# A cyclic flow on Teichmüller space

Francesco Bonsante

(joint work with G. Mondello and J.M. Schlenker)

May 15, 2012

# Landslides and smooth grafting

We introduce two new families of deformations on Teichmueller spaces called landslides and smooth grafting.
They can be regarded as a *smooth* version of earthquakes and grafting respectively.

- Earthquakes/grafting depend on the choice of a measured geodesic lamination.
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# Main idea

- The grafting of $S$ along $\lambda$ can be defined by applying some general recipe to surface obtained by bending $S$ along $\lambda$ in the hyperbolic space.
- The earthquake on $S$ along $\lambda$ can be defined by applying some (other) general recipe to the surface obtained by bending $S$ along $\lambda$ in the Anti de Sitter space,
- Landslides and smooth grafting are defined by replacing bent surfaces by constant curvature convex surfaces and applying the same recipes.

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# Goals

- Show that landslides share good properties as earthquakes.
- Prove that earthquakes can be regarded as a limit case of landslides.

# Notation

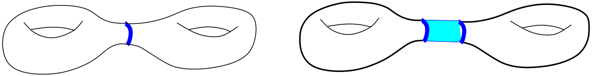
- $S =$ differentiable closed oriented surface of genus $g \geq 2$.
- $Teich(S)$={hyperbolic metrics on $S$}$/Diffeo_0(S) =$ {complex strutures on $S$}$/Diffeo_0(S)$.
- $\mathcal{ML}(S)$={measured geodesic laminations of $S$}.

Fix $\lambda$=measured geodesic lamination on $S$.
The grafting along $\lambda$ is a map

$$gr_\lambda : Teich(S) \to Teich(S)$$

If $\lambda = (c, a)$ and $h$ is a hyperbolic metric, $gr_\lambda([h])$ is constructed as follows

- Cut the surface along the $h$-geodesic representative of $c$.
- Insert a Euclidean annulus of width equal to $a$.



# 2-dimensional definition of grafting

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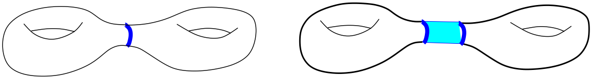

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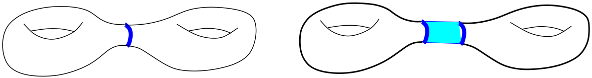

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The right (left) earthquake on $(S, h)$ along $\lambda$ is a map

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If $\lambda = (c, a)$ then $E^r_\lambda(h)$ is obtained as follows

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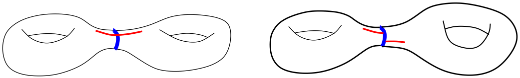

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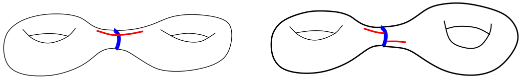

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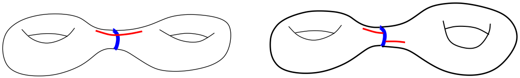
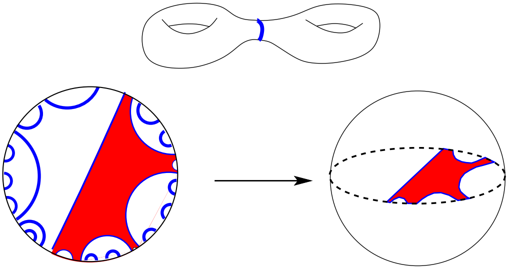


The bending of $(S, h)$ into the hyperbolic space along $\lambda$ is a map $\beta : \mathbb{H}^2 = \tilde{S} \to \mathbb{H}^3$ that is an isometric embedding on each region of $\tilde{S} \setminus \tilde{\lambda}$.
If $\lambda = (c, a)$, the map $\beta$ is construced in the following way:



# Bent surfaces in hyperbolic space

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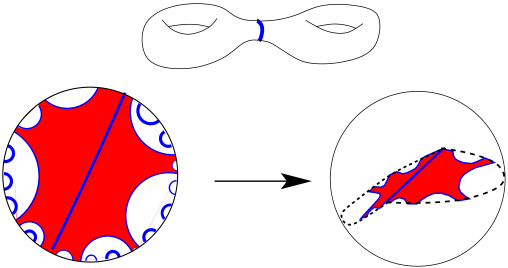

There exists a representation $\rho : \pi_1(S) \to PSL_2(\mathbb{C})$ such that $\beta(\gamma x) = \rho(\gamma)\beta(x)$ (the holonomy of the bending map).

