Fixed points of the composition of earthquakes

Francesco Bonsante

(joint work with J.-M. Schlenker)

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Earthquakes:definition

Let S be a closed orientable surface S of genus $g \geq 2$. Let us set

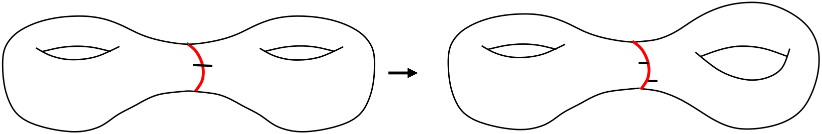
- $\mathcal{ML}_g$= space of measured geodesic laminations on S;
- $\mathcal{T}_g$= Teichmüller space of S = space of hyperbolic metrics on S up to isotopy.

Thurston defined two diffeomorphisms of $\mathcal{T}_g$ associated with $\lambda \in \mathcal{ML}_g$

$$E^r_\lambda, E^l_\lambda : \mathcal{T}_g \to \mathcal{T}_g \,.$$

Earthquakes:an example

If the lamination is a weighted curve, then E^r_λ and E^l_λ are fractional Dehn twists:



Earthquakes: main properties

- $E^r_\lambda = (E^l_\lambda)^{-1}$;
- The map $(t, x) \mapsto E^r_{t\lambda}(x)$ is a flow on $\mathcal{T}_g$.

THM (Kerckhoff, Thurston, Mess)

Given $\rho, \rho' \in \mathcal{T}_g$, there exists a unique pair $(\lambda, \mu) \in \mathcal{ML}^2_g$ such that

$$\rho' = E^r_\lambda(\rho) = E^l_\mu(\rho)\,.$$

Composition of earthquakes

Given two measured geodesic laminations λ and μ one can consider the composition

$$E^r_\mu \circ E^r_\lambda : \mathcal{T}_g \to \mathcal{T}_g \,.$$

- If λ and μ are disjoint, then the composition is simply the earthquake along $\lambda \cup \mu$.
- If λ and μ intersect, few things are known.

The result

THM 1 (B-Schlenker)

The composition of two right earthquakes $E^r_\lambda \circ E^r_\mu$ admits a fixed point in $\mathcal{T}_g$ iff λ and μ fill up the surface.

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Remark

There is some reason to believe that such fixed point is unique.

AdS space

- AdS_3=model of 3-dim Lorentzian geometry of const. curv. -1.
- $Isom(AdS_3) = PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$.
- AdS_3 is equipped with an asymptotic boundary $\partial_\infty AdS_3 = S^1 \times S^1$.
- The action of $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ extends on $\partial_\infty AdS_3$ to the product action.

GH AdS manifolds

Given $\rho, \rho' \in \mathcal{T}_g$ we consider the representation

$$h = (h_\rho, h_{\rho'}) : \pi_1(S) \to PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R}) = Isom(AdS_3)\,.$$

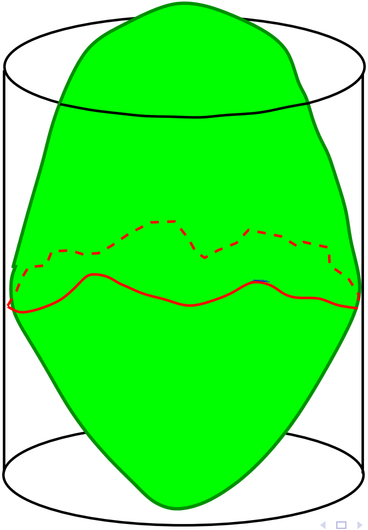
Prop (Mess)

There is a maximal convex open domain $\Omega \subset AdS_3$ such that

- *Ω is h-invariant;*
- *$M_{\rho,\rho'} = \Omega/h$ is a GH AdS spacetime diffeomorphic to $S \times \mathbb{R}$.*

The closure of Ω in $\partial_\infty AdS_3$ is an embedded curve Γ_h.

The convex core



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- The convex hull of Γ_h is an invariant domain in Ω that projects to the convex core of $M_{\rho,\rho'}$, that is the minimal convex deformation retract of $M_{\rho,\rho'}$.

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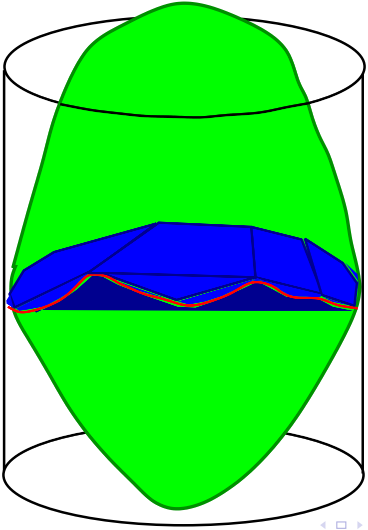
- The convex hull of Γ_h is an invariant domain in Ω that projects to the convex core of $M_{\rho,\rho'}$, that is the minimal convex deformation retract of $M_{\rho,\rho'}$.
- If $\rho = \rho'$, then K is a totally geodesic surface.
- If $\rho \neq \rho'$, the convex core is $\cong S \times [0, 1]$: its boundary components are called the upper and the lower boundary and are denoted by $\partial_+ K$ and $\partial_- K$.

