

# CCM, Part III (2)

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# Elliptic PDE's

One dimensional model problem ( $\Omega = ]a, b[$ )

$$\begin{cases} -u''(x) = f(x) & \text{in } \Omega \\ u(a) = u(b) = 0 \end{cases}$$

Boundary value problem (other boundary conditions possible)

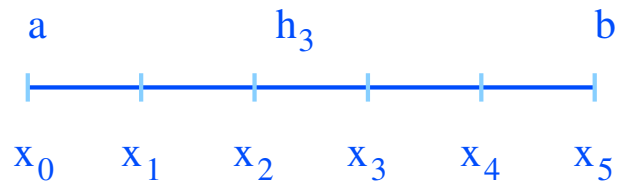
Generalization to  $\Omega \in \mathbb{R}^d$  with boundary  $\partial\Omega$

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Theorem: well-posedness (existence, uniqueness, stability)

# Finite differences

Summary: easy to design (approximate derivatives with difference quotients), easy to implement, very hard extension to general domains and boundary conditions



Here  $N = 5$ ,  $x_0 = a$ ,  $x_i = a + \sum_{j=1}^i h_j$ ,  $i = 1, \dots, N$

Denoting  $u_i = u(x_i)$ ,  $u'_i = u'(x_i)$ , first finite difference is

$$u'_i \simeq \frac{u_{i+1} - u_{i-1}}{h_i + h_{i+1}} \quad \text{second order accurate in } h \text{ (consistent)}$$

## Finite differences (cont'ed)

Approximation of second derivative

$$u''_i \simeq \frac{u'_{i+1/2} - u'_{i-1/2}}{\frac{h_i + h_{i+1}}{2}} \simeq \frac{\frac{u_{i+1} - u_i}{h_{i+1}} - \frac{u_i - u_{i-1}}{h_i}}{\frac{h_i + h_{i+1}}{2}}$$

If  $h_i = h$  (constant mesh size), simpler expression

$$u''_i \simeq \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} \quad \text{second order consistent}$$

Our approximate equation at  $x_i$  reads

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i, \quad i = 1, \dots, N-1$$

## Finite differences (cont'ed)

Putting things together we are led to the linear system

$$\begin{cases} u_0 = 0 \\ \dots \\ \frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i \\ \dots \\ u_N = 0 \end{cases}$$

$$AU = F \quad A = [\text{tridiag}(-1, 2, -1)]/h^2$$

# Weak formulations

Need for more general formulations.

Let's consider space  $V = H_0^1(a, b)$  consisting of continuous functions on  $[a, b]$ , piecewise differentiable with bounded derivative, and vanishing at endpoints.

Generalization to 2D requires Lebesgue integral and Hilbert spaces

$$H^1(\Omega) = \{v \in L^2(\Omega) \text{ s.t. } \vec{\text{grad}} v \in L^2(\Omega)\}$$

where

$$L^2(\Omega) = \left\{ v : \Omega \rightarrow \mathbb{R} \text{ integrable s.t. } \int_{\Omega} v^2 < \infty \right\}$$

## Weak formulations (cont'ed)

Take our model equation, multiply by a generic  $v \in V$  (test function), and integrate over  $(a, b)$

$$-\int_a^b u''(x)v(x) dx = \int_a^b f(x)v(x) dx$$

Integrating by parts gives

$$\int_a^b u'(x)v'(x) dx = \int_a^b f(x)v(x) dx$$

$$a : V \times V \rightarrow \mathbb{R}, F \in V^*$$

$$a(u, v) = \int_a^b u'(x)v'(x) dx, \quad F(v) = \int_a^b f(x)v(x) dx$$

# Weak formulations (cont'ed)

## Lax–Milgram Lemma

Find  $u \in V$  such that  $a(u, v) = F(v) \quad \forall v \in V$

This problem is well posed (exist., uniq., and stab.) provided

1.  $V$  Hilbert space
2.  $a$  bilinear, continuous,  $F$  linear, continuous
3.  $a$  coercive, that is there exists  $\alpha > 0$  s.t.

