Combinatorial and algebro-geometric cohomology classes on the moduli spaces of curves

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Introduction

Given a compact Riemann surface C, n points on it, and n positive real numbers ($n > \max(0, 2-2g)$), Strebel's theory of quadratic differentials [15] provides a canonical way of dissecting C into n poligons and assigning lengths to their sides. As Mumford first noticed, this can be used to give a combinatorial description of the moduli space $\mathcal{M}_{g,n}$ of n-pointed smooth curves of given genus g. If one looks at moduli spaces from this point of view, one can construct combinatorial cycles in them (cf. [7], for instance). It is then natural to ask how these may be related to the algebraic geometry of moduli space. It was first conjectured by Witten that the combinatorial cycles can be expressed in terms of Mumford-Morita-Miller classes. The first result in this direction is due to Penner [13]; we will comment on his work at the end of section 2. As we shall briefly explain now, and more extensively in section 3, our approach to the question has its origin in the papers [16], [17] and [7] by Witten and Kontsevich. The combinatorial cycles we are talking about will be denoted by the symbols $W_{m_*,n}$, where $m_* = (m_0, m_1, m_2, \dots)$ is an infinite sequence of non-negative integers, almost all zero, and n a positive integer. On the moduli space $\mathcal{M}_{g,n}$ live particular cohomology classes of degree two, denoted ψ_i, $i = 1, \dots, n$; by definition, ψ_i is the Chern class of the line bundle whose fiber at the point $[C; x_1, \dots, x_n] \in \mathcal{M}_{g,n}$ is the cotangent space to C at x_i. For the intersection numbers of the ψ_i along the $W_{m_*,n}$ one uses the notation

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{m_*} = \int_{W_{m_*,n}} \prod_{i=1}^{n} \psi_i^{d_i} \,.$$

These numbers are best organized as coefficients of an infinite series

$$F(t_0, t_1, \dots, s_0, s_1, \dots) = \sum \langle \tau_{d_1} \dots \tau_{d_n} \rangle_{m_*} t_{d_1} \dots t_{d_n} \prod s_i^{m_i} \,.$$

Kontsevich proves that $\exp F$ is an asymptotic expansion, as Λ^{-1} goes to zero, of the integral

$$\frac{\int_{\mathcal{H}_N} \exp\left(-\sqrt{-1}\sum_{j=0}^{\infty}\left(-\frac{1}{2}\right)^j s_j \frac{\operatorname{tr}(X^{2j+1})}{2j+1}\right) \exp\left(-\frac{1}{2}\operatorname{tr}(X^2\Lambda)\right) dX}{\int_{\mathcal{H}_N} \exp\left(-\frac{1}{2}\operatorname{tr}(X^2\Lambda)\right) dX} \,,$$

where $\mathcal{H}_N$ is the space of $N \times N$ hermitian matrices, dX is a $U(N)$-invariant measure, and Λ is a positive definite diagonal $N \times N$ matrix, linked to the t variables by the substitution $t_i = -(2i-1)!! \operatorname{tr}(\Lambda^{-2i-1})$. Using this result, Di Francesco, Itzykson and Zuber [3] showed

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that the derivatives of $\exp F$ with respect to the s variables, evaluated at $s_1 = 1$, $s_i = 0$ for $i \neq 1$, can be expressed as linear combinations of derivatives with respect to the t variables, evaluated at the same point. This had been previously conjectured, and proved in a few special cases, by Witten [17]. Our idea is that it is precisely this result which, when interpreted geometrically, should provide the sought-for link between combinatorial and algebro-geometric classes. In fact, this should be the case even on the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$. Why we believe this is explained in detail in section 3. We can show two things. First of all that our idea works in complex codimension 1 (and we are pretty sure it also works in codimension two). This is the content of section 4. Secondly that in all the cases when we have been able to make the Di Francesco, Itzykson and Zuber correpondence explicit (the first 11 cases, according to weight, as defined in section 3), this correspondence translates into identities of the type

$$\int_{W_{m_*,n}} \prod \psi_i^{d_i} = \int_{\overline{\mathcal{M}}_{g,n}} X_{m_*,n} \prod \psi_i^{d_i} ,$$

where the $X_{m_*,n}$ are explicit polynomials in the algebro-geometric classes, for any choice of $d_1, \ldots, d_n$. In other words, as linear functionals on a large subspace of the cohomology group of complementary degree, the classes $X_{m_*,n}$ behave as duals of the cycles $W_{m_*,n}$. The codimension one case is settled precisely by showing that this subspace is as large as it can possibly be, as soon as $n > 1$.

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1. Mumford classes

As is customary, we denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable n-pointed genus g algebraic curves and by $\mathcal{M}_{g,n}$ the moduli space of smooth ones. We shall consistently view these moduli spaces as orbifolds; likewise, morphisms between moduli spaces will always be morphism of orbifolds. We denote by Σ_k the subspace of $\overline{\mathcal{M}}_{g,n}$ consisting of all stable curves with exactly k singular points and their specializations; the codimension of Σ_k is k. This gives a stratification of $\overline{\mathcal{M}}_{g,n}$ whose codimension one stratum Σ_1 is nothing but the boundary $\partial\mathcal{M}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$. To describe the components of the Σ_k it is convenient to proceed as follows. Let C be a stable n-pointed genus g curve. Denote by Γ the dual graph of C. This is the directed graph whose vertices are the irreducible components of C and whose edges are the nodes of C, equipped with two additional data. If we let $V = V_\Gamma$ be the set of its vertices, the first is the map $p = p_\Gamma$ from V to the non-negative integers assigning to each vertex v the genus $p_v = p_{\Gamma,v}$ of the normalization of the corresponding component of C. The second is a partition of $\{1, \ldots, n\}$ indexed by V, that is, a map $P = P_\Gamma : V \to \mathcal{P}(\{1, \ldots, n\})$ such that $\{1, \ldots, n\}$ is the disjoint union of the $P(v)$, for $v \in V$. The partition in question is defined as follows: $P(v)$ is the set of all indices $i \in \{1, \ldots, n\}$ such that the i-th marked point belongs to the component corresponding to

v. We set $h_v = \#P(v)$, and denote by l_v the valency of v, that is, the cardinality of the set L_v of half-edges issuing from v. Clearly, the following hold

$$
\begin{aligned}
g &= \sum_{v\in V} p_v + h^1(\Gamma)\,, \\
n &= \sum_{v\in V} h_v\,.
\end{aligned}
\tag{1.1}
$$

Also, the stability of C translates into

$$
2p_v - 2 + h_v + l_v > 0 \tag{1.2}
$$

for every $v \in V$. It is clear that any Γ satisfying (1.1) and (1.2) arises as the dual graph of a stable n-pointed genus g curve. Now fix a Γ as above, and in addition choose, for each $v \in V$, an ordering of L_v. This determines a morphism

$$
\xi_\Gamma : \prod_{v\in V} \overline{\mathcal{M}}_{p_v, h_v+l_v} \to \overline{\mathcal{M}}_{g,n}\,, \tag{1.3}
$$

which is defined as follows. A point in the domain is the assignment of an (h_v+l_v)-pointed curve $(C_v; x_{v,1}, \dots, x_{v,h_v+l_v})$ of genus p_v for each $v \in V$. The image point under ξ_Γ is the stable n-pointed curve of genus g that one obtains by identifying two marked points x_{v,h_v+i} and x_{w,h_w+j} whenever the i-th element of L_v and the j-th element of L_w are the two halves of an edge; the marked points of this curve are the images of the points $x_{v,i}$ for $v \in V$ and $1 \le i \le h_v$, with the ordering induced by P.

The morphism ξ_Γ is a finite map onto an irreducible component of Σ_k, where k is the number of edges of Γ. As the reader may easily verify, this component does not depend on the choice of orderings of the L_v. This is a partial justification for omitting mention of these orderings in the notation for the map ξ_Γ. More importantly, our reason for introducing the morphisms ξ is to be able to describe (boundary) cohomology classes on $\overline{\mathcal{M}}_{g,n}$ as pushforwards of classes on $\prod_{v\in V} \overline{\mathcal{M}}_{p_v,h_v+l_v}$, and it will turn out that the classes we shall so obtain will always be independent of the choice of orderings of the L_v.

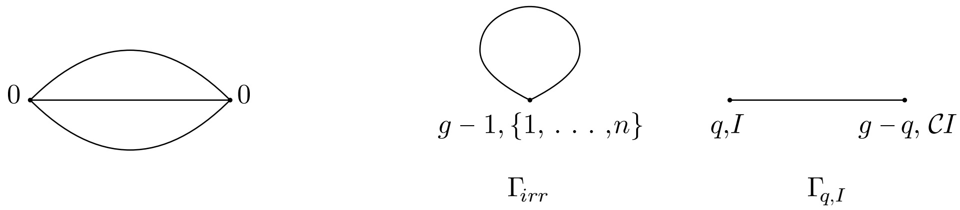
Let us denote by Δ_Γ the image of ξ_Γ. The degree of ξ_Γ, as a map to Δ_Γ, is precisely $\#\operatorname{Aut}(\Gamma)$. This has to be taken with a grain of salt, i. e., is true only if one regards ξ_Γ as a morphism of orbifolds. For example, given the graph Γ in Figure 1a), the corresponding map

$$
\xi : \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,3} \to \Delta_\Gamma \subset \overline{\mathcal{M}}_{2,0}
$$

is set-theoretically one-to-one, since both source and target consist of one point; on the other hand, while the unique point of $\overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,3}$ has no automorphisms, the automorphism group of its image in Δ_Γ has order twelve, as the automorphism group of Γ; thus the degree of ξ equals twelve in this case.

It is clear that the components of Σ_k are precisely the Δ_Γ for which Γ has k edges. Moreover Δ_Γ and $\Delta_{\Gamma'}$ are equal if and only if Γ and Γ' are isomorphic. We shall write δ_Γ to denote the fundamental class of Δ_Γ in the rational cohomology of $\overline{\mathcal{M}}_{g,n}$.

To exemplify, let's look at the codimension 1 case. The possible dual graphs are illustrated (with repetitions) in Figure 1b). The meaning of the labelling should be fairly clear; for instance, in the graph $\Gamma_{q,I}$ the function p assigns the integers q and $g-q$ to the two vertices, as indicated, while the partition $P_{\Gamma_{q,I}}$ is just $\{I, \mathcal{C}I\}$.



a) b)

Figure 1

To abbreviate, we shall write ξ_{irr}, $\xi_{q,I}$, δ_{irr}, $\delta_{q,I}$, instead of $\xi_{\Gamma_{irr}}$, $\xi_{\Gamma_{q,I}}$, ad so on. In addition, we set $\delta = \sum \delta_\Gamma$, where Γ runs through all *isomorphism classes* of codimension 1 dual graphs. For each of the dual graphs illustrated in Figure 1b), the underlying directed graph has an order two automorphism. This induces an automorphism of Γ_{irr} and an isomorphism between $\Gamma_{q,I}$ and $\Gamma_{g-q,\mathcal{C}I}$ (which, incidentally, is an automorphism precisely when $q = g/2$ and $n = 0$). It follows that

(1.4)
$$\delta = \frac{1}{2}\xi_{irr*}(1) + \frac{1}{2}\sum_{\substack{0\le q\le g\\ I\subset\{1,\ldots,n\}}} \xi_{q,I*}(1)\,.$$

Notice that the pushforwards in this formula are well defined since Poincaré duality with rational coefficients holds for all the moduli spaces involved. Next we look at the morphism "forgetting the last marked point"

$$\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}\,,$$

which we may also view as the universal curve $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. We denote by $\sigma_1, \ldots, \sigma_n$ the canonical sections of π_{n+1}, and by $D_1, \ldots, D_n$ the divisors in $\overline{\mathcal{M}}_{g,n+1}$ they define. A point of D_i corresponds to a stable $(n+1)$-pointed genus g curve obtained by attaching at one point a $\mathbb{P}^1$ to a curve C of genus g, with the i-th and $(n+1)$-th marked points on the $\mathbb{P}^1$, and the remaining ones on C. More exactly, D_i is just Δ_Γ, where Γ is the graph with one edge and two vertices $\{v, w\}$, $p_v = 0$, $p_w = g$, $P(v) = \{i, n+1\}$, and $P(w) = \{1, \ldots, i-1, \widehat{i}, i+1, \ldots, n\}$. We let $\omega_{\pi_{n+1}}$ be the relative dualizing sheaf. We set

$$\begin{aligned}
\psi_i &= c_1(\sigma_i^*(\omega_{\pi_{n+1}}))\,,\\
K &= c_1\left(\omega_{\pi_{n+1}}\left(\sum D_i\right)\right)\,,\\
\kappa_i &= \pi_{n+1*}(K^{i+1})\,.
\end{aligned}$$

Here, of course, Chern classes are taken to be in rational cohomology, and π_{n+1*} is well defined since, as we already observed, Poincaré duality holds for both the domain and the target of π_{n+1}. We shall call the classes κ_i *Mumford classes*; in fact, for $n = 0$, their analogues in the intersection ring were first introduced by Mumford in [12]. Notice that

$\kappa_0 = 2g - 2 + n$. Of course, a possible alternative generalization of Mumford's κ's to the case of n-pointed curves would be the classes

$$\tilde{\kappa}_i = \pi_{n+1*}(c_1(\omega_{\pi_{n+1}})^{i+1}) .$$

However, these are not as nicely behaved, from a functorial point of view, as the κ's, as we shall presently see. At any rate, the two are related by

$$\kappa_a = \tilde{\kappa}_a + \sum_{i=1}^{n} \psi_i^a . \tag{1.5}$$

The proof of this formula is based on the observation that, for any j, taking residues along D_j gives an isomorphism between the restriction of $\omega_{\pi_{n+1}}(\sum D_i)$ to D_j and $\mathcal{O}_{D_j}$. For brevity we set $\pi = \pi_{n+1}$ and $\tilde{K} = c_1(\omega_\pi)$. Then

$$\begin{aligned}
\kappa_a &= \pi_*((\tilde{K} + \sum D_i)^{a+1}) = \pi_*(\tilde{K}^{a+1}) + \sum_{l=0}^{a}\sum_{i=1}^{n} \binom{a+1}{l} \pi_*(\tilde{K}^l \cdot D_i^{a-l+1}) \\
&= \pi_*(\tilde{K}^{a+1}) + \sum_{l=0}^{a}\sum_{i=1}^{n} \binom{a+1}{l} \pi_* \left(\tilde{K}^l\big|_{D_i} \cdot (-\tilde{K})^{a-l}\big|_{D_i}\right) \\
&= \pi_*(\tilde{K}^{a+1}) + \sum_{l=0}^{a} (-1)^{a-l} \binom{a+1}{l} \cdot \sum_{i=1}^{n} \psi_i^a \\
&= \tilde{\kappa}_a + \sum_{i=1}^{n} \psi_i^a .
\end{aligned}$$

Some of the good properties enjoyed by the κ_i but not by the $\tilde{\kappa}_i$ are

$$\kappa_1 \text{ is ample} , \tag{1.6}$$

$$\pi_{n*}(\psi_1^{a_1} \dots \psi_{n-1}^{a_{n-1}} \psi_n^{a_n+1}) = \psi_1^{a_1} \dots \psi_{n-1}^{a_{n-1}} \kappa_{a_n} , \tag{1.7}$$

$$\xi_\Gamma^*(\kappa_a) = \sum_{v \in V} pr_v^*(\kappa_a) , \tag{1.8}$$

where ξ_Γ is as in (1.3) and pr_v stands for the projection from $\prod_{v\in V} \overline{\mathcal{M}}_{p_v, h_v+l_v}$ onto its v-th factor. That (1.6) holds is fairly well known; a short proof can be found in [2]. In view of (1.5), formula (1.7) can also be written in the form

$$\pi_{n*}(\psi_1^{a_1} \dots \psi_{n-1}^{a_{n-1}} \psi_n^{a_n+1}) = \psi_1^{a_1} \dots \psi_{n-1}^{a_{n-1}} \tilde{\kappa}_{a_n} + \sum_{j=1}^{n-1} \psi_1^{a_1} \dots \psi_j^{a_j+a_n} \dots \psi_{n-1}^{a_{n-1}} .$$

As such, with the obvious changes in notation, it is part of formula (1) in [8]; incidentally, the other part is the so-called string equation

$$\pi_{n*}(\psi_1^{a_1} \dots \psi_{n-1}^{a_{n-1}}) = \sum_{\{j : a_j > 0\}} \psi_1^{a_1} \dots \psi_j^{a_j-1} \dots \psi_{n-1}^{a_{n-1}} . \tag{1.9}$$

The proof of formula (1) of [8] is essentially given by Witten in section 2b) of [16]. We now come to (1.8). Consider the diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\ \eta\ } & \overline{\mathcal{C}}_{g,n} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi_{n+1}} \\ \prod_{v\in V} \overline{\mathcal{M}}_{p_v,h_v+l_v} & \xrightarrow{\ \xi_\Gamma\ } & \overline{\mathcal{M}}_{g,n} \end{array} ,$$

where

$$\mathcal{Y} = \coprod_{v\in V} \left(\overline{\mathcal{C}}_{p_v,h_v+l_v} \times \prod_{\substack{u\in V \\ u\neq v}} \overline{\mathcal{M}}_{p_u,h_u+l_u} \right)$$

and η is defined by glueing along sections in the manner prescribed by Γ. We also let π'_v and η_v be the restrictions of π' and η to $\overline{\mathcal{C}}_{p_v,h_v+l_v} \times \prod_{u\neq v} \overline{\mathcal{M}}_{p_u,h_u+l_u}$. The morphism π'_v is endowed with $h_v + l_v$ canonical sections $S_1, \ldots, S_{h_v+l_v}$. Now the point is that, by the very definition of dualizing sheaf, $\eta_v^*(K) = c_1(\omega_{\pi'_v}(\sum S_i))$. Property (1.8) follows.

