of C,
ii) the dimension of the space of abelian differentials on C,
iii) $\frac{1}{2}$(the number of zeros of an abelian differential on C)+1.

We shall now see that this holds for <u>every</u> RS. Consider the sequence of vector spaces and linear maps

(1.4) $$0 \to H^0(C,K_C) \to H^1(C,\mathbb{C}) \to H^1(C,\mathcal{O}_C) \xrightarrow{d} H^1(C,K_C) \to H^2(C,\mathbb{C}) \longrightarrow 0\ ,$$

where the meaning of the symbols is as follows:

- $H^0(C,K_C)$ (often abbreviated $H^0(K_C)$ or $H^0(K)$) stands for the vector space of abelian differentials on C.
- $H^1(C,K_C)$ (abbreviated $H^1(K_C)$ or $H^1(K)$) stands for $\dfrac{(1,1)\text{-forms}}{\bar\partial\text{-exact } (1,1)\text{-forms}}$.
- $H^i(C,\mathbb{C})$ (abbreviated $H^i(\mathbb{C})$) stands for $\dfrac{d\text{-closed } i\text{-forms}}{d\text{-exact } i\text{-forms}}$.
- $H^1(C,\mathcal{O}_C)$ (abbreviated $H^1(\mathcal{O}_C)$ or $H^1(\mathcal{O})$) stands for $\dfrac{(0,1)\text{-forms}}{\bar\partial\text{-exact } (0,1)\text{-forms}}$.

The maps are induced by the obvious ones on the form level; in particular d is induced by exterior differentiation. Of course, via the Dolbeault and deRham isomorphisms (more on these in section 2) (1.4) is just part of the long cohomology exact sequence of

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_C \xrightarrow{d} K_C \longrightarrow 0\ ,$$

where $\mathcal{O}_C$ is the sheaf of holomorphic functions on C and K_C the canonical sheaf, but this need not concern us now.

We claim that (1.4) is exact. The only point that deserves some attention is the fact that abelian differentials inject into $H^1(\mathbb{C})$. To see this, let α be an abelian differential, and suppose $\alpha = d\beta$. Then $\bar\partial\beta = 0$, so β is holomorphic, hence constant, and $\alpha = d\beta = 0$. Exactness of (1.4) at other places is straightforward. As an example, let's check exactness at $H^1(\mathbb{C})$. Suppose α is a d-closed 1-form, and write $\alpha = \beta + \gamma$, where β has type (1,0) and γ has type (0,1). To say that the class of α maps to zero in $H^1(\mathcal{O})$ means that we can write $\gamma = \bar\partial\eta$. Then

$$\bar\partial(\beta - \partial\eta) = \bar\partial\beta + \partial\bar\partial\eta = \bar\partial\beta + \partial\gamma = d\alpha = 0\ ,$$

$$\beta - \partial\eta = \beta + \gamma - d\eta = \alpha - d\eta\ .$$

The first identity says that $\beta - \partial\eta$ is an abelian differential, the second that it represents the same class in $H^1(\mathbb{C})$ as α.

Now consider the antilinear map

(1.5) $$H^0(C,K_C) \longrightarrow H^1(C,\mathcal{O}_C) \quad ; \quad \alpha \longmapsto \text{class of } \bar\alpha\ .$$

We wish to show that it is injective. Suppose in fact that $\bar\alpha = \bar\partial\beta$. Then

$$\partial\bar{\partial}\beta = \overline{\bar{\partial}\alpha} = 0 \ .$$

Now, if $z = x + iy$ is a local coordinate on C and x, y are its real and imaginary parts,

$$\frac{\partial^2\beta}{\partial z\,\partial\bar{z}} = \frac{\partial^2\beta}{\partial x^2} + \frac{\partial^2\beta}{\partial y^2} = \Delta\beta \ ,$$

so that $\partial\bar{\partial}\beta = \Delta\beta \cdot dz \wedge d\bar{z}$ is essentially the Laplacian of β. We know then that β is hamonic. But now the maximum principle (cf. the Appendix) says that, if a harmonic function has a local maximum, then it is constant in a neighbourhood of it. Since C is compact and connected, β must be constant, so α is zero.

To show that (1.5) is in fact an isomorphism is not as simple. What has to be shown is that, if α is a (0,1)-form, we can find β such that

$$\partial(\alpha - \bar{\partial}\beta) = 0 \ .$$

In other words, we must be able to solve the differential equation

$$\partial\bar{\partial}\beta = \partial\alpha \ .$$

That this is solvable follows from the following more precise result, which can be regarded as the *cornerstone of the theory of compact Riemann surfaces*.

(1.6) Theorem. Let C be a compact Riemann surface, and let ψ be a C^∞ 2-form on it. Then the equation

$$\partial\bar{\partial}u = \psi$$

can be solved if, and only if, v has zero mean, i. e.,

$$\int_C \psi = 0 \ .$$

To be able to apply (1.6) to our special problem, we need only notice that

$$\int_C \partial\alpha = \int_C d\alpha = 0 \ ,$$

by Stokes' theorem. As for the proof of (1.6), we refer to the Appendix; here we would just like to rephrase the theorem, so as to make it more plausible. Pick a volume form Φ on C, that is, a form that locally looks like

$$\frac{2}{i}\cdot a\cdot dz\wedge d\bar{z} = a\cdot dx\wedge dy\ ,$$

where a is a positive function, and define the Laplacian of u to be:

$$Lu = i\cdot(\partial\bar{\partial}u)/\Phi\ .$$

We also define an inner product on functions defined on C by setting

$$\langle u,v\rangle = \int_C u\bar{v}\,\Phi\ .$$

With respect to this inner product, L is a selfadjoint, strongly elliptic, partial differential operator. In fact

$$\langle Lu,v\rangle = i\int_C((\partial\bar{\partial}u)/\Phi)\bar{v}\Phi = i\int_C\partial\bar{\partial}u\cdot\bar{v} = i\int_C d(\bar{\partial}u\cdot\bar{v}) + i\int_C \bar{\partial}u\wedge\partial\bar{v}$$

$$= i\int_C \bar{\partial}u\wedge\partial\bar{v} = i\int_C d(u\cdot\partial\bar{v}) - i\int_C u\cdot\bar{\partial}\partial\bar{v} = -i\int_C u\cdot\bar{\partial}\partial\bar{v} = \langle u,Lv\rangle\ .$$

General theory then says that $Lu=v$ can be solved if, and only if, v is orthogonal to the kernel of the adjoint of L, that is, to the kernel of L. But this kernel consists precisely of the harmonic functions, which are all constant. Thus the condition for $Lu=v$ to be solvable is that

$$\int_C v\Phi = 0\ ,$$

and $\partial\bar{\partial}u = \psi$ can be solved if and only if

$$0 = \int_C(\psi/\Phi)\Phi = \int_C\psi\ .$$

Another consequence of (1.6) is that the homomorphism from $H^1(C,K_C)$ to $H^2(C,\mathbb{C})$ in (1.4) is an isomorphism: in fact, if α is a (1,1)-form and $\alpha = d\beta$, then we may write

$$\alpha = d\beta = \partial\bar{\partial}\gamma = -\bar{\partial}(\partial\gamma)\ ,$$

so α represents zero in $H^1(C,K_C)$. Moreover, $H^1(C,K_C)$ can be explicitly calculated. To see this, let α be a (1,1)-form: there is a constant k such that

$$\int_C\alpha = k\int_C\Phi\ .$$

By (1.6) this implies that we can find β such that

$$\alpha - k\Phi = \partial\bar{\partial}\beta = d\bar{\partial}\beta .$$

Thus every element of $H^1(C,K_C)$ is a multiple of the class of Φ, and $H^1(C,K_C)$ is one-dimensional.

In conclusion, if we write $h^i(\ ,\ )$ to denote the complex dimension of $H^i(\ ,\ )$, the exact sequence (1.4) and the corollaries of Theorem (1.6) tell us that:

$$\frac{1}{2}\cdot h^1(C,\mathbb{C}) = h^0(C,K_C) = h^1(C,\mathcal{O}_C)\ , \tag{1.7}$$

$$h^1(C,K_C) = h^2(C,\mathbb{C}) = 1\ . \tag{1.8}$$

Of these equalities, the first is certainly the most remarkable, for it tells us that two invariants of C that are defined in terms of the holomorphic structure, such as $h^0(K_C)$ and $h^1(\mathcal{O}_C)$, are in fact purely differential invariants, because such is $h^2(\mathbb{C})$. Moreover, since we have explicitly calculated, for at least one RS for each genus, that $h^0(K)$ equals the genus, and since all RS of the same genus are differentiably the same, we find that

$$\frac{1}{2}\cdot h^1(C,\mathbb{C}) = h^0(C,K_C) = h^1(C,\mathcal{O}_C) = g(C)\ , \tag{1.9}$$

where $g(C)$ stands for the genus of C.

2. Riemann-Roch and duality. Let C be a compact connected RS. We recall that a holomorphic line bundle L on C can be described by giving a covering of C by open sets U_α and holomorphic transition functions $g_{\alpha\beta}$ such that

$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$$

for any choice of α, β, and γ. More exactly, L can be built from the disjoint union

$$\coprod_\alpha U_\alpha \times \mathbb{C}$$

via the identifications

$$U_\alpha \times \mathbb{C} \ni (x, \zeta_\alpha) \sim (x, \zeta_\beta) \in U_\beta \times \mathbb{C} \qquad \text{if} \qquad \zeta_\alpha = g_{\alpha\beta}\zeta_\beta \,.$$

A holomorphic section (or simply a section) of L is a collection $s = \{s_\alpha\}$ of holomorphic functions $U_\alpha \to \mathbb{C}$ such that

$$s_\alpha = g_{\alpha\beta} s_\beta \qquad \text{on } U_\alpha \cap U_\beta \,. \tag{2.1}$$

One similarly defines meromorphic sections or C^∞ sections of L. Likewise, an L-valued (0,1)-form is a collection $\varphi = \{\varphi_\alpha\}$ of C^∞ (0,1)-forms on the open sets U_α such that

$$\varphi_\alpha = g_{\alpha\beta}\varphi_\beta \qquad \text{on } U_\alpha \cap U_\beta \,.$$

Let s be a C^∞ section of L. Differentiating (2.1) with respect to $\bar\partial$ gives

$$\bar\partial s_\alpha = g_{\alpha\beta} \bar\partial s_\beta \,,$$

since $g_{\alpha\beta}$ is holomorphic. Thus $\{\bar\partial s_\alpha\}$ is an L-valued form, which we will denote by $\bar\partial s$.

(2.2) Warning. If L is a line bundle we shall often refer to the sheaf of its holomorphic sections (cf. Gomez-Mont's notes in these proceedings) as a line bundle and denote it by the same symbol L.

(2.3) Example. We denote by $\mathcal{O}_C$, or more simply by $\mathcal{O}$, the trivial line bundle $C \times \mathbb{C}$ (or, more precisely, its sheaf of sections, i. e., the sheaf of holomophic functions on C). We denote by K_C, or more simply by K, the canonical line bundle on C, i. e., the line bundle whose transition functions, relative to a covering $\{U_\alpha\}$ by coordinate open sets with coordinates z_α, are the functions dz_β/dz_α. A holomorphic section of K is nothing but an abelian differential: in fact, if $a=\{a_\alpha\}$ is a section of K, then $a_\alpha \cdot dz_\alpha = a_\beta \cdot dz_\beta$ on $U_\alpha \cap U_\beta$, so that $\{a_\alpha \cdot dz_\alpha\}$ is an abelian differential.