$$a(v, v) \geq \alpha \|v\|_V^2, \quad \forall v \in V$$

$$\|u\|_V \leq \frac{1}{\alpha} \|F\|_{V^*} \quad \text{Stability estimate}$$

## Weak formulations (cont'ed)

In our case hypotheses of LM Lemma OK (Poincaré inequality)

**Theorem** If  $f$  is smooth enough, the unique solution to weak formulation solves the original equation as well (strong solution)

More general situation

$$\begin{cases} -\operatorname{div}(\varepsilon \vec{\operatorname{grad}} u) + \vec{\beta} \cdot \vec{\operatorname{grad}} u + \sigma u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$a(u, v) = \int_{\Omega} \varepsilon \vec{\operatorname{grad}} u \cdot \vec{\operatorname{grad}} v \, d\vec{x} + \int_{\Omega} v \vec{\beta} \cdot \vec{\operatorname{grad}} u \, d\vec{x} + \int_{\Omega} \sigma uv \, d\vec{x}$$

## Weak formulations (cont'ed)

In general problem in weak form, when  $a$  is symmetric, is equivalent to the following variational problem:

Find  $u \in V$  such that

$$J(u) = \min_{v \in V} J(v), \quad J(v) = \frac{1}{2}a(v, v) - F(v)$$

In the one dimensional model problem, we have

$$J(v) = \frac{1}{2} \int_a^b (v'(x))^2 dx - \int_a^b f(x)v(x) dx$$

# Finite elements (Galerkin method)

Consider a finite dimensional subspace  $V_h \subset V$  ( $h$  refers to a mesh parameter).

$$\text{Find } u_h \in V_h \text{ such that } a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$$

Problem is solvable by Lax–Milgram

Suppose that  $V_h = \text{span}\{\varphi_1, \dots, \varphi_N(h)\}$ , so  $u_h = \sum_{i=1}^N u_i \varphi_i$

Problem can be written: find  $\{u_i\}$  s.t. for any  $j$

$$a\left(\sum_{i=1}^N u_i \varphi_i, \varphi_j\right) = F(\varphi_j)$$

## Galerkin method (cont'ed)

Bilinearity of  $a$  gives

$$\sum_{i=1}^N u_i a(\varphi_i, \varphi_j) = F(\varphi_j), \quad j = 1, \dots, N$$

Let's denote by  $A$  the *stiffness* matrix  $A_{ji} = a(\varphi_i, \varphi_j)$  and by  $b$  the *load* vector  $b_j = F(\varphi_j)$ . Then we have the matrix form of discrete problem

$$A\mathbf{u} = b$$

$a$  symmetric and coercive implies  $A$  symmetric positive definite

# Galerkin method (cont'ed)

Existence and uniqueness (Lax–Milgram)

Convergence = Consistency + Stability

Stability:

$$\|u_h\|_V \leq \frac{1}{\alpha} \|F\|_{V^*}$$

Strong consistency

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h$$

# Galerkin method (cont'ed)

Error estimate (Céa's Lemma)

$$\begin{aligned}\alpha \|u - u_h\|_V^2 &\leq a(u - u_h, u - u_h) = a(u - u_h, u - v_h) \\ &\leq M \|u - u_h\|_V \|u - v_h\|_V\end{aligned}$$

$$\|u - u_h\|_V \leq \frac{M}{\alpha} \inf_{v \in V_h} \|u - v_h\|_V$$

Error bounded by best approximation

Need for good choice of  $V_h$ !