Another useful property of the classes κ is that, on $\overline{\mathcal{M}}_{g,n}$, one has

(1.10) $$\kappa_a = \pi_n^*(\kappa_a) + \psi_n^a .$$

To prove this, look at the diagram

$$\begin{array}{ccc} \overline{\mathcal{C}}_{g,n} & \xrightarrow{\ \lambda\ } & \overline{\mathcal{C}}_{g,n-1} \\ {\scriptstyle \varphi_{n+1}}\downarrow & & \downarrow{\scriptstyle \varphi_n} \\ \overline{\mathcal{M}}_{g,n} & \xrightarrow{\ \pi_n\ } & \overline{\mathcal{M}}_{g,n-1} \end{array} ,$$

and denote by $D_1, \ldots, D_n$ (resp., $D'_1, \ldots, D'_{n-1}$) the canonical sections of φ_{n+1} (resp., φ_n). We claim that

(1.11) $$\lambda^* \left(\omega_{\varphi_n} \left(\sum D'_i \right) \right) = \omega_{\varphi_{n+1}} \left(\sum_{i<n} D_i \right) .$$

In fact, there is a natural homomorphism from $\lambda^*(\omega_{\varphi_n}(\sum D'_i))$ to $\omega_{\varphi_{n+1}}(\sum D_i)$; we wish to see that this is an isomorphism onto $\omega_{\varphi_{n+1}}(\sum_{i<n} D_i)$. The question is local in the orbifold sense. It is therefore sufficient to prove (1.11) when universal curves over moduli are replaced by Kuranishi families. To keep things simple we shall use the same notation in this new setup. Therefore, from now on, $\varphi_n = \pi_n : \mathcal{C} \to B$ will stand for a Kuranishi family of stable $(n-1)$-pointed curves, and $D'_1, \ldots, D'_{n-1}$ for its canonical sections. A suitable blow-up $\mathcal{C}'$ of $\mathcal{C} \times_B \mathcal{C}$ provides a Kuranishi family $\varphi_{n+1} : \mathcal{C}' \to \mathcal{C}$ of stable n-pointed curves, whose canonical sections we shall denote by $D_1, \ldots, D_n$. The diagram we shall look at is

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\ \lambda\ } & \mathcal{C} \\ {\scriptstyle \varphi_{n+1}}\downarrow & & \downarrow{\scriptstyle \varphi_n} \\ \mathcal{C} & \xrightarrow{\ \pi_n\ } & B \end{array} .$$

Observe that, in order to prove that the natural homomorphism from $\lambda^*(\omega_{\varphi_n}(\sum D_i'))$ to $\omega_{\varphi_{n+1}}(\sum D_i)$ is an isomorphism onto $\omega_{\varphi_{n+1}}(\sum_{i<n} D_i)$, it suffices to do so fiber by fiber. Now a fiber of φ_n is nothing but an $(n-1)$-pointed curve $(C; x_1, \dots, x_{n-1})$. The inverse image via λ of this fiber, which we denote by X, can be described as follows. Set $Y = C \times C$, $D_i'' = C \times \{x_i\}$ for $i < n$, and let D_n'' be the diagonal. Then X is the blow-up of Y at the points where D_n'' hits a node or one of the D_i'', for $i < n$. We denote the exceptional curves arising from points of this second type by $E_1, \dots, E_{n-1}$. We also observe that the proper transform of each D_i'' is just the intersection of D_i with X. But then

$$\lambda^*(\omega_{\varphi_n}(\textstyle\sum D_i'))\big|_X = \lambda^*(\omega_{\varphi_n})(\sum_{i<n} D_i)\big|_X(\sum E_i) \, .$$

On the other hand it is easy to prove (cf. [16]) that $\lambda^*(\omega_{\varphi_n}) = \omega_{\varphi_{n+1}}(-\Delta)$, where Δ is the divisor in $\mathcal{C}'$ defined as follows. Look at fibers of φ_{n+1} containing a smooth rational component meeting the rest of the fiber at only one point; we will refer to such a component as a "rational tail". Then Δ is the divisor swept out by the rational tails containing only two marked points, one of which is the n-th point. Notice that the divisor cut out by Δ on X is $\sum E_i$. Coupled with the formula above, this implies that

$$\lambda^*(\omega_{\varphi_n}(\textstyle\sum D_i'))\big|_X = \omega_{\varphi_{n+1}}(\sum_{i<n} D_i)\big|_X \, ,$$

which is what we had to show. To prove (1.10) we now argue exactly as in the proof of (1.5), but applying ${\varphi_{n+1}}_*$ to the $(a+1)$-st selfintersection of both sides of (1.11) instead of using the definition of κ_a.

Using (1.10) and the push-pull formula, we can apply formula (1.7) repeatedly to obtain

$$\begin{aligned}(\pi_{n-1}\pi_n)_*(\psi_1^{a_1}\dots\psi_{n-2}^{a_{n-2}}\psi_{n-1}^{a_{n-1}+1}\psi_n^{a_n+1}) &= {\pi_{n-1}}_*(\psi_1^{a_1}\dots\psi_{n-2}^{a_{n-2}}\psi_{n-1}^{a_{n-1}+1}\kappa_{a_n}) \\ &= {\pi_{n-1}}_*(\psi_1^{a_1}\dots\psi_{n-1}^{a_{n-1}+1}({\pi_{n-1}}^*(\kappa_{a_n}) + \psi_{n-1}^{a_n})) \\ &= \psi_1^{a_1}\dots\psi_{n-2}^{a_{n-2}}(\kappa_{a_{n-1}}\kappa_{a_n} + \kappa_{a_{n-1}+a_n}) \, ,\end{aligned}$$

$$\begin{aligned}(\pi_{n-2}\pi_{n-1}\pi_n)_*(\psi_1^{a_1}\dots\psi_{n-3}^{a_{n-3}}\psi_{n-2}^{a_{n-2}+1}\psi_{n-1}^{a_{n-1}+1}\psi_n^{a_n+1}) = (\psi_1^{a_1}\dots\psi_{n-3}^{a_{n-3}})\times \\ \times(\kappa_{a_{n-2}}\kappa_{a_{n-1}}\kappa_{a_n} + \kappa_{a_{n-2}}\kappa_{a_{n-1}+a_n} + \kappa_{a_{n-1}}\kappa_{a_{n-2}+a_n} + \kappa_{a_n}\kappa_{a_{n-2}+a_{n-1}} + 2\kappa_{a_{n-2}+a_{n-1}+a_n}) \, ,\end{aligned}$$

and so on. In general, one finds formulas

$$(\pi_{k+1}\dots\pi_n)_*(\psi_1^{a_1}\dots\psi_k^{a_k}\psi_{k+1}^{a_{k+1}+1}\dots\psi_n^{a_n+1}) = \psi_1^{a_1}\dots\psi_k^{a_k} R_{a_{k+1}\dots a_n} \, , \tag{1.12}$$

where $R_{a_{k+1}\dots a_n}$ is a polynomial in the Mumford classes. A compact expression for $R_{b_1\dots b_l}$, which we learned from Carel Faber, is

$$R_{b_1\dots b_l} = \sum_{\sigma\in\mathcal{S}_l} \kappa_\sigma \, , \tag{1.13}$$

where κ_σ is defined as follows. Write the permutation σ as a product of $\nu(\sigma)$ disjoint cycles, including 1-cycles: $\sigma = \alpha_1 \dots \alpha_{\nu(\sigma)}$, where we think of $\mathcal{S}_l$ as acting on the l-tuple $(b_1, \dots, b_l)$. Denote by $|\alpha|$ the sum of the elements of a cycle α. Then

$$\kappa_\sigma = \kappa_{|\alpha_1|}\kappa_{|\alpha_2|}\dots\kappa_{|\alpha_{\nu(\sigma)}|} \, .$$

Formula (1.12), together with the string equation, expresses the remarkable fact that the intersection theory of the classes ψ_i and κ_i on a fixed $\overline{\mathcal{M}}_{g,n}$ is completely determined by the intersection theory of the ψ_i alone on all the $\overline{\mathcal{M}}_{g,\nu}$ with $\nu \geq n$, and conversely. A special case of this is Witten's remark [16] that knowing the intersection numbers of the κ's on $\overline{\mathcal{M}}_{g,0}$ is equivalent to knowing the intersection numbers of the ψ's on all the $\overline{\mathcal{M}}_{g,n}$. Using the "correct" classes κ_i makes all of this particularly transparent.

A final remark has to do with Wolpert's formula [18] stating that, on $\overline{\mathcal{M}}_{g,0}$,

$$\kappa_1 = \frac{1}{2\pi^2}[WP] \,,$$

where WP is the Weil-Petersson Kähler form. It may be observed that this carries over with no formal changes to $\overline{\mathcal{M}}_{g,n}$, for any n. To prove the formula (including the case considered by Wolpert), one may proceed as follows. The "restriction phenomenon" (page 502 of Wolpert's paper) amounts to the statement that the analogue of (1.8) above holds for the class of the Weil-Petersson Kähler form. Arguing by induction on the genus and the number of marked points, we may then assume that the difference between κ_1 and $\frac{1}{2\pi^2}[WP]$ restricts to zero on any component of the boundary of $\overline{\mathcal{M}}_{g,n}$. One then proves a general lemma to the effect that a degree two cohomology class with this property actually vanishes on $\overline{\mathcal{M}}_{g,n}$, except in the cases when $\overline{\mathcal{M}}_{g,n}$ is one-dimensional; these are the initial cases of the induction and are dealt with by direct computation. The general lemma is proved, although not formally stated, in [1], for $n = 0$; similar ideas can be used to deal with the case when $n > 0$.

2. Combinatorial classes

Following Kontsevich [7], whose notation we shall adhere to throughout this section, we consider connected ribbon graphs with metric and with valency of each vertex greater than or equal to three such that the corresponding noncompact surface has genus g and n punctures, numbered by $\{1, \ldots, n\}$. We let $\mathcal{M}_{g,n}^{comb}$ be the space of equivalence classes of such graphs, endowed with its natural orbifold structure. Recall that the map

$$\mathcal{M}_{g,n} \times \mathbb{R}_+^n \to \mathcal{M}_{g,n}^{comb}$$

which associates to a smooth n-punctured curve and an n-tuple of positive real numbers the critical graph of the corresponding canonical Strebel quadratic differential is a homeomorphism or orbifolds. As Kontsevich has indicated, the above map extends to a map of "orbispaces"

$$\overline{\mathcal{M}}_{g,n} \times \mathbb{R}_+^n \to \overline{\mathcal{M}}_{g,n}^{comb} \,,$$

where $\overline{\mathcal{M}}_{g,n}^{comb}$ is a suitable partial compactification of $\mathcal{M}_{g,n}^{comb}$. This map, however, is no longer one-to-one, as a certain amount of contraction takes place at the boundary. More specifically, $\overline{\mathcal{M}}_{g,n}^{comb}$ is isomorphic to $\overline{\mathcal{M}}'_{g,n} \times \mathbb{R}_+^n$, where $\overline{\mathcal{M}}'_{g,n}$ equals $\overline{\mathcal{M}}_{g,n}$ modulo the *closure* of the following equivalence relation. Two stable n-pointed curves are considered equivalent if there is a homeomorphism of pointed curves between them which is complex analytic on all components containing at least one marked point. We let

$$\alpha : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}'_{g,n}$$

be the natural projection.

Now fix a sequence $m_* = (m_0, m_1, \dots)$ of non-negative integers almost all of which are zero. We denote by $\mathcal{M}_{m_*,n}$ the space of equivalence classes of connected numbered ribbon graphs with metric having n boundary components, m_i vertices of valency $2i+1$ for each i, and no vertices of even valency. The dimension of $\mathcal{M}_{m_*,n}$ is nothing but the number of edges of such a graph, and hence

$$\dim_{\mathbb{R}} \mathcal{M}_{m_*,n} = \frac{1}{2} \sum_i m_i(2i+1) \,.$$

When $m_0 = 0$, the space $\mathcal{M}_{m_*,n}$ naturally lies inside $\mathcal{M}^{comb}_{g,n}$, where g is given by the formula

$$2g - 2 + n = \frac{1}{2} \sum_i m_i(2i-1) \,.$$

More generally, the Strebel construction always gives a map from $\mathcal{M}_{m_*,n}$ to $\mathcal{M}_{g,n} \times \mathbb{R}^n_+$, even for $m_0 \neq 0$, so that in particular the classes ψ_i can be pulled back to $\mathcal{M}_{m_*,n}$. In all cases we have

$$\dim_{\mathbb{R}} \mathcal{M}_{m_*,n} = 6g - 6 + 2n - \sum_i m_i(i-1) = \dim_{\mathbb{R}} \mathcal{M}_{g,n} - \sum_i m_i(i-1) \,.$$

On each component of $\mathcal{M}_{m_*,n}$ one can put a natural orientation, as explained on page 11 of [7]. When $m_0 = 0$ it can be seen that, with this orientation, $\mathcal{M}_{m_*,n}$ is a cycle with non-compact support in $\overline{\mathcal{M}}^{comb}_{g,n}$. As such, it defines a class

$$[\mathcal{M}_{m_*,n}] \in H^{non-compact}_{d+n-2k}(\overline{\mathcal{M}}^{comb}_{g,n}, \mathbb{Q}) \,,$$

where $d = 6g - 6 + 2n$ and $k = \sum_i m_i(i-1)$, hence an element of the dual of

$$H^{d+n-2k}_c(\overline{\mathcal{M}}^{comb}_{g,n}, \mathbb{Q}) = H^{d-2k}(\overline{\mathcal{M}}'_{g,n}, \mathbb{Q}) \,.$$

This can be also viewed as an element $W_{m_*,n} \in H_{d-2k}(\overline{\mathcal{M}}'_{g,n}, \mathbb{Q})$

It has been conjectured by Kontsevich [7] (and previously, in a somewhat more restricted form, by Witten) that the classes $W_{m_*,n}$ "can be expressed in terms of the Mumford-Miller classes". We next give a possible interpretation of this sentence and a more precise form of the conjecture. The statement is made a bit clumsy by the fact that it is not a priori clear whether the classes $W_{m_*,n}$ lift to classes in $H_*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, as would happen, for instance, if Poincaré duality held on $\overline{\mathcal{M}}'_{g,n}$. What is certainly true is that, given a cohomology class $x \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$,

$$\varphi \mapsto \int_{\overline{\mathcal{M}}_{g,n}} x \cup \alpha^*(\varphi)$$

defines a linear functional on $H^{d-2k}(\overline{\mathcal{M}}'_{g,n},\mathbb{Q})$. What may be conjectured is that this functional equals $W_{m_*,n}$ for an x of the form

$$x = P_{m_*,n}(\kappa_1,\kappa_2,\dots) + \beta_{m_*,n}\,,$$

where $P_{m_*,n}$ is a weighted-homogeneous polynomial in the Mumford classes and $\beta_{m_*,n}$ is supported on the boundary of moduli. In what follows we shall often take the liberty of writing $W_{m_*,n} \equiv x$ to express this, when no confusion seems likely. One may be more precise about $P_{m_*,n}$ and $\beta_{m_*,n}$. Define the *level* of a monomial $\prod_{a\geq 1}\kappa_a^{h_a}$ in the Mumford classes to be $\sum_a h_a$. Then $P_{m_*,n}$ should be of the form

$$(2.1)\qquad \prod_{i=2}^{\infty}\frac{(2^i(2i-1)!!)^{m_i}}{m_i!}\prod_{i\geq 2}\kappa_{i-1}^{m_i} + \text{ a linear combination of monomials of lower level}\,.$$

As for $\beta_{m_*,n}$ it should be a linear combination of classes of the form $\xi_{\Gamma *}(y)$, where y is a monomial in the Mumford classes and in the $pr_v^*(\psi_i)$, for $i > h_v$, where of course we have freely used the notation established in section 1. As a special case, one should have

$$(2.2)\qquad W_{(0,m_1,0,\dots,0,m_j=1,0,\dots),n} \equiv 2^j(2j-1)!!\kappa_{j-1} + \text{ boundary terms}\,.$$

In the next section we shall give evidence for these conjectures. In section 4 we shall prove (2.2) in the codimension 1 case for $n > 1$; more exactly, we shall show that

$$(2.3)\qquad W_{(0,m_1,1,\dots),n} \equiv 12\kappa_1 - \delta\,,$$

where δ is the usual class of the boundary. We shall also see that, for $n > 1$, this formula includes as a special case the main result of Penner [13], with the following caveat. In our notation, what Penner claims is that $W_{(0,m_1,1,\dots),n} = 6\tilde{\kappa}_1$ on the open moduli space $\mathcal{M}_{g,n}$, while the correct formula is $W_{(0,m_1,1,\dots),n} = 12\kappa_1$. It should be said that Penner's argument, which, by the way, is entirely different from ours, is completely correct, except for two minor mistakes in the interpretation of what has been proved. The first mistake is that, as we have noticed in section 1, the class of the Weil-Petersson Kähler form is κ_1 and not $\tilde{\kappa}_1$. The second mistake actually occurs in Theorem A.2 of [14], where the explicit expression of the Weil-Petersson Kähler form should be divided by two. In fact, if one looks at how this is gotten, one sees that it is computed as the pull-back of the Weil-Petersson Kähler form on $\mathcal{M}_{2g+n-1,0}$ via the doubling map which associates to a genus g smooth n-pointed curve C the curve obtained by attaching at the punctures two identical copies of C. But now, since one is doubling, one must also divide by two, as the resulting curve carries the extra automorphism which exchanges the two components. An advantage of our method over Penner's is perhaps that, in addition to giving a certain amount of control over the boundary, it is not special to the codimension one case but provides, at least in principle, a mechanism for dealing with classes of higher codimension.

