Finite element approximation of eigenvalue problems

Daniele Boffi

Dipartimento di Matematica “F. Casorati”, Università di Pavia
http://www-dimat.unipv.it/boffi

Outline of the course

General theory and surveys

- I. Babuška and J. Osborn (1991), Eigenvalue problems, in *Handbook of numerical analysis, Vol. II*, Handb. Numer. Anal., II, North-Holland, Amsterdam, pp. 641–787
- D. Boffi, Finite element approximation of eigenvalue problems, *Acta Numerica*, 19 (2010), 1–120
- D. Boffi, F. Gardini, and L. Gastaldi, Some remarks on eigenvalue approximation by finite elements, In *Frontiers in Numerical Analysis* – Durham 2010, Springer Lecture Notes in Computational Science and Engineering, 85 (2012), 1–77
- D. Boffi, F. Brezzi, and M. Fortin, Mixed Finite Element Methods and Applications, *Springer Series in Comp. Math.*, Vol. 44, 2013
- R. Hiptmair (2002*b*), 'Finite elements in computational electromagnetism', *Acta Numer.* **11**, 237–339
- P. Monk (2003), *Finite element methods for Maxwell's equations*, Numerical Mathematics and Scientific Computation, Oxford University Press, New York

References for ellipic problems

- G. Strang, G.J. Fix, An analysis of the finite element method. *Prentice-Hall Series in Automatic Computation*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1973
- I. Babuška and J. E. Osborn (1989), 'Finite element-Galerkin approximation of the eigenvalues and eigenvectors of selfadjoint problems', *Math. Comp.* **52**(186), 275–297
- J. H. Bramble and J. E. Osborn (1973), 'Rate of convergence estimates for nonselfadjoint eigenvalue approximations', *Math. Comp.* **27**, 525–549
- W. G. Kolata (1978), 'Approximation in variationally posed eigenvalue problems', *Numer. Math.* **29**(2), 159–171
- A. V. Knyazev and J. E. Osborn (2006), 'New a priori FEM error estimates for eigenvalues', *SIAM J. Numer. Anal.* **43**(6), 2647–2667

References for mixed formulation

- R. S. Falk and J. E. Osborn (1980), 'Error estimates for mixed methods', *RAIRO Anal. Numér.* **14**(3), 249–277
- B. Mercier, J. Osborn, J. Rappaz and P.-A. Raviart (1981), 'Eigenvalue approximation by mixed and hybrid methods', *Math. Comp.* **36**(154), 427–453
- D. Boffi, F. Brezzi and L. Gastaldi (1997), 'On the convergence of eigenvalues for mixed formulations', *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **25**(1-2), 131–154 (1998)
- D. Boffi, F. Brezzi and L. Gastaldi (2000*a*), 'On the problem of spurious eigenvalues in the approximation of linear elliptic problems in mixed form', *Math. Comp.* **69**(229), 121–140

Maxwell's eigenvalues I

- F. Kikuchi (1987), 'Mixed and penalty formulations for finite element analysis of an eigenvalue problem in electromagnetism', *Comput. Methods Appl. Mech. Engrg.* **64**(1-3), 509–521

- F. Kikuchi (1989), 'On a discrete compactness property for the Nédélec finite elements', *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **36**(3), 479–490

- D. Boffi, P. Fernandes, L. Gastaldi and I. Perugia (1999*b*), 'Computational models of electromagnetic resonators: analysis of edge element approximation', *SIAM J. Numer. Anal.* **36**(4), 1264–1290

- D. Boffi (2000), 'Fortin operator and discrete compactness for edge elements', *Numer. Math.* **87**(2), 229–246

- D. Boffi (2001), 'A note on the de Rham complex and a discrete compactness property', *Appl. Math. Lett.* **14**(1), 33–38

Maxwell's eigenvalues II

- P. Monk and L. Demkowicz (2001*a*), 'Discrete compactness and the approximation of Maxwell's equations in $\mathbb{R}^3$', *Math. Comp.* **70**(234), 507–523
- D. Boffi, L. Demkowicz and M. Costabel (2003), 'Discrete compactness for p and hp 2D edge finite elements', *Math. Models Methods Appl. Sci.* **13**(11), 1673–1687
- D. Boffi, M. Costabel, M. Dauge and L. Demkowicz (2006*b*), 'Discrete compactness for the hp version of rectangular edge finite elements', *SIAM J. Numer. Anal.* **44**(3), 979–1004
- D. Boffi, M. Costabel, M. Dauge, L. Demkowicz, R. Hiptmair, Discrete compactness for the p-version of discrete differential forms. *SIAM J. Numer. Anal.* 49 (2011), no. 1, 135–158

Other papers related to the course

Spurious modes in the square

D. Boffi, R. G. Durán and L. Gastaldi (1999*a*), 'A remark on spurious eigenvalues in a square', *Appl. Math. Lett.* **12**(3), 107–114

Equivalence of DCP and mixed conditions

D. Boffi (2007), 'Approximation of eigenvalues in mixed form, discrete compactness property, and application to *hp* mixed finite elements', *Comput. Methods Appl. Mech. Engrg.* **196**(37-40), 3672–3681

Finite Element Exterior Calculus

D.N. Arnold, R.S. Falk, R. Winther, Finite element exterior calculus: from Hodge theory to numerical stability. *Bull. Amer. Math. Soc. (N.S.)* 47 (2010), no. 2, 281–354

Some initial computations

1D Laplacian

$$\begin{cases} -u''(t) = \lambda u(t) & t \in [0, \pi] \\ u(0) = u(\pi) = 0 \end{cases}$$

Find $\lambda \in \mathbb{R}$ and non-vanishing $u \in H^1_0(0, \pi)$ such that

$$\int_0^\pi u'(t)v'(t)\,dt = \lambda \int_0^\pi u(t)v(t)\,dt \quad \forall v \in H^1_0(0, \pi)$$

