ALGEBRAIC GEOMETRY AND PATH INTEGRALS FOR CLOSED STRINGS ☆

Roberto CATENACCI [1], Maurizio CORNALBA
Dipartimento di Matematica, via Strada Nuova 65, I-27100 Pavia, Italy

Maurizio MARTELLINI [1] and Cesare REINA
Dipartimento di Fisica, via Celoria 16, I-20133 Milan, Italy

Received 11 February 1986

The pth loop contribution to the partition function for closed strings is studied by applying recent mathematical results on the geometry of the moduli space $\mathcal{M}_p$ of smooth algebraic curves of genus p. By reasoning on determinants of operators and line bundles over $\mathcal{M}_p$, we get a geometric explanation of the critical dimensions 26 and 10. The extension of path integrals for strings to the compactified moduli space $\overline{\mathcal{M}}_p$ of stable curves is also discussed. While the Weil–Petersson measure has a continuous extension on $\overline{\mathcal{M}}_p$, the bosonic path integral has a bad behaviour on the boundary $\overline{\mathcal{M}}_p - \mathcal{M}_p$. Instead, the functional approach to the spinning string of Ramond–Neveu–Schwarz seems to yield a finite pth loop contribution to the partition function.

The path integral approach to the bosonic string theory has been recently studied by several authors [1–3]. The partition function $Z = \Sigma_p \exp(ap)Z_p$ involves the sum of integrals over the space M_p of metrics on compact Riemann surfaces of genus p. Now, M_p carries the principal action of the group of diffeomorphisms and of Weyl transformations. Since these are classical invariance groups of the string action, one tries as usual to factor out the integration on them which formally contributes to Z_p with a divergent multiplicative constant. When this can be done, that is in the critical dimension, the functional integration reduces to a finite dimensional ordinary integral in terms of the so-called Teichmuller parameters. Actual computations beyond one loop seem however to be quite hard to come because there is little control of the explicit dependence of the integrand on Teichmuller coordinates. Besides and more importantly, one needs a deep knowledge of the topology of the domain of integration $\mathcal{T}_p/\Gamma_p$, where $\mathcal{T}_p$ is the Teichmuller space and Γ_p is the mapping class group. Indeed taking the quotient of $\mathcal{T}_p$ by Γ_p is needed to avoid infrared divergencies.

In this letter we give a short account of a geometric approach to both these problems, while full details will be published elsewhere. The starting point is that the domain $\mathcal{M}_p = \mathcal{T}_p/\Gamma_p$ is the moduli space of smooth algebraic curves of genus p; it is a quasi-projective variety about which much is known in algebraic geometry. Recall that a quasi-projective manifold is the complement of a Zariski closed set in a projective algebraic variety. A natural compactification $\overline{\mathcal{M}}_p$ of $\mathcal{M}_p$ has been constructed by Deligne and Mumford [4] by considering a larger class of curves besides the smooth ones; $\overline{\mathcal{M}}_p$ will be briefly described below. The important fact for our concern is that the functional measure induces the Weil–Petersson volume element ω_{WP} on the moduli space $\mathcal{M}_p$. By a result of Wolpert [5], this measure extends to $\overline{\mathcal{M}}_p$, in such a way that $\overline{\mathcal{M}}_p$ has a finite Weil–Petersson volume.

Our strategy is then to study the induced integral over $\overline{\mathcal{M}}_p$. In particular, we show that the invariance of this integral under the group of orientation preserv-

☆ Work partially sypported by Gruppo Nazionale di Fisica Matematica, CNR, Gruppo Nazionale per le Strutture Algebriche e Geometriche e Applicazioni, CNR and Progetto Nazionale "Geometria e Fisica" MPI.

[1] Also at INFN, Sezione di Pavia, Pavia, Italy.