(2.4) Example: the line bundle associated to a divisor. Consider a divisor $D=\sum n_i p_i$, and let $\{U_\alpha\}$ be an open covering of C. For each α choose a defining equation f_α for D in U_α (e. g., if U_α contains only one point of D, say p_i, and z is a coordinate centered at p_i, then we may take $f_\alpha = z^{n_i}$, while if U_α does not contain any point of D, then we may take $f_\alpha = 1$). Set

$$g_{\alpha\beta} = f_\alpha / f_\beta \,,$$

and let $\mathcal{O}(D)$ be the corresponding line bundle; it is easy to check that changing covering or defining functions yields equivalent transition functions (cf. Gomez-Mont's notes in these proceedings), so that $\mathcal{O}(D)$ is well defined.

Let $s=\{s_\alpha\}$ be a section of $\mathcal{O}(D)$. Thus

$$s_\alpha / f_\alpha = s_\beta / f_\beta \quad \text{on } U_\alpha \cap U_\beta \,,$$

so that s_α / f_α is the restriction to U_α of a globally defined meromorphic function F; moreover the order of pole of F at p_i does not exceed n_i, while at points of C not belonging to D, F is holomorphic. Conversely, given a meromorphic function with these properties, $\{f_\alpha F\}$ is a section of $\mathcal{O}(D)$. Put otherwise, if we denote by $H^0(C,L)$ the vector space of sections of a line bundle L, and by $\mathcal{L}(D)$ ($\mathcal{M}(D)$ in Gomez-Mont's notation) the vector space of those meromorphic functions whose order of pole at each p_i does not exceed n_i, then

$$H^0(C, \mathcal{O}(D)) = \mathcal{L}(D) \,.$$

Suppose L is given by transition functions $g_{\alpha\beta}$, and has a nonzero meromorphic section $s=\{s_\alpha\}$. The divisor of s is the divisor

$$D = \sum_{p \in C} n_p p \,,$$

where n_p stands for the order of zero of s at p, poles counting as zeros of negative order; notice that we are dealing with a finite sum. Now s_α is clearly a local defining equation for D; since

$$g_{\alpha\beta} = s_\alpha / s_\beta \,,$$

L is isomorphic to $\mathcal{O}(D)$. In particular, to show that a line bundle is of the form $\mathcal{O}(D)$ it suffices to show that it has a nonzero meromorphic section.

Line bundles can be tensored: if L and M are given, with respect to the same covering (we can always reduce to this situation by refining the original coverings), by transition functions $g_{\alpha\beta}$ and $f_{\alpha\beta}$, then the tensor product $L\otimes M$ is the line bundle with transition functions $g_{\alpha\beta}f_{\alpha\beta}$. We shall normally write LM instead of $L\otimes M$. A section of L can be multiplied by a section of M, the result being a well defined section of LM. The dual, or inverse, of L, written L^{-1}, is the line bundle with transition functions $g_{\alpha\beta}^{-1}$. Clearly, the tensor product of L with L^{-1} is trivial. Tensoring line bundles corresponds to adding divisors; more precisely

$$\mathcal{O}(D+D') = \mathcal{O}(D)\otimes\mathcal{O}(D') \qquad , \qquad \mathcal{O}(D)^{-1} = \mathcal{O}(-D) \ .$$

We shall often write $L(D)$ to denote the tensor product $L\otimes\mathcal{O}(D)$.

Let L be a line bundle, and let s and t be two nonzero meromorphic sections of L. Then the quotient s/t is a meromorphic function on C, and the residue theorem shows that it has as many zeros as it has poles. Thus the numbers of zeros of s and t are the same (poles count as zeros with negative multiplicity). It then makes sense to define the degree of L as

$$\deg(L) = \text{number of zeros minus number of poles of a meromorphic section of } L.$$

In other words:

$$\deg(\mathcal{O}(D)) = \deg(D)\ .$$

We shall see later that any line bundle on C is of the form $\mathcal{O}(D)$ for some divisor D, so that the degree of L is defined for any line bundle L.

(2.5) The Dolbeault isomorphism. Let L be a line bundle on C. Consider the sheaf sequence

$$0 \longrightarrow L \longrightarrow \mathcal{A}^0(L) \xrightarrow{\bar\partial} \mathcal{A}^1(L) \longrightarrow 0\ , \qquad (2.6)$$

where $\mathcal{A}^0(L)$ is the sheaf of C^∞ sections of L, and $\mathcal{A}^1(L)$ the sheaf of L-valued (0,1)-forms. We claim that it is exact. The only problem is to show that $\bar\partial$ is onto. This is a purely local question. What has to be shown is that, if φ is a (0,1)-form defined in a neighbourhood of a point $p\in\mathbb{C}$, then the equation

$$\varphi = \bar\partial\xi$$

can be solved in a neighbourhood of p. To do this, one first solves

$$\partial\varphi = \partial\bar\partial\eta\,. \tag{2.7}$$

Assuming that this is possible one then notices that $d(\varphi - \bar\partial\eta) = 0$, so that we can locally write

$$\varphi - \bar\partial\eta = d\zeta = \bar\partial\zeta\,,$$

or, equivalently,

$$\varphi = \bar\partial(\eta + \zeta)\,.$$

That (2.7) is solvable in a neighbourhood of p can be seen in a number of ways. For example, one may argue as follows. Since we are looking for a local solution, we may suppose we are on a torus T and not on $\mathbb{C}$. Let χ be a C^∞ function that is equal to 1 in a neighbourhood of p and equal to zero outside of a neighbourhood U of p. Let ψ be equal to $\chi\varphi$ on U and equal to zero elsewhere. Then we may solve (2.7) by solving, on all of T,

$$\partial\psi = \partial\bar\partial\eta\,.$$

This is possible because of (1.6), whose proof, incidentally, is very easy on a torus.

Now that (2.6) has been shown to be exact, one gets from it a long exact cohomology sequence

$$0 \to H^0(C,L) \to H^0(C,\mathcal{A}^0(L)) \to H^0(C,\mathcal{A}^1(L)) \to H^1(C,L) \to H^1(C,\mathcal{A}^0(L)) \to \cdots\,.$$

We shall show that $H^1(C,\mathcal{A}^0(L))$ vanishes, so that one has the <u>Dolbeault isomorphism</u>

$$H^1(C,L) = \frac{L\text{-valued }(0,1)\text{-forms}}{\bar\partial\text{-exact }L\text{-valued }(0,1)\text{-forms}}\,. \tag{2.8}$$

To see that $H^1(C,\mathcal{A}^0(L)) = 0$, let $\{\xi_{ij}\}$ be an $\mathcal{A}^0(L)$-valued 1-cocycle relative to an open covering $\{U_i\}$. Thus

$$\xi_{ij} + \xi_{jk} = \xi_{ik} \qquad \text{on } U_i \cap U_j \cap U_k\,.$$

Taking $i = j$, one gets that $\xi_{ii} = 0$ on $U_i \cap U_k$; then, taking $i = k$, one gets that $\xi_{ij} = -\xi_{ji}$ on $U_i \cap U_j$. Now choose a C^∞ partition of unity $\{\lambda_i\}$ subordinated to $\{U_i\}$. Set

$$\eta_i = \sum_h \lambda_h \xi_{hi}\,,$$

where it is understood that $\lambda_h\xi_{hi}$ is extended to zero on $U_i-(U_i\cap U_h)$. Then

$$\eta_j-\eta_i=\sum_h\lambda_h\xi_{hj}-\sum_h\lambda_h\xi_{hi}=\sum_h\lambda_h(\xi_{ih}+\xi_{hj})=(\sum_h\lambda_h)\xi_{ij}=\xi_{ij}\,.$$

Thus $\{\xi_{ij}\}$ represents zero in $H^1(C,\mathcal{A}^0(L))$. This completes the proof of (2.8).

In what follows, we shall not make any use of sheaf cohomology. The only reason for discussing the Dolbeault isomorphism has been to indicate that our arguments also have a cohomological interpretation.

The Riemann-Roch problem is the problem of computing the dimension of $H^0(C,L)$, where L is a line bundle on the RS C. A partial answer is provided by the Riemann-Roch theorem.

(2.9) Riemann-Roch theorem. Let C be a RS of genus g and L a holomorphic line bundle on it. The vector spaces $H^0(C,L)$ and $H^1(C,L)$ are finite-dimensional and their dimensions are related by the Riemann-Roch formula:

$$h^0(C,L)-h^1(C,L)=\deg(L)+1-g\,.$$

To begin with, the theorem is clearly true if $L=\mathcal{O}$. Let p be a point of C: consider the sequence

(2.10) $$0\to H^0(C,L(-p))\to H^0(C,L)\xrightarrow{\alpha}L_p\xrightarrow{\beta}H^1(C,L(-p))\to H^1(C,L)\to 0\,,$$

where L_p is the fiber of L at p, α is evaluation at p, and β is obtained as follows. If $c\in L_p$, there is a section s of L on a neighbourhood U of p such that $s(p)=c$. Pick a C^∞ function χ that is equal to 1 in a neighbourhood of p and equal to zero outside of U (as we will have to use these functions often, we shall agree to call such a χ a "bump function in U around p"). Then $\beta(c)$ is the class of $\bar\partial(\chi s)$ extended to zero outside U; notice that $\bar\partial(\chi s)$ is identically zero in a neighbourhood of p.

The sequence (2.10) is exact. This is quite easy to check. For example, let's verify exactness at L_p. With the notations we have just established, if $\beta(c)=0$, then $\bar\partial(\chi s)=\bar\partial\psi$, where ψ is an L-valued form that vanishes at p. Thus $\chi s-\psi$ is a holomorphic section of L whose value at p is c, i. e.,

$$\alpha(\chi s-\psi)=c\,.$$

It is equally easy to show that $H^1(C,L(-p))$ maps onto $H^1(C,L)$. Given an L-valued (0,1)-form φ, we can write, near p, $\varphi=\bar\partial\eta$. Then, if χ is a bump function around p, $\varphi - \bar\partial(\chi\eta)$ is an L(−p)-valued form whose class maps to the class of φ.

Since the alternating sum of the dimensions in an exact sequence of vector spaces vanishes, and the degree of L(−p) is one less than the degree of L, (2.10) shows that

$$\text{Riemann-Roch holds for } L \iff \text{it holds for } L(-p).$$

Since we have remarked that the Riemann-Roch theorem is valid for $L = \mathcal{O}$, this proves the Riemann-Roch theorem for any $\mathcal{O}(D)$. If we grant that any line bundle on C is of the form $\mathcal{O}(D)$, this finishes the proof of Riemann-Roch.

(2.11) Corollary. The degree of the canonical bundle of a RS of genus g is $2g-2$.

This follows by applying the Riemann-Roch formula to K.

To make full use of the Riemann-Roch formula it is necessary to give an alternate description of $H^1(C,L)$. Consider an L-valued (0,1)-form φ, and denote by $[\varphi]$ its class in $H^1(C,L)$; φ is a collection $\{\varphi_\alpha\}$ of ordinary (0,1)-forms on open sets U_α, tied by

$$\varphi_\alpha = g_{\alpha\beta}\varphi_\beta ,$$

where the $g_{\alpha\beta}$ are transition functions for L. Now let ψ be a section of KL^{-1}: we can view it as a collection $\{\psi_\alpha\}$ of holomorphic (1,0)-forms tied by

$$\psi_\alpha = g_{\alpha\beta}^{-1}\psi_\beta .$$

Thus

$$\psi_\alpha \wedge \varphi_\alpha = \psi_\beta \wedge \varphi_\beta \quad \text{on } U_\alpha \cap U_\beta .$$

Define $\psi\wedge\varphi$ to be the (1,1)-form that on each U_α restricts to $\psi_\alpha\wedge\varphi_\alpha$. We define a pairing $\langle , \rangle$ between $H^0(C,KL^{-1})$ and $H^1(C,L)$ by setting

$$\langle \psi, [\varphi] \rangle = \int_C \psi\wedge\varphi .$$

The basic result about this pairing is

(2.12) Duality theorem. $\langle , \rangle$ is a perfect pairing between $H^0(C,KL^{-1})$ and $H^1(C,L)$.