## Galerkin method (cont'ed)

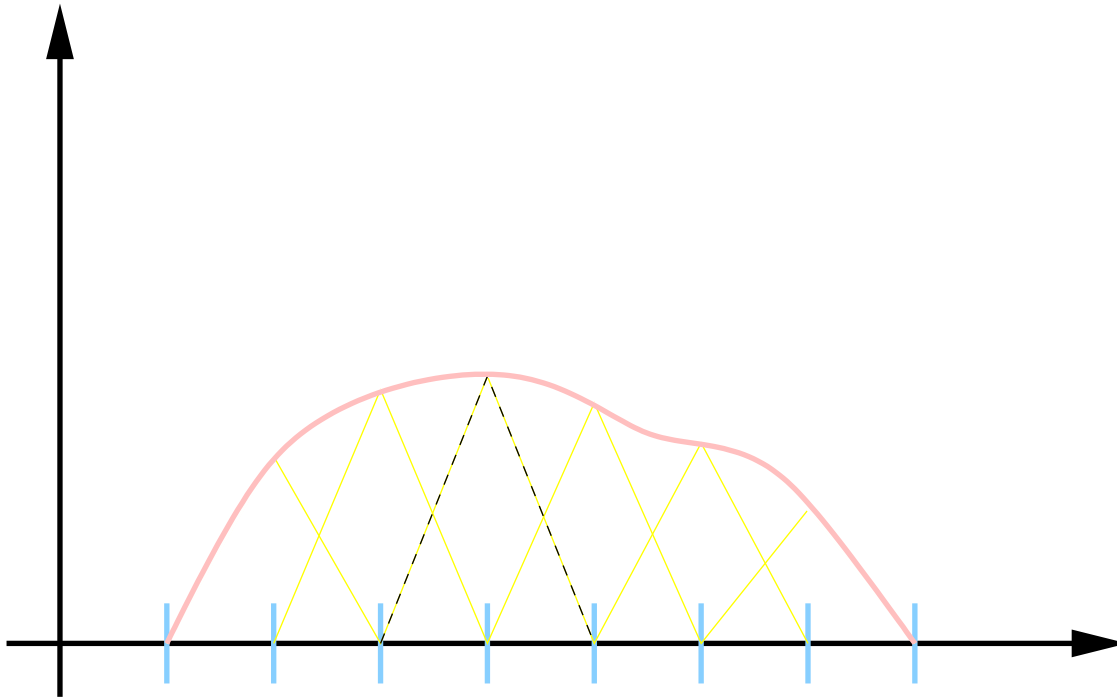
Moreover, when  $a$  is symmetric, we have the variational property

$$J(u_h) = \min_{v_h \in V_h} J(v_h)$$

Since  $V_h \subset V$ , in particular, we have

$$J(u) \leq J(u_h)$$

# Finite elements



One dimensional p/w linear approximation.  
Shape (or basis) functions:  
hat functions.

A finite element is defined by:

- 1) a domain (interval, triangle, tetrahedron, . . . ),
- 2) a finite dimensional (polynomial) space,
- 3) a set of degrees of freedom.

# Finite elements (cont'ed)

## One dimensional finite elements

- 1) domain: interval
- 2) space:  $\mathbb{P}_p$
- 3) d.o.f.'s: depend on polynomial order

linear element: endpoints (2)

quadratic element: endpoints + midpoint (3)

...

Set  $\{a_j\}_{j=1}^N$  of degrees of freedom is *unisolvent*, that is, given  $N$  numbers  $\alpha_1, \dots, \alpha_N$ , there exists a unique polynomial  $\varphi$  in  $\mathbb{P}_p$  s. t.

$$\varphi(a_j) = \alpha_j, \quad j = 1, \dots, N$$

# Finite elements (cont'ed)

Approximation properties of one dimensional finite elements

$$\inf_{v_h \in V_h} \|u - v_h\|_{H^k} \leq Ch^{p+1-k} |u|_{H^{p+1}} \quad k = 0, 1$$

Remark on  $hp$  FEM

- ▶ Refine in  $h$  where solution is singular
- ▶ Refine in  $p$  where solution is regular

# End of Part Two