Exact solution:

$\lambda_k = k^2$
$u_k(t) = \sin(kt)$ $\qquad (k = 1, 2, 3, \dots)$

Discretization

Conforming approximation $V_h \subset V = H^1_0(0,\pi)$

Find $\lambda_h \in \mathbb{R}$ and non-vanishing $u_h \in V_h$ such that

$$\int_0^\pi u_h'(t)v'(t)\,dt = \lambda_h \int_0^\pi u_h(t)v(t)\,dt \quad \forall v \in V_h$$

$$\boxed{Ax = \lambda Mx}$$

Approximation with p/w linear finite elements

	$n = 8$	$n = 16$	$n = 32$
1	1.0129160450588	1.0032168743567	1.0008034482562
4	4.2095474481529	4.0516641802355	4.0128674974272
9	10.0802909335883	9.2631305555446	9.0652448637285
16	19.4536672593288	16.8381897926118	16.2066567209423
25	33.2628304890884	27.0649225609802	25.5059230069702

	$n = 64$	$n = 128$	$n = 256$
1	1.0002008137390	1.0000502004122	1.0000125499161
4	4.0032137930241	4.0008032549556	4.0002008016414
9	9.0162763381719	9.0040668861371	9.0010165838380
16	16.0514699897078	16.0128551720960	16.0032130198251
25	25.1257489536113	25.0313903532369	25.0078446408520

Approximation with quadratic finite elements

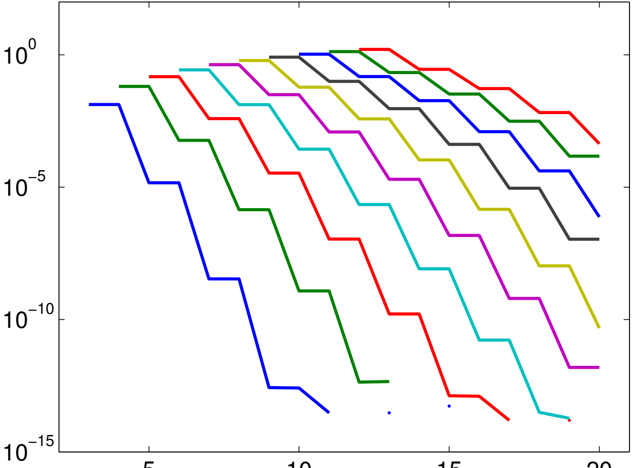
	Computed eigenvalue (rate)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
1	1.0000	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)
4	4.0020	4.0001 (4.0)	4.0000 (4.0)	4.0000 (4.0)	4.0000 (4.0)
9	9.0225	9.0015 (3.9)	9.0001 (4.0)	9.0000 (4.0)	9.0000 (4.0)
16	16.1204	16.0082 (3.9)	16.0005 (4.0)	16.0000 (4.0)	16.0000 (4.0)
25	25.4327	25.0307 (3.8)	25.0020 (3.9)	25.0001 (4.0)	25.0000 (4.0)
36	37.1989	36.0899 (3.7)	36.0059 (3.9)	36.0004 (4.0)	36.0000 (4.0)
49	51.6607	49.2217 (3.6)	49.0148 (3.9)	49.0009 (4.0)	49.0001 (4.0)
64	64.8456	64.4814 (0.8)	64.0328 (3.9)	64.0021 (4.0)	64.0001 (4.0)
81	95.7798	81.9488 (4.0)	81.0659 (3.8)	81.0042 (4.0)	81.0003 (4.0)
100	124.9301	101.7308 (3.8)	100.1229 (3.8)	100.0080 (3.9)	100.0005 (4.0)
#	15	31	63	127	255

Remark

Convergence from above
Double order of convergence (w.r.t. approximation properties in energy norm)

Spectral elements

Exponential convergence



Spectral method

Convergence of fifth eigenvalue

p	DOF	Computed
7	5	35.5593555378041
8	6	35.5593555378041
9	7	25.7779168651921
10	8	25.7779168651921
11	9	25.0306605127133
12	10	25.0306605127132
13	11	25.0004945052929
14	12	25.0004945052929
15	13	25.0000037734250
16	14	25.0000037734250
17	15	25.0000000156754
18	16	25.0000000156756
19	17	25.0000000000389
20	18	25.0000000000389

Two dimensional Laplacian

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega =]0,\pi[\times]0,\pi[\\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Find $\lambda \in \mathbb{R}$ and non-vanishing $u \in H_0^1(\Omega)$ such that

$$(\nabla u \cdot \nabla v) = \lambda(u,v) \quad \forall v \in H_0^1(\Omega)$$

Exact solution:

$$\lambda_{m,n} = m^2 + n^2$$
$$u_{m,n}(x,y) = \sin(mx)\sin(ny) \qquad (m,n = 1,2,3,\dots)$$

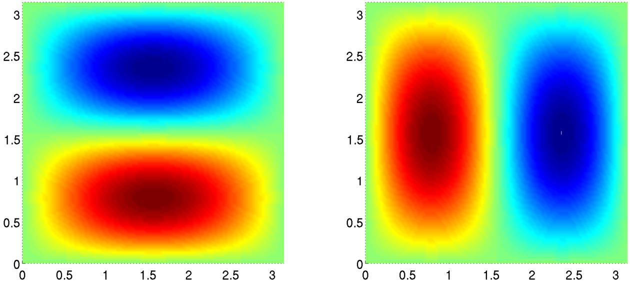
Approximation with p/w linear finite elements