To prove the theorem, notice first that it holds for $L=\mathcal{O}$ and for $L=K$. For what concerns $\mathcal{O}$, it has been shown in section 1 that $\alpha \to \bar{\alpha}$ gives an antiisomorphism between $H^0(C,K)$ and $H^1(C,\mathcal{O})$, and one has

$$\langle \alpha, i[\bar{\alpha}]\rangle = \int_C \alpha \wedge i\bar{\alpha} > 0 \qquad \text{if } \alpha \neq 0 .$$

As regards K, we know that $H^1(C,K)$ is generated by the class of a volume form Φ, and we have

$$\langle 1, [\Phi]\rangle = \int_C \Phi > 0 .$$

To prove the duality theorem in general, we shall show that, for any point p in C:

$$\text{Duality holds for } L \text{ and } KL^{-1} \iff \text{it holds for } L(-p) \text{ and } KL^{-1}(p) .$$

To do this, consider the exact sequence (2.10) and its analogue for $KL^{-1}(p)$:

$$\begin{array}{l} 0 \to H^0(L(-p)) \to H^0(L) \xrightarrow{\alpha} L_p \xrightarrow{\beta} H^1(L(-p)) \to H^1(L) \to 0 \\ 0 \leftarrow H^1(KL^{-1}(p)) \leftarrow H^1(KL^{-1}) \xleftarrow{\gamma} KL^{-1}(p)_p \xleftarrow{\delta} H^0(KL^{-1}(p)) \leftarrow H^0(KL^{-1}) \leftarrow 0 . \end{array}$$

Each vector space in the upper sequence, except the middle one, is paired with the space just below it; the pairings are compatible with the connecting homomorphisms. By the "five lemma", to conclude it will suffice to define a pairing between L_p and $KL^{-1}(p)_p$ that is compatible, up to sign, with the pairings between $H^0(L)$ and $H^1(KL^{-1})$, $H^1(L(-p))$ and $H^0(KL^{-1}(p))$. Suppose c belongs to L_p and e belongs to $KL^{-1}(p)_p$. Let z be a local coordinate centered at p. There are a section s of L and a section t of L^{-1} on a neighbourhood of p such that

$$c = s(p) \qquad ; \qquad e = t(p)\cdot \frac{dz}{z} .$$

We then set

$$\langle e, c\rangle = 2\pi i \cdot t(p)\cdot s(p) .$$

We shall now check compatibility. Let c be as above, and let φ be a section of $KL^{-1}(p)$. Near p, φ may be written in the form $t\cdot(dz/z)$, with t a local section of L^{-1}. We wish to show that

$$\langle \delta\varphi, c\rangle = \langle \varphi, \beta c\rangle .$$

The left-hand side equals $2\pi i\cdot s(p)\cdot t(p)$. To compute the right-hand side, recall that βc is the class of $\bar\partial(\chi s)$, where χ is a bump function around p. Then, if Γ is a small circle around p,

$$\langle \varphi, \beta c\rangle = \int_C \varphi\wedge\bar\partial(\chi s) = -\int_C d(\chi s\varphi) = \int_\Gamma s\cdot t\cdot(dz/z) = 2\pi i\cdot s(p)\cdot t(p)\,.$$

A similar argument proves compatibility, up to sign, with α and γ. This concludes the proof of the duality theorem.

The duality theorem makes it possible to rewrite the Riemann-Roch formula in the following more useful form:

$$h^0(C,L)-h^0(C,KL^{-1}) = \deg(L)+1-g\,. \tag{2.13}$$

The Riemann-Roch and duality theorems have really been proved only for line bundles of the form $\mathcal{O}(D)$; as we have already said, however, <u>every</u> line bundle on a RS is of this type. We shall not give a complete proof of this here, but we limit ourselves to the following simple remark. Let L be a line bundle of degree d on the RS C, and suppose we know that

$$h^1(C,L) < +\infty\,. \tag{2.14}$$

Then we claim that L is of the form $\mathcal{O}(D)$, for some divisor D. In fact, consider the exact sequence

$$0\to H^0(C,L)\to H^0(C,L(p))\to L(p)_p\to H^1(C,L)\to H^1(C,L(p))\to 0\,,$$

where p is a point of C. If $h^0(C,L(p))>0$, we are done, because then $L(p)$ has a nonzero holomorphic section, hence L has a nonzero meromorphic section s, and $L=\mathcal{O}(D)$, where D is the divisor of zeros of s (a pole being counted as a zero of negative multiplicity). We may then suppose that $h^0(C,L(p))=0$, so that

$$h^1(C,L(p)) = h^1(C,L)-1\,.$$

Now we repeat the same argument, replacing L with $L(p)$, and so on. Since $h^1(C,L)$ is finite, we conclude that there is an integer n such that $h^0(C,L(n\cdot p))$ is not zero. Thus L has a nonzero meromorphic section, and hence is of the form $\mathcal{O}(D)$.

It remains to prove (2.14): for this we refer to the Appendix.

3. Applications of Riemann-Roch. In this section we shall discuss a number of elementary applications of the Riemann-Roch and duality theorems for a line bundle L of degree d on a RS C of genus g. A trivial remark is that

$$h^0(C,L) = 0 \qquad \text{if } d < 0 \,,$$
$$h^1(C,L) = 0 \qquad \text{if } d > 2g-2 \,.$$

The first of these follows from the fact that the number of zeros of a nonzero holomorphic section of L equals the degree of L. The second can be obtained by applying the first identity to KL^{-1}, since the degree of K equals $2g-2$ and $h^1(C,L)=h^0(C,KL^{-1})$, by duality.

The Riemann-Roch formula thus "solves" the Riemann-Roch problem when $d<0$ or $d>2g-2$. In particular, we have that

$$h^0(C,L) = d+1-g \qquad \text{if } d > 2g-2 \,.$$

The "borderline" cases $d=0$ and $d=2g-2$ are also easily dealt with. If $d=0$ and L has a nonzero section, this cannot vanish anywhere, so L is trivial. If $d=2g-2$ and $h^1(C,L)$ is not zero, then, by duality, KL^{-1} must be trivial. In conclusion

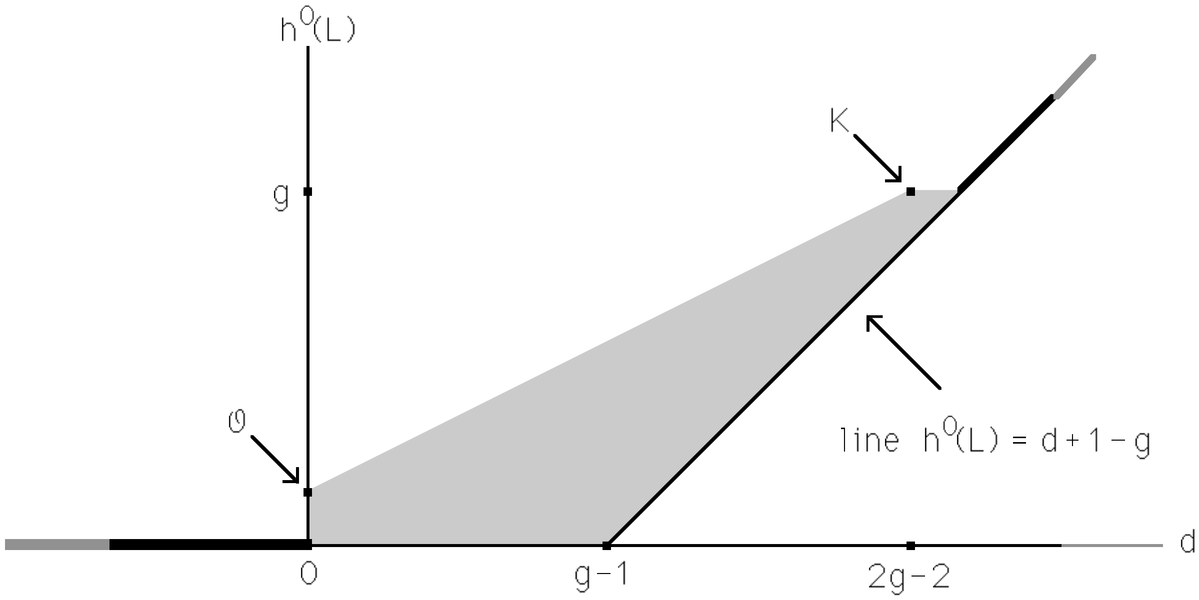
when $d=0$, <u>either</u> $L = \mathcal{O}$ and $h^0(L) = 1$ <u>or</u> $h^0(L) = 0$;

when $d=2g-2$, <u>either</u> $L = K$ and $h^0(L) = g$ <u>or</u> $h^0(C,L) = g-1$.

In the range $0<d<2g-2$ things are not so simple. All we can say, for the time being, is that, in this range,

$$h^0(C,L) \geq d+1-g \,.$$

Thus, in the diagram below, the possible values of $(d,h^0(L))$ lie on the heavy lines or in the shaded region.



We shall be able to be more precise about the upper boundary of the shaded area later, when we discuss Clifford's theorem.

There is a close relationship between line bundles and maps into projective spaces. Suppose C is a RS of genus g. A linear system on C is the projectivization

$$|V| = \mathbb{P}V$$

of a vector subspace V of $H^0(L)$, for some line bundle L on C. The degree d of $|V|$ is the degree of L, and the dimension of $|V|$ is the projective dimension, that is, one less than the dimension of V. If the dimension of $|V|$ is r, it is customary to say that $|V|$ is a g^r_d. If L is a line bundle and D a divisor, $|L|$ is an abbreviation for $|H^0(L)|$, and $|D|$ one for $|\mathcal{O}(D)|$: a linear system of this sort is said to be <u>complete</u>. One says that $|V|$ has no base points if, for any point p of C, there is an element s of V such that $s(p) \neq 0$.

Suppose then that $|V|$ is a g^r_d with no base points: one may then define a holomorphic map

$$\Phi_{|V|}: C \longrightarrow \mathbb{P}(V^\vee) ,$$

where $V^\vee$ is the dual of V, by setting

$$\Phi_{|V|}(p) = \text{the hyperplane } \{s \in V \mid s(p) = 0\} .$$

If one chooses coordinates for $\mathbb{P}(V^\vee)$, i. e., a basis $s_0, \cdots, s_r$ for V, then $\Phi_{|V|}$ is given by

$$p \longmapsto [s_0(p): \cdots : s_r(p)] ,$$

where $[s_0(p):\cdots:s_r(p)]$ is the point in $\mathbb{P}^r$ that is defined as follows: pick a generator t for the fiber of L at p; write $s_i(p)=f_i t$. Then

$$[s_0(p):\cdots:s_r(p)] = [f_0:\cdots:f_r]\ .$$

This is well defined since the right-hand side does not depend on the choice of t.

(3.1) Example. Let p be a point on the RS C; then $h^0(\mathcal{O}(p))$ equals 1 or 2. The latter occurs precisely when there is a meromorphic function f on C whose only singularity is a simple pole at p. Then $|\mathcal{O}(p)|$ has no base points and hence defines a map Φ from C to the Riemann sphere $\mathbb{P}^1$; thus

$$\Phi(q) = [1:f(q)]\ ,$$

where it is understood that p maps to the point at infinity $[0:1]$. If $a \in \mathbb{C}$, the section $f-a$ of $\mathcal{O}(p)$ has only one zero: this means that Φ is one-to-one. Moreover, since the zero of $f-a$ has to be simple, Φ is an immersion away from p; it is an immersion at p as well since f has a simple pole. In other words, C is isomorphic to the Riemann sphere. In particular, since the Riemann-Roch formula says that, when C has genus zero, then $h^0(\mathcal{O}(p))=2$, we conclude that the only RS of genus zero is the Riemann sphere, up to isomorphism.