3. Geometrical consequences of a result of Di Francesco, Itzykson and Zuber

Following Witten [16] and Kontsevich [7], given a sequence of non-negative integers $\underline{d} = (d_1, \dots, d_n)$ and an infinite sequence $m_* = (m_0, m_1, m_2, \dots)$ of non-negative integers, almost all zero, we set

$$\langle \tau_{\underline{d}} \rangle_{m_*} = \langle \tau_{d_1} \dots \tau_{d_n} \rangle_{m_*} = \int_{\mathcal{M}_{m_*,n}} \prod_{i=1}^{n} \psi_i^{d_i} \times [\mathbb{R}_+^n] \,,$$

where $[\mathbb{R}_+^n]$ stands for the fundamental class with compact support of $\mathbb{R}_+^n$. This integral is zero unless $\sum_i d_i = \frac{1}{2} \dim \mathcal{M}_{m_*,n} = \frac{1}{4} \sum_i m_i(2i+1)$. Notice that, when $m_0 = 0$, one can also write

$$\langle \tau_{\underline{d}} \rangle_{m_*} = \int_{W_{m_*,n}} \prod_{i=1}^{n} \psi_i^{d_i} \,,$$

and again this is zero unless

$$\sum_i d_i = \frac{1}{2} \dim W_{m_*,n} = 3g - 3 + n - \sum_i (i-1) m_i \,,$$

where $2g - 2 + n = (1/2) \sum_i m_i(2i-1)$. We also set

$$\langle \tau_{\underline{d}} \rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} \,.$$

It is clear that $\langle \tau_{\underline{d}} \rangle_{g,n} = \langle \tau_{\underline{d}} \rangle_{m_*}$ for $m_* = (0, 4g-4+2n, 0, 0, \dots)$. The symbol $\langle \tau_{\underline{d}} \rangle$, with no subscripts, stands for $\langle \tau_{\underline{d}} \rangle_{g,n}$ when the number g *defined* by $3g - 3 + n = \sum d_i$ is a non-negative integer, and is set to zero otherwise. Sometimes the abbreviated notation $\tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \dots$ is used in place of

$$\underbrace{\tau_0 \dots \tau_0}_{n_0 \text{ times}} \underbrace{\tau_1 \dots \tau_1}_{n_1 \text{ times}} \underbrace{\tau_2 \dots \tau_2}_{n_2 \text{ times}} \dots$$

One then considers the formal power series

$$F(t_*, s_*) = \sum_{n_*, m_*} \langle \tau_{\underline{d}} \rangle_{m_*} \frac{t_*^{n_*}}{n_*!} s_*^{m_*} \,,$$
$$Z(t_*, s_*) = \exp(F(t_*, s_*)) \,,$$

where the following notational conventions are adopted. First of all

$$t_* = (t_0, t_1, t_2, \dots) \,, \qquad s_* = (s_0, s_1, s_2, \dots) \,,$$

are infinite sequences of indeterminates, and

$$m_* = (m_0, m_1, m_2, \dots) \,, \qquad n_* = (n_0, n_1, n_2, \dots) \,,$$

are infinite sequences of non-negative integers, almost all zero. We have also set

$$n_*! = \prod_{i=0}^{\infty} n_i! \,, \qquad t_*^{n_*} = \prod_{i=0}^{\infty} t_i^{n_i} \,,$$

and similarly for $s_*^{m_*}$. Finally, if $n = \sum_i n_i$, the sequence of non-negative integers $\underline{d} = (d_1, \dots, d_n)$ is determined (up to their order, which is irrelevant) by the requirement that n_i equal the number of j's such that $i = d_j$; in other words, one could also have written $\langle \prod_{i=0}^{\infty} \tau_i^{n_i} \rangle_{m_*}$ instead of $\langle \tau_{\underline{d}} \rangle_{m_*}$.

Now let $\mathcal{H}_N$ be the space of $N \times N$ hermitian matrices, and consider on it the $U(N)$-invariant measure

$$dX = \prod_{1 \le i \le N} dX_{ii} \prod_{1 \le i < j \le N} d\,\mathrm{Re}\, X_{ij}\, d\,\mathrm{Im}\, X_{ij} \,.$$

For any positive definite $N \times N$ diagonal matrix Λ we also consider the measure

$$d\mu_\Lambda = c_{\Lambda,N} \exp\left(-\frac{1}{2} \operatorname{tr}(X^2 \Lambda) \right) dX \,,$$

where $c_{\Lambda,N}$ is the constant such that $\int d\mu_\Lambda = 1$. It has been shown by Kontsevich that, for any fixed s_*, and with the substitution

$$t_i = -(2i-1)!! \operatorname{tr}(\Lambda^{-2i-1}) \,,$$

the series $Z(t_*, s_*)$ is an asymptotic expansion of the integral

$$\int_{\mathcal{H}_N} \exp\left(-\sqrt{-1} \sum_{j=0}^{\infty} \left(-\frac{1}{2} \right)^j s_j \frac{\operatorname{tr}(X^{2j+1})}{2j+1} \right) d\mu_\Lambda \tag{3.1}$$

as Λ^{-1} goes to zero (notice the minus sign in front of the argument of the exponential, which is missing in the formula given in [7]). To simplify notations, we set

$$\langle f \rangle_\Lambda = \int_{\mathcal{H}_N} f d\mu_\Lambda \,,$$

$$\langle\langle f \rangle\rangle_\Lambda = \int_{\mathcal{H}_N} f \exp\left(\frac{\sqrt{-1} \operatorname{tr} X^3}{6} \right) d\mu_\Lambda \,.$$

Let us now fix non-negative integers $m_2, m_3, \dots$, almost all equal to zero, and set $\hat{s}_* = (0, 1, 0, 0, 0 \dots)$. It follows from the definitions that

$$\prod_{i \ge 2} \frac{1}{m_i!} \left(\frac{\partial}{\partial s_i} \right)^{m_i} F(t_*, s_*) \,\Big|_{s_* = \hat{s}_*} = \sum_{n_*, m_1} \langle \tau_{\underline{d}} \rangle_{(0, m_1, m_2, m_3, \dots)} \frac{t_*^{n_*}}{n_*!} \,. \tag{3.2}$$

In other words, the coefficients of the above derivative of F are just the intersection numbers

$$\int_{W_{m_*,n}} \psi_1^{d_1} \dots \psi_n^{d_n} \,, \tag{3.3}$$

where we have written m_* for $(0, m_1, m_2, m_3, \dots)$. If the conjecture formulated in section 2 holds true, it should be possible to write these intersection numbers under the form

$$\int_{\overline{\mathcal{M}}_{g,n}} (P_{m_*,n}(\kappa_1, \kappa_2, \dots) + \beta_{m_*,n})\psi_1^{d_1} \dots \psi_n^{d_n} , \tag{3.4}$$

where $P_{m_*,n}$ and $\beta_{m_*,n}$ are as in that section. We contend that this result should be implicitly contained in a theorem, conjectured by Witten, and proved by Di Francesco, Itzykson, and Zuber [3]. To explain this, the first step is to observe that, by differentiating (3.1), we obtain asymptotic expansions

$$\left\langle\left\langle \prod_i \left(-\sqrt{-1} \left(\frac{-1}{2}\right)^i \frac{\operatorname{tr} X^{2i+1}}{2i+1} \right)^{\nu_i} \right\rangle\right\rangle_\Lambda \sim \prod_i \left(\frac{\partial}{\partial s_i}\right)^{\nu_i} Z(t_*, s_*) \,\big|_{s_*=\hat{s}_*} ,$$

for any sequence $\nu_* = (\nu_0, \nu_1, \dots)$ of non-negative integers such that $\nu_i = 0$ for large enough i. Now the theorem of Di Francesco, Itzykson, and Zuber (henceforth referred to as the DFIZ theorem) states that, given any polynomial Q in the *odd* traces of X, there exists a differential polynomial $R_Q = R_Q\left(\frac{\partial}{\partial t_0}, \frac{\partial}{\partial t_1}, \dots\right)$ such that

$$\langle\langle Q \rangle\rangle_\Lambda = R_Q Z(t_*) ,$$

where $Z(t_*)$ stands for $Z(t_*, \hat{s}_*)$. Putting this together with the previous remark shows that

$$\prod_i \left(\frac{\partial}{\partial s_i}\right)^{\nu_i} Z(t_*, s_*) \,\big|_{s_*=\hat{s}_*} = U_{\nu_*}\left(\frac{\partial}{\partial t_0}, \frac{\partial}{\partial t_1}, \dots\right) Z(t_*) ,$$

where U_{ν_*} is a polynomial. In terms of F, this amounts to saying that

$$\prod_i \left(\frac{\partial}{\partial s_i}\right)^{\nu_i} F(t_*, s_*) \,\big|_{s_*=\hat{s}_*} = \widetilde{U}_{\nu_*} ,$$

where $\widetilde{U}_{\nu_*}$ is a polynomial in the partial derivatives of $F(t_*, s_*)$ with respect to the t variables, evaluated at $s_* = \hat{s}_*$. The expression that Di Francesco, Itzykson, and Zuber give for U_{ν_*}, and hence implicitly for $\widetilde{U}_{\nu_*}$, is quite complicated. However, if we define the *weight* of a partial derivative

$$\prod_i \left(\frac{\partial}{\partial t_i}\right)^{\nu_i} F(t_*, s_*)$$

to be $\sum(2i+1)\nu_i$, and the weight of a product of partial derivatives to be the sum of the weights of its factors, then what can be said is that

$$\begin{aligned} &\prod_i \left(\frac{\partial}{\partial s_i}\right)^{\nu_i} F(t_*, s_*) \,\big|_{s_*=\hat{s}_*} = \\ &\qquad \prod_i \left(2^i(2i-1)!!\frac{\partial}{\partial t_i}\right)^{\nu_i} F(t_*) + \text{ terms of lower weight} , \end{aligned} \tag{3.5}$$

where $F(t_*)$ is defined to be equal to $F(t_*, \hat{s}_*)$. In addition, only terms whose weight is congruent to $\sum(2i+1)\nu_i$ modulo 3 appear in (3.5).

In the case when $\nu_* = (0, 0, m_2, m_3, \dots)$ we have already explained how the left-hand side of (3.5) is linked to the intersection theory of products of classes ψ_i with the $W_{m_*,n}$; it remains to explain the geometric significance of the right-hand side. Consider the series

$$\prod \left(\frac{\partial}{\partial t_i}\right)^{\mu_i} F(t_*) = \sum_{n_*} a_{\underline{d}} \frac{t_*^{n_*}}{n_*!}\,.$$

Then it is easy to show that

$$a_{\underline{d}} = \left\langle \prod \tau_i^{\mu_i} \tau_{\underline{d}} \right\rangle\,.$$

In a certain sense one can say that differentiating $F(t_*)$ with respect to the t_i variable μ_i times, for $i = 0, 1, \dots$, corresponds to the insertion of $\prod \tau_i^{\mu_i}$ in the coefficients of $F(t_*)$. Now fix a positive integer n, $\underline{d} = (d_1, \dots, d_n)$, and $m_* = (0, m_1, m_2, \dots)$. Then $W_{m_*,n}$ is a cycle in $\overline{\mathcal{M}}'_{g,n}$, for a well-determined g. Setting $\nu_* = (0, 0, m_2, m_3, \dots)$ and equating coefficients in (3.5) one finds that $\langle \tau_{\underline{d}} \rangle_{m_*}$ is a linear combination, with rational coefficients, of terms of the form

$$\left\langle \prod_i \tau_i^{\lambda_{i,1}} \tau_{\underline{d}_{I_1}} \right\rangle \cdots \left\langle \prod_i \tau_i^{\lambda_{i,k}} \tau_{\underline{d}_{I_k}} \right\rangle\,,$$

where $\{I_1, \dots, I_k\}$ is a partition of $\{1, \dots, n\}$ and, for any subset I of $\{1, \dots, n\}$, we set $\langle \tau_{\underline{d}_I} \rangle = \langle \prod_{i \in I} \tau_{d_i} \rangle$. Moreover, if we set $\mu_i = \sum_j \lambda_{i,j}$, then $\sum(2i+1)\mu_i$ is not greater than $\sum_{i \geq 2}(2i+1)m_i$, and congruent to it modulo 3. For instance, the term coming from the highest weight part of the right-hand side of (3.5) is simply

$$\prod_{i \geq 2} (2^i(2i-1)!!)^{m_i} \left\langle \prod \tau_i^{m_i} \tau_{\underline{d}} \right\rangle_{g, n+\sum m_i} =$$

$$\prod_{i \geq 2} (2^i(2i-1)!!)^{m_i} \int_{\overline{\mathcal{M}}_{g,n}} \left(\prod_{a \geq 1} \kappa_a^{m_{a+1}} + \dots \right) \prod \psi_i^{d_i}\,,$$

where we have used the formulas for the pushforwards of products of classes ψ_i given in section 1. The lower weight terms are considerably more messy. In particular there is, a priori, no reason why it should be possible to write each one of them under the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \alpha \prod \psi_i^{d_i}\,,$$

where α is a suitable cohomology class. That this indeed happens, at least in all the cases we have been able to compute, depends on some remarkable cancellations, as we shall presently see. At any rate, we have that

$$\int_{W_{m_*,n}} \psi_1^{d_1} \dots \psi_n^{d_n} = \left[\int_{\overline{\mathcal{M}}_{g,n}} \left(\prod_{i \geq 2} \frac{1}{m_i!} (2^i(2i-1)!! \kappa_{i-1})^{m_i} \right) \psi_1^{d_1} \dots \psi_n^{d_n} \right] + \cdots\,,$$

which can be viewed as a first step in writing the intersection number (3.3) in the form (3.4).