The convex core



The geometry of the boundary of K

- $\partial\tilde{K}$ is the union of spacelike totally geodesic convex ideal polygons bent along a lamination.
- $\partial_{\pm}K$ carries a hyperbolic structure $\mu_{\pm}$
- The bending locus is a geodesic lamination $\lambda_{\pm}$ equipped with a transverse measure that encodes the amount of bending.

The bending map

We consider the map

$$B : \mathcal{T} \times \mathcal{T} \setminus \Delta \to \mathcal{ML}_g \times \mathcal{ML}_g$$

where $B(\rho, \rho') = (\lambda_+, \lambda_-)$ are the bending laminations of $M_{\rho,\rho'}$.

THM 2 (B-Schlenker)

The image of B is the set $\mathcal{FML}_g$ of pairs of measured geodesic laminations that fill up the surface.

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Conjecture

B is a 1-to-1 correspondence between $\mathcal{T} \times \mathcal{T} \setminus \Delta$ and $\mathcal{FML}_g$.

Comparison with the quasi-Fuchsian case

We consider the map

$$B_H : \mathcal{T}_g \times \mathcal{T}_g \setminus \Delta \to \mathcal{ML}_g \times \mathcal{ML}_g$$

defined by associating ρ, ρ' with the pairs of bending laminations of the Quasi-Fuchsian manfold corresponding to ρ, ρ' through the Bers parameterization.

THM (Bonahon-Otal)

The image of B_H is the set of pairs of laminations that fill the surface which have no closed curve with weight bigger than π.

Remark

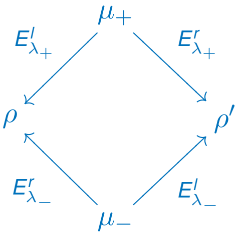
- *In Lorentzian geometry the angle between two spacelike planes is a well-defined number in* $[0, +\infty)$.
- *The maps* B *and* B_H *have a very different behavior.*

Mess diagram

Let ρ, ρ' be two hyperbolic structures on S and consider

- the hyperbolic structures μ_+, μ_- on the boundary of the convex core of $M_{\rho,\rho'}$;
- the bending laminations λ_+, λ_-.

Mess discovered the following relation between these objects:



Consequence of Mess diagram

From Mess diagram we have

$$\rho' = E^r_{2\lambda_+}(\rho) = E^l_{2\lambda_-}(\rho)$$

These relations uniquely determine λ_+ and λ_-.

Equivalence between Thm 1 and Thm 2.

Prop

The pair (λ, μ) lies in the image of B $\Leftrightarrow E^r_{2\mu} \circ E^r_{2\lambda}$ admits a fixed point.

$(\Rightarrow)$

- Suppose that there are ρ, ρ' such that λ, μ are the bending laminations of $M_{\rho,\rho'}$
- We have that

$$E^r_{2\lambda}(\rho) = E^l_{2\mu}(\rho) = \rho'$$

- In particular $\rho = E^r_{2\mu} \circ E^r_{2\lambda}(\rho)$.

The image of *B* is contained in $\mathcal{FML}_g$

The bending laminations λ_+, λ_- of an AdS manifold *M* fill up the surface:

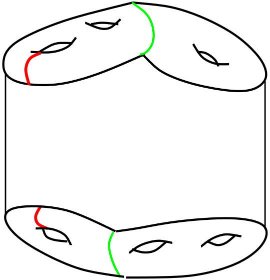
The image of B is contained in $\mathcal{FML}_g$

The bending laminations λ_+, λ_- of an AdS manifold M fill up the surface:

We have to prove that any loop c must intersect either λ_+ or λ_-.

By contradiction suppose that $\iota(c, \lambda) = \iota(c, \mu) = 0$. Let c_+ and c_- denote the geodesic representative of c in $\partial_+ K, \partial_- K$

c_+ and c_- are geodesic of M and are freely homotopic.



The scheme of the proof

Step 1 Given $(\lambda, \mu) \in \mathcal{FML}_g$, there is $\epsilon > 0$ such that $(t\lambda, t\mu)$ are realized as bending laminations of some GH AdS space for every $t < \epsilon$.

Step 2 The map $B : \mathcal{T}_g \times \mathcal{T}_g \setminus \Delta \to \mathcal{FML}_g$ is proper.

Step 3 Given $(\lambda, \mu) \in \mathcal{FML}_g$ there is $\epsilon' > 0$ such that $(t\lambda, t\mu)$ are uniquely realized as bending laminations of some GH AdS manifold for every $t < \epsilon'$.

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Conclusion: since the map is proper, the degree can be defined. By step 3, the degree is equal to 1 and the surjectivity follows

Step 1

- Analog of Bonahon result for quasi-Fuchsian manifolds.
- The proof uses hyperbolic geometry, in particular Kerckhoff results on the length

Step 2

It is based on the following estimate obtained by studying the geometry of the convex core

Lemma

Given $\lambda, \mu \in \mathcal{ML}_g$ and $\rho \in \mathcal{T}_g$ such that

$$E^r_\lambda(\rho) = E^l_\mu(\rho)$$

then we have

- *If $l_\lambda(\rho) \geq 1$ then $l_\lambda(\rho) \leq C\iota(\lambda, \mu)$.*
- *If $l_\lambda(\rho) < 1$ then $l^2_\lambda(\rho) \leq C\iota(\lambda, \mu)$.*

Step 3

- Analog of Series result for quasi-Fuchsian manifolds.
- The proof uses the second part of the estimate stated in the previous slice.