(3.2) Remark. In section 4 we shall see that any RS of genus one is isomorphic to $\mathbb{C}/\Lambda$ for some lattice Λ.

(3.3) Example: the canonical mapping. A corollary of (3.1) is that, on any RS of genus $g \geq 1$, the canonical linear system $|K|$ has no base points. In fact, (3.1) shows that, for any p,

$$1 = h^0(\mathcal{O}(p)) = h^1(K(-p))\ ,$$

so that, by Riemann-Roch,

$$h^0(K(-p)) = g-1\ .$$

The mapping

$$\alpha = \Phi_{|K|} : C \longrightarrow \mathbb{P}^{g-1}$$

is called the <u>canonical</u> <u>map</u>.

As an example, let's describe what the canonical map looks like for the RS C of example (1.3); we keep the same notations as in (1.3). We showed that the abelian differentials on C are the linear combinations of φ, $y\varphi$, $y^2\varphi, \cdots, y^{g-1}\varphi$. Thus the canonical map for C is

$$(x,y) \longmapsto [1:y:y^2:\cdots:y^{g-1}] .$$

In other words, the canonical map is obtained by composing the projection $(x,y) \mapsto y$ onto the Riemann sphere with the $(g-1)^{st}$ Veronese embedding

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^{g-1} ,$$
$$y \longmapsto [1:y:y^2:\cdots:y^{g-1}]$$

(whose image is the so-called rational normal curve of degree $g-1$). In particular, the canonical map, in this case, is generically 2-1 and not 1-1.

From now on we assume that $g>1$. Suppose that the canonical map α is not injective, i. e., that there are two distinct points, p and q, such that $\alpha(p)=\alpha(q)$. This means that, if a section of K vanishes at p, it automatically vanishes at q, and conversely, so that

$$h^0(K(-p-q)) = g-1 ,$$

or, by Riemann-Roch,

$$h^0(\mathcal{O}(p+q)) = 2 .$$

Suppose instead that the canonical map is not an immersion at some point $p \in C$. We choose a basis $s_0, \cdots, s_{g-1}$ for the abelian differentials in such a way that $s_1, \cdots, s_{g-1}$ all vanish at p. Since we are assuming that the canonical map is not an immersion at p, they must in fact vanish <u>doubly</u> at p; we then have

$$h^0(K(-2p)) = g-1 \qquad , \qquad h^0(\mathcal{O}(2p)) = 2 .$$

In conclusion, if the canonical mapping is not an isomorphism of C onto its image, there are two points (distinct or coincident) p and q on C such that

$$h^0(\mathcal{O}(p+q)) = 2 .$$

In other words, C possesses a g^1_2, or, which is the same, there is a meromorphic function f on C with only two poles (or one double pole). The map

$$\Phi_{|\mathcal{O}(p+q)|} : C \longrightarrow \mathbb{P}^1,$$
$$x \longmapsto [1:f(x)]$$

is then generically 2-1, for the section $f-a$ of $\mathcal{O}(p+q)$ has two zeros (or one double zero) for every $a\in\mathbb{C}$; as one says, C is a two-sheeted covering of $\mathbb{P}^1$. A RS of positive genus that is a two-sheeted covering of $\mathbb{P}^1$ is said to be hyperelliptic. We can then say that, unless the RS in question is hyperelliptic, the canonical map is an isomorphism onto its image; the image of the canonical map is then called the canonical curve. The RS constructed in example (1.3) are all hyperelliptic. We shall see in a moment that in fact all hyperelliptic RS have been constructed in (1.3). The analysis of the canonical map for these RS, which has been carried out above, then shows that for a hyperelliptic RS C the canonical map is the composition of the double covering

$$C \longrightarrow \mathbb{P}^1$$

and of the Veronese embedding

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^{g-1} .$$

This seems to be a good spot to discuss an important formula, due to Riemann and Hurwitz, relating the genus of a RS and of a covering.

(3.4) The Riemann-Hurwitz formula. Let C and Γ be two RS of genera g and γ, and suppose there is a non-constant holomorphic map

$$f : C \longrightarrow \Gamma .$$

We denote by d the degree of f, that is, the number of points in a general fiber $f^{-1}(x)$; one says that C is a d-sheeted (ramified) covering of Γ. Let p be a point of C, and set $q=f(p)$; choose coordinates z and w centered at p and q, respectively. Near p, the map f is given, in these coordinates, by

$$w = z^n \cdot b(z) ,$$

where $b(0)\neq 0$. Replacing the local coordinate z with the new local coordinate $z\cdot\sqrt[n]{b(z)}$, we may actually suppose that

$$w = z^n . \tag{3.5}$$

The number $n-1$ is called the ramification index of p and denoted by r_p. If $r_p>0$ we say that p is a ramification point and q a branch point. Clearly there are only finitely many ramification points. The Riemann-Hurwitz formula then says that

(3.6) $$2g-2 = d(2\gamma-2) + \sum_{p\in C} r_p .$$

To see this, let ω be a nonzero meromorphic differential on Γ. Then if, as usual, we count poles as zeros of negative multiplicity, the number of zeros of ω is $2\gamma-2$. We may choose ω so that its zeros and poles are not at branch points of f. Now suppose $p\in C$ is a ramification point of f, and let f be given, near p, by (3.5). At $q=f(p)$,

$$\omega = a(w)\cdot dw \ , \qquad a(0)\neq 0 \ ,$$

so that

$$f^*(\omega) = a(z^{r_p+1})\cdot(r_p+1)z^{r_p}dz \ ,$$

that is, $f^*(\omega)$ has an r_p-fold zero at p. Thus the number of zeros of $f^*(\omega)$ is

$$d(2\gamma-2) + \sum_{p\in C} r_p .$$

Since this number also equals $2g-2$, the Riemann-Hurwitz formula (3.6) is proved.

Now suppose C is hyperelliptic of genus g. Then there is a two-sheeted covering $f: C\longrightarrow \mathbb{P}^1$, and the Riemann-Hurwitz formula says that

$$2g-2 = -4+r \ ,$$

where r is the number of branch points of f. Hence f is branched at $2g+2$ points $\zeta_1, \zeta_2, \cdots, \zeta_{2g+2}$, and we may choose coordinates y, $\eta=(1/y)$ on $\mathbb{P}^1$ so that none of them equals ∞. Then C is isomorphic to the (completion of the) RS with equation

$$x^2 = \prod_{1\le i\le 2g+2} (y-\zeta_i) :$$

in fact, the function $\prod(y-\zeta_i)$ has a single-valued square root x on C and, keeping the notation of example (1.3),

$$p \longmapsto (x(p), y(p))$$

yields an isomorphism of $C-f^{-1}(\infty)$ onto Z which extends to an isomorphism of C onto the completion of Z.

(3.7) Every RS is algebraic. Let L be a line bundle on the RS C; let g be the genus of C and d the degree of L. In order for $\Phi_{|L|}$ to be defined it is necessary that $|L|$ have no base points: this means that, for any $p \in C$, we must have

$$h^0(L(-p)) = h^0(L) - 1 .$$

When $d \geq 2g$ this is a consequence of the Riemann-Roch formula. It also follows from our discussion of the canonical mapping that, In order for $\Phi_{|L|}$ to be an embedding, we must have

$$h^0(L(-p-q)) = h^0(L) - 2$$

for any two points $p, q \in C$. When $d > 2g$ this is again a consequence of the Riemann-Roch formula. We then conclude that $|L|$ always yields an embedding of C in some projective space $\mathbb{P}^n$ when the degree of L is larger than $2g$; in particular, any RS can be realized as a complex submanifold of some $\mathbb{P}^n$.

At this point, we could appeal to a basic theorem of Chow which says that any compact complex subvariety of $\mathbb{P}^n$ is defined by homogeneous algebraic equations, i. e., is an algebraic subvariety of $\mathbb{P}^n$, to conclude that every RS is isomorphic to an algebraic curve. If, however, we wish to avoid Chow's theorem, we can proceed as follows. Let C be a RS, which we view as embedded in some $\mathbb{P}^n$ via a linear system $|V|$. In particular, C is not contained in any hyperplane. Consider the subvariety X of $\mathbb{P}^n$ which is the union of all the projective lines joining couples of points of C (the line joining a point to itself is the tangent to C at the point). The dimension of X is at most three, so, if $n \geq 4$, there is a point $x \in \mathbb{P}^n$ not contained in X. By the definition of X, then, projection from x to $\mathbb{P}^{n-1}$ maps C isomorphically onto its image. Notice also that the projection of C from x to $\mathbb{P}^{n-1}$ is given by the linear system $|V'|$, where V' is the subspace of V consisting of all $s \in V$ such that the corresponding hyperplane in $\mathbb{P}^n$ passes through x. Repeating the same procedure, if necessary, we may then suppose that $n = 3$. Now consider the two subvarieties Y and Z of $\mathbb{P}^3$ defined as follows: Y is the union of all tangent lines to C, while Z is the union of all lines passing through a fixed point $p \in C$ and some other point of C. Both Y and Z are at most two-dimensional, so there is a point $x \in \mathbb{P}^3$ not belonging to any of them. Let Γ be the image of C under the projection from x into $\mathbb{P}^2$. Since x does not belong to $Y \cup Z$, the map from C to $\mathbb{P}^2$ is an immersion and is generically 1-1 onto Γ. In particular, every singular point of Γ is the union of a finite number of smooth branches. Now the map from C to $\Gamma \subset \mathbb{P}^2$ is obtained by setting

$$X_i = s_i \ , \quad i = 0, 1, 2 \ ,$$

where X_0, X_1, X_2 are homogeneous coordinates and s_0, s_1, s_2 are sections of some line bundle L. By Riemann-Roch, for large k the dimension of $H^0(C, L^k)$ is a linear function of k; in particular, the number of linearly independent monomials of degree k in s_0, s_1, s_2, which are sections of L^k, is $O(k)$. On the other hand, the number of independent monomials of degree k in X_0, X_1, X_2 is $\frac{k(k+1)}{2}$. This implies that, for large enough k, there must be a homogeneous polynomial $P(X_0, X_1, X_2)$ of degree k such that $P(s_0, s_1, s_2)$ vanishes identically on C; in other words, $P(X_0, X_1, X_2)$ vanishes identically on Γ. Let

$$P = \prod Q_i$$

be the decomposition of P into irreducible factors. Since C is connected, one of the factors, call it Q, must vanish identically on Γ. Denote by E the zero locus of Q in $\mathbb{P}^2$. Since Q is irreducible, the complement of the singular locus of E (a finite set of points) is connected. Hence E equals Γ. By blowing up its singular points, Γ can be desingularized (cf. Bardelli's notes), and the projection map $C \to \Gamma$ lifts to a surjective map $\pi : C \to \tilde{\Gamma}$, where $\tilde{\Gamma}$ is the desingularization of Γ. Since π is generically 1-1, it has no branch points, hence is an immersion, hence an isomorphism. As $\tilde{\Gamma}$ is an algebraic subvariety of a suitable projective space, we are done.

(3.7) A geometric version of the Riemann-Roch theorem. The Riemann-Roch formula can be "read" in the geometry of the canonical map. For simplicity, we shall do this only for non-hyperelliptic RS. Let C be one such, and view it as being embedded in $\mathbb{P}^{g-1}$ via the canonical map. A linear function on $\mathbb{P}^{g-1}$ corresponds to an abelian differential, so the divisor cut out on C by a hyperplane is the divisor of an abelian differential, and conversely. Now let $D = p_1 + p_2 + \cdots + p_d$ be an effective divisor on C, and let $r = \dim(|D|)$ be the dimension of the corresponding complete linear system. There are exactly

$$h^0(K(-D)) = r + g - d$$

linearly independent hyperplanes containing D, so the linear span of D, written $\bar{D}$, has dimension

$$\dim(\bar{D}) = d - r - 1 . \tag{3.8}$$

In a way, this is a geometric version of the Riemann-Roch theorem. Of course, when D contains multiple points, we have to be cautious about what we mean by "span of D". The definition of span that makes (3.8) work is the following. Write $D = \sum n_i p_i$, where the p_i are distinct; then the span of D is defined to be the subspace of $\mathbb{P}^1$ spanned by the osculating subspaces to C of order (n_i-1) at the p_i.