To illustrate the procedure we just described we shall work out three examples. The first one deals with the cycle $W_{(0,m_1,1,0,0,0\dots),n}$. This is the only codimension one cycle among the $W_{m_*,n}$ and corresponds to ribbon graphs having at least one five-valent vertex. In this case (3.2) reads

$$\frac{\partial}{\partial s_2}F(t_*,s_*)\ \big|_{s_*=\hat{s}_*} = \sum_{n_*,m_1}\langle\tau_{\underline{d}}\rangle_{(0,m_1,1,0,0,\dots)}\frac{t_*^{n_*}}{n_*!}\,.$$

On the other hand the DFIZ theorem tells us, in this case, that

$$\frac{\partial}{\partial s_2}Z(t_*,s_*)\ \big|_{s_*=\hat{s}_*} = \left(12\frac{\partial}{\partial t_2}-\frac{1}{2}\frac{\partial^2}{\partial t_0^2}\right)Z(t_*) \tag{3.6}$$

As $Z=\exp F$, we then get, upon dividing by Z,

$$\frac{\partial}{\partial s_2}F(t_*,s_*)\ \big|_{s_*=\hat{s}_*} = 12\frac{\partial F}{\partial t_2}(t_*)-\frac{1}{2}\frac{\partial^2 F}{\partial t_0^2}(t_*)-\frac{1}{2}\left(\frac{\partial F}{\partial t_0}(t_*)\right)^2\,.$$

Comparing coefficients, this amounts to

$$\begin{aligned}\langle\tau_{\underline{d}}\rangle_{(0,m_1,1,0,0,\dots)} &= 12\langle\tau_2\tau_{\underline{d}}\rangle_{g,n+1}-\frac{1}{2}\langle\tau_0\tau_0\tau_{\underline{d}}\rangle_{g-1,n+2}\\ &\quad -\frac{1}{2}\sum_{I\subset\{1,\dots,n\}}\langle\tau_0\tau_{\underline{d}_I}\rangle_{p,h+1}\langle\tau_0\tau_{\underline{d}_{\complement I}}\rangle_{g-p,n-h+1}\,,\end{aligned}$$

where

$$\tau_{\underline{d}_I}=\prod_{i\in I}\tau_{d_i}\,,$$

g is given by

$$\sum_{i=1}^n d_i = 3g-3+n-\frac{1}{2}\operatorname{codim}(W_{(0,m_1,1,0,0,\dots),n}) = 3g-3+n-1\,,$$

m_1 by

$$m_1 = 4g-7+2n\,,$$

and h and p by

$$h=\#I\,,\qquad 3p-3+h+1=\sum_{i\in I}d_i\,.$$

Using (1.7), this can be rewritten as

$$\begin{aligned}\int_{W_{(0,m_1,1,0,0,\dots),n}}\psi_1^{d_1}\dots\psi_n^{d_n} &= 12\int_{\overline{\mathcal{M}}_{g,n+1}}\psi_1^{d_1}\dots\psi_n^{d_n}\psi_{n+1}^2-\frac{1}{2}\int_{\overline{\mathcal{M}}_{g-1,n+2}}\xi_{irr}^*(\psi_1^{d_1}\dots\psi_n^{d_n})\\ &\quad -\frac{1}{2}\sum_{\substack{0\le p\le g\\ I\subset\{1,\dots,n\}}}\int_{\overline{\mathcal{M}}_{p,h+1}\times\overline{\mathcal{M}}_{g-p,n-h+1}}\xi_{p,I}^*(\psi_1^{d_1}\dots\psi_n^{d_n})\\ &= 12\int_{\overline{\mathcal{M}}_{g,n}}\kappa_1\psi_1^{d_1}\dots\psi_n^{d_n}-\frac{1}{2}\int_{\overline{\mathcal{M}}_{g,n}}\xi_{irr*}(1)\psi_1^{d_1}\dots\psi_n^{d_n}\\ &\quad -\frac{1}{2}\sum_{\substack{0\le p\le g\\ I\subset\{1,\dots,n\}}}\int_{\overline{\mathcal{M}}_{g,n}}\xi_{p,I*}(1)\psi_1^{d_1}\dots\psi_n^{d_n}\,,\end{aligned}$$

or, in view of formula (1.4),

$$\int_{W_{(0,m_1,1,0,0,\dots),n}} \psi_1^{d_1}\dots\psi_n^{d_n} = \int_{\overline{\mathcal{M}}_{g,n}} (12\kappa_1-\delta)\psi_1^{d_1}\dots\psi_n^{d_n}\,. \tag{3.7}$$

Now let's turn to the codimension 2 case. Among the $W_{m_*,n}$ there are two codimension 2 classes, corresponding to $m_* = (0, m_1, 0, 1, 0, \dots)$ and to $m_* = (0, m_1, 2, 0, \dots)$. The first one corresponds to ribbon graphs with at least one 7-valent vertex, the second one to ribbon graphs with at least two 5-valent vertices. To make notations lighter, from now on we shall adopt the following convention. Whenever identities between derivatives of Z or F will be given, these will always be meant to hold for $s_* = \hat{s}_* = (0, 1, 0, \dots)$, unless otherwise specified. With this notation, the DFIZ theorem gives

$$\frac{\partial Z}{\partial s_3} = 120\frac{\partial Z}{\partial t_3} - 6\frac{\partial^2 Z}{\partial t_0\partial t_1} + \frac{5}{4}\frac{\partial Z}{\partial t_0}\,, \tag{3.8}$$

$$\frac{\partial^2 Z}{\partial s_2^2} = 144\frac{\partial^2 Z}{\partial t_2^2} - 840\frac{\partial Z}{\partial t_3} - 12\frac{\partial^3 Z}{\partial t_0^2\partial t_2} + 24\frac{\partial^2 Z}{\partial t_0\partial t_1} + \frac{1}{4}\frac{\partial^4 Z}{\partial t_0^4} - 3\frac{\partial Z}{\partial t_0}\,. \tag{3.9}$$

In terms of derivatives of F these translate into

$$\frac{\partial F}{\partial s_3} = 120\frac{\partial F}{\partial t_3} - 6\frac{\partial^2 F}{\partial t_0\partial t_1} - 6\frac{\partial F}{\partial t_0}\frac{\partial F}{\partial t_1} + \frac{5}{4}\frac{\partial F}{\partial t_0} \tag{3.10}$$

$$\begin{aligned}\frac{\partial^2 F}{\partial s_2^2} + \left(\frac{\partial F}{\partial s_2}\right)^2 = {} & 144\frac{\partial^2 F}{\partial t_2^2} + 144\left(\frac{\partial F}{\partial t_2}\right)^2 - 840\frac{\partial F}{\partial t_3} - 12\frac{\partial^3 F}{\partial t_0^2\partial t_2} \\ & - 12\frac{\partial^2 F}{\partial t_0^2}\frac{\partial F}{\partial t_2} - 24\frac{\partial F}{\partial t_0}\frac{\partial^2 F}{\partial t_0\partial t_2} - 12\left(\frac{\partial F}{\partial t_0}\right)^2\frac{\partial F}{\partial t_2} + 24\frac{\partial^2 F}{\partial t_0\partial t_1} \\ & + 24\frac{\partial F}{\partial t_0}\frac{\partial F}{\partial t_1} + \frac{1}{4}\frac{\partial^4 F}{\partial t_0^4} + \frac{\partial^3 F}{\partial t_0^3}\frac{\partial F}{\partial t_0} + \frac{3}{2}\frac{\partial^2 F}{\partial t_0^2}\left(\frac{\partial F}{\partial t_0}\right)^2 \\ & + \frac{1}{4}\left(\frac{\partial F}{\partial t_0}\right)^4 + \frac{3}{4}\left(\frac{\partial^2 F}{\partial t_0^2}\right)^2 - 3\frac{\partial F}{\partial t_0}\end{aligned}$$

Taking into account (3.6), the second of these yields

$$\begin{aligned}\frac{\partial^2 F}{\partial s_2^2} = {} & 144\frac{\partial^2 F}{\partial t_2^2} - 840\frac{\partial F}{\partial t_3} - 12\frac{\partial^3 F}{\partial t_0^2\partial t_2} - 24\frac{\partial F}{\partial t_0}\frac{\partial^2 F}{\partial t_0\partial t_2} + 24\frac{\partial^2 F}{\partial t_0\partial t_1} \\ & + 24\frac{\partial F}{\partial t_0}\frac{\partial F}{\partial t_1} + \frac{1}{4}\frac{\partial^4 F}{\partial t_0^4} + \frac{\partial^3 F}{\partial t_0^3}\frac{\partial F}{\partial t_0} + \frac{\partial^2 F}{\partial t_0^2}\left(\frac{\partial F}{\partial t_0}\right)^2 + \frac{1}{2}\left(\frac{\partial^2 F}{\partial t_0^2}\right)^2 - 3\frac{\partial F}{\partial t_0}\end{aligned} \tag{3.11}$$

As the reader may notice, the right-hand sides of (3.8) and (3.9) contain considerably fewer terms than one would a priori expect, based on the general statement of the DFIZ theorem in the form given by (3.5). In fact, $\partial^4 Z/\partial t_0^4$ is missing from (3.8), while $\partial^7 Z/\partial t_0^7$, $\partial^3 Z/\partial t_0\partial t_1^2$ and $\partial^5 Z/\partial t_0^4\partial t_1$ are not present in (3.9). In addition to this phenomenon, further unexpected cancellations occur when passing from derivatives of Z to derivatives of F. For instance, $(\partial F/\partial t_0)^4$ and $(\partial F/\partial t_2)^2$ do not appear in (3.11). We shall see in a moment that these remarkable phenomena have geometrical significance. Indeed, in equating coefficients in the two sides of (3.10) and (3.11), it is precisely these facts that make it possible to interpret the resulting identities as relations between intersection numbers on a specific moduli space $\overline{\mathcal{M}}_{g,n}$, rather than relations involving intersection numbers on different moduli spaces.

Term by term, (3.10) translates into

$$\begin{aligned}
\langle\tau_{\underline{d}}\rangle_{(0,m_1,0,1,0,\dots)} &= 120\langle\tau_3\tau_{\underline{d}}\rangle_{g,n+1} - 6\langle\tau_0\tau_1\tau_{\underline{d}}\rangle_{g-1,n+2} \\
&\quad - 6\sum_{I\subset\{1,\dots,n\}}\langle\tau_1\tau_{\underline{d}_I}\rangle_{p,h+1}\langle\tau_0\tau_{\underline{d}_{CI}}\rangle_{g-p,n-h+1} + \frac{5}{4}\langle\tau_0\tau_{\underline{d}}\rangle_{g-1,n+1} \\
&= 120\langle\tau_3\tau_{\underline{d}}\rangle_{g,n} - 6\langle\tau_0\tau_1\tau_{\underline{d}}\rangle_{g-1,n+2} \\
&\quad - 6\sum_{I\subset\{1,\dots,n\}}\langle\tau_1\tau_{\underline{d}_I}\rangle_{p,h+1}\langle\tau_0\tau_{\underline{d}_{CI}}\rangle_{g-p,n-h+1} + 30\langle\tau_1\rangle\langle\tau_0\tau_{\underline{d}}\rangle_{g-1,n+1}\,,
\end{aligned}$$

where $3g-3+n-2=\sum d_i$, $m_1=4g-9+2n$, $h=\#I$, $3p-3+h+1=\sum_{i\in I}d_i$, and we have used the fact that $\langle\tau_1\rangle=1/24$. Proceeding exactly as in the derivation of (3.7), we conclude that

$$\text{(3.12)}\qquad \int_{W_{(0,m_1,0,1,0,\dots),n}}\psi_1^{d_1}\dots\psi_n^{d_n} = \int_{\overline{\mathcal{M}}_{g,n}}(120\kappa_2+\beta)\psi_1^{d_1}\dots\psi_n^{d_n}\,,$$

where

$$\text{(3.13)}\qquad \begin{aligned}\beta = -6\xi_{irr\,*}(\psi_{n+1}) - 6\sum_{\substack{0\le p\le g\\ I\subset\{1,\dots,n\}}}\xi_{p,I\,*}(\psi_{h+1}\times 1)\\ + 30\xi_{1,\emptyset\,*}(\psi_1\times 1)\,.\end{aligned}$$

The reader should be warned that β is not unambiguously defined, or, more exactly, that (3.12) holds also for a different choice of boundary term β. To see this notice that, using (1.7), we can write

$$\langle\tau_0\tau_1\tau_{\underline{d}}\rangle_{g-1,n+2} = (2(g-1)-2+n+1)\langle\tau_0\tau_{\underline{d}}\rangle_{g-1,n+1} = 24(2g+n-3)\langle\tau_1\rangle\langle\tau_0\tau_{\underline{d}}\rangle_{g-1,n+1}\,.$$

Since $\kappa_0=2g+n-3$ on $\overline{\mathcal{M}}_{g-1,n+1}$, this means that we could have chosen β to be given by

$$\text{(3.13}'\text{)}\qquad \begin{aligned}\beta = -144\xi_{1,\emptyset\,*}(\psi_1\times\kappa_0) - 6\sum_{\substack{0\le p\le g\\ I\subset\{1,\dots,n\}}}\xi_{p,I\,*}(\psi_{h+1}\times 1)\\ + 30\xi_{1,\emptyset\,*}(\psi_1\times 1)\,.\end{aligned}$$

This kind of ambiguity will be present in all the formulas for combinatorial classes that we shall give; however, it will be confined to some of the boundary terms.

Let's turn to formula (3.11). This gives

$$
\begin{aligned}
2\langle\tau_{\underline{d}}\rangle_{(0,m_1,2,0,0,\dots)} &= 144\langle\tau_2^2\tau_{\underline{d}}\rangle - 840\langle\tau_3\tau_{\underline{d}}\rangle - 12\langle\tau_0^2\tau_2\tau_{\underline{d}}\rangle\\
&- 24\sum_{I\sqcup J=\{1,\dots,n\}}\langle\tau_0\tau_{\underline{d}_I}\rangle\langle\tau_0\tau_2\tau_{\underline{d}_J}\rangle + 24\langle\tau_0\tau_1\tau_{\underline{d}}\rangle\\
&+ 24\sum_{I\sqcup J=\{1,\dots,n\}}\langle\tau_0\tau_{\underline{d}_I}\rangle\langle\tau_1\tau_{\underline{d}_J}\rangle + \frac{1}{4}\langle\tau_0^4\tau_{\underline{d}}\rangle\\
&+ \sum_{I\sqcup J=\{1,\dots,n\}}\langle\tau_0^3\tau_{\underline{d}_I}\rangle\langle\tau_0\tau_{\underline{d}_J}\rangle\\
&+ \sum_{I\sqcup J\sqcup K=\{1,\dots,n\}}\langle\tau_0^2\tau_{\underline{d}_I}\rangle\langle\tau_0\tau_{\underline{d}_J}\rangle\langle\tau_0\tau_{\underline{d}_K}\rangle\\
&+ \frac{1}{2}\sum_{I\sqcup J=\{1,\dots,n\}}\langle\tau_0^2\tau_{\underline{d}_I}\rangle\langle\tau_0^2\tau_{\underline{d}_J}\rangle - 3\langle\tau_0\tau_{\underline{d}}\rangle\,.
\end{aligned}
\tag{3.14}
$$

We wish to see that this can be interpreted as an identity among intersection numbers on $\overline{\mathcal{M}}_{g,n}$, where g, m_1 and the d_i are related by $\sum d_i = 3g-5+n$ and $m_1 = 4g-10+2n$. The left-hand side of (3.14) is twice the integral of $\psi_1^{d_1}\dots\psi_n^{d_n}$ over $W_{(0,m_1,2,0,0,\dots),n}$. As for the right-hand side, it is convenient to examine each summand separately. The first two terms cause no trouble for, using (1.12), they can be written as

$$
\begin{aligned}
144\langle\tau_2\tau_2\tau_{\underline{d}}\rangle_{g,n+2} - 840\langle\tau_3\tau_{\underline{d}}\rangle_{g,n+1} &= 144\int_{\overline{\mathcal{M}}_{g,n}}(\kappa_1^2+\kappa_2)\psi_1^{d_1}\dots\psi_n^{d_n}\\
&\quad - 840\int_{\overline{\mathcal{M}}_{g,n}}\kappa_2\psi_1^{d_1}\dots\psi_n^{d_n}\\
&= \int_{\overline{\mathcal{M}}_{g,n}}(144\kappa_1^2 - 696\kappa_2)\psi_1^{d_1}\dots\psi_n^{d_n}\,.
\end{aligned}
$$

Now look at the remaining terms.

- Term 3.

$$
\langle\tau_0^2\tau_2\tau_{\underline{d}}\rangle = \langle\tau_0^2\tau_2\tau_{\underline{d}}\rangle_{g-1,n+3} = \int_{\overline{\mathcal{M}}_{g-1,n+2}}\kappa_1\prod_{i=1}^{n}\psi_i^{d_i} = \int_{\overline{\mathcal{M}}_{g,n}}\xi_{irr*}(\kappa_1)\prod\psi_i^{d_i}\,.
$$

- Term 4. Disregarding the coefficient, this is

$$
\sum_{I\sqcup J=\{1,\dots,n\}}\langle\tau_0\tau_{\underline{d}_I}\rangle_{q,h+1}\langle\tau_0\tau_2\tau_{\underline{d}_J}\rangle_{r,k+2}\,,
$$

where

$$
h=\#I\,,\quad k=\#J\,,\quad \sum_{i\in I}d_i = 3q-3+h+1\,,\quad \sum_{j\in J}d_j = 3r-3+k+2\,.
$$

Since $h+k=n$ and, as we observed above, $\sum d_i = 3g-5+n$, this gives $q+r=g$. Hence

$$\begin{aligned}\sum_{I\sqcup J=\{1,\dots,n\}} \langle\tau_0\tau_{\underline{d}_I}\rangle\langle\tau_0\tau_2\tau_{\underline{d}_J}\rangle &= \sum_{I\sqcup J=\{1,\dots,n\}} \int_{\overline{\mathcal{M}}_{q,h+1}} \prod_{i\in I}\psi_i^{d_i} \int_{\overline{\mathcal{M}}_{r,k+1}} \kappa_1 \prod_{j\in J}\psi_j^{d_j} \\ &= \sum_{\substack{0\le q\le g\\ I\subset\{1,\dots,n\}}} \int_{\overline{\mathcal{M}}_{g,n}} \xi_{q,I\,*}(1\times\kappa_1)\prod_{i=1}^n \psi_i^{d_i}\,.\end{aligned}$$

- Term 5.

$$\langle\tau_0\tau_1\tau_{\underline{d}}\rangle = \langle\tau_0\tau_1\tau_{\underline{d}}\rangle_{g-1,n+2} = \int_{\overline{\mathcal{M}}_{g,n}} \xi_{irr\,*}(\psi_{n+2})\prod_{i=1}^n \psi_i^{d_i}\,.$$

This is an expression which appeared also in the formula for $W_{(0,m_1,0,1,0,\dots),n}$. As in that case, it could have been interpreted, alternatively, as

$$24\int_{\overline{\mathcal{M}}_{g,n}} \xi_{1,\emptyset\,*}(\psi_1\times\kappa_0)\prod_{i=1}^n \psi_i^{d_i}\,.$$

- Term 6. The relevant part is

$$\sum_{I\sqcup J=\{1,\dots,n\}} \langle\tau_0\tau_{\underline{d}_I}\rangle_{q,h+1}\langle\tau_1\tau_{\underline{d}_J}\rangle_{r,k+1}\,,$$

where $q+r=g$, and thus it can be rewritten as

$$\sum_{\substack{0\le q\le g\\ I\subset\{1,\dots,n\}}} \int_{\overline{\mathcal{M}}_{g,n}} \xi_{q,I\,*}(1\times\psi_{k+1})\prod_{i=1}^n \psi_i^{d_i}\,.$$

- Term 7. We have

$$\langle\tau_0^4\tau_{\underline{d}}\rangle = \langle\tau_0^4\tau_{\underline{d}}\rangle_{g-2,n+4} = \int_{\overline{\mathcal{M}}_{g,n}} \xi_{A\,*}(1)\prod_{i=1}^n \psi_i^{d_i}\,,$$

where the graph A is depicted in Figure 2.