0370-2693/86/$ 03.50 © Elsevier Science Publishers B.V. (North-Holland Physics Publishing Division)

ing diffeomorphisms is equivalent to the vanishing of the first Chern class of a suitable complex line bundle G. If we work on the interior $\mathcal{M}_p$ of $\overline{\mathcal{M}}_p$, the triviality of G can be only achieved in the critical dimension $D = 26$. The extension of the same argument to the spinning strings yields the critical dimension $D = 10$. So we see a posteriori that absence of global anomalies under the group of diffeomorphisms is sufficient to make the theory free of trace anomalies as well. When one considers the whole of $\overline{\mathcal{M}}_p$, one finds that for the bosonic string G is never trivial. As a byproduct we see that the integrand in Z_p has a pole on degenerate surfaces belonging to the boundary $\overline{\mathcal{M}}_p - \mathcal{M}_p$. The order of this pole is entirely controlled by the complex analytic structures of both $\overline{\mathcal{M}}_p$ and G. On the contrary, a preliminary computation suggests that the "anomaly cancellation" for the spinning string in $D = 10$ extends to whole moduli space $\overline{\mathcal{M}}_p$. This physically would imply the finiteness of Z_p at any loop order $p \geqslant 2$.

To be more definite, we denote by C an algebraic curve (i.e. a closed Riemann surface) of genus $p \geqslant 2$. A point $[C]$ in $\mathcal{M}_p$ is simply a biholomorphic equivalence class of curves. In real terms, $[C]$ is an orbit of the group of orientation preserving diffeomorphisms in the space of hyperbolic metrics on a closed two-dimensional surface of genus p. The natural compactification of $\mathcal{M}_p$ is given by considering equivalence classes of stable curves [6]. The boundary $\Delta = \overline{\mathcal{M}}_p - \mathcal{M}_p$ can be described as follows. Let C be a stable curve, and q a singular point of C (if any). It is known that q is a node and we say that it is a node of type i ($1 \leqslant i \leqslant [p/2]$) if $C - \{q\}$ is the union of two connected components of genera i and $p - i$. If $C - \{q\}$ is connected, we say that q is a node of type 0. The boundary Δ is the union of irreducible components Δ_i ($0 \leqslant i \leqslant [p/2]$) parametrizing stable curves with at least a node of type i.

As $\overline{\mathcal{M}}_p$ is a nice domain of integration with a "good" measure on it, the next question about the definition of the pth loop contribution to the partition function concerns the integrability of the product W of the determinants of certain operators. Generally speaking, these are real operators but, as noticed by Alvarez [7], there is some merit to splitting them into complex operators. The reason for this procedure is that W can then be interpreted as the "square modulus" of a holomorphic section of an hermitean line bundle on $\mathcal{M}_p$.

To see how this goes, let us first work on $\mathcal{M}_p$, and consider for instance the Laplace operator d^+d acting on complex valued functions on a curve C. Notice that $d^+d = 2\partial^+\partial$, where ∂ is the holomorphic differential. As usual, one sets

$$\det(\tfrac{1}{2}d^+d) = \det(\partial^+\partial) =: \exp[-\zeta'(0)] \ , \tag{1}$$

where ζ' is the derivative of the ζ-function of the operator $\partial^+\partial$. The same result can be obtained as follows. The operator ∂ maps complex valued functions into one-forms of type $(1, 0)$ on C. Now $\ker \partial = \mathbf{C}$ is the space of constant functions, while $\operatorname{coker} \partial = H^0(C, \Omega) = \mathbf{C}^p$ is the space of holomorphic forms of type $(1, 0)$. Building on the ideas of ref. [8], we define $\det \partial|_C$ to be the element of the linear space $L_C = (\Lambda^{\max} \ker \partial)^{-1} \otimes (\Lambda^{\max} \operatorname{coker} \partial) = \Lambda^p H^0(C, \Omega)$ given by $\det \partial|_C =: \exp[-\frac{1}{2}\zeta'(0)]\, \psi_C$, where Λ denotes exterior power and ψ_C is a unitary basis in L_C. Notice that we are taking advantage of the fact that we shall need a definition of $\det \partial$ up to a phase factor. If we compute the norm

$$|\det \partial|_C|^2 = \langle \exp[-\tfrac{1}{2}\zeta'(0)]\, \psi_C | \exp[-\tfrac{1}{2}\zeta'(0)]\, \psi_C \rangle = \exp[-\zeta'(0)] \langle \psi_C | \psi_C \rangle = \exp[-\zeta'(0)] \ , \tag{2}$$

where $\langle \ | \ \rangle$ is the hermitean inner product on L_C, we get exactly the same result as (1).