The Riemann-Roch theorem, in this geometric form, has an interesting consequence. Notice that, since there are exactly g independent abelian differentials on C, the canonical curve is not contained in any hyperplane. Thus, if $p_1, p_2, \cdots, p_d$ are general points of C, they span a linear subspace of $\mathbb{P}^{g-1}$ of dimension $\min(d-1, g-1)$. This means that, for a general effective divisor D of degree d, one has

$$h^0(\mathcal{O}(D)) = 1 \qquad \text{if } d \leq g\,,$$

$$h^0(\mathcal{O}(D)) = d+1-g \qquad \text{if } d \geq g\,.$$

Now let p be a general point of C, and D' a general effective divisor of degree $d \leq g$. Clearly, $h^0(\mathcal{O}(D'-p)) = 0$; hence, if D is a general divisor of degree d, one has

$$h^0(\mathcal{O}(D)) = 0 \qquad \text{if } d < g\,,$$

$$h^0(\mathcal{O}(D)) = d+1-g \qquad \text{if } d \geq g\,.$$

Notice that these are the minimum possible values allowed by Riemann-Roch.

We have noticed that the Riemann-Roch formula yields a lower bound for $h^0(\mathcal{O}(D))$ in the range $0 \leq \deg(D) \leq 2g-2$, and have promised to give an upper bound. This is provided by

(3.9) Clifford's theorem. Let C be a RS of genus g and let L be a line bundle of degree d on C, with $0 \leq d \leq 2g-2$. Set $r = h^0(L) - 1$. Then

$$d \geq 2\cdot r\,.$$

Moreover, if $d = 2\cdot r$, then one of the following occurs:

i) $L = \mathcal{O}$,

ii) $L = K$,

iii) C is hyperelliptic.

We shall prove only the first assertion. Notice, to begin with, that there is nothing to prove if $r \leq 0$ or $h^1(L) = 0$; in this second case, in fact, Riemann-Roch gives

$$r = (d-1)-(g-1) \leq (d-1)-(d/2) < d/2\,.$$

In the remaining cases we shall rely on the following general

(3.10) Remark. Let L be a line bundle on C. Then

$$h^0(L) \geq h+1 \quad \Leftrightarrow \quad \begin{cases} \text{for any } h\text{-uple of points } p_1,\cdots,p_h \text{ of } C, \\ \text{there is a nonzero section } s \in H^0(L) \\ \text{such that } s(p_i)=0,\ i=1,\cdots,h. \end{cases}$$

To see this, notice that $s(p_1)=s(p_2)=\cdots=s(p_h)=0$ is a system of h linear equations in s. If $h^0(L) \geq h+1$, it has a nonzero solution. Now let's prove the converse. Let k be the dimension of $h^0(L)$; since, by our assumption, $k>0$, there is a point p_1 of C where not all sections of L vanish. Thus

$$h^0(L(-p_1)) = h^0(L)-1 = k-1 .$$

If $h>0$, we can find a point p_2 such that $h^0(L(-p_1-p_2))=h^0(L(-p_1))-1$, and so on. After h steps we find points $p_1,p_2,\cdots,p_h$ such that

$$h^0(L(-p_1-p_2-\cdots-p_h)) = k-h .$$

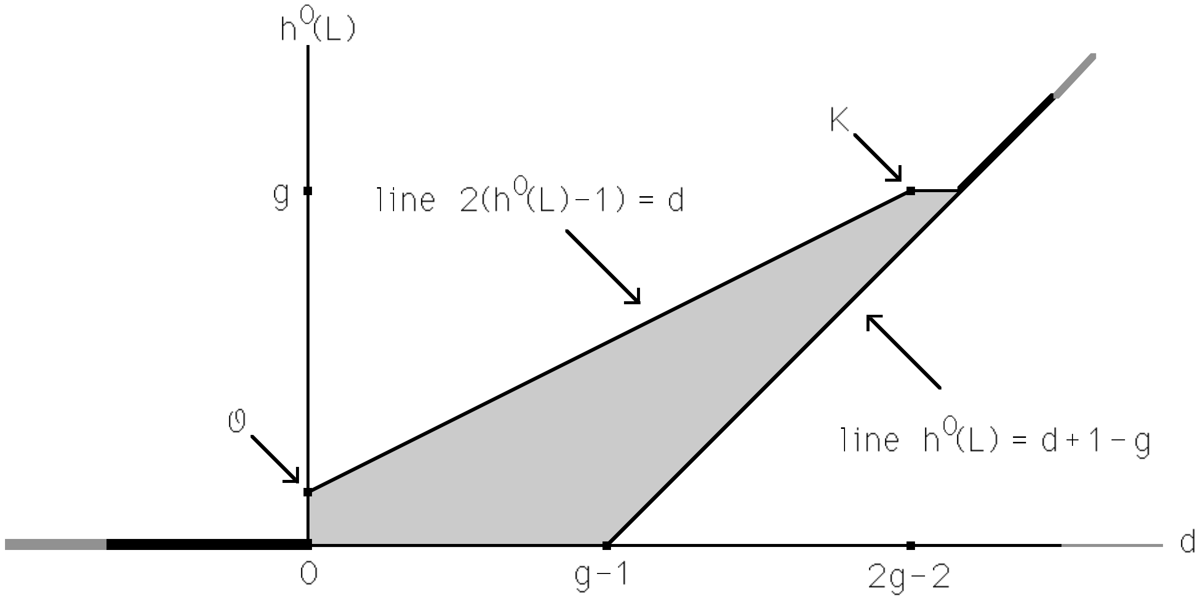
Our assumption then says that $k-h>0$, that is, $h^0(L)\geq h+1$.

Now we return to the proof of Clifford's theorem. Pick any $r+j-1$ points $p_1,\cdots,p_r,\cdots,p_{r+j-1}$ on C, where $j=h^1(L)=h^0(KL^{-1})$. By (3.10), we may find nonzero sections s of L and t of KL^{-1} such that s vanishes at $p_1,\cdots,p_r$, while t vanishes at $p_{r+1},\cdots,p_{r+j-1}$; hence st is a nonzero section of K that vanishes at $p_1,\cdots,p_r,\cdots,p_{r+j-1}$. By (3.10) and by Riemann-Roch, then,

$$g = h^0(K) \geq r+j = r+r+g-d ,$$

that is, $2\cdot r \leq d$.

We conclude this section by noticing that the theorems of Riemann-Roch and Clifford imply that, in the $(d,h^0(L))$-plane (figure below), for fixed genus g, the values of $(d,h^0(L))$ that may actually occur lie on the heavy lines or in the shaded region only.

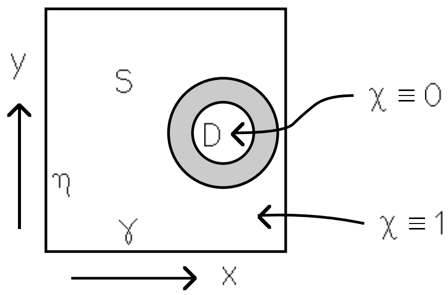


h0(L)
g
K
line 2(h0(L)-1) = d
𝒪
line h0(L) = d+1-g
d
0
g-1
2g-2

4. Abel's theorem. In the real (x,y) plane, consider the square

$$S = \{(x,y) \mid 0 \le x \le 1\,,\ 0 \le y \le 1\}\ .$$

In the interior of S, draw two concentric disks, and call the inner one D. Now let χ be a C^∞ function that is identically zero on D, and identically equal to 1 outside of the larger disk.



Also, let T be the torus obtained by identifying opposite sides of S, and let γ and η be the oriented closed paths in T which are the images of the sides $\{y=0\}$ and $\{x=0\}$ of S, oriented in the direction of increasing x's or y's. Then

$$\int_S d(\chi x)\wedge d(\chi y) = \int_S d(\chi x\cdot d(\chi y)) = \int_{\partial S} \chi x\cdot d(\chi y) = \int_{\partial S} x\cdot dy = \int_S dx\wedge dy = 1\ .$$

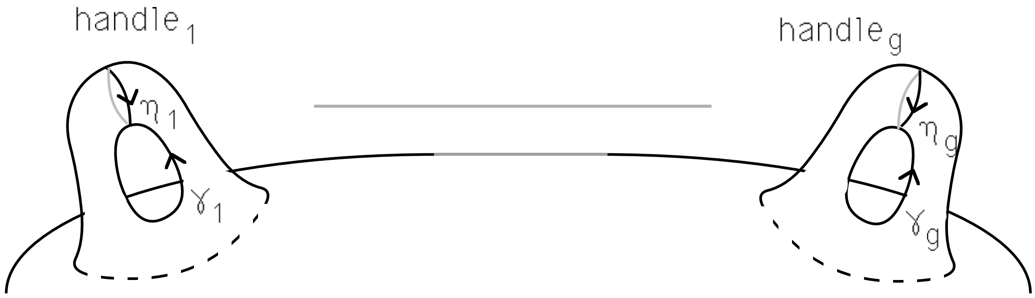
Now the differentials $d(\chi x)$ and $d(\chi y)$ induce differentials α and β on T, and the above computation says that

$$\int_C \alpha\wedge\beta = 1\ . \tag{4.1}$$

Notice also that α and β vanish on D. Moreover

$$\int_\gamma \alpha = \int_\eta \beta = 1 \quad ; \quad \int_\eta \alpha = \int_\gamma \beta = 0\ . \tag{4.2}$$

Now consider a RS C of genus g. Differentiably, it is simply a sphere with g handles attached; number these from 1 to g. Each handle is diffeomorphic to the torus T with the disk D punched out (see figure below). Hence, for each i, the differentials α and β induce differentials on the i-th handle, which can be extended to zero on the rest of C: denote by α_i and β_i the resulting differentials on C. Also, let γ_i and η_i be the images of γ and η on the i-th handle (see figure).



Then (4.1) says that

(4.3) $$\int_C \alpha_i \wedge \beta_j = \delta_{ij} \quad ; \quad \int_C \alpha_i \wedge \alpha_j = \int_C \beta_i \wedge \beta_j = 0 \ ,$$

while (4.2) says that

(4.4) $$\int_{\gamma_i} \alpha_j = \int_{\eta_i} \beta_j = \delta_{ij} \quad ; \quad \int_{\eta_i} \alpha_j = \int_{\gamma_i} \beta_j = 0 \ .$$

The loops $\gamma_1, \cdots, \gamma_g, \eta_1, \cdots, \eta_g$ constitute a basis for the integral homology group $H_1(C, \mathbb{Z})$. For each d-closed 1-form φ on C, denote by $[\varphi]$ the corresponding class in $H^1(C, \mathbb{C})$. We define the integral cohomology group $H^1(C, \mathbb{Z})$ to be

$$H^1(C, \mathbb{Z}) = \{[\varphi] \in H^1(C, \mathbb{C}) \big| \int_\gamma \varphi \in \mathbb{Z} \ \ \forall\, \gamma \in H_1(\mathbb{Z})\} \ .$$

Clearly, a class $[\varphi]$ belongs to $H^1(C, \mathbb{Z})$ if and only if its integral over each of the basic loops γ_i, η_i is an integer. Our definition of $H^1(C, \mathbb{Z})$ is of course justified by the fact that, under the deRham isomorphism, it corresponds exactly to the usual integral cohomology; we won't need this, however.