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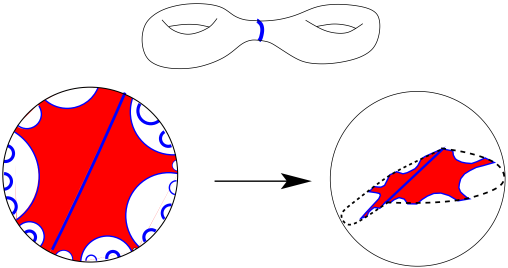

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- Let $\sigma : \tilde{S} \to \mathbb{H}^3$ be an equivariant locally convex $C^1$-immersion.
- For $x \in \tilde{S}$, let $d(x) \in S^2_\infty$ endpoint of the geodesic ray from $\sigma(x)$ orthogonal to $\sigma(\tilde{S})$ and pointing in the concave side.
- The map $d : \tilde{S} \to S^2_\infty$ is an equivariant locally homeomorphism. A conformal structure is induced on $S$ by $d$.
- Applying this construction on the bending map $\beta$, the conformal structure obtained is $gr_\lambda(S)$.

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### Problem

The map $\beta$ is not $C^1$, so in general $d$ cannot be defined.

### How to fix the problem

A normal vector $v$ of $\beta$ at $x$ of the bending map is a unit vector of $T_{\beta(x)}\mathbb{H}^3$ which is a (local)support plane for $\beta(S)$.
$\tilde{U}$= set of couples $(x, v)$ with $x \in \tilde{S}$ and $v$ normal vector of $\beta$ at $x$.
The map $d : \tilde{U} \to S^2_\infty$ can be defined. Moreover $\tilde{U}/\pi_1(S) \cong S$.

# The Anti de Sitter space

$AdS_3$= Lorentz space-form of constant curvature $-1$.

- $Iso_0(AdS_3) = PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$.
- Space-like planes of $AdS_3$ are isometric to $\mathbb{H}^2$.
- A notion of angle between space-like planes is defined. The angle is a number in $[0, +\infty)$.

Given a hyperbolic metric $h$ on $S$ and a measured geodesic lamination $\lambda$, the bending of $S$ into $AdS_3$ can be defined

$$\alpha : \tilde{S} \to AdS_3$$

The map $\alpha$ is always an embedding.

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**THM (Mess)**

*Let $\lambda$ be a measured geodesic lamination on S. For any hyperbolic metric h, let $(\rho_l, \rho_r) : \pi_1(S) \to Isom(AdS_3)$ be the holonomy of the bending map $\alpha(h, \lambda)$. Then $\rho_l$ and $\rho_r$ are Fuchsian representations and*

$$\mathbb{H}^2/\rho_l = E^r_\lambda(h) \qquad \mathbb{H}^2/\rho_r = E^l_\lambda(h)$$

# Codazzi operators

Given a (hyperbolic) metric $h$ on $S$, $\nabla$= Levi Civita connection of $h$.
A Codazzi operator $b : TS \to TS$ is a solution of Codazzi equation:

$$d^{\nabla} b = 0, \text{ where } (d^{\nabla} b)(v, w) = \nabla_v(bw) - \nabla_w(bv) - b[v, w].$$

- The shape operators of surfaces in 3 Riemann manifolds are examples of Codazzi operators.
- If $b$ is a non degenerate Codazzi tensor, then the curvature of $h(b, b)$ is simply $-K_h / \det b$.