Unstructured mesh

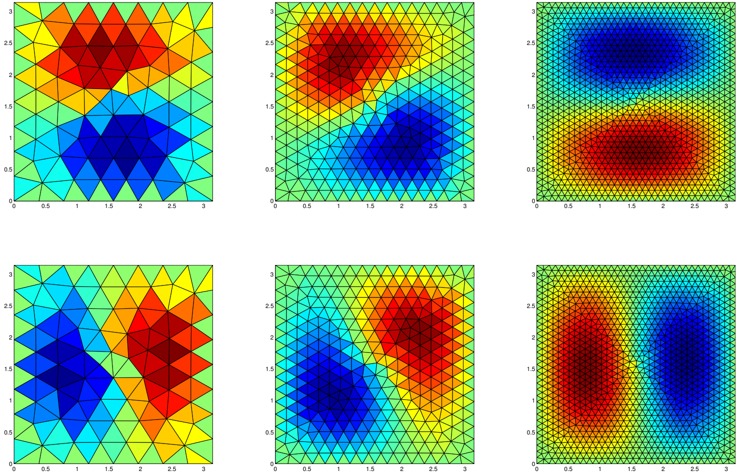
	Computed eigenvalue (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.2468	2.0463 (2.4)	2.0106 (2.1)	2.0025 (2.1)	2.0006 (2.0)
5	6.5866	5.2732 (2.5)	5.0638 (2.1)	5.0154 (2.0)	5.0038 (2.0)
5	6.6230	5.2859 (2.5)	5.0643 (2.2)	5.0156 (2.0)	5.0038 (2.0)
8	10.2738	8.7064 (1.7)	8.1686 (2.1)	8.0402 (2.1)	8.0099 (2.0)
10	12.7165	11.0903 (1.3)	10.2550 (2.1)	10.0610 (2.1)	10.0152 (2.0)
10	14.3630	11.1308 (1.9)	10.2595 (2.1)	10.0622 (2.1)	10.0153 (2.0)
13	19.7789	14.8941 (1.8)	13.4370 (2.1)	13.1046 (2.1)	13.0258 (2.0)
13	24.2262	14.9689 (2.5)	13.4435 (2.2)	13.1053 (2.1)	13.0258 (2.0)
17	34.0569	20.1284 (2.4)	17.7468 (2.1)	17.1771 (2.1)	17.0440 (2.0)
17		20.2113	17.7528 (2.1)	17.1798 (2.1)	17.0443 (2.0)
#	9	56	257	1106	4573

Multiple eigenfunctions

Exact solutions ($5 = 1^2 + 2^2 = 2^2 + 1^2$)



Multiple eigenfunctions (discrete)

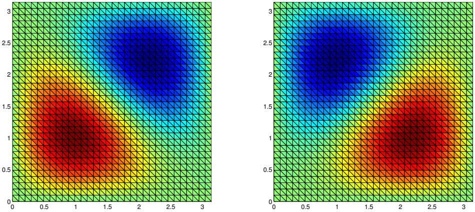


Uniform mesh

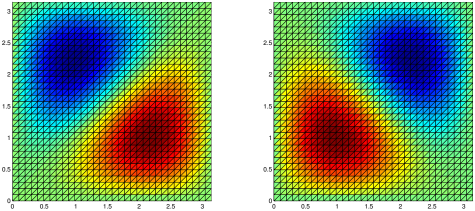
	Computed eigenvalue (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.3168	2.0776 (2.0)	2.0193 (2.0)	2.0048 (2.0)	2.0012 (2.0)
5	6.3387	5.3325 (2.0)	5.0829 (2.0)	5.0207 (2.0)	5.0052 (2.0)
5	7.2502	5.5325 (2.1)	5.1302 (2.0)	5.0324 (2.0)	5.0081 (2.0)
8	12.2145	9.1826 (1.8)	8.3054 (2.0)	8.0769 (2.0)	8.0193 (2.0)
10	15.5629	11.5492 (1.8)	10.3814 (2.0)	10.0949 (2.0)	10.0237 (2.0)
10	16.7643	11.6879 (2.0)	10.3900 (2.1)	10.0955 (2.0)	10.0237 (2.0)
13	20.8965	15.2270 (1.8)	13.5716 (2.0)	13.1443 (2.0)	13.0362 (2.0)
13	26.0989	17.0125 (1.7)	13.9825 (2.0)	13.2432 (2.0)	13.0606 (2.0)
17	32.4184	21.3374 (1.8)	18.0416 (2.1)	17.2562 (2.0)	17.0638 (2.0)
17		21.5751	18.0705 (2.1)	17.2626 (2.0)	17.0653 (2.0)
#	9	49	225	961	3969

Multiple eigenfunctions (uniform meshes)

Uniform mesh



Uniform mesh (reversed)



Multiple eigenfunctions (symmetric mesh)