- Term 8. This is

$$\sum_{I\sqcup J=\{1,\dots,n\}} \langle\tau_0^3\tau_{\underline{d}_I}\rangle_{q,h+3}\langle\tau_0\tau_{\underline{d}_J}\rangle_{r,k+1}\,,$$

with $q+r=g-1$. Clearly it can also be written as

$$\sum_{p,P}\int_{\overline{\mathcal{M}}_{g,n}} \xi_{(B,p,P)_*}(1)\prod_{i=1}^n \psi_i^{d_i}\,,$$

where the directed graph B is illustrated in Figure 2, and p (resp., P) runs through all possible assignments of genera to the vertices of B subject to the condition that their sum be equal to $g-1$ (resp., all partitions of $\{1,\dots,n\}$ indexed by the vertices of B).

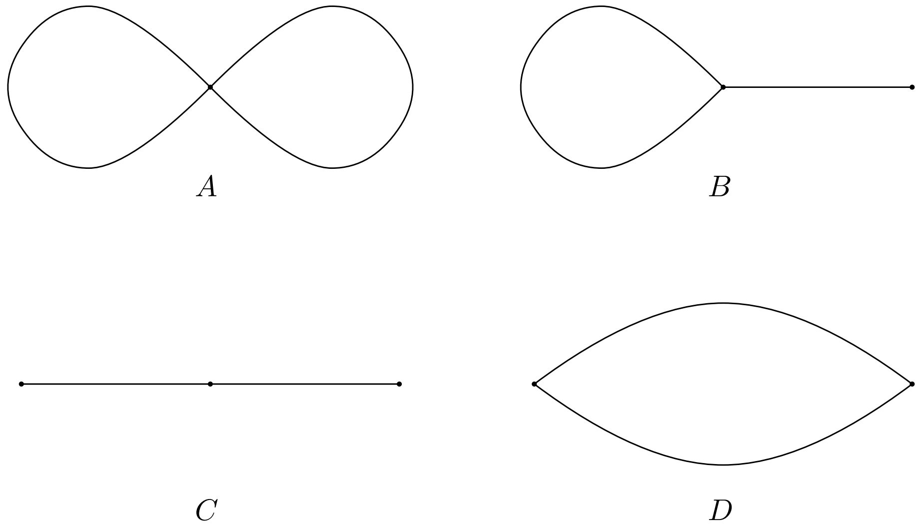


Figure 2

- Term 9. This is

$$\sum_{I\sqcup J\sqcup K=\{1,\dots,n\}} \langle\tau_0^2\tau_{\underline{d}_I}\rangle_{q,h+2}\langle\tau_0\tau_{\underline{d}_J}\rangle_{r,k+1}\langle\tau_0\tau_{\underline{d}_K}\rangle_{s,l+1}\ ,$$

where q, r and s add to g. As in the preceding case, then, this term can also be written

$$\sum_{p,P}\int_{\overline{\mathcal{M}}_{g,n}} \xi_{(C,p,P)_*}(1)\prod_{i=1}^{n}\psi_i^{d_i}\ ,$$

where the directed graph C is illustrated in Figure 2, and p (resp., P) runs through all possible assignments of genera to the vertices of C subject to the condition that their sum be equal to g (resp., all partitions of $\{1,\dots,n\}$ indexed by the vertices of C).

- Term 10. This is handled similarly to the two preceding ones. Consider the directed graph D in Figure 2. Then

$$\sum_{I\sqcup J=\{1,\dots,n\}} \langle\tau_0^2\tau_{\underline{d}_I}\rangle\langle\tau_0^2\tau_{\underline{d}_J}\rangle=\sum_{p,P}\int_{\overline{\mathcal{M}}_{g,n}} \xi_{(D,p,P)_*}(1)\prod_{i=1}^{n}\psi_i^{d_i}\ ,$$

where p runs through all assignments of genera to the vertices of D adding to $g-1$ and P through all partitions of $\{1,\dots,n\}$ indexed by the vertices of D.

- Term 11. The expression $\langle\tau_0\tau_{\underline{d}}\rangle$ already appears in in the formula for $W_{(0,m_1,0,1,0,\dots),n}$, and we have seen that it equals

$$24\int_{\overline{\mathcal{M}}_{g,n}} \xi_{1,\emptyset_*}(\psi_1\times 1)\prod_{i=1}^{n}\psi_i^{d_i}\ .$$

What all the above computation suggests is that a reasonable candidate for an expression of $W_{(0,m_1,2,0,\dots),n}$ in terms of the standard algebro-geometric classes might be

$$
\begin{aligned}
&72\kappa_1^2 - 348\kappa_2 - 6\xi_{irr*}(\kappa_1) - 12\sum_{\substack{0\le q\le g\\ I\subset\{1,\dots,n\}}}\xi_{q,I*}(1\times\kappa_1) + 12\xi_{irr*}(\psi_{n+2})\\
&+12\sum_{\substack{0\le q\le g\\ I\subset\{1,\dots,n\}}}\xi_{q,I*}(1\times\psi_{k+1}) + \frac{1}{8}\xi_{A*}(1) + \frac{1}{2}\sum_{p,P}\xi_{(B,p,P)_*}(1)\\
&+\frac{1}{2}\sum_{p,P}\xi_{(C,p,P)_*}(1) + \frac{1}{4}\sum_{p,P}\xi_{(D,p,P)_*}(1) - 36\xi_{1,\emptyset *}(\psi_1\times 1)\,,
\end{aligned}
\tag{3.15}
$$

up to the ambiguity noticed in the analysis of term 5.

As we remarked after formula (3.11), due to a number of remarkable cancellations, in the expressions of the derivatives of F with respect to the s variables in terms of derivatives with respect to the t variables many derivatives of F that would a priori be allowed by the DFIZ theorem are not present. For example, in codimension 2 the terms

$$\left(\frac{\partial F}{\partial t_0}\right)^4\,,\quad \left(\frac{\partial F}{\partial t_2}\right)^2\,,$$

among many others, are missing. This is wonderful, for otherwise our formulas would have been ruined. In fact, when studying codimension 2 classes in genus g, we would have gotten terms of the sort

$$
\begin{aligned}
&\langle\tau_0\tau_{\underline{d}_{I_1}}\rangle_{q_1,h_1+1}\langle\tau_0\tau_{\underline{d}_{I_2}}\rangle_{q_2,h_2+1}\langle\tau_0\tau_{\underline{d}_{I_3}}\rangle_{q_3,h_3+1}\langle\tau_0\tau_{\underline{d}_{I_4}}\rangle_{q_4,h_4+1}\,,\\
&\langle\tau_2\tau_{\underline{d}_{I_1}}\rangle_{r_1,h_1+1}\langle\tau_2\tau_{\underline{d}_{I_2}}\rangle_{r_2,h_2+1}\,,
\end{aligned}
$$

respectively. Now it is easy to see that we must have $q_1+q_2+q_3+q_4 = g+1 = r_1+r_2$. Thus it would have been impossible to write these terms as intersection numbers on $\overline{\mathcal{M}}_{g,n}$.

In the same vein, but with considerably more effort, we could have given similar formulas for some classes $W_{m_*,n}$ of higher codimension, including in particular all those of codimension 3. In the Appendix we have listed the expressions of the derivatives of F with respect to the s variables in terms of those with respect to the t variables that are needed to carry out these computations, which are otherwise left to the reader.

The resulting identities are of the form

$$\int_{W_{m_*,n}}\prod\psi_i^{d_i} = \int_{\overline{\mathcal{M}}_{g,n}} X_{m_*,n}\prod\psi_i^{d_i}\,,$$

where g is given by $4g-4+2n=\sum m_i(2i-1)$ and $X_{m_*,n}$ is a polynomial in the Mumford classes and the boundary classes. The reader is invited to check that this is indeed true, using the methods employed in this section. Of course, the point is to verify that all the terms that one gets can be interpreted as intersection numbers on $\overline{\mathcal{M}}_{g,n}$, and not in higher genus. The formulas one finds, arranged by increasing codimension, look as follows

- Codimension 1

$$X_{(0,m_1,1,0,\dots),n} = 12\kappa_1 + \cdots$$

- Codimension 2

$$X_{(0,m_1,0,1,0,\dots),n} = 120\kappa_2 + \cdots$$
$$X_{(0,m_1,2,0,\dots),n} = 72\kappa_1^2 - 348\kappa_2 + \cdots$$

- Codimension 3

$$X_{(0,m_1,0,0,1,0,\dots),n} = 1680\kappa_3 + \cdots$$
$$X_{(0,m_1,1,1,0,\dots),n} = 1440\kappa_1\kappa_2 - 13680\kappa_3 + \cdots$$
$$X_{(0,m_1,3,0,\dots),n} = 288\kappa_1^3 - 4176\kappa_1\kappa_2 + 20736\kappa_3 + \cdots$$

- Codimension 4

$$X_{(0,m_1,0,0,0,1,0,\dots),n} = 30240\kappa_4 + \cdots$$
$$X_{(0,m_1,1,0,1,0\dots),n} = 20160\kappa_1\kappa_3 - 312480\kappa_4 + \cdots$$
$$X_{(0,m_1,0,2,0\dots),n} = 7200\kappa_2^2 - 159120\kappa_4 + \cdots$$

- Codimension 5

$$X_{(0,m_1,0,0,0,0,1,0,\dots),n} = 665280\kappa_5 + \cdots$$

- Codimension 6

$$X_{(0,m_1,0,0,0,0,0,1,0,\dots),n} = 17297280\kappa_6 + \cdots$$

The dots stand for boundary classes, which are well determined up to ambiguities of the kinds previously described. This list is complete up to codimension 3 included.

The same remarkable cancellations that we observed for codimension two classes occur, to an even greater extent, in higher codimension. For instance, the expression for $\partial^3 F/\partial s_2^3$ given in the appendix involves 41 terms while, a priori, up to 585 might have been expected from the statement of the DFIZ theorem.

4. The codimension one case

Our main goal in this section is to complete the study of the codimension one class $W_{(0,m_1,1,0,\dots),n}$. For simplicity, this will be denoted simply by W throughout the section. We shall prove the following

PROPOSITION 1. *When $n \geq 2$, for any class $\gamma \in H^{6g-6+2n-2}(\overline{\mathcal{M}}'_{g,n}, \mathbb{Q})$ one has*

$$\int_W \gamma = \int_{\overline{\mathcal{M}}_{g,n}} \alpha^*(\gamma)(12\kappa_1 - \delta)\,,$$

where α is the natural map from $\overline{\mathcal{M}}_{g,n}$ to $\overline{\mathcal{M}}'_{g,n}$.

In section 3 we have shown that the proposition holds when γ is a product of classes ψ_i. Notice that both W and $12\kappa_1 - \delta$ are invariant under the natural action of the symmetric group $\mathcal{S}_n$ on $\overline{\mathcal{M}}_{g,n}$. Moreover, it is reasonable to expect that W can be "lifted", non-uniquely, to a homology class on $\overline{\mathcal{M}}_{g,n}$. Proving this, however, requires a little argument that will be given later. Granting this, the proposition is then a direct consequence of the following lemma.

LEMMA 2. *Let $x \in H^2(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ be an $\mathcal{S}_n$ invariant class, with $n \geq 2$. Suppose that*

$$\int_{\overline{\mathcal{M}}_{g,n}} x \cup \prod_{i=1}^{n} \psi_i^{d_i} = 0$$

for all choices of $d_1, \dots, d_n$. Then x is a linear combination of classes $\delta_{p,\emptyset}$.

Incidentally, it is obvious that the converse of the statement of the lemma is true. Moreover, the $\delta_{p,\emptyset}$ are precisely the classes of those components of the boundary that are partially contracted by α. We now prove the lemma.

It follows from a well-known theorem of Harer [4] that the second cohomology group of $\overline{\mathcal{M}}_{g,n}$ is generated by the classes $\kappa_1, \psi_1, \dots, \psi_n$ and by the boundary classes δ_Γ, where Γ runs through all isomorphisms classes of dual graphs having only one edge. We set $\psi = \sum_{i=1}^{n} \psi_i$. Given integers p and h, with $0 \leq p \leq g$ and $0 \leq h \leq n$, we denote by $\delta_{p,h}$ the sum $\sum \delta_\Gamma$, where Γ runs through all isomorphism classes of dual graphs of curves with exactly two components meeting at one point, one of which has genus p and carries h marked points. In terms of these, one may write the class of the boundary as

$$\delta = \sum_{\substack{0 \leq p \leq g/2 \\ 0 \leq h \leq n \\ h \leq n/2 \text{ if } p = g/2}} \delta_{p,h} \,.$$

Clearly, any invariant class in the second cohomology group of $\overline{\mathcal{M}}_{g,n}$ is a linear combination of κ_1, ψ, δ_{irr}, and the $\delta_{p,h}$. If $g = 2$ (resp., $g = 1$, resp., $g = 0$) one can do without κ_1 (resp., κ_1 and ψ, resp., κ_1, ψ and δ_{irr}). We then write

$$x = a\kappa_1 + b\psi + c\delta_{irr} + \sum c_{p,h}\delta_{p,h} \,.$$

We will first show that $a = b = c = 0$. Set

$$\alpha_s = \prod_{i=1}^{n-2} \psi_i \psi_{n-1}^s \psi_n^d \,,$$

where $d = 3g-s-2$, and notice that, if $s \not\equiv 2 \mod 3$, then $\int \delta_{p,h}\alpha_s = 0$. In fact, $\int \delta_{p,h}\alpha_s$ is a linear combination of terms of the form $\langle \tau_0\tau_1^a \rangle \langle \tau_0\tau_1^b\tau_s\tau_d \rangle$ or of the form $\langle \tau_0\tau_1^a\tau_s \rangle \langle \tau_0\tau_1^b\tau_d \rangle$. But since $a-(a+1)$ and $a+s-(a+2)$ are not divisible by 3 ($s \not\equiv 2 \mod 3$), both $\langle \tau_0\tau_1^a \rangle$ and $\langle \tau_0\tau_1^a\tau_s \rangle$ vanish. We now wish to compute $\int \kappa_1\alpha_s$, $\int \psi\alpha_s$, and $\int \delta_{irr}\alpha_s$. For this we are going to use the following well-known formulae [16].

$$\text{(4.1)} \qquad \begin{aligned} \langle \tau_0\tau_{d_1}\dots\tau_{d_n} \rangle &= \sum_{d_i > 0} \langle \tau_{d_1}\dots\tau_{d_i - 1}\dots\tau_{d_n} \rangle \,, \\ \langle \tau_1\tau_{d_1}\dots\tau_{d_n} \rangle &= (2g-2+n)\langle \tau_{d_1}\dots\tau_{d_n} \rangle \,, \end{aligned}$$

where $3g-3+n = \sum d_i$. The first formula is just the string equation (1.9), while the second is a special case of (1.7). Setting $r = 2g-2$ we then have

$$\int \kappa_1\alpha_s = \langle \tau_2\tau_1^{n-2}\tau_s\tau_d \rangle = \frac{(r+n)!}{(r+2)!}\langle \tau_2\tau_s\tau_d \rangle_{g,3} \,,$$

$$\int \psi\alpha_s = (n-2)\langle\tau_2\tau_1^{n-3}\tau_s\tau_d\rangle + \langle\tau_1^{n-2}\tau_{s+1}\tau_d\rangle + \langle\tau_1^{n-2}\tau_s\tau_{d+1}\rangle$$
$$= \frac{(r+n-1)!}{(r+2)!}(n-2)\langle\tau_2\tau_s\tau_d\rangle_{g,3} + \frac{(r+n-1)!}{(r+1)!}\langle\tau_{s+1}\tau_d\rangle_{g,2}$$
$$+ \frac{(r+n-1)!}{(r+1)!}\langle\tau_s\tau_{d+1}\rangle_{g,2}\,,$$

$$2\int \delta_{irr}\alpha_s = \langle\tau_0^2\tau_1^{n-2}\tau_s\tau_d\rangle = \frac{(r+n-1)!}{(r+1)!}\langle\tau_0^2\tau_s\tau_d\rangle$$
$$= \frac{(r+n-1)!}{(r+1)!}\left(\langle\tau_{s-2}\tau_d\rangle_{g-1,2} + 2\langle\tau_{s-1}\tau_{d-1}\rangle_{g-1,2} + \langle\tau_s\tau_{d-2}\rangle_{g-1,2}\right)\,.$$

To simplify these expressions we shall use the fundamental fact [7] that the function $Z(t_*) = \exp F(t_*)$ satisfies the KdV equation. It is convenient to set

$$\varphi(g) = \begin{cases} \langle\tau_{3g-2}\rangle & \text{if } g \geq 1\,, \\ \langle\tau_0^3\rangle = 1 & \text{if } g = 0\,, \end{cases}$$

$$T(s,r) = \frac{\langle\tau_s\tau_d\rangle_{g,2}}{\varphi(g)}\,,$$

$$U(s,r) = \frac{\langle\tau_2\tau_s\tau_d\rangle_{g,3}}{\varphi(g)}\,.$$

Writing the KdV equation in Lax form (also referred to as Gel'fand-Dikii form), one gets in particular (cf. [16], page 251) that

$$\varphi(g) = \frac{1}{24g}\varphi(g-1) = \frac{1}{12(r+2)}\varphi(g-1)\,.$$

It follows that

$$(4.2)\qquad \begin{cases} \displaystyle\int \kappa_1\alpha_s = \frac{(r+n-1)!}{(r+2)!}\varphi(g)(r+n)U(s,r)\,, \\ \displaystyle\int \psi\alpha_s = \frac{(r+n-1)!}{(r+2)!}\varphi(g)\left[(n-2)U(s,r)\right. \\ \qquad\qquad \left. +(r+2)T(s+1,r) + (r+2)T(s,r)\right]\,, \\ \displaystyle\int \delta_{irr}\alpha_s = \frac{(r+n-1)!}{(r+2)!}\varphi(g)6(r+2)^2\left[T(s-2,r-2)\right. \\ \qquad\qquad \left. +2T(s-1,r-2) + T(s,r-2)\right]\,. \end{cases}$$

To calculate $T(s,r)$ and $U(s,r)$ we use again the fact that Z satisfies the KdV equation, but expressing this by saying that Z is annihilated by the Virasoro operators L_k, for $k \geq -1$.