Since everything is equivariant under biholomorphisms, $\det \partial$ descends to a section of the "Hodge line bundle" $L = \Lambda^p H^0(\cdot, \Omega)$ over the moduli space $\mathcal{M}_p$. One should actually work in terms of line bundles "over the moduli stack", but we will not mention such subtleties here. Summing up,

$$\det(\partial^+\partial) = |\det \partial|^2 \ , \tag{3}$$

where $\det \partial \in \Gamma(L)$ and $|\ \ |^2$ is the hermitean fibre metric induced on L by $\langle \ | \ \rangle$.

To see where "anomalies" come from in the definition (3), notice that one can locally choose the unitary section ψ_C in such a way that $\det \partial$ is a holomorphic section of L. Accordingly, if we trivialize L on a covering $\{U_\alpha\}$ of $\mathcal{M}_p$, we have in general that

$$\det \partial_\alpha = g_{\alpha\beta} \det \partial_\beta \ , \tag{4}$$

where $g_{\alpha\beta}$ is a holomorphic transition function with values in $\mathrm{Gl}(1, \mathbf{C})$. Then

$$|\det \partial_\alpha|^2 = |g_{\alpha\beta}|^2 |\det \partial_\beta|^2 \ . \tag{5}$$

Now $|\det \partial_\alpha|^2$ is a section of the real line bundle $|L|^2$, defined by the transition functions $|g_{\alpha\beta}|^2$. Clearly, $|L|^2$ is topologically trivial over $\mathcal{M}_p$, but its trivializations cannot be induced by holomorphic trivializations of L, because L is non-trivial [9] and one cannot choose $g_{\alpha\beta} = 1$. Incidentally, no positive power of L is holomorphically trivial. From the physical point of view, eq. (5) tells us that $\zeta'_\alpha(0) = \zeta'_\beta(0) - \ln|g_{\alpha\beta}|^2$, i.e. we have an anomaly.

This kind of argument does not depend on the particular nature of L. So we see that a necessary and sufficient condition for absence of anomalies in the square modulus of holomorphic sections of a line bundle G on $\mathcal{M}_p$ is that the first Chern class $c_1(G)$ vanishes. In fact the group of holomorphic isomorphism classes of topologically trivial line bundles $\mathrm{Pic}_0(\mathcal{M}_p)$ of the moduli space is known to be trivial [10,11]. This answers the question of Alvarez [7] as to why one can control anomalies for real operators by means of the family index theorem for certain associated complex operators. By the way, notice that this result heavily relies on holomorphic structures, so there is little hope of extending it to higher dimensional cases. We can be more precise on the kind of anomalies whose absence is controlled by the topological condition $c_1(G) = 0$. Recall that $\mathcal{M}_p = \mathcal{T}_p/\Gamma_p$. Since the Teichmuller space is contractible, any line bundle is trivial on it. So we see that non-trivial line bundles over $\mathcal{M}_p$ can be considered as trivial line bundles over $\mathcal{T}_p$ carrying a non-trivial action of the mapping class group Γ_p. So, our anomalies are nothing but global anomalies of the group of diffeomorphisms.

Besides the Laplace operator, in studying the partition function of the bosonic string one is interested in the traceless part of the Lie derivative of the metric. This is the operator

$$\overline{\nabla} + \nabla = Q\colon \Gamma(T'\,C) \oplus \Gamma(T''\,C) \to \Gamma(\overline{2K}) \oplus \Gamma(2K)\,, \tag{6}$$

where $T'\,C$ ($T''\,C$) is the (anti)holomorphic tangent bundle and $2K$ is the bundle of quadratic differentials of type (2, 0). Locally Q is given by