Now formula (4.4) says that the basis $[\alpha_1], \cdots, [\alpha_g], [\beta_1], \cdots, [\beta_g]$ of $H^1(\mathbb{Z})$ is a dual basis of $\gamma_1, \cdots, \gamma_g, \eta_1, \cdots, \eta_g$. On the other hand, (4.3) says that, relative to $[\alpha_1], \cdots, [\alpha_g], [\beta_1], \cdots, [\beta_g]$, the matrix of the intersection form

$$\langle [\varphi], [\psi] \rangle = \int_C \varphi \wedge \psi$$

is

$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \ ,$$

where I_g stands for the $g \times g$ identity matrix. In particular, the intersection form is unimodular on integral cohomology, so that the prescription

$$\int_\gamma \varphi = \int_C P(\gamma)\wedge\varphi = \langle P(\gamma),[\varphi]\rangle \tag{4.5}$$

defines an <u>isomorphism</u>

$$P : H_1(C,\mathbb{Z}) \longrightarrow H^1(C,\mathbb{Z}) .$$

In our context, this is the content of Poincaré duality. Notice that

$$P(\eta_i) = [\alpha_i] \quad ; \quad P(\gamma_i) = -[\beta_i] .$$

We now turn to Abel's theorem. We denote by $\mathrm{Div}_0(C)$ the group of degree zero divisors on C, by $\mathrm{Pic}(C)$ the group of line bundles on C, and by $\mathrm{Pic}_0(C)$ the group of degree zero line bundles. The map $D \longmapsto \mathcal{O}(D)$ is a group homomorphism from $\mathrm{Div}_0(C)$ to $\mathrm{Pic}_0(C)$. We also define the Jacobian variety $J(C)$ to be the quotient

$$J(C) = H^0(C,K)^\vee / j(H_1(C,\mathbb{Z})) ,$$

where $^\vee$ denotes dual vector space and j associates to each $\gamma \in H_1(C,\mathbb{Z})$ the integration functional

$$\omega \longmapsto \int_\gamma \omega .$$

Define a map

$$u : \mathrm{Div}_0(C) \longrightarrow J(C)$$

by setting

$$u(\sum p_i - \sum q_i) = \sum_i \int_{q_i}^{p_i} .$$

The ambiguity involved in the choice of integration paths from q_i to p_i is precisely compensated by the fact that we are modding out by integrals on closed paths. If we choose a basis $\omega_1,\cdots,\omega_g$ for the abelian differentials, the Jacobian and the map u can be more concretely described as follows

$$J(C) = \mathbb{C}^g/\Lambda \quad , \quad \Lambda = \{(\int_\gamma \omega_1,\cdots,\int_\gamma \omega_g) \,|\, \gamma \in H_1(C,\mathbb{Z})\} ,$$

$$u(\sum p_i - \sum q_i) = (\sum_i \int_{q_i}^{p_i} \omega_1,\cdots,\sum_i \int_{q_i}^{p_i} \omega_g) .$$

(4.6) Abel's theorem. There is an isomorphism $\xi : \mathrm{Pic}_0(C) \rightarrow J(C)$ which renders the following diagram commutative

$$\begin{array}{ccc} & & \mathrm{Pic}_0(C) \\ & \nearrow^{\mathcal{O}(\ )} & \\ \mathrm{Div}_0(C) & & \downarrow \xi \\ & \searrow_{u} & \\ & & J(C) \end{array}$$

The strategy of the proof is as follows. First of all, we remark that there is an isomorphism

(4.7) $$\mathrm{Pic}_0(C) = H^1(C, \mathcal{O}) / H^1(C, \mathbb{Z}) .$$

This follows, of course, from the long cohomology exact sequence of

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp(2\pi i\ )} \mathcal{O}^{\times} \longrightarrow 0 ,$$

that is, from

$$\cdots \rightarrow H^1(C, \mathbb{Z}) \rightarrow H^1(C, \mathcal{O}) \rightarrow H^1(C, \mathcal{O}^{\times}) = \mathrm{Pic}(C) \xrightarrow{\deg} H^2(C, \mathbb{Z}) = \mathbb{Z} \rightarrow 0$$

(cf. Gomez-Mont's notes in these proceedings). Since, however, we have been using differential forms and essentially no cohomological machinery all along, we shall give a direct description of the isomorphism (4.7) that does not rely on the exponential sheaf sequence. To do this, let φ be a differential form of type (0,1); we can find a covering $\{U_\alpha\}$ of C and functions u_α such that

$$\varphi = \bar{\partial} u_\alpha \quad \text{on } U_\alpha .$$

In particular, $u_\beta - u_\alpha$ is holomorphic. We then set

$$g_{\alpha\beta} = \exp(2\pi i (u_\beta - u_\alpha)) .$$

It is clear that the $g_{\alpha\beta}$ satisfy the cocycle condition, and hence are transition functions for a line bundle L. It is also straightforward to check that the isomorphism class of L does not depend on the choice of the covering and of the u_α; in particular, if $\varphi = \bar{\partial} u$ on all of C, that is, if φ represents zero in $H^1(C, \mathcal{O})$, L is trivial. We thus have a well defined homomorphism

$$f : H^1(C,\mathcal{O}) \longrightarrow \mathrm{Pic}(C) ;$$

what has to be shown is that

i) the image of f is contained in $\mathrm{Pic}_0(C)$,

ii) f maps $H^1(C,\mathcal{O})$ onto $\mathrm{Pic}_0(C)$,

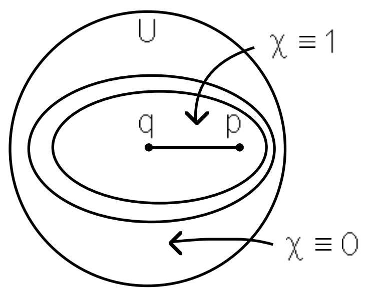
iii) the kernel of f is $H^1(C,\mathbb{Z})$.

We shall prove ii) and then, assuming i) and iii), Abel's theorem. Finally we will go back to i) and iii), and prove them.

Consider then a line bundle L of degree zero, i. e., one of the form $L = \mathcal{O}(\sum(p_i - q_i))$: we wish to show that it is in the image of f. By linearity, it will be enough to do this when

$$L = \mathcal{O}(p - q) .$$

Moreover, always by linearity, we may suppose that p and q are "very close". For us, this will mean that there is a coordinate disk U with coordinate z such that $p, q \in U$, $z(q) = 0$, $z(p) = 1$. The function $\log(\frac{z-1}{z})$ has a single-valued determination in the complement of the segment joining q to p. Now let χ be a C^∞ function on C which is identically equal to 1 in a neighbourhood V of the segment $\overline{qp}$ and identically equal to zero outside of a closed neighbourhood of the segment $\overline{qp}$ contained in U.



The form

$$\varphi = \frac{1}{2\pi i}\bar{\partial}(\chi \log(\frac{z-1}{z})) , \tag{4.8}$$

extended to zero outside of U and across the segment $\overline{qp}$, is then a C^∞ (0,1)-form on C. We claim that it maps to $\mathcal{O}(p-q)$. In fact, consider the covering $\{V, W\}$ of C, where $W = C - \overline{qp}$. A local defining equation for $p - q$ in W is 1, while a local defining equation in V is $(z-1)/z$: thus

$$g_{VW} = \frac{z-1}{z}$$

is the transition function for $\mathcal{O}(p-q)$. On the other hand,

$$\bar\partial 0 = \varphi \qquad\qquad \text{on } V\,,$$

$$\bar\partial\Big(\frac{1}{2\pi i}\chi\log\big(\frac{z-1}{z}\big)\Big) = \varphi \qquad \text{on } W\,,$$

therefore the class of φ in $H^1(C,\mathcal{O})$ maps to a line bundle with transition function

$$\gamma_{VW} = \exp\Big(2\pi i\big(\frac{1}{2\pi i}\chi\log\big(\frac{z-1}{z}\big) - 0\big)\Big) = \frac{z-1}{z}\,,$$

i. e., to $\mathcal{O}(p-q)$.

Assuming that we have proved that $\mathrm{Pic}_0(C) = H^1(C,\mathcal{O})/H^1(C,\mathbb{Z})$, we now define ξ and prove Abel's theorem. Suppose the line bundle L comes from the (0,1)-form ψ. We then let $\xi(L)$ be the functional

$$\omega \longmapsto -\int_C \psi\wedge\omega\,.$$

If ψ comes from $H^1(C,\mathbb{Z})$, i. e., if $\psi+\bar\psi$ is integral, then, by Poincaré duality, $[\psi+\bar\psi]$ is equal to $P(\gamma)$, for some γ in $H_1(C,\mathbb{Z})$, hence

$$\int_C \psi\wedge\omega = \int_C (\psi+\bar\psi)\wedge\omega = \int_C P(\gamma)\wedge\omega = \int_\gamma \omega\,. \tag{4.9}$$

Thus ξ is a well defined homomorphism from $\mathrm{Pic}_0(C)$ to $J(C)$. To show that ξ is injective suppose that (4.9) is satisfied for every abelian differential ω. Then, since $\psi+\bar\psi$ is real,

$$\int_C (\psi+\bar\psi)\wedge\bar\omega = \int_\gamma \bar\omega$$

for every ω. As the abelian differentials and their conjugates span $H^1(C,\mathbb{C})$, this shows that $[\psi+\bar\psi] = P(\gamma)$.

To finish the proof of Abel's theorem it now suffices to show that

$$\xi(\mathcal{O}(D)) = u(D)$$

for any divisor of degree zero D. By linearity, we may suppose that $D = p-q$, where p and q are "very close", so that $\mathcal{O}(D)$ comes from the differential form φ given by (4.8). Let α be a loop around $\overline{qp}$, oriented clockwise. For any abelian differential ω,

$$2\pi i\int_C \varphi\wedge\omega = \int_C d(\chi\log((z-1)/z)\omega)$$
$$= \int_\alpha \log((z-1)/z)\omega = \int_0^1 \log((z-1)/z)_{\mathrm{upper}}\omega - \int_0^1 \log((z-1)/z)_{\mathrm{lower}}\omega\ ,$$

where $\log((z-1)/z)_{\mathrm{upper}}$ and $\log((z-1)/z)_{\mathrm{lower}}$ stand for the upper and lower determinations of $\log((z-1)/z)$ along $\overline{qp}$. As the difference between these two determinations is $2\pi i$, we get

$$-2\pi i\int_C \varphi\wedge\omega = \int_0^1 2\pi i\omega = 2\pi i\int_q^p \omega\ .$$

This exactly says that $\xi(\mathcal{O}(p-q)) = u(p-q)$.

We have proved Abel's theorem assuming it has been shown that

$$\mathrm{Pic}_0(C) = H^1(C,\mathcal{O})/H^1(C,\mathbb{Z}).$$

We have already given a recipe to construct a line bundle from a class in $H^1(C,\mathcal{O})$, and have proved that any degree zero line bundle can be so obtained. Now suppose that the line bundle L comes from a (0,1)-form φ. We wish to show that L has degree zero. We may, and will, suppose that φ is the conjugate of an abelian differential, and hence is closed. Fix a base point q and set

$$u(x) = \int_q^x \varphi\ .$$

This is a multi-valued function: its value at x depends only on the homotopy class of the path used to join q to x, and any two of its determinations differ by a constant, namely by the integral of φ over a suitable closed loop. Since $\bar\partial u = \varphi$, L has transition functions of the form

$$g_{\alpha\beta} = \exp(2\pi i(u_\beta - u_\alpha))\ ,$$

where u_α and u_β are determinations of u. Therefore, L has constant transition functions. Moreover, since $\bar u_\alpha = \int\bar\varphi$ is holomorphic, an equivalent set of transition functions is

$$h_{\alpha\beta} = \exp(2\pi i(u_\beta + \bar u_\beta - u_\alpha - \bar u_\alpha)) = \exp(2\pi i\int_\gamma(\varphi+\bar\varphi))\ , \tag{4.10}$$

where γ is a suitable closed loop. These transition functions are <u>constants</u> <u>of absolute value</u> 1. Now let $s = \{s_\alpha\}$ be a meromorphic section. Then $d\log(s_\alpha)$

is a globally defined meromorphic differential, so the sum of its residues is zero. Since this sum is precisely the degree of L, we are done.