Criss-cross mesh

Exact	Computed eigenvalue (rate)				
	$N=4$	$N=8$	$N=16$	$N=32$	$N=64$
2	2.0880	2.0216 (2.0)	2.0054 (2.0)	2.0013 (2.0)	2.0003 (2.0)
5	5.6811	5.1651 (2.0)	5.0408 (2.0)	5.0102 (2.0)	5.0025 (2.0)
5	5.6811	5.1651 (2.0)	5.0408 (2.0)	5.0102 (2.0)	5.0025 (2.0)
8	9.4962	8.3521 (2.1)	8.0863 (2.0)	8.0215 (2.0)	8.0054 (2.0)
10	12.9691	10.7578 (2.0)	10.1865 (2.0)	10.0464 (2.0)	10.0116 (2.0)
10	12.9691	10.7578 (2.0)	10.1865 (2.0)	10.0464 (2.0)	10.0116 (2.0)
13	17.1879	14.0237 (2.0)	13.2489 (2.0)	13.0617 (2.0)	13.0154 (2.0)
13	17.1879	14.0237 (2.0)	13.2489 (2.0)	13.0617 (2.0)	13.0154 (2.0)
17	25.1471	19.3348 (1.8)	17.5733 (2.0)	17.1423 (2.0)	17.0355 (2.0)
17	38.9073	19.3348 (3.2)	17.5733 (2.0)	17.1423 (2.0)	17.0355 (2.0)
18	38.9073	19.8363 (3.5)	18.4405 (2.1)	18.1089 (2.0)	18.0271 (2.0)
20	38.9073	22.7243 (2.8)	20.6603 (2.0)	20.1634 (2.0)	20.0407 (2.0)
20	38.9073	22.7243 (2.8)	20.6603 (2.0)	20.1634 (2.0)	20.0407 (2.0)
25	38.9073	28.7526 (1.9)	25.8940 (2.1)	25.2201 (2.0)	25.0548 (2.0)
25	38.9073	28.7526 (1.9)	25.8940 (2.1)	25.2201 (2.0)	25.0548 (2.0)
DOF	25	113	481	1985	8065

Babuška–Osborn example

$$
\begin{aligned}
&-\left(\frac{1}{\varphi'(x)}u'(x)\right)' = \lambda\varphi'(x)u(x) \quad \text{in } (-\pi,\pi)\\
&u(-\pi) = u(\pi)\\
&\frac{1}{\varphi'(-\pi)}u'(-\pi) = \frac{1}{\varphi'(\pi)}u'(\pi)
\end{aligned}
$$

$$
\varphi(x) = \pi^{-\alpha}|x|^{1+\alpha}\text{sign}(x), \qquad 0 < \alpha < 1
$$

Exact solution

$\lambda = 0$ ($u(x) \equiv 1$)

$\lambda = k^2$ ($k = 1, 2, \dots$) double eigenvalues

$u(x) = \sin(k\varphi(x))$ regularity $(3 + \alpha)/2$

$u(x) = \cos(k\varphi(x))$ regularity $(5 + 3\alpha)/2$

$\alpha = .9$

Exact	Relative error (rate)				
	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
1	1.7e-03	4.2e-04 (2.00)	1.0e-04 (2.00)	2.6e-05 (2.00)	6.6e-06 (1.98)
1	4.6e-03	1.3e-03 (1.88)	3.4e-04 (1.88)	9.2e-05 (1.88)	2.5e-05 (1.88)
4	6.2e-03	1.5e-03 (2.00)	3.9e-04 (2.00)	9.7e-05 (2.00)	2.4e-05 (2.00)
4	9.0e-03	2.4e-03 (1.94)	6.2e-04 (1.94)	1.6e-04 (1.93)	4.2e-05 (1.93)
9	1.4e-02	3.4e-03 (2.00)	8.5e-04 (2.00)	2.1e-04 (2.00)	5.3e-05 (2.00)
9	1.7e-02	4.2e-03 (1.97)	1.1e-03 (1.96)	2.8e-04 (1.96)	7.1e-05 (1.96)
16	2.4e-02	6.0e-03 (2.00)	1.5e-03 (2.00)	3.8e-04 (2.00)	9.4e-05 (2.00)
16	2.7e-02	6.8e-03 (1.98)	1.7e-03 (1.98)	4.4e-04 (1.98)	1.1e-04 (1.97)
25	3.7e-02	9.4e-03 (2.00)	2.3e-03 (2.00)	5.9e-04 (2.00)	1.5e-04 (2.00)
25	4.0e-02	1.0e-02 (1.99)	2.6e-03 (1.99)	6.5e-04 (1.98)	1.6e-04 (1.98)
DOF	64	128	256	512	1024

Table: Error in the eigenvalues computed with linear elements and $\alpha = 0.9$.

$\alpha = .5$

Exact	Relative error (rate)				
	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
1	1.2e-03	3.0e-04 (2.00)	7.4e-05 (2.00)	1.9e-05 (2.00)	4.7e-06 (2.00)
1	4.3e-03	1.4e-03 (1.59)	4.8e-04 (1.57)	1.7e-04 (1.55)	5.7e-05 (1.53)
4	4.5e-03	1.1e-03 (2.00)	2.8e-04 (2.00)	7.0e-05 (2.00)	1.8e-05 (2.00)
4	7.5e-03	2.2e-03 (1.75)	6.8e-04 (1.71)	2.2e-04 (1.67)	7.0e-05 (1.63)
9	1.0e-02	2.5e-03 (2.00)	6.2e-04 (2.00)	1.6e-04 (2.00)	3.9e-05 (2.00)
9	1.3e-02	3.6e-03 (1.86)	1.0e-03 (1.81)	3.0e-04 (1.77)	9.1e-05 (1.72)
16	1.8e-02	4.4e-03 (2.00)	1.1e-03 (2.00)	2.7e-04 (2.00)	6.9e-05 (2.00)
16	2.1e-02	5.5e-03 (1.91)	1.5e-03 (1.88)	4.2e-04 (1.84)	1.2e-04 (1.80)
25	2.7e-02	6.8e-03 (2.00)	1.7e-03 (2.00)	4.3e-04 (2.00)	1.1e-04 (2.00)
25	3.0e-02	7.9e-03 (1.94)	2.1e-03 (1.91)	5.7e-04 (1.88)	1.6e-04 (1.85)
DOF	64	128	256	512	1024

Table: Error in the eigenvalues computed with linear elements and $\alpha = 0.5$.