$$Q(X^z \partial/\partial_z + \overline{X^z}\partial/\partial_{\bar z})$$
$$= h_{z\bar z}\partial_{\bar z} X^z \,\mathrm{d}\bar z \otimes \mathrm{d}\bar z + h_{z\bar z}\partial_z \overline{X^z}\,\mathrm{d}z \otimes \mathrm{d}z\ , \tag{7}$$

where $h_{z\bar z}$ is a metric on C. To compute the determinant of Q we work as before by considering the complex operator

$$\nabla\colon \Gamma(T''\,C) \to \Gamma(2K)\,, \tag{8}$$

given by

$$\nabla(\overline{X^z}\partial/\partial_{\bar z}) = h_{z\bar z}\partial_z \overline{X^z}\,\mathrm{d}z \otimes \mathrm{d}z\ . \tag{9}$$

Notice that $\ker \nabla = 0$, since there are no antiholomorphic vector fields on C for $p \geqslant 2$. On the other hand $\mathrm{coker}\, \nabla = \{a_{zz}\,\mathrm{d}z \otimes \mathrm{d}z | \partial_{\bar z} a_{zz} = 0\}$ is the space $\mathrm{H}^0(C, 2K) = \mathbb{C}^{3p-3}$ of holomorphic quadratic differentials on C. Then, as above, $\det \nabla$ goes down to a section of the line bundle $K = \Lambda^{\max}\mathrm{H}^0(\cdot, 2K)$ over the moduli space. Incidentally, K is the canonical line bundle of the moduli space $\mathcal{M}_p$. As we did for the Laplace operator, the determinant of the Faddeev–Popov operator Q can be written as

$$\det Q = |\det \nabla|^2\ . \tag{10}$$

We are now in the position of explaining the cancellation of the anomaly of the bosonic string in the critical dimension. The integrand of the pth loop contribution to the partition function reads

$$W = [\det(\tfrac{1}{2}\mathrm{d}^+\mathrm{d})]^{-D/2} \det Q$$
$$= |\det \partial|^{-D} |\det \nabla|^2\ , \tag{11}$$

where D is the real dimension of the target space. Recall that the determinant of the Laplace operator comes from the gaussian integration over string fields, with zero modes removed. Now $s = (\det \partial)^{-D/2} \times (\det \nabla)$ is a section of the line bundle $L^{-D/2} \otimes K$ over $\mathcal{M}_p$, and it is such that $W = |s|^2$ with respect to the induced fibre metric. So, anomaly cancellation requires that $c_1(L^{-D/2} \otimes K) = -\frac{1}{2}Dc_1(L) + c_1(K)$ vanishes. As is customary we set

$$c_1(L) = \lambda\ . \tag{12}$$

Recall [11] that λ is a generator of the infinite cyclic group $\mathrm{Pic}(\mathcal{M}_p)$ of line bundles on the moduli stack $\mathcal{M}_p$. This means that any line bundle over $\mathcal{M}_p$ is a multiple of L. The non trivial fact for our concern is that K is actually isomorphic to L^{13}. This is proved for example in refs. [6,9] by means of Grothendieck's relative Riemann–Roch theorem, which is nothing but the Atiyah–Singer index theorem applied to the family of operators ∂ and ∇ parametrized by $\mathcal{M}_p$. So,

$$c_1(K) = 13\lambda\ . \tag{13}$$

This fact is responsible for the critical dimension: indeed we have

$$c_1(L^{-D/2} \otimes K) = (-D/2 + 13)\lambda\,, \tag{14}$$

which vanishes if and only if $D = 26$.

So far we have been working over uncompactified moduli space $\mathcal{M}_p$, and now we come to the problem of extending integration to $\overline{\mathcal{M}}_p$, as required by physics. Our first result is that anomaly cancellation for the bosonic string does not survive compactification. Let Δ be the boundary $\overline{\mathcal{M}}_p - \mathcal{M}_p$, which is of complex codimension 1 in $\overline{\mathcal{M}}_p$, and δ the first Chern class of the line bundle $[\Delta]$ on the moduli stack associated to Δ. Now det ∇ and det ∂ are sections of suitable line bundles K^c and L^c, whose Chern classes are given by [9]