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# Labourie operators

- A Labourie operator on a hyperbolic surface $(S, h)$ is an operator $b : TS \to TS$ such that
  1. $b$ is $h$-self-adjoint positive operator.
  2. $\det b = 1$.
  3. $b$ solves the Codazzi equation for $h$: $d^{\nabla} b = 0$.
- If $b$ is a Labourie operator, then $h^{\star} = h(b\cdot, b\cdot)$ is hyperbolic.

**THM (Labourie)**

*Given $h, h'$ hyperbolic metric on $S$, there is a unique $h$-Labourie operator $b$ on $S$ such that $h(b\cdot, b\cdot)$ is isotopic to $h'$.*

- Given two hyperbolic metrics $(h, h')$ on $S$, the Labourie operator of the pair $(h, h')$ is the $h$-Labourie operator $b$ such that $h(b\cdot, b\cdot) \cong h'$.

# Definition of landslides

Fix a point $[h^\star] \in Teich(S)$ and $\theta \in \mathbb{R}$.
Given a hyperbolic metric $h$ on $S$,
$J$= complex structure induced by $h$.
$b = b(h, h^\star)$= Labourie operator of the pair $(h, h^\star)$.
Define $b_\theta = \cos(\theta/2)Id + \sin(\theta/2)Jb$
$b_\theta$ is a Codazzi operator and $\det b_\theta = 1$.
So the metric

$$L_{h^\star,\theta}(h) = h(b_\theta, b_\theta)$$

is hyperbolic.

**Remark**

*$L_{h^\star,\theta}$ is $2\pi$-periodic in $\theta$, and $L_{h^\star,\pi}(h) = h^\star$.*

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# Definition of smooth grafting

With the notation of the previous slice,
let $b'_s = \cosh(s)Id + \sinh(s)b$.
Then the smooth grafting of $S$ along $(h^\star, s)$ is the conformal structure induced by the metric

$$sgr_{h^\star,s}(h) = h(b'_s, b'_s)$$

**Remark**

*Since* $\det b'_s$ *is not constant the curvature of* $sgr_{h^\star,s}(h)$ *is not constant. Indeed it is equal to* $-1/\det(b'_s)$.

# Convex constant curvature surfaces in hyperbolic 3-manifolds

A K- hyperbolic immersion is an equivariant locally convex $C^2$-immersion

$$\sigma : \tilde{S} \to \mathbb{H}^3$$

such that the induced first fundamental form has constant curvature $K$.
If $\sigma$ is a K-hyperbolic immersion then

- $K \in [-1, 0)$.
- For $K \in (-1, 0)$ the shape operator $B$ is a positive self-adjoint operator which solves Codazzi equation and such that $\det B = 1 + K$.

The third fundamental form $III = I(B, B)$ has constant curvature $K/(1 + K)$.

# Description of $K$- hyperbolic immersions

**Prop (Labourie)**

*Let us fix $K \in (-1, 0)$. For every pair of hyperbolic metrics $h$ and $h^\star$ on $S$ there is a unique K- hyperbolic immersion $\sigma_K(h, h^\star) : \tilde{S} \to \mathbb{H}^3$ such that the first fundamental form I is proportional to h and the third fundamental form is proportional to $h^\star$.*

**Proof.**

Let $b$ be the Labourie operator of $(h, h^\star)$ and define

$$I = \frac{1}{K} h \qquad B = (1 + K)^{1/2} b$$

They are the embedding data of a K immersion which verifies the conditions of the theorem. □

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$$\{\text{K- hyperbolic immersions}\} \leftrightarrow Teich(S) \times Teich(S)$$

A $\kappa$-isometric $AdS_3$ immersion is an equivariant map

$$\tau : \tilde{S} \to AdS_3$$

such that the induced first metric is Riemannian of constant curvature $\kappa$.

If $\tau$ is a $\kappa$- AdS immersion then

- $\kappa \in (-\infty, -1]$.
- For $\kappa \in (-\infty, -1)$ the shape operator $B$ is a positive self-adjoint operator which solves Codazzi equation and such that $\det B = -\kappa - 1$.