$\alpha = .1$

Exact	Relative error (rate)				
	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
1	8.5e-04	2.1e-04 (2.00)	5.3e-05 (2.00)	1.3e-05 (2.00)	3.3e-06 (2.00)
1	1.3e-03	4.2e-04 (1.59)	1.6e-04 (1.45)	6.2e-05 (1.33)	2.6e-05 (1.24)
4	3.4e-03	8.4e-04 (2.00)	2.1e-04 (2.00)	5.2e-05 (2.00)	1.3e-05 (2.00)
4	3.7e-03	1.0e-03 (1.85)	3.1e-04 (1.75)	1.0e-04 (1.62)	3.6e-05 (1.49)
9	7.5e-03	1.9e-03 (2.00)	4.7e-04 (2.00)	1.2e-04 (2.00)	2.9e-05 (2.00)
9	7.8e-03	2.1e-03 (1.93)	5.6e-04 (1.87)	1.6e-04 (1.78)	5.2e-05 (1.67)
16	1.3e-02	3.3e-03 (2.01)	8.3e-04 (2.00)	2.1e-04 (2.00)	5.2e-05 (2.00)
16	1.4e-02	3.5e-03 (1.96)	9.2e-04 (1.92)	2.5e-04 (1.86)	7.4e-05 (1.78)
25	2.1e-02	5.2e-03 (2.01)	1.3e-03 (2.00)	3.2e-04 (2.00)	8.1e-05 (2.00)
25	2.1e-02	5.3e-03 (1.98)	1.4e-03 (1.95)	3.7e-04 (1.91)	1.0e-04 (1.84)
DOF	64	128	256	512	1024

Table: Error in the eigenvalues computed with linear elements and $\alpha = 0.1$.

L-shaped domain

P1 elements (Neumann boundary conditions)

	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
0	-0.0000	0.0000	-0.0000	-0.0000	-0.0000
1.48	1.6786	1.5311 (1.9)	1.4946 (1.5)	1.4827 (1.4)	1.4783 (1.4)
3.53	3.8050	3.5904 (2.3)	3.5472 (2.1)	3.5373 (2.0)	3.5348 (2.0)
9.87	12.2108	10.2773 (2.5)	9.9692 (2.0)	9.8935 (2.1)	9.8755 (2.0)
9.87	12.5089	10.3264 (2.5)	9.9823 (2.0)	9.8979 (2.0)	9.8767 (2.0)
11.39	13.9526	12.0175 (2.0)	11.5303 (2.2)	11.4233 (2.1)	11.3976 (2.1)
#	20	65	245	922	3626

Nonconforming P1

	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	1.9674	1.9850 (1.1)	1.9966 (2.1)	1.9992 (2.0)	1.9998 (2.0)
5	4.4508	4.9127 (2.7)	4.9787 (2.0)	4.9949 (2.1)	4.9987 (2.0)
5	4.7270	4.9159 (1.7)	4.9790 (2.0)	4.9949 (2.0)	4.9987 (2.0)
8	7.2367	7.7958 (1.9)	7.9434 (1.9)	7.9870 (2.1)	7.9967 (2.0)
10	8.5792	9.6553 (2.0)	9.9125 (2.0)	9.9792 (2.1)	9.9949 (2.0)
10	9.0237	9.6663 (1.5)	9.9197 (2.1)	9.9796 (2.0)	9.9950 (2.0)
13	9.8284	12.4011 (2.4)	12.8534 (2.0)	12.9654 (2.1)	12.9914 (2.0)
13	9.9107	12.4637 (2.5)	12.8561 (1.9)	12.9655 (2.1)	12.9914 (2.0)
17	10.4013	15.9559 (2.7)	16.7485 (2.1)	16.9407 (2.1)	16.9853 (2.0)
17	11.2153	16.0012 (2.5)	16.7618 (2.1)	16.9409 (2.0)	16.9854 (2.0)
#	40	197	832	3443	13972

Mixed approximation of Laplacian

Find $\lambda \in \mathbb{R}$ and $u \in L^2(0,\pi)$ such that for some $s \in H^1(0,\pi)$

$$\begin{cases} (s,t) + (t',u) = 0 & \forall t \in H^1(0,\pi) & s = u' \\ (s',v) = -\lambda(u,v) & \forall v \in L^2(0,\pi) & s' = -\lambda u \end{cases}$$

After conforming discretization $\Sigma_h \subset \Sigma = H^1(0,\pi)$ and $U_h \subset U = L^2(0,\pi)$ the discrete problem has the following matrix form

$$\begin{bmatrix} A & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} 0 & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The good element

$P_1 - P_0$ scheme (in general, $P_{k+1} - P_k$)

Same eigenvalues as for the standard Galerkin P_1 scheme

$$\lambda_h^{(k)} = \frac{6}{h^2} \cdot \frac{1 - \cos kh}{2 + \cos kh}$$

$$u_h^{(k)}|_{]ih,(i+1)h[} = \frac{s_h^{(k)}(ih) - s_h^{(k)}((i+1)h)}{h\lambda_h^{(k)}}$$