$$c_1(L^c) = \lambda\,, \quad c_1(K^c) = 13\lambda - \delta\,. \tag{15}$$

So the condition of anomaly cancellation on the compactified moduli space requires the vanishing of

$$c_1((L^c)^{-D/2} \otimes K^c) = (-D/2 + 13)\lambda - \delta\,. \tag{16}$$

Unfortunately this is not possible even in the critical dimension $D = 26$ because of the contribution from the boundary. Indeed, for $D = 26$ the line bundle $(L^c)^{-D/2} \otimes K^c$ has no global holomorphic sections, s, on $\overline{\mathcal{M}}_p$. Since its first Chern class is $-\delta$, any section holomorphic on $\mathcal{M}_p$ has an extension to $\overline{\mathcal{M}}_p$ which is necessarily meromorphic with a pole of order one on the boundary.

Summing up, our geometric argument shows that one cannot consistently cancel anomalies on the boundary Δ, that is on the locus of curves with certain basic loops pinched down to a node. The relation between the presence of poles in the integrand of Z_p and the boundary components has been first considered in ref. [2], by studying the behaviour of the Selberg trace formula for the relevant determinants in Fenchel–Nielsen coordinates. A pole of the same order has been also found in ref. [12]. These authors also suggest a possible algebraic geometrical explanation of the meromorphic structure described above, and give a direct computation of the order of the pole by relying on the complex analytic structure of $\overline{\mathcal{M}}_p$. In any case, this pole is related to the higher order divergence in the integrand of Z_p, which physically signals the presence of the tachyon in the theory.

Our final aim is to look for string models which are anomaly free on the compactified moduli space. A standard way of treating local anomalies is to extend the matter content of the model in order to achieve complete anomaly cancellation. Here, the obvious candidate is a "supersymmetric string" a la Ramond–Neveu–Schwarz, since the tachyon is absent because of "supersymmetry".

The path integral formulation [13] for the spinning string requires the sum over all spin structures on the world sheet [14]. Besides, it requires also integration over the space of gravitini. This would lead to the so-called supermoduli spaces. Here we restrict our attention to the integral over the moduli $\mathcal{S}_p$ of spin structures on algebraic curves of genus p. It is well known [15] that on a smooth curve there are 4^p inequivalent spin structures, so $\mathcal{S}_p$ is an unbranched 4^p-sheeted covering of $\mathcal{M}_p$. Also, $\mathcal{S}_p$ is the union of two connected components $\mathcal{S}_p^{\pm}$, called even and odd spin structures, according to the parity of the dimension of the kernel of the chiral Dirac operator. The integrand now contains the determinants of two more operators [7], namely the chiral Dirac operator $\not\partial$ and the fermionic Faddeev–Popov operator $\not\nabla$, and reads

$$W = |\det \partial|^{-D} |\det \nabla|^2 |\det \not\partial|^D |\det \not\nabla|^{-2}\,. \tag{17}$$

This is the square modulus of a section of the line bundle

$$G = L^{-D/2} \otimes K \otimes E^{D/2} \otimes F^{-1}\,, \tag{18}$$

where E and F are the determinant line bundles of the index bundles of the operators $\not\partial$ and $\not\nabla$. One finds that

$$c^1(E) = -\lambda/2\,, \quad c_1(F) = 11\lambda/2\,. \tag{19}$$

As above we check anomaly cancellation in the critical dimension on $\mathcal{S}_p$, by computing

$$c_1(G) = -\tfrac{1}{2}Dc_1(L) + c_1(K) + \tfrac{1}{2}Dc_1(E) - c_1(F)$$

$$= [(30 - 3D)/4]\lambda\,, \tag{20}$$

which vanishes for $D = 10$. So, also in this case we recover the critical dimension by a topological argument.