The third fundamental form has constant curvature $-\kappa/(\kappa + 1)$

# Description of K- AdS immersions

**Prop**

*Let us fix $\kappa \in (-\infty, -1)$. For every pair of hyperbolic metrics h and $h^\star$ on Teich$(S)$ there is a unique $\kappa$- AdS embedding $\tau_\kappa(h, h^\star) : \tilde{S} \to \mathbb{H}^3$ such that the first fundamental form I is proportional to h and the third fundamental form is proportional to $h^\star$*

$$\{\kappa\text{- AdS immersions } \kappa\} \leftrightarrow Teich(S) \times Teich(S)$$

Let us fix $[h^\star] \in Teich(S)$ and $s > 0$.
Given a hyperbolic metric $h$ let us consider the K- hyperbolic immersion

$$\sigma_K(h, h^\star) : \tilde{S} \to \mathbb{H}^3$$

for $K = -\cosh(s/2)^{-1}$.
For any $x \in \tilde{S}$ let $d(x) \in S^2_\infty$ be the final point of the ray through $\sigma_K(x)$ orthogonal to the immersion.
$d : \tilde{S} \to S^2_\infty$ is an equivariant map, so it induces a complex structure on $S$.

**Lemma**

*This complex structure is isomorphic to $sgr_{s,h^*}(h)$.*

# The smooth grafting : a 3-dimensional characterization

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## The landslide: a 3-dimensional characterization

Fix $[h^\star] \in Teich(S)$, $\theta \in [0, \pi)$ and $h$ a hyperbolic metric on $S$.
Let us consider the $\kappa$-AdS embedding

$$\tau_\kappa(h, h^\star) : \tilde{S} \to AdS_3$$

for $\kappa = \cos(\theta/2)^{-1}$.

**Lemma**

*The left and the right holonomies of $\tau_\kappa(h, b)$ are Fuchsian representations $\rho_l$ and $\rho_r$ and*

$$\mathbb{H}^2/\rho_l = L_{h^*,-\theta}(h) \qquad \mathbb{H}^2/\rho_r = L_{h^*,\theta}(h).$$

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# Earthquake flow

The earthquake deformation verifies this simple semigroup law

$$E^l_{t\lambda} \circ E^l_{s\lambda}(h) = E^l_{(t+s)\lambda}(h)$$

$Teich(S) \times \mathcal{ML}(S)$= trivial fiber bundle on $Teich(S)$.

**Remark**

*There is an $\mathbb{R}$-action on Teich$(S) \times \mathcal{ML}(S)$ defined by*

$$E_t(h,\lambda) = \begin{cases} (E^l_{t\lambda}(h), \lambda) & \text{if } t \geq 0 \\ (E^r_{t\lambda}(h), \lambda) & \text{if } t \leq 0 \end{cases}$$

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# Flow properties of landslides

For any $\theta \in \mathbb{R}/2\pi\mathbb{Z} = S^1$ let us consider

$$L_\theta : Teich(S) \times Teich(S) \to Teich(S) \times Teich(S)$$

defined by $L_\theta(h, h^\star) = (L_{h^\star,\theta}(h), L_{h,\theta}(h^\star))$

**Lemma**

$L_\theta \circ L_{\theta'} = L_{\theta+\theta'}$.

**Remark**

$L_\pi(h, h^\star) = (h^\star, h)$.

We may suppose that $h^\star = h(b\cdot, b\cdot)$.

- The Labourie operator of the pair $(h^\star, h)$ is $b^{-1}$.
- $L_{h,\theta}(h^\star) = L_{h^\star,\pi+\theta}(h)$.
- If $h_\theta = L_{h^\star,\theta}(h)$ then the Labourie operator of $L_\theta(h, h^\star) = (h_\theta, h_{\pi+\theta})$ is $b_\theta^{-1} \circ b \circ b_\theta$.