The equations $L_{-1}Z = L_0 Z = 0$ are equivalent to the (4.1). The Virasoro operator L_k is given, for $k > 0$, by

$$L_k = -\frac{(2k+3)!!}{2}\frac{\partial}{\partial t_{k+1}} + \frac{1}{2}\sum_{i=0}^{\infty}(2k+2i+1)(2k+2i-1)\dots(2i+1)t_i\frac{\partial}{\partial t_{i+k}}$$
$$+\frac{1}{4}\sum_{r+s+1=k}(2r+1)!!(2s+1)!!\frac{\partial^2}{\partial t_r \partial t_s}$$

Recalling that

$$F(t_*) = \sum\langle\tau_{d_1}\dots\tau_{d_n}\rangle t_{d_1}\dots t_{d_n}\,,$$

to say that $L_k Z = 0$ translates into

$$\langle\tau_{k+1}\tau_{\underline{d}}\rangle = \frac{1}{(2k+3)!!}\left[\sum_{j=1}^{n}\frac{(2k+2d_j+1)!!}{(2d_j-1)!!}\langle\tau_{d_1}\dots\tau_{d_{j+k}}\dots\tau_{d_n}\rangle\right.$$
$$+\frac{1}{2}\sum_{r+s=k-1}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\tau_{\underline{d}}\rangle$$
$$\left.+\frac{1}{2}\sum_{r+s=k-1}(2r+1)!!(2s+1)!!\sum_{I\subset\{1,\dots,n\}}\langle\tau_r\tau_{\underline{d}_I}\rangle\langle\tau_s\tau_{\underline{d}_{CI}}\rangle\right]$$

for any $\underline{d} = (d_1,\dots,d_n)$. It follows that, provided $s \not\equiv 2 \mod 3$,

$$U(s,r) = \frac{1}{3\cdot 5}\left[(2s+3)(2s+1)T(s+1,r) + (3r-2s+5)(3r-2s+3)T(s,r)\right.$$
$$\left.+6(r+2)(T(s-2,r-2)+2T(s-1,r-2)+T(s,r-2))\right]\,.$$

Furthermore we have

$$T(s,r) = 0 \qquad \text{if } s<0\,,$$
$$T(0,r) = 1\,,$$
$$T(1,r) = r+1\,,$$
$$T(2,r) = \frac{1}{3\cdot 5}\left[(3r+3)(3r+1)+6(r+2)\right]\,,$$
$$T(3,r) = \frac{1}{3\cdot 5\cdot 7}\left[(3r+3)(3r+1)(3r-1)+3\cdot 12r(r+2)+\frac{3}{2}(r+2)\right]\,,$$

$$T(4,r) = \frac{1}{3\cdot 5\cdot 7\cdot 9}\left[(3r+3)(3r+1)(3r-1)(3r-3)\right.$$
$$\left.+12(r+2)\left(3\cdot 5+\frac{9}{2}r+\frac{3}{8}\right)T(1,r-2)+3\cdot 5\cdot 12(r+2)T(2,r-2)\right]\,,$$
$$T(5,r) = \frac{1}{3\cdot 5\cdot 7\cdot 9\cdot 11}\left[(3r+3)(3r+1)(3r-1)(3r-3)(3r-5)\right.$$
$$\left.+12(r+2)\left(\left(3\cdot 5\cdot 7+3\cdot 3\cdot 5\cdot r+\frac{3\cdot 5}{8}\right)T(2,r-2)+3\cdot 5\cdot 7\cdot T(3,r-2)\right)\right]\,.$$

Now consider the system of three linear equations in the unknowns a, b and c given by

$$\mathrm{Eq}_s) \qquad \frac{(r+2)!}{\varphi(g)(r+n-1)!}\int x\alpha_s = 0\,, \qquad s = 0,1,3\,.$$

The coefficients are implicitly given by (4.2), an can be calculated using the formulas for $U(s,r)$ and $T(s,r)$ we have just given. An algebraic calculation (best done by computer) shows that the determinant of this system equals

$$\frac{36}{875}(r-2)r^2(r+2)^5(n+r)(4r+17)\,.$$

Recall that n is an integer greater or equal to 2, and that r is an integer greater or equal to -2; thus our determinant vanishes only for $r=-2,0,2$, that is, for $g=0,1,2$. This shows that $a=b=c=0$ for $g\geq 3$. If $g=2$, the class κ_1 is linearly dependent on the others, so that we may set $a=0$ and view Eq_0 and Eq_1 as a linear system in the unknowns b and c. The determinant of this system equals $(1536/5)(n+2)$, which is non-zero in our situation. If $g=1$ one may set $a=b=0$, and the coefficient of c in Eq_0 is 24. If $g=0$ the classes κ_1, ψ and δ_{irr} $(=0)$ are linear combinations of the $\delta_{p,h}$.

We have thus shown that, for any value of the genus g, the class x is a linear combination

$$x = \sum c_{p,h}\delta_{p,h}\,.$$

We wish to show that $c_{p,h}$ vanishes unless $h=0$ or $h=n$. Set

$$\beta_{0,j} = \psi_1^{s-j}\prod_{h=2}^{j-1}\psi_h\,, \qquad s = 3g-2+n\,,$$

$$\beta_{q,j} = \psi_1^{s-j}\psi_2^{3q-1}\prod_{h=3}^{j+1}\psi_h\,, \qquad s = 3(g-q)-2+n \qquad \text{if } q>0\,.$$

In what follows we shall always assume, as we may, that $p\leq g/2$. The intersection number $\int\delta_{p,h}\beta_{0,j}$ is, a priori, a linear combinations of terms of the form

$$\langle\tau_0^{a+1}\tau_1^c\rangle_{p,h+1}\langle\tau_0^{b+1}\tau_1^d\tau_{3g-2+n-j}\rangle_{g-p,n-h+1}$$

or

$$\langle\tau_0^{a+1}\tau_1^c\tau_{3g-2+n-j}\rangle_{p,h+1}\langle\tau_0^{b+1}\tau_1^d\rangle_{g-p,n-h+1}\,.$$

However, since $\langle\tau_0^{b+1}\tau_1^d\rangle_{g-p,n-h+1}$ is non-zero only if $b=2$, in which case $g-p=0$, there are no terms of the second kind. As for those of the first kind, they may be non-zero only if $a=2$, $p=0$, $h=a+c$, $c+d=j-2$, so that $h\leq a+j-2=j$. In conclusion

$$(4.3) \qquad \int\delta_{p,h}\beta_{0,j} = 0 \qquad \text{if } p>0 \text{ or } p=0,\ h>j\,;$$

moreover

$$(4.4) \qquad \int\delta_{0,j}\beta_{0,j} \neq 0 \qquad \text{if } 1<j<n\,,$$

since this number is a multiple, with positive coefficient, of $\langle\tau_0^3\tau_1^{j-2}\rangle\langle\tau_0^{n-j}\tau_{3g-2+n-j}\rangle \neq 0$. Let us now compute the intersection number

$$\int \delta_{p,h}\beta_{q,j}$$

when $q > 0$ and, as usual, $p \leq g/2$. This is, a priori, a linear combination of terms of the form

$$\langle\tau_0^{a+1}\tau_1^c\tau_{3q-1}\rangle_{p,h+1}\langle\tau_0^{b+1}\tau_1^d\tau_{s-j}\rangle_{g-p,n-h+1}\,,$$

or

$$\langle\tau_0^{a+1}\tau_1^c\tau_{s-j}\rangle_{p,h+1}\langle\tau_0^{b+1}\tau_1^d\tau_{3q-1}\rangle_{g-p,n-h+1}\,,$$

or else

$$\langle\tau_0^{a+1}\tau_1^c\rangle_{p,h+1}\langle\tau_0^{b+1}\tau_1^d\tau_{3q-1}\tau_{s-j}\rangle_{g-p,n-h+1}\,,$$

or, finally,

$$\langle\tau_0^{a+1}\tau_1^c\tau_{3q-1}\tau_{s-j}\rangle_{p,h+1}\langle\tau_0^{b+1}\tau_1^d\rangle_{g-p,n-h+1}\,.$$

We already saw that a term of this last type is not equal to zero only if $g - p = 0$, which is impossible. Terms of the third type are different from zero only if $p = 0$, $a = 2$, $h = c + 2 \leq j + 1$. Let us analyze terms of the first type. These are non-zero only if $3p - 3 + h + 1 = 3q - 1 - c$. On the other hand $h = a + c + 1$, so that $3q = 3p + a$. Furthermore $c + d = j - 1$, so that $h \leq j + a$. The same argument shows that there are no non-zero terms of the second type. We conclude that

$$\int \delta_{p,h}\beta_{q,j} = 0 \qquad \text{if } p > q \text{ or } p \leq q,\ h > j + 3(q-p)\,; \tag{4.5}$$

moreover

$$\int \delta_{p,h}\beta_{p,h} \neq 0 \qquad \text{if } 0 < h < n\,, \tag{4.6}$$

since this number is a positive multiple of $\langle\tau_0\tau_1^{h-1}\tau_{3p-1}\rangle\langle\tau_0^{n-h}\tau_{3(g-p)-3+n-h+1}\rangle \neq 0$. Arguing by double induction on p and h, it follows from (4.3), (4.4), (4.5), and (4.6) that $c_{p,h} = 0$ for all p and all h different from 0 and n. This proves the lemma.

As we announced, to complete the proof of Proposition 1 it remains to compare the (co)homology of $\overline{\mathcal{M}}'_{g,n}$ with that of $\overline{\mathcal{M}}_{g,n}$. Rational coefficients will be used throughout. We shall show that

Lemma 3. *There is an exact sequence*

$$0 \to H^{6g-6+2n-2}(\overline{\mathcal{M}}'_{g,n}) \xrightarrow{\alpha^*} H^{6g-6+2n-2}(\overline{\mathcal{M}}_{g,n}) \to A \to 0\,,$$

where $\alpha : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}'_{g,n}$ is the natural map and A is the vector space freely generated by the boundary classes $\delta_{p,\emptyset}$, with $1 \leq p \leq g$ (or $1 \leq p \leq g-1$ for $n = 1$).

A consequence of this lemma is that the functional defined by integration on W lifts to an element $\widetilde{W}$ of $H^{6g-6+2n-2}(\overline{\mathcal{M}}_{g,n})^\vee \cong H_{6g-6+2n-2}(\overline{\mathcal{M}}_{g,n})$, which we may choose to be $\mathcal{S}_n$-invariant, since W is. Proposition 1 follows by applying Lemma 2 to the difference between $12\kappa_1 - \delta$ and the Poincaré dual of $\widetilde{W}$.

We now prove the lemma. Look at the commutative diagram

$$
\begin{array}{ccccc}
 & & H^{d-2}(\overline{\mathcal{M}}') & & \\
 & \overset{\rho}{\nearrow} & & \overset{\alpha^*}{\searrow} & \\
H_c^{d-2}(\overline{\mathcal{M}}' \setminus \Sigma') & \overset{\cong}{\longleftarrow} & H_c^{d-2}(\overline{\mathcal{M}} \setminus \Sigma) & \longrightarrow & H^{d-2}(\overline{\mathcal{M}}) \\
\downarrow & & \downarrow & & \downarrow \\
H_2(\overline{\mathcal{M}}' \setminus \Sigma') & \longleftarrow & H_2(\overline{\mathcal{M}} \setminus \Sigma) & \overset{\sigma}{\longrightarrow} & H_2(\overline{\mathcal{M}}) \ ,
\end{array}
$$

where we have set $d = 6g - 6 + 2n$, $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{M}}' = \overline{\mathcal{M}}'_{g,n}$, and Σ (resp., Σ') stands for the union of the components of $\partial\overline{\mathcal{M}}$ of the form $\Delta_{p,\emptyset}$ (resp., the image of Σ in $\overline{\mathcal{M}}'$). The three vertical arrows are isomorphisms by Poincaré duality. The map ρ is a piece of the exact sequence of cohomology with compact support

$$\cdots \to H^{d-3}(\Sigma') \to H_c^{d-2}(\overline{\mathcal{M}}' \setminus \Sigma') \xrightarrow{\rho} H^{d-2}(\overline{\mathcal{M}}') \to H^{d-2}(\Sigma') \to \cdots .$$

On the other hand $H^{d-3}(\Sigma')$ and $H^{d-2}(\Sigma')$ vanish since the dimension of Σ' is strictly smaller than $d-3$, so ρ is an isomorphism. It follows in particular that α^* is injective if and only if σ is, and that its cokernel can be identified with the one of σ. Passing to duals, we have to look at $\sigma^\vee : H^2(\overline{\mathcal{M}}) \to H^2(\overline{\mathcal{M}} \setminus \Sigma)$, which fits into the exact sequence

$$\cdots \to H^2(\overline{\mathcal{M}}; \overline{\mathcal{M}} \setminus \Sigma) \to H^2(\overline{\mathcal{M}}) \xrightarrow{\sigma^\vee} H^2(\overline{\mathcal{M}} \setminus \Sigma) \to H^3(\overline{\mathcal{M}}; \overline{\mathcal{M}} \setminus \Sigma) \to \cdots .$$

Now the Thom isomorphism implies that $H^3(\overline{\mathcal{M}}; \overline{\mathcal{M}} \setminus \Sigma)$ vanishes, since $H^1(\overline{\mathcal{M}}_{p,h})$ does, for any p and h, and that, moreover, $H^2(\overline{\mathcal{M}}; \overline{\mathcal{M}} \setminus \Sigma)$ is freely generated by the classes of the components of Σ. Since the images of these are independent in $H^2(\overline{\mathcal{M}})$, the conclusion follows.

5. Examples and comments

It is instructive to work out a couple of simple examples. It should be clear from them how intricate a direct attack on the problem would be. We begin by checking formula (2.3) on $\overline{\mathcal{M}}_{1,1}$ and on $\overline{\mathcal{M}}_{0,4}$. One thing that simplifies matters in these cases is that, for these values of g and n, one has $\overline{\mathcal{M}}'_{g,n} = \overline{\mathcal{M}}_{g,n}$. We shall write M for $\mathcal{M}_{(0,m_1,1,0,\dots),n}$ and W for $W_{(0,m_1,1,0,\dots),n}$. In general, if v and l stand for the numbers of vertices and edges of a ribbon graph of genus g with n boundary components all of whose vertices are trivalent save for a pentavalent one, one has

$$v = 2n + 4g - 6\,, \qquad l = 3n + 6g - 8\,.$$

In the case of $\overline{\mathcal{M}}_{1,1}$ this yields $v = 0$; thus, $W = 0$. To check (2.3) in this case we must then show that $12\kappa_1 = \delta$. But now one knows that $\tilde{\kappa}_1 = 12\lambda - \delta$ vanishes on $\overline{\mathcal{M}}_{1,1}$ (cf. [6], for instance). On the other hand $\psi = \lambda$. Thus $12\kappa_1 = 12\psi = 12\lambda = \delta$, as desired.

The case of $\overline{\mathcal{M}}_{0,4}$ is more entertaining. The formulas above give $v = 2$, $l = 4$; in particular, M is 4-dimensional and hence W zero-dimensional. The possible graphs in M are those of type a), b), and c) in Figure 3; their degenerations are illustrated in d), e), f), and g). Among these, the first two are internal to moduli, while the last two correspond to points in the boundary.