$$s_h^{(k)}(ih) = k\cos(kih)$$

$i = 0, \dots, N \quad (N = \text{number of intervals})$

$k = 1, \dots, N$

A troublesome element

$P_1 - P_1$ scheme

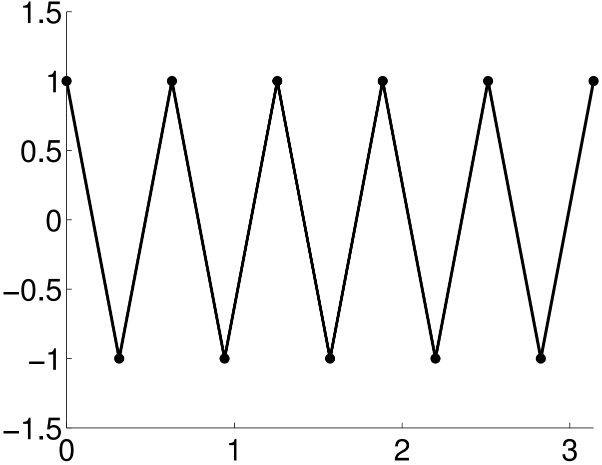
	Computed eigenvalue (rate)				
	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
0	0.0000	-0.0000	-0.0000	-0.0000	-0.0000
1	1.0001	1.0000 (4.1)	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)
4	3.9660	3.9981 (4.2)	3.9999 (4.0)	4.0000 (4.0)	4.0000 (4.0)
9	7.4257	8.5541 (1.8)	8.8854 (2.0)	8.9711 (2.0)	8.9928 (2.0)
9	8.7603	8.9873 (4.2)	8.9992 (4.1)	9.0000 (4.0)	9.0000 (4.0)
16	14.8408	15.9501 (4.5)	15.9971 (4.1)	15.9998 (4.0)	16.0000 (4.0)
25	16.7900	24.5524 (4.2)	24.9780 (4.3)	24.9987 (4.1)	24.9999 (4.0)
36	38.7154	29.7390 (-1.2)	34.2165 (1.8)	35.5415 (2.0)	35.8846 (2.0)
36	39.0906	35.0393 (1.7)	35.9492 (4.2)	35.9970 (4.1)	35.9998 (4.0)
49		46.7793	48.8925 (4.4)	48.9937 (4.1)	48.9996 (4.0)

Remark

Now the eigenvalues are not always approximated from above

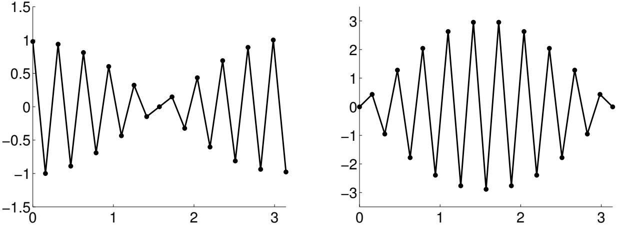
First spurious eigenfunction

$\boxed{\lambda = 0}$

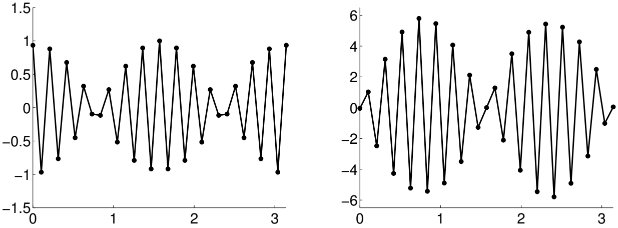


Higher order spurious eigenfunctions

$\lambda \simeq 9$



$\lambda \simeq 36$

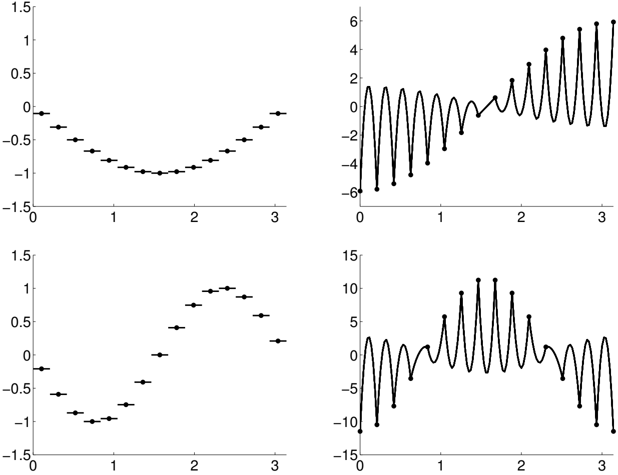


An intriguing element

$P_2 - P_0$ scheme

	Computed eigenvalue (rate with respect to 6λ)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
1	5.7061	5.9238 (1.9)	5.9808 (2.0)	5.9952 (2.0)	5.9988 (2.0)
4	19.8800	22.8245 (1.8)	23.6953 (1.9)	23.9231 (2.0)	23.9807 (2.0)
9	36.7065	48.3798 (1.6)	52.4809 (1.9)	53.6123 (2.0)	53.9026 (2.0)
16	51.8764	79.5201 (1.4)	91.2978 (1.8)	94.7814 (1.9)	95.6925 (2.0)
25	63.6140	113.1819 (1.2)	138.8165 (1.7)	147.0451 (1.9)	149.2506 (2.0)
36	71.6666	146.8261 (1.1)	193.5192 (1.6)	209.9235 (1.9)	214.4494 (2.0)
49	76.3051	178.6404 (0.9)	253.8044 (1.5)	282.8515 (1.9)	291.1344 (2.0)
64	77.8147	207.5058 (0.8)	318.0804 (1.4)	365.1912 (1.8)	379.1255 (1.9)
81		232.8461	384.8425 (1.3)	456.2445 (1.8)	478.2172 (1.9)
100		254.4561	452.7277 (1.2)	555.2659 (1.7)	588.1806 (1.9)
#	8	16	32	64	128

Eigenfunctions for the $P_2 - P_0$ element



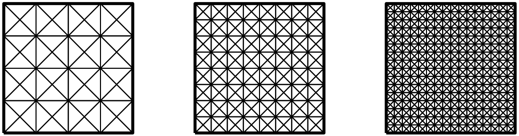
Another intriguing example in 2D

Neumann eigenvalue problem for the Laplacian

Find $\lambda \in \mathbb{R}$ and $u \in L^2_0(\Omega)$ such that for some $\boldsymbol{\sigma} \in \mathbf{H}_0(\mathrm{div};\Omega)$

$$\begin{cases} (\boldsymbol{\sigma},\boldsymbol{\tau}) + (\operatorname{div}\boldsymbol{\tau}, u) = 0 & \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathrm{div};\Omega) \\ (\operatorname{div}\boldsymbol{\sigma}, v) = -\lambda(u,v) & \forall v \in L^2_0(\Omega) \end{cases}$$