# Earthquake theorem

THM (Kerckhoff/Thurston/Mess)

*Given* $[h]$ *and* $[h']$ *in Teich*$(S)$ *there exists a unique lamination* $\lambda$ *such that*

$$E^l_\lambda([h]) = [h']$$

# Earthquake theorem: reformulation

**THM (Kerckhoff/Thurston/Mess)**

*Given* $[h]$ *and* $[h']$ *in Teich*$(S)$ *and* $x \in \mathbb{R}$*, there exists a unique lamination* $\lambda$ *such that*

$$E_x([h], \lambda) = ([h'], \lambda)$$

**THM (B-Mondello-Schelnker)**

*Given* $[h]$ *and* $[h']$ *in* $Teich(S)$ *and* $\theta \in S^1$, *there exists a unique hyperbolic metric* $h^\star$ *such that*

$$L_{h^\star,\theta}(h) = h'$$

# Landslide theorem: proof

- Mess proved that there exists an *AdS* spacetime $M(h,h') = S \times \mathbb{R}$ such that $\mathbb{H}^2/\rho_l = (S,h)$ and $\mathbb{H}^2/\rho_r = (S,h')$.
- Barbot, Beguin, Zeghib proved that $M$ contains a unique convex surface $S$ of constant curvature $\kappa = -1/\cos^2(\theta/4)$.
- Let $h_+ = \frac{1}{\cos^2\theta/4} I_S$ and $h_+^\star = \frac{1}{\sin^2\theta/4} III_S$. $h_+$ and $h_+^\star$ are hyerbolic metrics.
- We have $L_{h_+^\star,-\theta/2}(h_+) = h$ and $L_{h_+^\star,\theta/2}(h_+) = h'$.
- By the flow properties, if we put $h^\star = L_{h_+,-\theta/2}(h_+^*)$ we have that $L_{h^\star,\theta}(h) = h'$

# Landslide theorem: proof

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**THM (McMullen)**

*Fix a hyperbolic surface h and a measured geodesic lamination $\lambda$. If $\mathbb{H}$ denotes the upper half plane of $\mathbb{C}$, the map*

$$E_c(h, \lambda) : \mathbb{H} \ni z = t + is \mapsto gr_{s\lambda}(E^r_{t\lambda}(h)) \in Teich(S)$$

*is holomorphic.*

If we put $gr_s : Teich(S) \times \mathcal{ML}(S) \ni (h, \lambda) \mapsto gr_{s\lambda}(h) \in Teich(S)$,we can write

$$E_c(h, \lambda)(z) = gr_s \circ E_{-t}(h, \lambda)$$

# Complex landslides

Let us fix two hyperbolic metrics $h$ and $h^\star$ on $S$.

**THM (B-Mondello-Schlenker)**

*Let us put* $L_\theta(h, h^\star) = (h_\theta, h^\star_\theta)$ *The map*

$$L_C(h, h^\star) : S^1 \times [0, +\infty) \ni \theta + is \mapsto sgr_{h^\star_{-\theta}, s}(h_{-\theta}) \in Teich(S)$$

*is a holomorphic embedding.*

- If we put $sgr_s(h, h^\star) = sgr_{s,h^\star}(h)$ we have $L_C(h, h^\star)(\theta + is) = sgr_s \circ L_{-\theta}(h, h^\star)$ .
- Notice that $L_C(L_{\theta_0}(h, h^\star))(\theta + is) = L_C(h, h^*)((\theta - \theta_0) + is)$.

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# Earthquakes and grafting as limit of landslides and smooth grafting

**THM (B-Mondello-Schlenker)**

*Let $h^\star_n$ be a diverging sequence in Teichmuller space converging to a point $[\lambda]$ in the Thurston boundary of Teich$(S)$.*
*Take $\theta_n \to 0$ such that*
$\theta_n \ell_{h^\star_n}(\gamma) \to \iota(\lambda, \gamma)$ *for every* $\gamma \in \pi_1(S)$
*Then*

$$L_{h^\star_n,\theta_n}(h) \to E^l_{\lambda/2}(h) \qquad sgr_{h^\star_n,\theta_n}(h) \to gr_{\lambda/2}(h)$$