It is clear from (4.10) that L is trivial if $\varphi+\bar{\varphi}$ is integral. It remains to show the converse. Suppose then that L has a nonzero holomorphic section $s=\{s_\alpha\}$. Taking absolute values in $s_\alpha=h_{\alpha\beta}s_\beta$, we conclude that $|s_\alpha|=|s_\beta|$, so that, by the maximum principle for holomorphic functions, s_α must be constant for every α. Thus, if γ is any closed loop with endpoints at q, and $h_{\alpha\beta}$ is given by (4.10), by analytic continuation along γ we get that $s_\alpha(q)=s_\beta(q)$. Since, however, $s_\alpha=h_{\alpha\beta}s_\beta$, $h_{\alpha\beta}$ must be equal to 1. This means that

$$\int_\gamma(\varphi+\bar{\varphi})\in\mathbb{Z}$$

for every γ, i. e., that $[\varphi+\bar{\varphi}]$ is integral. The proof of Abel's theorem is thus complete.

Two final remarks. If we denote by $H^1(C,\mathbb{R})$ the subgroup of $H^1(C,\mathbb{C})$ consisting of the classes of real 1-forms, then $H^1(C,\mathbb{Z})$ is a lattice in $H^1(C,\mathbb{R})$, i. e., a discrete subgroup of maximal rank: this follows, for example, from the intersection relations (4.3). On the other hand, the projection mapping $H^1(C,\mathbb{R})\to H^1(C,\mathcal{O})$ is an isomorphism of <u>real</u> vector spaces. In fact, the two spaces have the same dimension and every element of $H^1(C,\mathcal{O})$ is the class of the conjugate of an abelian differential ω, so $\omega+\bar{\omega}$ is a d-closed real 1-form whose class maps to the class of $\bar{\omega}$ in $H^1(C,\mathcal{O})$. Thus $\mathrm{Pic}_0(C)=J(C)=H^1(C,\mathcal{O})/H^1(C,\mathbb{Z})$ is a complex torus of dimension g; actually, it is a principally polarized abelian variety, but this is another story.

Now suppose $g>0$, and choose a base point $q\in C$, together with a basis $\omega_1,\cdots,\omega_g$ for the abelian differentials. By Abel's theorem the map

$$v: C\longrightarrow J(C)\ ,$$

$$v(p)=(\int_q^p\omega_1,\cdots,\int_q^p\omega_g)\ ,$$

is injective. Since $|K|$ has no base points, it is also an immersion. In particular, when $g=1$, we conclude that C is isomorphic to $J(C)$.

Appendix. In this appendix we have collected a few results about the Laplace operator on a RS; in particular we give a proof of the basic theorem (1.6).

A.1. The maximum principle. Let $u(z)$ be a harmonic function in a neighbourhood of $0 \in \mathbb{C}$; notice that the real and imaginary parts of u are both harmonic. We have that

$$0 = \int_{|z| \le \varepsilon} \partial\bar\partial u = \int_{|z| \le \varepsilon} d(\bar\partial u) = \int_{|z| \le \varepsilon} \bar\partial u \ .$$

Using this and Stokes' theorem we find that

$$0 = \int_{\alpha \le |z| \le \beta} \partial\bar\partial u \cdot \log|z|^2 = \int_{|z| = \beta} \bar\partial u \cdot \log|\beta|^2 - \int_{|z| = \alpha} \bar\partial u \cdot \log|\alpha|^2 + \int_{\alpha \le |z| \le \beta} \bar\partial u \wedge \partial(\log|z|^2)$$

$$= \int_{\alpha \le |z| \le \beta} \bar\partial u \wedge \partial(\log|z|^2) = \int_{|z| = \beta} u \cdot \partial(\log|z|^2) - \int_{|z| = \alpha} u \cdot \partial(\log|z|^2) \ ,$$

since $\log|z|^2$ is harmonic. On the other hand, if we write $z = r \cdot e^{i\vartheta}$,

$$\partial \log|z|^2 = \frac{\bar z\, dz}{|z|^2} = i d\vartheta \qquad \text{on } |z| = r \ ,$$

so we have proved that

$$\int_0^{2\pi} u(r \cdot e^{i\vartheta}) d\vartheta$$

is <u>independent</u> <u>of</u> r, and hence, by continuity, equal to $2\pi u(0)$.

Now suppose u is real-valued and has a maximum at 0. We claim that u is constant in a neighbourhood of 0. In fact, since the integrand in

$$\int_0^{2\pi} (u(0) - u(r \cdot e^{i\vartheta}))\, d\vartheta = 2\pi u(0) - \int_0^{2\pi} u(r \cdot e^{i\vartheta})\, d\vartheta = 0$$

is continuous and non-negative, it must be identically zero for every r.

Now suppose u is a harmonic function on the compact connected RS C. Since C is compact, the real and imaginary parts of u both have a maximum. As we have just shown, the locus where the maximum is attained is open. Since it is also closed, and C is connected, u must be constant.

A.2. A proof of Theorem (1.6). Let C be a RS. We choose, once and for all, a volume form Φ on C and a finite covering $\{U_i\}$ of C with coordinate disks; let z_i be a coordinate on U_i. By slightly shrinking the U_i, if necessary, we may suppose that the coordinate changes and their derivatives of all orders are bounded functions. In the sequel, when we speak of derivatives of a function on C, we shall mean derivatives with respect to the z_i coordinates. We may also suppose, possibly by rescaling the z_i, that for each $p \in C$ there is an i such that $p \in U_i$ and the disk $|z_i - w_i| \le 2$ is contained in U_i. The diagonal in $C \times C$ is covered by the open sets $U_i \times U_i$; we denote the composition of z_i with projection to the second factor by w_i, and the composition with projection to the first factor again by z_i. Then (z_i, w_i) is a system of local coordinates in $U_i \times U_i$. For each positive integer n let $E_{i,n}$ be the region $\{|z_i - w_i| \le 1/n\}$, and choose C^∞ functions $\lambda_{i,n}$ on $C \times C$ such that

- $0 \le \lambda_{i,n}$,
- $\lambda_{i,n}(p,q) = \lambda_{i,n}(q,p)$,
- the support of $\lambda_{i,n}$ is contained in $U_i \times U_i$,
- $\sum_i \lambda_{i,n} - \quad \sum_i \lambda_{i,n} - \quad \sum_i \lambda_{i,n} \le 1$.

We then define

$$\tilde{h}_n(p,q) = \sum_i \lambda_{i,n} \cdot |z_i - w_i|^2 + (1 - \sum_i \lambda_{i,n}) ,$$

$$h_n(p,q) = \log(\tilde{h}_n(p,q)) \qquad \text{for } p \ne q .$$

The function h_n has the following properties:

$h_n(p,q) = h_n(q,p)$,

(A.2.1) $h_n(p,q) \le 0$,

(A.2.2) $h_n(p,q)$ is supported in $\bigcup U_i \times U_i$, $h_n(p,q)$ (A.2.3) $h_n(p,q) = \log|z_i - w_i|^2 +$ a C^∞ function in $U_i \times U_i$.

We also set

$$k_n(p,q) = \partial_1 \bar{\partial}_1 h_n(p,q) ;$$

here and in the following, the subscript "1" (resp. "2") means differentiation with respect to the first set of variables (resp., the second set of variables). Notice that, by (A.2.3), $k_n(p,q)$ extends in a C^∞ way across the diagonal.

We shall denote by L_2 the space of square-integrable functions on C with the inner product

$$(f,g) = \int_C f\bar{g}\Phi .$$

The symbol $\|f\|$ will denote the norm of f with respect to this inner product. We define continuous operators

$$H_n : L_2 \longrightarrow L_2$$

$$K_n : L_2 \longrightarrow L_2$$

by

$$H_n f(p) = \frac{1}{2\pi i} \int_{q\in C} h_n(p,q) f(q)\Phi(q) \ ,$$

$$K_n f(p) = \frac{1}{2\pi i\,\Phi(p)} \int_{q\in C} k_n(p,q) f(q)\Phi(q) \ .$$

Notice that, since $k_n(p,q)$ is C^∞, $K_n f$ can be infinetely differentiated under the integral sign, so that it is a C^∞ function for any f in L_2. The same is true for the adjoint of K_n, denoted K_n^*, which is given by

$$K_n^* f(q) = \int_{p\in C} \partial_1\bar{\partial}_1 k_n(p,q) f(p) .$$

The operator H_n, on the other hand, carries C^∞ functions to C^∞ functions. To see this, let $\{\chi_i\}$ be any partition of unity. Then, if $H_n(\chi_i u)$ is C^∞ for each i, then $H_n(u)$ is C^∞, by linearity. By choosing a sufficiently fine partition of unity, then, we are reduced to proving our statement when the support of u is "small". By this we mean that there is an i such that the support of u is contained in U_i and, for any $p \in \text{supp}(u)$, the disk $|z_i - w_i| \le 1$ is contained in U_i. This will make it possible to do all of our computations in $U_i \times U_i$, in the z_i and w_i variables. In fact, it implies that, if $p \notin U_i$, $H_n u(p)$ (and also $K_n u(p)$) vanishes. Since we will be working on $U_i \times U_i$, we drop the subscript i and write z and w for z_i and w_i; we shall also write h and k instead of h_n and k_n. As we observed, in $U_i \times U_i$ one has

$$h(z,w) = g(z,w) + \log(|z-w|^2) \ ,$$

where g is C^∞. Then what really has to be shown is that

$$\int_{\mathbb{C}} \log(|z-w|^2)v(w)\,dw\wedge d\bar{w}$$

is a C^∞ function of z for any C^∞ v with "small" support. First of all this function is certainly continuous. Then an integration by parts yields

$$\frac{\partial}{\partial z}\int \log(|z-w|^2)v(w)\,dw\wedge d\overline{w} = -\int \frac{1}{w-z}v(w)\,dw\wedge d\overline{w}$$

$$= -\int \frac{\partial}{\partial w}\log(|z-w|^2)\,v(w)\,dw\wedge d\overline{w} = \int \log(|z-w|^2)\frac{\partial v}{\partial w}(w)\,dw\wedge d\overline{w} \ .$$

A similar formula holds for the derivative with respect to $\bar{z}$. What the formulas say is that, if $\int \log(|z-w|^2)v(w)\,dw\wedge d\bar{w}$ is C^s for any C^∞ function v with "small" support, then it is also C^{s+1}; the inescapable conclusion is that it must be C^∞.