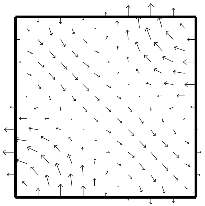
Criss-cross mesh sequence, $P_1 - \mathrm{div}(P_1)$ scheme



	Computed eigenvalue (rate)				
	$N = 2$	$N = 4$	$N = 8$	$N = 16$	$N = 32$
1	1.0662	1.0170 (2.0)	1.0043 (2.0)	1.0011 (2.0)	1.0003 (2.0)
1	1.0662	1.0170 (2.0)	1.0043 (2.0)	1.0011 (2.0)	1.0003 (2.0)
2	2.2035	2.0678 (1.6)	2.0171 (2.0)	2.0043 (2.0)	2.0011 (2.0)
4	4.8634	4.2647 (1.7)	4.0680 (2.0)	4.0171 (2.0)	4.0043 (2.0)
4	4.8634	4.2647 (1.7)	4.0680 (2.0)	4.0171 (2.0)	4.0043 (2.0)
5	6.1338	5.3971 (1.5)	5.1063 (1.9)	5.0267 (2.0)	5.0067 (2.0)
5	6.4846	5.3971 (1.9)	5.1063 (1.9)	5.0267 (2.0)	5.0067 (2.0)
6	6.4846	5.6712 (0.6)	5.9229 (2.1)	5.9807 (2.0)	5.9952 (2.0)
8	11.0924	8.8141 (1.9)	8.2713 (1.6)	8.0685 (2.0)	8.0171 (2.0)
9	11.0924	10.2540 (0.7)	9.3408 (1.9)	9.0864 (2.0)	9.0217 (2.0)
9	11.1164	10.2540 (0.8)	9.3408 (1.9)	9.0864 (2.0)	9.0217 (2.0)
10		10.9539	10.4193 (1.2)	10.1067 (2.0)	10.0268 (2.0)
10		10.9539	10.4193 (1.2)	10.1067 (2.0)	10.0268 (2.0)
13		11.1347	13.7027 (1.4)	13.1804 (2.0)	13.0452 (2.0)
13		11.1347	13.7027 (1.4)	13.1804 (2.0)	13.0452 (2.0)
15		9.4537	13.9639 (2.1)	14.7166 (1.9)	14.9272 (2.0)
15		19.4537	13.9639 (2.1)	14.7166 (1.9)	14.9272 (2.0)
16		19.7860	17.0588 (1.8)	16.2722 (2.0)	16.0684 (2.0)
16		19.7860	17.0588 (1.8)	16.2722 (2.0)	16.0684 (2.0)
17		20.9907	18.1813 (1.8)	17.3073 (1.9)	17.0773 (2.0)
dof	11	47	191	767	3071

Spurious modes

7



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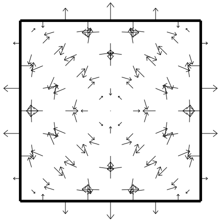

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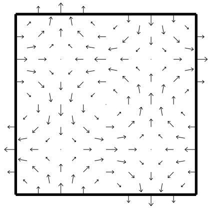
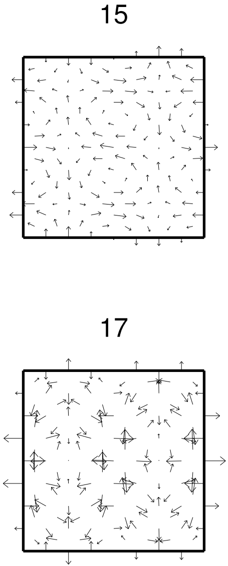


More spurious modes



Source vs. eigenvalue problem in mixed form

⟨B.–Brezzi–Gastaldi ’00⟩

N.B.

The criss-cross $P_1 - \mathrm{div}(P_1)$ element is a good element for the *source* problem (inf-sup condition OK!)

The $Q_1 - P_0$ scheme

⟨B.–Durán–Gastaldi '99⟩

The discrete eigenvalues can be explicitly computed:

$$\lambda_h^{(mn)} = \frac{4}{h^2} \frac{\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2}) - 2\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}{1 - \frac{2}{3}(\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2})) + \frac{4}{9}\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}$$

$$\boldsymbol{\sigma}_h^{(mn)} = (\sigma_1^{(mn)}, \sigma_2^{(mn)})$$

$$\sigma_1^{(m,n)}(x_i, y_j) = \frac{2}{h}\sin\left(\frac{mh}{2}\right)\cos\left(\frac{nh}{2}\right)\sin(mx_i)\cos(ny_j)$$
$$\sigma_2^{(m,n)}(x_i, y_j) = \frac{2}{h}\cos\left(\frac{mh}{2}\right)\sin\left(\frac{nh}{2}\right)\cos(mx_i)\sin(ny_j)$$

Does it converge?