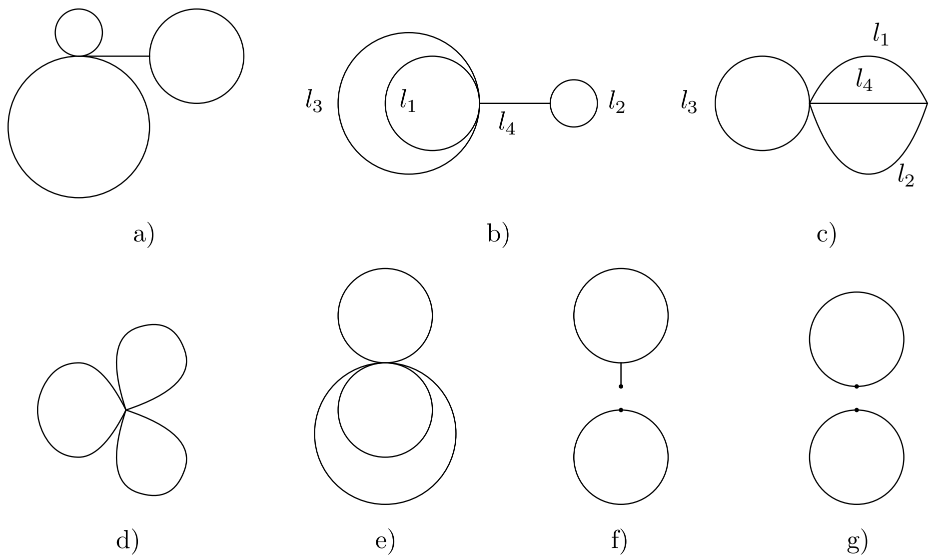


Figure 3

Now let us consider the projection

$$\eta : \overline{\mathcal{M}}_{0,4}^{comb} = \overline{\mathcal{M}}_{0,4} \times \mathbb{R}^4 \to \mathbb{R}^4$$

which associates to any numbered ribbon graph with metric the quadruple of positive real numbers given by the lengths of its four boundary components. Clearly, a cycle Z in $\overline{\mathcal{M}}_{0,4}$ representing W can be obtained by cutting M with a section $\eta = (P_1, \dots, P_4)$, where the P_i are positive constants. We choose $P_1, \dots, P_4$ in such a way that $P_{i+1} \geq 10P_i$, for $i = 1, 2, 3$. Since for graphs f) and g) two of the perimeters necessarily coincide, Z is entirely contained in the interior of moduli. We now show that graphs of types c), d), and e) cannot occur in Z. In fact, for graphs of type d) one of the perimeters equals the sum of the remaining three, and this is forbidden by our choice of P's. In e) one of the perimeters, which we may assume to be the longest, equals the sum of two of the other perimeters minus the remaining one. This too is incompatible with our choices. Let us now look at graph c), where the edges have been labelled with their respective lengths $l_1, \dots, l_4$. Up to the numbering of the boundary components the perimeters are

$$p_1 = l_1 + l_4\,, \quad p_2 = l_2 + l_4\,, \quad p_3 = l_3\,, \quad p_4 = l_1 + l_2 + l_3\,.$$

We also have the obvious inequalities

$$p_4 < p_1 + p_2 + p_3\,, \quad p_3 < p_4\,, \quad p_1 < p_2 + p_4\,, \quad p_2 < p_1 + p_4\,.$$

These inequalities imply, in order, that neither p_4, nor p_3, nor p_2 or p_1 can equal P_4. This excludes case c). We next examine case a). It is clear that the longest perimeter is the "external" one and that, except for this restriction, the perimeters can be arbitrarily assigned. Therefore this case accounts for $6 = 3!$ points of Z, one for each choice of labelling of the three "internal" boundary components by $\{1, 2, 3\}$.

We now examine graph b). Up to the numbering of the boundary components the perimeters are

$$p_1 = l_1\,, \quad p_2 = l_2\,, \quad p_3 = l_3 + l_1\,, \quad p_4 = l_2 + l_3 + 2l_4\,.$$

The following inequalities hold

$$p_1 < p_3\,, \quad p_2 < p_4\,, \quad p_3 < p_1 + p_4\,, \quad p_1 + p_2 < p_3 + p_4\,.$$

From the first three inequalities we get in particular that we must have $p_4 = P_4$. The only possibilities for (p_1, p_2, p_3, p_4) are

$$(P_1, P_2, P_3, P_4)\,, \quad (P_1, P_3, P_2, P_4)\,, \quad (P_2, P_1, P_3, P_4)\,.$$

In conclusion, the support of Z consists of 9 points. We claim that Z is the sum of these points, taken with the positive sign. This follows immediately from Kontsevich's recipe (cf. [7], page 11) for the orientation of Z. In his notation, this is given by Ω^d, where d is the dimension of Z, i.e., by the constant 1. It follows that W is 9 times the fundamental class of $\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$. To prove formula (2.3) in the present case it now suffices to show that $12\kappa_1 - \delta$ has degree 9. This follows if we can show that

$$\deg \delta = 3\,, \quad \deg \psi = 4\,, \quad \deg \tilde{\kappa}_1 = -\deg \delta = -3\,.$$

We briefly indicate how to do it. Set $X' = \mathbb{P}^1 \times \mathbb{P}^1$, denote by f' the projection onto the second factor, and by σ'_i, $i = 1, 2, 3$, three constant sections of f'. Blow up X' at the three points where these sections meet the diagonal, to obtain a family $f : X \to \mathbb{P}^1$ together with four distinct sections $\sigma_1, \ldots, \sigma_4$, which are the proper transforms of $\sigma'_1, \ldots, \sigma'_3$ and of the diagonal. This is the universal curve over $\overline{\mathcal{M}}_{0,4}$. Denote by E_1, E_2, E_3 the exceptional curves of the blow-up, and set $D_i = \sigma_i(\mathbb{P}^1)$. Then

$$\begin{aligned} 0 &= (D_i + E_i)^2 = D_i^2 + 2 + E_i^2 = D_i^2 + 1 \qquad \text{if } i \le 3 \\ 2 &= (D_4 + \textstyle\sum E_i)^2 = D_4^2 + 6 - 3 = D_4^2 + 3\,. \end{aligned}$$

In all cases $\deg \psi_i = -D_i^2 = 1$, so that ψ has degree 4. Clearly, $\lambda = 0$, and $\deg \delta = 3$, since the universal family contains exactly three singular fibers. The result follows.

With the next example in mind, we now make a general remark. Fix a sequence $m_* = (0, m_1, \ldots)$ and a positive integer n, and denote by v, l, and g the number of vertices, of edges, and the genus of any graph belonging to $\mathcal{M}^{comb}_{m_*,n}$. The real dimension of $W_{m_*,n}$ equals $l - n$, while g is given by $2 - 2g = v - l + n$. It follows that $v \le 0$, and hence $W_{m_*,n}$ is empty, as soon as the real codimension of $W_{m_*,n}$ equals or exceeds $4g - 4 + 2n$. In particular, in this range, the formulas we are after would amount to expressing certain polynomial in the Mumford classes as linear combinations of boundary classes. Moreover, if indeed these formulas were given by the DFIZ theorem, one could conclude, by induction on the level, that all monomials in the Mumford classes vanish on $\mathcal{M}_{g,n}$ in real codimension at least $4g-4+2n$. This is indeed true, as follows from the observation by Harer (cf. [5], for instance) that $\mathcal{M}_{g,n}$ has the homotopy type of a CW-complex of dimension $4g-4+n$.

We next look at the codimension two classes $W_{m_*,n}$ on $\overline{\mathcal{M}}_{1,2}$. The remark we just made implies in particular that these are both zero. The corresponding conjectural formulas coming from the DFIZ theorem are

$$0 = 120\kappa_2 - 6\xi_{irr*}(\psi_1) - 6\xi_{1,\emptyset *}(\psi_1 \times 1) + 30\xi_{1,\emptyset *}(\psi_1 \times 1)\,, \tag{5.1}$$

$$\begin{aligned} 0 &= 72\kappa_1^2 - 348\kappa_2 - 6\xi_{irr*}(\kappa_1) - 12\xi_{1,\emptyset *}(\kappa_1 \times 1) + 12\xi_{irr*}(\psi_4) + 12\xi_{1,\emptyset *}(\psi_1 \times 1) \\ &\quad + \frac{1}{2}\sum_{p,P}\xi_{(B,p,P)_*}(1) + \frac{1}{4}\sum_{p,P}\xi_{(D,p,P)_*}(1) - 36\xi_{1,\emptyset *}(\psi_1 \times 1)\,. \end{aligned} \tag{5.2}$$

This last formula is a special case of formula (3.15). We have used the fact that graphs of type A and B cannot occur in our situation. It should also be observed that the term corresponding to graph B consists of a single summand, for the two marked points must of necessity be on the rational tail. On the other hand, two distinct summands appear in the term corresponding to graph D. In fact, this corresponds geometrically to two smooth rational curves joined at two points, each component carrying a marked point, and these can be labelled in two different ways. Now, using the computations of the two preceding examples, we have that

$$\begin{aligned} &\int_{\overline{\mathcal{M}}_{1,2}} \xi_{irr*}(\kappa_1) = \int_{\overline{\mathcal{M}}_{0,4}} \kappa_1 = 1\,, &&\int_{\overline{\mathcal{M}}_{1,2}} \xi_{irr*}(\psi_4) = \int_{\overline{\mathcal{M}}_{0,4}} \psi_4 = 1\,, \\ &\int_{\overline{\mathcal{M}}_{1,2}} \xi_{1,\emptyset *}(\psi_1 \times 1) = \int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}\,, &&\int_{\overline{\mathcal{M}}_{1,2}} \xi_{1,\emptyset *}(\kappa_1 \times 1) = \int_{\overline{\mathcal{M}}_{1,1}} \kappa_1 = \frac{1}{24}\,, \\ &\int_{\overline{\mathcal{M}}_{1,2}} \xi_{(B,p,P)_*}(1) = \int_{\overline{\mathcal{M}}_{0,3}\times\overline{\mathcal{M}}_{0,3}} 1 = 1\,, &&\int_{\overline{\mathcal{M}}_{1,2}} \xi_{(D,p,P)_*}(1) = \int_{\overline{\mathcal{M}}_{0,3}\times\overline{\mathcal{M}}_{0,3}} 1 = 1\,. \end{aligned}$$

On the other hand one has

$$\int_{\overline{\mathcal{M}}_{1,2}} \kappa_2 = \frac{1}{24}\,, \qquad \int_{\overline{\mathcal{M}}_{1,2}} \kappa_1^2 = \frac{1}{8}\,.$$

This follows either from a simple algebro-geometric calculation or, alternatively, by noticing that the integrals to be computed are just

$$\langle \tau_0^2\tau_3 \rangle = \langle \tau_1 \rangle = \frac{1}{24}$$

and

$$\langle \tau_0^2\tau_2^2 \rangle - \langle \tau_0^2\tau_3 \rangle = 2\langle \tau_0\tau_1\tau_2 \rangle - \langle \tau_1 \rangle = 2\langle \tau_0\tau_2 \rangle + 2\langle \tau_1\tau_1 \rangle - \langle \tau_1 \rangle = 3\langle \tau_1 \rangle = \frac{1}{8}\,.$$

Substituting these values in the right-hand sides of (5.1) and (5.2) gives zero, as desired.

We end this section with a few remarks. The first one concerns possible generalizations of Lemma 2, and hence of Proposition 1, of section 4 to higher codimension. It is clear that an essential ingredient in the proof of that lemma is the possibility of writing every degree two cohomology class as a linear combination of standard ones. The analogue of this is only known to hold in degree four, although it is a standard conjecture that it should in fact hold in every degree, provided the genus is sufficiently large. However, it would not be without interest, and perhaps provable with the same methods we have used in this section, that an analogue of Lemma 2 holds in all degrees, provided attention is restricted only to those cohomology classes which can be expressed as linear combinations of standard ones.

The second remark has to do with relations among standard classes. There is a set of conjectures, due to Faber [unpublished], dealing with the relations that the classes κ_i satisfy in the rational cohomology of $\mathcal{M}_{g,n}$. It is known [9][10] that there are no such relations in degree less than $g/6$. For higher degrees Faber provides an explicit algebro-geometric recipe to generate relations which, conjecturally, should yield all relations. It occurred to us that perhaps an alternative way of obtaining relations could be via a recent result of Mulase [11] which states that the function $Z(t_*, s_*)$ satisfies the KdV hierarchy as a function of s_*, for any fixed t_*. Making these equations explicit would yield relations among the derivatives of F with respect to the s variables, and we have explained how these could be translated into relations among the κ_i.

Finally, it is clear that one needs to understand better the DFIZ theorem. In particular, one should try and systematically explain the marvellous cancellations that experimentally occur in all the cases we have been able to compute.

Appendix

Below are listed the expressions of the derivatives of F with respect to the s variables in terms of derivatives with respect to t variables that are relevant to the problem of expressing classes $W_{m_*,n}$ in terms of algebro-geometric classes, up to weight 15. We recall that the weight of a partial derivative $\prod(\partial/\partial s_i)^{m_i}$ is defined to be $\sum m_i(2i+1)$. Of course the equalities below hold *only at* $s_* = \hat{s}_* = (0, 1, 0, \dots)$.

$$\frac{\partial F}{\partial s_2} = 12\,\frac{\partial F}{\partial t_2} - \frac{1}{2}\,\frac{\partial^2 F}{\partial t_0^2} - \frac{1}{2}\left(\frac{\partial F}{\partial t_0}\right)^2$$

$$\frac{\partial F}{\partial s_3} = 120\,\frac{\partial F}{\partial t_3} - 6\,\frac{\partial^2 F}{\partial t_0 \partial t_1} - 6\,\frac{\partial F}{\partial t_1}\,\frac{\partial F}{\partial t_0} + \frac{5}{4}\,\frac{\partial F}{\partial t_0}$$

$$\frac{\partial F}{\partial s_4} = 1680\,\frac{\partial F}{\partial t_4} - 18\,\frac{\partial^2 F}{\partial t_1^2} - 18\left(\frac{\partial F}{\partial t_1}\right)^2 - 60\,\frac{\partial^2 F}{\partial t_0 \partial t_2} - 60\,\frac{\partial F}{\partial t_2}\,\frac{\partial F}{\partial t_0} + \frac{7}{6}\,\frac{\partial^3 F}{\partial t_0^3} + \frac{7}{2}\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0^2} + \frac{7}{6}\left(\frac{\partial F}{\partial t_0}\right)^3 + \frac{49}{2}\,\frac{\partial F}{\partial t_1} - \frac{35}{96}$$

$$\frac{\partial^2 F}{\partial s_2^2} = 144\,\frac{\partial^2 F}{\partial t_2^2} - 840\,\frac{\partial F}{\partial t_3} - 12\,\frac{\partial^3 F}{\partial t_0^2 \partial t_2} - 24\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0 \partial t_2} + 24\,\frac{\partial^2 F}{\partial t_0 \partial t_1} + 24\,\frac{\partial F}{\partial t_1}\,\frac{\partial F}{\partial t_0}$$
$$+ \frac{1}{4}\,\frac{\partial^4 F}{\partial t_0^4} + \frac{\partial F}{\partial t_0}\,\frac{\partial^3 F}{\partial t_0^3} + \frac{1}{2}\left(\frac{\partial^2 F}{\partial t_0^2}\right)^2 + \left(\frac{\partial F}{\partial t_0}\right)^2 \frac{\partial^2 F}{\partial t_0^2} - 3\,\frac{\partial F}{\partial t_0}$$

$$\frac{\partial F}{\partial s_5} = 30240\,\frac{\partial F}{\partial t_5} - 360\,\frac{\partial^2 F}{\partial t_1 \partial t_2} - 360\,\frac{\partial F}{\partial t_2}\,\frac{\partial F}{\partial t_1} - 840\,\frac{\partial^2 F}{\partial t_0 \partial t_3} - 840\,\frac{\partial F}{\partial t_3}\,\frac{\partial F}{\partial t_0}$$
$$+ 27\,\frac{\partial^3 F}{\partial t_0^2 \partial t_1} + 27\,\frac{\partial F}{\partial t_1}\,\frac{\partial^2 F}{\partial t_0^2} + 54\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0 \partial t_1} + 27\,\frac{\partial F}{\partial t_1}\left(\frac{\partial F}{\partial t_0}\right)^2 + 585\,\frac{\partial F}{\partial t_2}$$
$$- \frac{105}{8}\,\frac{\partial^2 F}{\partial t_0^2} - \frac{105}{8}\left(\frac{\partial F}{\partial t_0}\right)^2$$

$$\frac{\partial^2 F}{\partial s_2 \partial s_3} = 1440\,\frac{\partial^2 F}{\partial t_2 \partial t_3} - 15120\,\frac{\partial F}{\partial t_4} - 60\,\frac{\partial^3 F}{\partial t_0^2 \partial t_3} - 120\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0 \partial t_3} - 72\,\frac{\partial^3 F}{\partial t_0 \partial t_1 \partial t_2}$$
$$- 72\,\frac{\partial F}{\partial t_1}\,\frac{\partial^2 F}{\partial t_0 \partial t_2} - 72\,\frac{\partial^2 F}{\partial t_1 \partial t_2}\,\frac{\partial F}{\partial t_0} + 90\,\frac{\partial^2 F}{\partial t_1^2} + 90\left(\frac{\partial F}{\partial t_1}\right)^2 + 375\,\frac{\partial^2 F}{\partial t_0 \partial t_2}$$
$$+ 360\,\frac{\partial F}{\partial t_2}\,\frac{\partial F}{\partial t_0} + 3\,\frac{\partial^4 F}{\partial t_0^3 \partial t_1} + 6\,\frac{\partial^2 F}{\partial t_0 \partial t_1}\,\frac{\partial^2 F}{\partial t_0^2} + 9\,\frac{\partial F}{\partial t_0}\,\frac{\partial^3 F}{\partial t_0^2 \partial t_1} + 3\,\frac{\partial F}{\partial t_1}\,\frac{\partial^3 F}{\partial t_0^3}$$
$$+ 6\,\frac{\partial F}{\partial t_1}\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0^2} + 6\left(\frac{\partial F}{\partial t_0}\right)^2 \frac{\partial^2 F}{\partial t_0 \partial t_1} - \frac{45}{8}\,\frac{\partial^3 F}{\partial t_0^3} - \frac{65}{4}\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0^2} - 5\left(\frac{\partial F}{\partial t_0}\right)^3$$
$$- \frac{165}{2}\,\frac{\partial F}{\partial t_1} + \frac{29}{32}$$