# Convergence of constant curvature surfaces to bent surfaces

- Let $h_n^*$ be a sequence of hyperbolic metrics converging to a point $[\lambda] \in P\mathcal{ML}(S) = \partial\mathcal{T}$.
- Take $\theta_n \to 0$ such $\theta_n \ell_{h_n^\star}(\gamma) \to \iota(\lambda, c)$ for every $\gamma \in \pi_1(S)$
- Define $k_n = -1 + \theta_n^2/2$ and $\kappa_n = -1 - \theta_n^2/2$. Then
  - $\sigma_{k_n}(h, h_n^\star) : \tilde{S} \to \mathbb{H}^3$ converges to the bending map $\beta(h, \lambda)$.
  - $\tau_{\kappa_n}(h, h_n^\star) : \tilde{S} \to AdS_3$ converges to the bending map $\alpha(h, \lambda)$.

# Degeneration of distance on a couple of normalized metrics

**THM (B-Mondello-Schlenker)**

*Take a diverging sequence of Labourie operatrs $b_n$ such that $h_n^\star = h(b_n, b_n)$ converges to $[\lambda]$ and take $\theta_n \to 0$ as above.*
*Then, for any arc c transverse to the h-realization of $\lambda$, the $h_n^\star$-length of c rescaled by $\theta_n$ converges to the intersection of c with the h-realization of the lamination $\lambda$.*

# Definition of the center

Notice that $S^1 \times [0, +\infty) \cong \Delta^*$.
Given $h, h^\star$ we have defined

$$L_C(h, h^\star) : \Delta^* \to Teich(S)$$

**Lemma**

*The map $CL(h, h^\star)$ extends to 0.*

The center of $h, h^\star$ is the point $c(h, h^\star) = CL(h, h^\star)(0)$.

**Remark**

*The center is fixed by the $S^1$-action:*

$$c(L_\theta(h, h^\star)) = c(h, h^\star).$$

# A characterization of the center

## A 2-dimensional characterization

The point $c = c(h, h^\star)$ is characterized by the property that the Hopf differential of the harmonic maps

$$(S, c) \to (S, h) \qquad (S, c) \to (S, h^\star)$$

are opposite.

## A 3-dimensional characterization

The point $c = c(h, h^\star)$ represents the conformal class of the second fundamental form of the AdS immersion $\tau_k(h, h^\star)$.

Given a point $c \in Teich(S)$ and a quadratic differential $\phi \in T^*(Teich(S))$ we denote by $h(c, \phi)$ the hyperbolic metric on $S$ such that the Hopf differential of the harmonic map $(S, c) \to (S, h(c, \phi))$ is $\phi$ [$h(c, \phi)$ is well defined by a result of Wolf]

**Prop**

*The map*

$$T^*(Teich(S)) \ni (c, \phi) \to (h(c, -\phi), h(c, \phi)) \in Teich(S) \times Teich(S)$$

*is a diffeomrophism conjugating the $S^1$-landslide action on $Teich(S) \times Teich(S)$ with the natural $S^1$ action on $T^*(Teich(S))$.*

# Comparison with earthquake case

The $S^1$ action on $T^*Teich(S)$ extends to a $SL_2(\mathbb{R})$-action. Consider the unipotent subrgoup $U(2) \cong \mathbb{R}$ in $SL_2(\mathbb{R})$. The restriction of the action of $U(2)$ on $T^*Teich(S)$ is called the unipotent flow.

**THM (Mirzakhani)**

*The unipotent flow on $T^*Teich(S)$ is measurably conjugated to the earthquake flow on $Teich(S) \times \mathcal{ML}(S)$.*

# The landslide flow is Hamiltonian

Consider on $Teich(S) \times Teich(S)$ the symplectic form $\omega = \omega_{WP} \oplus \omega_{WP}$.
Let $E : Teich(S) \times Teich(S) \to \mathbb{R}$ be the function

$$E(h, h^\star) = \text{energy of the harmonic map } (S, c) \to (S, h)$$

**Prop**

*$L_\theta$ is the Hamiltonian flow of $E$.*

# Convexity of $E$

**Prop**

*For any $h^\star$ fixed, the function $E(\cdot, h^\star)$ is strictly convex on WP geodesics.*