The extension of this approach to a compactification $\overline{\mathcal{S}}_p$ of spin structures on stable curves requires a full control of the behaviour of "spin structures" on curves with nodes as singularities, which is not available at present. However, a preliminary computation seems to show that the relevant line bundle G^c on $\overline{\mathcal{S}}_p$

has first Chern class

$$c_1(G^c) = (10 - D)(12\lambda - \delta)/16\,. \tag{21}$$

This suggests, as expected, that for $D = 10$ one achieves a complete anomaly cancellation on the whole of $\bar{\mathcal{S}}_p$ at any loop. Clearly, this would signal absence of the tachyon, which physically occurs in the even G-parity sector of the spinning string. It is tempting to relate G-parity with the two components $\mathcal{S}_p^{\pm}$ of even and odd spin structures [15]. A full proof of these results for spinning strings will be the matter of future investigations.

The meaning of eq. (21) is that the integrand of the partition function for the spinning string is at least continuous on the compact space $\bar{\mathcal{S}}_p$. Notice also that the Weil–Petersson volume element on $\overline{\mathcal{M}}_p$ is pulled back by the covering map π: $\bar{\mathcal{S}}_p \to \overline{\mathcal{M}}_p$ to a continuous measure on $\bar{\mathcal{S}}_p$. Physically this would imply that the pth loop contribution Z_p to the partition function for the spinning string is finite for any $p \geqslant 2$. As a word of caution, we stress that this has nothing to do with the finiteness of the physical Green's functions which naively require the "insertion" of vertices in path integral.

Still one faces the problem of proving the convergence of the series $Z = \Sigma_p \exp(ap) Z_p$, which gives the full partition function for the superstring. Some hints may come from the study of the projective embeddings of the compactified moduli space for very high genera p.

We thank very much D. Amati, E. Arbarello, S. Fubini, L. Girardello, M.A. Namazie and R. Stora for useful discussions. We are also grateful to E. Gava, R. Iengo and R. Ramachandran for suggesting several improvements to the first draft of the present paper and for drawing our attention to ref. [14].

References

[1] A.M. Polyakov, Phys. Lett. B 103 (1981) 207;
O. Alvarez, Nucl. Phys. B 216 (1983) 125;
G. Moore and P. Nelson, Absence of non local anomalies in the Polyakov string, preprint HUTP-85/A057 (1985);
A. Cohen, G. Moore, P. Nelson and J. Polchinski, An off-shell propagator for string theory, preprint HUTP-85/A058 (1985);
J. Polchinski, Evaluation of the one-loop string path integral, preprint UTTG-13-85 (1985);
G. Gilbert, String theory path integral: genus two and higher, preprint UTTG-23-85 (1985);
M.A. Namazie and S. Rajeev, On multiloop computations in Polyakov's string theory, preprint CERN-TH.4327/85 (1985).

[2] E. Gava, R. Jengo, T. Jayaraman and R. Ramachandran, Multiloop divergencies in the closed bosonic string theory, preprint ICTP-90/85/E.P. (1985).

[3] E. D'Hoker and D.H. Phong, Multiloop amplitudes for the bosonic Polyakov string, preprint CU-TP (1985).

[4] P. Deligne and D. Mumford, Publ. IHES 36 (1969) 75.

[5] S. Wolpert, On the Weil–Petersson geometry of the moduli space of curves, Am. J. Math. (1985), to be published.

[6] D. Mumford, L'Ens. Math. 23 (1977) 39.

[7] O. Alvarez, Conformal anomalies and the index theorem, preprint UCB-PTH-85/39 (1985).

[8] D. Quillen, Determinants of Cauchy–Riemann operators over a Riemann surface, preprint IHES/P/84/37 (1984).

[9] J. Harris and D. Mumford, Invent. Math. 67 (1982) 23.

[10] J. Harer, Invent. Math. 72 (1983) 221; and The cohomology of the moduli space of curves, C.I.M.E. notes (Montecatini, 1985).

[11] E. Arbarello and M. Cornalba, The Picard groups of the moduli spaces of curves, preprint (1985).

[12] A.A. Belavin and V.K. Knizhnik, Algebraic geometry and the geometry of the quantum strings, Landau Institute preprint.

[13] A.M. Polyakov, Phys. Lett. B 103 (1981) 211.

[14] S.J. Avis and C.J. Isham, Nucl. Phys. B 156 (1979) 441.

[15] M.F. Atiyah, Ann. Scien. Ec. Norm. Sup. 4 (1971) 47.