$$\frac{\partial F}{\partial s_6} = 665280\,\frac{\partial F}{\partial t_6} - 1800\,\frac{\partial^2 F}{\partial t_2^2} - 1800\left(\frac{\partial F}{\partial t_2}\right)^2 - 5040\,\frac{\partial^2 F}{\partial t_1 \partial t_3} - 5040\,\frac{\partial F}{\partial t_3}\,\frac{\partial F}{\partial t_1}$$
$$- 15120\,\frac{\partial^2 F}{\partial t_0 \partial t_4} - 15120\,\frac{\partial F}{\partial t_4}\,\frac{\partial F}{\partial t_0} + 16170\,\frac{\partial F}{\partial t_3} + 198\,\frac{\partial^3 F}{\partial t_0 \partial t_1^2} + 396\,\frac{\partial F}{\partial t_1}\,\frac{\partial^2 F}{\partial t_0 \partial t_1}$$
$$+ 198\,\frac{\partial^2 F}{\partial t_1^2}\,\frac{\partial F}{\partial t_0} + 198\left(\frac{\partial F}{\partial t_1}\right)^2 \frac{\partial F}{\partial t_0} + 330\,\frac{\partial^3 F}{\partial t_0^2 \partial t_2} + 330\,\frac{\partial F}{\partial t_2}\,\frac{\partial^2 F}{\partial t_0^2}$$
$$+ 660\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0 \partial t_2} + 330\,\frac{\partial F}{\partial t_2}\left(\frac{\partial F}{\partial t_0}\right)^2 - \frac{33}{8}\,\frac{\partial^4 F}{\partial t_0^4} - \frac{33}{2}\,\frac{\partial F}{\partial t_0}\,\frac{\partial^3 F}{\partial t_0^3} - \frac{99}{8}\left(\frac{\partial^2 F}{\partial t_0^2}\right)^2$$
$$- \frac{99}{4}\left(\frac{\partial F}{\partial t_0}\right)^2 \frac{\partial^2 F}{\partial t_0^2} - \frac{33}{8}\left(\frac{\partial F}{\partial t_0}\right)^4 - \frac{891}{2}\,\frac{\partial^2 F}{\partial t_0 \partial t_1} - \frac{891}{2}\,\frac{\partial F}{\partial t_1}\,\frac{\partial F}{\partial t_0} + \frac{1155}{32}\,\frac{\partial F}{\partial t_0}$$

$$
\begin{aligned}
\frac{\partial^2 F}{\partial s_2 \partial s_4} = {} & 20160\,\frac{\partial^2 F}{\partial t_2 \partial t_4} - 332640\,\frac{\partial F}{\partial t_5} - 216\,\frac{\partial^3 F}{\partial t_1^2 \partial t_2} - 432\,\frac{\partial F}{\partial t_1}\,\frac{\partial^2 F}{\partial t_1 \partial t_2} - 840\,\frac{\partial^3 F}{\partial t_0^2 \partial t_4} \\
& - 1680\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0 \partial t_4} - 720\,\frac{\partial^3 F}{\partial t_0 \partial t_2^2} - 720\,\frac{\partial^2 F}{\partial t_2^2}\,\frac{\partial F}{\partial t_0} - 720\,\frac{\partial F}{\partial t_2}\,\frac{\partial^2 F}{\partial t_0 \partial t_2} \\
& + 6720\,\frac{\partial^2 F}{\partial t_0 \partial t_3} + 6720\,\frac{\partial F}{\partial t_3}\,\frac{\partial F}{\partial t_0} + 2814\,\frac{\partial^2 F}{\partial t_1 \partial t_2} + 2520\,\frac{\partial F}{\partial t_2}\,\frac{\partial F}{\partial t_1} \\
& + 9\,\frac{\partial^4 F}{\partial t_0^2 \partial t_1^2} + 18\,\frac{\partial F}{\partial t_1}\,\frac{\partial^3 F}{\partial t_0^2 \partial t_1} + 18\left(\frac{\partial^2 F}{\partial t_0 \partial t_1}\right)^2 + 18\,\frac{\partial F}{\partial t_0}\,\frac{\partial^3 F}{\partial t_0 \partial t_1^2} \\
& + 36\,\frac{\partial F}{\partial t_1}\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0 \partial t_1} + 44\,\frac{\partial^4 F}{\partial t_0^3 \partial t_2} + 30\,\frac{\partial F}{\partial t_2}\,\frac{\partial^3 F}{\partial t_0^3} + 102\,\frac{\partial^2 F}{\partial t_0 \partial t_2}\,\frac{\partial^2 F}{\partial t_0^2} \\
& + 132\,\frac{\partial F}{\partial t_0}\,\frac{\partial^3 F}{\partial t_0^2 \partial t_2} + 102\left(\frac{\partial F}{\partial t_0}\right)^2\frac{\partial^2 F}{\partial t_0 \partial t_2} + 60\,\frac{\partial F}{\partial t_2}\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0^2} - 2835\,\frac{\partial F}{\partial t_2} \\
& - \frac{637}{4}\,\frac{\partial^3 F}{\partial t_0^2 \partial t_1} - 147\,\frac{\partial F}{\partial t_1}\left(\frac{\partial F}{\partial t_0}\right)^2 - 147\,\frac{\partial F}{\partial t_1}\,\frac{\partial^2 F}{\partial t_0^2} - \frac{637}{2}\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0 \partial t_1} \\
& - \frac{7}{12}\,\frac{\partial^5 F}{\partial t_0^5} - \frac{35}{12}\,\frac{\partial F}{\partial t_0}\,\frac{\partial^4 F}{\partial t_0^4} - \frac{21}{4}\,\frac{\partial^2 F}{\partial t_0^2}\,\frac{\partial^3 F}{\partial t_0^3} - 7\,\frac{\partial F}{\partial t_0}\left(\frac{\partial^2 F}{\partial t_0^2}\right)^2 \\
& - \frac{21}{4}\left(\frac{\partial F}{\partial t_0}\right)^2\frac{\partial^3 F}{\partial t_0^3} - \frac{7}{2}\left(\frac{\partial F}{\partial t_0}\right)^3\frac{\partial^2 F}{\partial t_0^2} + \frac{385}{8}\,\frac{\partial^2 F}{\partial t_0^2} + \frac{385}{8}\left(\frac{\partial F}{\partial t_0}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 F}{\partial s_3^2} = {} & 14400\,\frac{\partial^2 F}{\partial t_3^2} - 332640\,\frac{\partial F}{\partial t_5} - 1440\,\frac{\partial^3 F}{\partial t_0 \partial t_1 \partial t_3} - 1440\,\frac{\partial^2 F}{\partial t_1 \partial t_3}\,\frac{\partial F}{\partial t_0} - 1440\,\frac{\partial F}{\partial t_1}\,\frac{\partial^2 F}{\partial t_0 \partial t_3} \\
& + 7020\,\frac{\partial^2 F}{\partial t_0 \partial t_3} + 6720\,\frac{\partial F}{\partial t_3}\,\frac{\partial F}{\partial t_0} + 2160\,\frac{\partial^2 F}{\partial t_1 \partial t_2} + 2160\,\frac{\partial F}{\partial t_2}\,\frac{\partial F}{\partial t_1} + 36\,\frac{\partial^4 F}{\partial t_0^2 \partial t_1^2} \\
& + 36\left(\frac{\partial^2 F}{\partial t_0 \partial t_1}\right)^2 + 36\,\frac{\partial^2 F}{\partial t_1^2}\,\frac{\partial^2 F}{\partial t_0^2} + 72\,\frac{\partial F}{\partial t_1}\,\frac{\partial^3 F}{\partial t_0^2 \partial t_1} + 72\,\frac{\partial F}{\partial t_0}\,\frac{\partial^3 F}{\partial t_0 \partial t_1^2} \\
& + 36\left(\frac{\partial F}{\partial t_1}\right)^2\frac{\partial^2 F}{\partial t_0^2} + 72\,\frac{\partial F}{\partial t_1}\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0 \partial t_1} + 36\,\frac{\partial^2 F}{\partial t_1^2}\left(\frac{\partial F}{\partial t_0}\right)^2 - 2400\,\frac{\partial F}{\partial t_2} \\
& - 165\,\frac{\partial^3 F}{\partial t_0^2 \partial t_1} - 165\,\frac{\partial F}{\partial t_1}\,\frac{\partial^2 F}{\partial t_0^2} - 150\,\frac{\partial F}{\partial t_1}\left(\frac{\partial F}{\partial t_0}\right)^2 - 315\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0 \partial t_1} \\
& + \frac{725}{16}\,\frac{\partial^2 F}{\partial t_0^2} + \frac{175}{4}\left(\frac{\partial F}{\partial t_0}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F}{\partial s_7} = {} & 17297280\,\frac{\partial F}{\partial t_7} - 50400\,\frac{\partial^2 F}{\partial t_2 \partial t_3} - 50400\,\frac{\partial F}{\partial t_3}\,\frac{\partial F}{\partial t_2} - 90720\,\frac{\partial^2 F}{\partial t_1 \partial t_4} - 90720\,\frac{\partial F}{\partial t_4}\,\frac{\partial F}{\partial t_1} \\
& - 332640\,\frac{\partial^2 F}{\partial t_0 \partial t_5} - 332640\,\frac{\partial F}{\partial t_5}\,\frac{\partial F}{\partial t_0} + 468\,\frac{\partial^3 F}{\partial t_1^3} + 1404\,\frac{\partial F}{\partial t_1}\,\frac{\partial^2 F}{\partial t_1^2} + 468\left(\frac{\partial F}{\partial t_1}\right)^3 \\
& + 4680\,\frac{\partial^3 F}{\partial t_0 \partial t_1 \partial t_2} + 4680\,\frac{\partial F}{\partial t_1}\,\frac{\partial^2 F}{\partial t_0 \partial t_2} + 4680\,\frac{\partial F}{\partial t_2}\,\frac{\partial^2 F}{\partial t_0 \partial t_1} + 4680\,\frac{\partial^2 F}{\partial t_1 \partial t_2}\,\frac{\partial F}{\partial t_0} \\
& + 4680\,\frac{\partial F}{\partial t_2}\,\frac{\partial F}{\partial t_1}\,\frac{\partial F}{\partial t_0} + 5460\,\frac{\partial^3 F}{\partial t_0^2 \partial t_3} + 5460\,\frac{\partial F}{\partial t_3}\,\frac{\partial^2 F}{\partial t_0^2} + 10920\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0 \partial t_3} \\
& + 5460\,\frac{\partial F}{\partial t_3}\left(\frac{\partial F}{\partial t_0}\right)^2 + 507780\,\frac{\partial F}{\partial t_4} - 10725\,\frac{\partial^2 F}{\partial t_0 \partial t_2} - 10725\,\frac{\partial F}{\partial t_2}\,\frac{\partial F}{\partial t_0} \\
& - \frac{5577}{2}\,\frac{\partial^2 F}{\partial t_1^2} - \frac{5577}{2}\left(\frac{\partial F}{\partial t_1}\right)^2 - 143\,\frac{\partial^4 F}{\partial t_0^3 \partial t_1} - 143\,\frac{\partial F}{\partial t_1}\,\frac{\partial^3 F}{\partial t_0^3} - 429\,\frac{\partial^2 F}{\partial t_0 \partial t_1}\,\frac{\partial^2 F}{\partial t_0^2} \\
& - 429\,\frac{\partial F}{\partial t_0}\,\frac{\partial^3 F}{\partial t_0^2 \partial t_1} - 429\left(\frac{\partial F}{\partial t_0}\right)^2 \frac{\partial^2 F}{\partial t_0 \partial t_1} - 429\,\frac{\partial F}{\partial t_1}\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0^2} - 143\,\frac{\partial F}{\partial t_1}\left(\frac{\partial F}{\partial t_0}\right)^3 \\
& + \frac{1001}{8}\,\frac{\partial^3 F}{\partial t_0^3} + \frac{3003}{8}\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0^2} + \frac{1001}{8}\left(\frac{\partial F}{\partial t_0}\right)^3 + \frac{27027}{16}\,\frac{\partial F}{\partial t_1} - \frac{5005}{384}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 F}{\partial s_2^3} = {} & 1728\,\frac{\partial^3 F}{\partial t_2^3} - 30240\,\frac{\partial^2 F}{\partial t_2 \partial t_3} - 216\,\frac{\partial^4 F}{\partial t_0^2 \partial t_2^2} - 432\left(\frac{\partial^2 F}{\partial t_0 \partial t_2}\right)^2 - 432\,\frac{\partial F}{\partial t_0}\,\frac{\partial^3 F}{\partial t_0 \partial t_2^2} \\
& + 864\,\frac{\partial^3 F}{\partial t_0 \partial t_1 \partial t_2} + 864\,\frac{\partial F}{\partial t_1}\,\frac{\partial^2 F}{\partial t_0 \partial t_2} + 864\,\frac{\partial^2 F}{\partial t_1 \partial t_2}\,\frac{\partial F}{\partial t_0} + 1260\,\frac{\partial^3 F}{\partial t_0^2 \partial t_3} \\
& + 2520\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0 \partial t_3} + 9\,\frac{\partial^5 F}{\partial t_0^4 \partial t_2} + 36\,\frac{\partial^2 F}{\partial t_0 \partial t_2}\,\frac{\partial^3 F}{\partial t_0^3} + 36\,\frac{\partial F}{\partial t_0}\,\frac{\partial^4 F}{\partial t_0^3 \partial t_2} + 36\,\frac{\partial^2 F}{\partial t_0^2}\,\frac{\partial^3 F}{\partial t_0^2 \partial t_2} \\
& + 72\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0 \partial t_2}\,\frac{\partial^2 F}{\partial t_0^2} + 36\left(\frac{\partial F}{\partial t_0}\right)^2 \frac{\partial^3 F}{\partial t_0^2 \partial t_2} + 151200\,\frac{\partial F}{\partial t_4} - 576\,\frac{\partial^2 F}{\partial t_1^2} - 576\left(\frac{\partial F}{\partial t_1}\right)^2 \\
& - 2628\,\frac{\partial^2 F}{\partial t_0 \partial t_2} - 2520\,\frac{\partial F}{\partial t_2}\,\frac{\partial F}{\partial t_0} - 36\,\frac{\partial^4 F}{\partial t_0^3 \partial t_1} - 108\,\frac{\partial F}{\partial t_0}\,\frac{\partial^3 F}{\partial t_0^2 \partial t_1} - 36\,\frac{\partial F}{\partial t_1}\,\frac{\partial^3 F}{\partial t_0^3} \\
& - 72\,\frac{\partial^2 F}{\partial t_0 \partial t_1}\,\frac{\partial^2 F}{\partial t_0^2} - 72\,\frac{\partial F}{\partial t_1}\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0^2} - 72\left(\frac{\partial F}{\partial t_0}\right)^2 \frac{\partial^2 F}{\partial t_0 \partial t_1} - \frac{1}{8}\,\frac{\partial^6 F}{\partial t_0^6} - \frac{3}{4}\,\frac{\partial F}{\partial t_0}\,\frac{\partial^5 F}{\partial t_0^5} \\
& - \frac{3}{2}\,\frac{\partial^2 F}{\partial t_0^2}\,\frac{\partial^4 F}{\partial t_0^4} - \frac{5}{4}\left(\frac{\partial^3 F}{\partial t_0^3}\right)^2 - 6\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0^2}\,\frac{\partial^3 F}{\partial t_0^3} - \left(\frac{\partial^2 F}{\partial t_0^2}\right)^3 - \frac{3}{2}\left(\frac{\partial F}{\partial t_0}\right)^2 \frac{\partial^4 F}{\partial t_0^4} \\
& - 3\left(\frac{\partial F}{\partial t_0}\right)^2 \left(\frac{\partial^2 F}{\partial t_0^2}\right)^2 - \left(\frac{\partial F}{\partial t_0}\right)^3 \frac{\partial^3 F}{\partial t_0^3} + \frac{63}{2}\,\frac{\partial^3 F}{\partial t_0^3} + 90\,\frac{\partial F}{\partial t_0}\,\frac{\partial^2 F}{\partial t_0^2} + 27\left(\frac{\partial F}{\partial t_0}\right)^3 \\
& + 378\,\frac{\partial F}{\partial t_1} - \frac{63}{20}
\end{aligned}
$$

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