We let L be the differential operator defined by

$$Lu = \frac{\partial\bar{\partial}u}{\Phi} \ .$$

We claim that, for any C^∞ function u, one has

(A.2.4) $$LH_n u = K_n u - u \ .$$

This is a simple residue computation. By linearity, it will suffice to prove this when u has "small" support. Accordingly, we shall adopt all the conventions used in showing that $H_n u$ is C^∞ when u is. As we observed, in $U_i\times U_i$ one has

$$h(z,w) = g(z,w) + \log(|z-w|^2) \ ,$$

$$k(z,w) = \frac{\partial^2 g}{\partial z\partial\overline{z}}\,dz\wedge d\overline{z} \ ,$$

where g is C^∞. Now, for any C^∞ function v with "small" support in U_i, one has

$$\frac{\partial^2}{\partial z\partial\overline{z}}\int h(z,w)v(w)\,dw\wedge d\overline{w} =$$

$$= \int \frac{\partial^2 g}{\partial z\partial\overline{z}}(z,w)v(w)\,dw\wedge d\overline{w} + \frac{\partial^2}{\partial z\partial\overline{z}}\int \log(|z-w|^2)v(w)\,dw\wedge d\overline{w} \ .$$

On the other hand, we have seen that

$$\frac{\partial}{\partial \bar{z}} \int \log(|z-w|^2) v(w)\, dw \wedge d\bar{w} = \int \log(|z-w|^2) \frac{\partial v}{\partial \bar{w}}(w)\, dw \wedge d\bar{w} ,$$

so that, if we denote by D_r the disk of radius r centered at z:

$$\frac{\partial^2}{\partial z \partial \bar{z}} \int \log(|z-w|^2) v(w)\, dw \wedge d\bar{w} = - \int \frac{1}{w-z} \frac{\partial v}{\partial \bar{w}}(w)\, dw \wedge d\bar{w}$$

$$= \lim_{r \to 0} \int_{U_i - D_r} d\left(\frac{v(w)}{w-z} dw\right) = - \lim_{r \to 0} \int_{\partial D_r} v(w) \frac{dw}{w-z} = -2\pi i v(z) .$$

In conclusion

$$\frac{\partial^2}{\partial z \partial \bar{z}} \int h(z,w) v(w)\, dw \wedge d\bar{w} = \int \frac{\partial^2 g}{\partial z \partial \bar{z}}(z,w) v(w)\, dw \wedge d\bar{w} - 2\pi i v(z) .$$

Taking $v(w)\, dw \wedge d\bar{w} = u(w)\Phi(w)$, this says that

$$\partial\bar{\partial} H_n u = \Phi K_n u - \Phi u ,$$

which is exactly what had to be proved.

We are now essentially ready to prove (1.6). The proof will use two simple results of functional analysis. The first is

(A.2.5) Fact. A normed complex vector space is locally compact if and only if it is finite-dimensional.

The second is the Ascoli-Arzel**à** theorem in the following version.

(A.2.6) Fact. Let $\{f_n\}$ be a sequence of C^∞ functions on C. Suppose that the f_n and their first derivatives are uniformly bounded. Then $\{f_n\}$ has a uniformly convergent subsequence.

It is a consequence of (A.2.6) that K_n is a compact operator, i. e., that it carries bounded sequences in L_2 to sequences with a convergent subsequence. Suppose in fact that $\{f_n\}$ is a bounded sequence in L_2. Then the fact that k_n and its first derivatives are bounded, plus the Schwartz inequality, shows that the (C^∞) functions $K_n f_h$ and their first derivatives are uniformly bounded. Thus a subsequence of $\{K_n f_h\}$ converges uniformly, and hence in L_2. The adjoint of K_n is also compact. This follows from general theory or, alternatively, by an argument similar to the one used for K_n.

A consequence of these considerations is that the kernels of $K_n - I$ and $K_n^* - I$ are finite-dimensional and made up entirely of C^∞ functions. Let's do this for K_n. Firstly, if

$$K_n u - u = 0 ,$$

then u is C^∞ since $K_n u$ is. Secondly, if $u_h \in \ker(K_n - I)$ and $\{u_h\}$ is bounded, then the sequence $u_h = K_n u_h$ has a convergent subsequence so, by (A.2.5), the kernel of $K_n - I$ is finite dimensional.

The next remark is that

$$\bigcap_{n\in\mathbb{N}} \ker(K_n^* - I) = \{\text{constant functions}\} .$$

Suppose in fact that $K_n^* u = u$ for every n (so that, in particular, u is C^∞). Then

$$0 = (K_n^* - I)u = H_n^* L^* u = H_n L u .$$

If u is not constant, then $v = Lu$ is not zero, so that there is a point p where $v(p) \neq 0$; say $v(p) > 0$. Choose a neighbourhood V of p where the sign of v does not change. Then, for large enough n, $h_n(p,q) = 0$ if $q \notin V$, so, by (A.2.1),

$$H_n v(p) = \int_V h_n(p,q)\, v(q)\, \Phi(q) < 0 ,$$

a contradiction. The same kind of argument applies if $v(p) < 0$.

Since $\ker(K_n^* - I)$ is finite-dimensional, for large n we have

$$\ker(K_1^* - I) \cap \cdots \cap \ker(K_n^* - I) = \{\text{constants}\} .$$

Therefore, if $\int f \Phi = 0$, i. e., if $(f,1) = 0$, we may write

$$\begin{aligned}
f &= f_1 + f_1' , & f_1 &\perp \ker(K_1^* - I) , \; f_1' \in \ker(K_1^* - I) , \\
f_1' &= f_2 + f_2' , & f_2 &\perp \ker(K_2^* - I) , \; f_2' \in \ker(K_1^* - I) \cap \ker(K_2^* - I) , \\
& & &\cdots\cdots \\
f_{n-1}' &= f_n , & f_n &\perp \ker(K_n^* - I) .
\end{aligned}$$

Thus $\sum f_i = f$. Moreover, if f is C^∞, so is f_i for every i. Now suppose we can solve

$$\text{(A.2.7)} \qquad (K_i - I) v_i = f_i , \qquad i = 1, \cdots, n .$$

If f is C^∞, so is v_i, since $K_i v_i$ is C^∞ for every i. Moreover

$$LH_i v_i = K_i v_i - v_i = f_i\ ,$$

so that

$$L(\sum H_i v_i) = \sum f_i = f\ ,$$

and the equation

$$Lu = f$$

is solved by the C^∞ function $u = \sum H_i v_i$.

It remains to solve (A.2.7), that is, to show that, if we set $R = K_i - I$, $Rv = f$ can be solved whenever $f \perp \ker(R^*)$. This is standard. We first claim that, if $\alpha \perp \ker(R^*)$, then there is a positive constant N such that

$$\|\alpha\| \le N\|R^*\alpha\|\ .$$

If not, there would be a sequence α_j with

$$\|\alpha_j\| = 1\ ,\ \ \alpha_j \perp \ker(R^*)\ ,\ \ \|R^*\alpha_j\| \to 0\ \ .$$

Since K_i^* is compact, passing to a subsequence we would have $K_i^*\alpha_j \to \beta$, hence $\alpha_j \to \beta$, $\|\beta\| = 1$, but also $R^*\beta = 0$, $\beta \perp \ker(R^*)$, so $\beta = 0$, a contradiction. Now consider the functional

$$R^*(L_2) \longrightarrow \mathbb{C}$$
$$R^*\alpha \longmapsto (\alpha, f)\quad ,\qquad \alpha \perp \ker(R^*)\ .$$

Then

$$|(\alpha, f)| \le \|f\|\cdot\|\alpha\| \le N\cdot\|f\|\cdot\|R^*\alpha\|\ ,$$

so our functional is continuous, hence, by the Riesz representation theorem, there is $v \in \overline{R^* L_2}$ such that

$$(\alpha, f) = (R^*\alpha, v)\quad ,\qquad \alpha \perp \ker(R^*)\ .$$

Since this is true by hypothesis when $\alpha \in \ker(R^*)$, we conclude that $f = Rv$.

We have shown that $Lu = f$ is solvable when $\int f\Phi = 0$. Thus $\partial\bar\partial u = \varphi$ is solvable when $\int\varphi = \int(\varphi/\Phi)\Phi = 0$. Theorem (1.6) is thus completely proved.

A.3 The finiteness of $h^1(L)$. Let L be a holomorphic line bundle on the RS C; we wish to show that $h^1(C,L)$ is finite. We begin by remarking that $h^0(C,L)$ is finite: in fact, if L has no nonzero sections, there is nothing to prove, while, if it has one, $L=\mathcal{O}(D)$ for some D, and Riemann-Roch, in the form we have proved, applies. Pick a hermitian metric on L, i. e., a smoothly varying hermitian metric on the fibers of L (that a hermitian metric always exists can be seen by pasting together local hermitian metrics by means of a partition of unity): given sections s and t, we shall write $\langle s,t\rangle$ to indicate the inner product of s and t with respect to this metric and $|s|$ to indicate the length of s. Suppose L is given by transition functions $g_{\alpha\beta}$, and $s=\{s_\alpha\}$ is a section: then

$$|s|^2 = \mu_\alpha|s_\alpha|^2 = \mu_\beta|s_\beta|^2 ,$$

where the μ_α are positive C^∞ functions. Clearly the μ_α must satisfy

$$\mu_\beta = |g_{\alpha\beta}|^2\mu_\alpha ;$$

conversely, to give positive functions satisfying these relations is equivalent to giving a metric on L. Suppose $\varphi=\{\varphi_\alpha\}$ is an L-valued (0,1)-form. Set $\#\varphi=\{\psi_\alpha\}$, where

$$\psi_\alpha = \mu_\alpha\bar{\varphi}_\alpha .$$

Since $\psi_\alpha = \mu_\alpha\bar{\varphi}_\alpha = |g_{\alpha\beta}|^{-2}\mu_\beta g_{\alpha\beta}\bar{\varphi}_\beta = g^{-1}_{\alpha\beta}\mu_\beta\bar{\varphi}_\beta$, $\#\varphi$ is a C^∞ section of KL^{-1}, which is holomorphic if and only if $\vartheta\varphi=0$, where ϑ is the differential operator defined by

$$s_\alpha = \frac{\partial(\mu_\alpha\varphi_\alpha)}{\mu_\alpha\Phi}$$

One easily checks that $s=\{s_\alpha\}$ is a section of L. If we define the inner products of two sections s, t of L and two L-valued (0,1)-forms $\varphi=\{\varphi_\alpha\}$ and $\psi=\{\psi_\alpha\}$ by

$$(s,t) = \int\langle s,t\rangle\Phi \qquad ; \qquad (\varphi,\psi) = \int\varphi\wedge\#\psi ,$$

where $\varphi\wedge\#\psi$ is the global (1,1)-form with local expressions $\mu_\alpha\varphi_\alpha\wedge\bar{\psi}_\alpha$, then ϑ is the adjoint of $\bar{\partial}$.

Now suppose that we can solve the differential equation

(A.3.1) $$\vartheta\bar{\partial}s = \vartheta\varphi$$

for any L-valued $(0,1)$-form φ. Then any class in $H^1(C,L)$ has a ϑ-closed representative, and this is unique, since $\vartheta\bar\partial s=0$ implies that $(\bar\partial s,\bar\partial s)=0$, so $\bar\partial s$ must vanish. Moreover the map

$$\vartheta\text{-closed } L\text{-valued } (0,1)\text{-forms} \longrightarrow \{\text{holomorphic sections of } KL^{-1}\},$$

$$\varphi \longmapsto \#\varphi ,$$

is an antiisomorphism between $H^1(C,L)$ and $H^0(C,KL^{-1})$ that transforms the inner product on $H^1(C,L)$ into the standard pairing between $H^1(C,L)$ and $H^0(C,KL^{-1})$: this gives another proof of duality and shows that $H^1(C,L)$ is finite, since $H^0(C,KL^{-1})$ is.

That (A.3.1) is always solvable can be shown by much the same methods that we used to prove (1.6), applied not to the standard Laplacian, but to the operator $\vartheta\bar\partial$. We omit the details.

Bibliographical note. The point of view adopted in these notes is somewhat intermediate between the classical one used in the books by Weyl, Siegel, Springer, Farkas and Kra, and the more modern one of Gunning. The study of compact Riemann surfaces as algebraic varieties is further developed in the books by Griffiths and Harris and by Arbarello, Cornalba, Griffiths and Harris.

In the present volume, the reader will find background material on complex varieties, Riemann surfaces, and sheaves in Gomez-Mont's lectures, while an ideal continuation of these notes is constituted by Roy Smith's lectures.

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