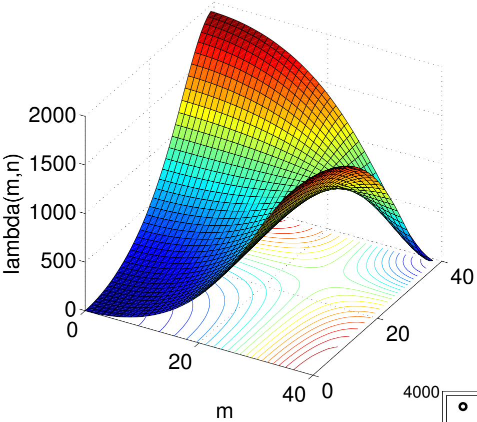
	Computed eigenvalue (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
2	1.9909	1.9995 (4.1)	2.0000 (4.0)	2.0000 (4.0)	2.0000 (4.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
5	5.3896	5.1129 (1.8)	5.0288 (2.0)	5.0072 (2.0)	5.0018 (2.0)
5	5.3896	5.1129 (1.8)	5.0288 (2.0)	5.0072 (2.0)	5.0018 (2.0)
8	7.2951	7.9636 (4.3)	7.9978 (4.1)	7.9999 (4.0)	8.0000 (4.0)
9	8.7285	10.0803 (-2.0)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
9	11.2850	10.0803 (1.1)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
10	11.2850	10.8308 (0.6)	10.2066 (2.0)	10.0515 (2.0)	10.0129 (2.0)
10	12.5059	10.8308 (1.6)	10.2066 (2.0)	10.0515 (2.0)	10.0129 (2.0)
13	12.5059	13.1992 (1.3)	13.0736 (1.4)	13.0197 (1.9)	13.0050 (2.0)
13	12.8431	13.1992 (-0.3)	13.0736 (1.4)	13.0197 (1.9)	13.0050 (2.0)
16	12.8431	14.7608 (1.3)	16.8382 (0.6)	16.2067 (2.0)	16.0515 (2.0)
16		17.5489	16.8382 (0.9)	16.2067 (2.0)	16.0515 (2.0)
17		19.4537	17.1062 (4.5)	17.1814 (-0.8)	17.0452 (2.0)
17		19.4537	17.7329 (1.7)	17.1814 (2.0)	17.0452 (2.0)
18		19.9601	17.7329 (2.9)	17.7707 (0.2)	17.9423 (2.0)
18		19.9601	17.9749 (6.3)	17.9985 (4.0)	17.9999 (4.0)
20		21.5584	20.4515 (1.8)	20.1151 (2.0)	20.0289 (2.0)
20		21.5584	20.4515 (1.8)	20.1151 (2.0)	20.0289 (2.0)

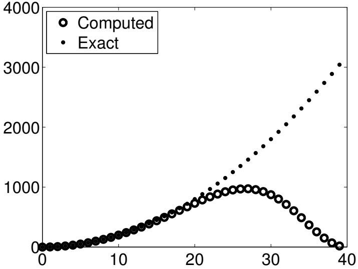
Wrong proof?

$$\lambda_h^{(mn)} = \frac{4}{h^2}\frac{\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2}) - 2\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}{1 - \frac{2}{3}(\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2})) + \frac{4}{9}\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}$$

Indeed, if $h = \pi/N$, we have:

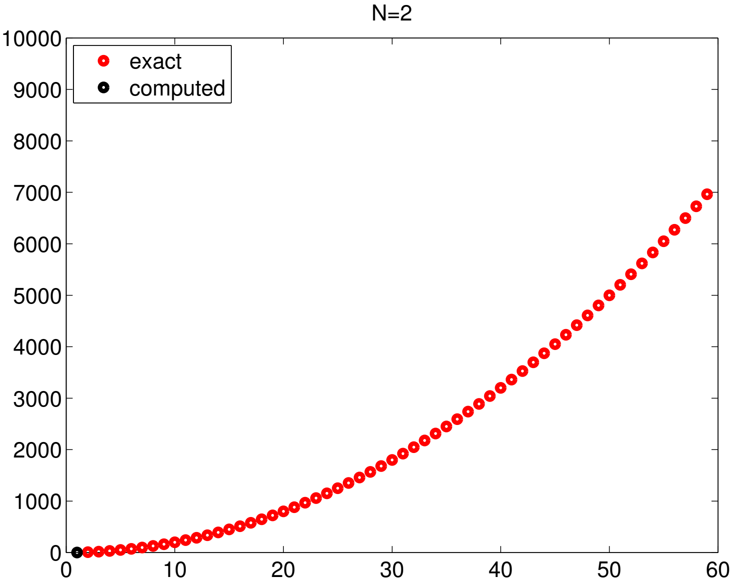
$$\lim_{N\to\infty}\lambda_h^{(N-1,N-1)} = 18$$


lambda(m,n)
2000
1500
1000
500
0
0
20
40
m
0
20
40

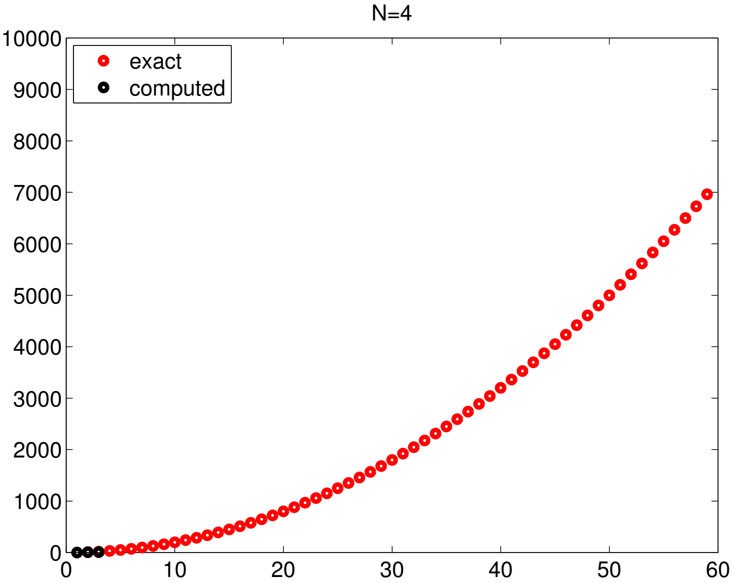

Computed
Exact
4000
3000
2000
1000
0
0
10
20
30
40

Plot for $m = n$

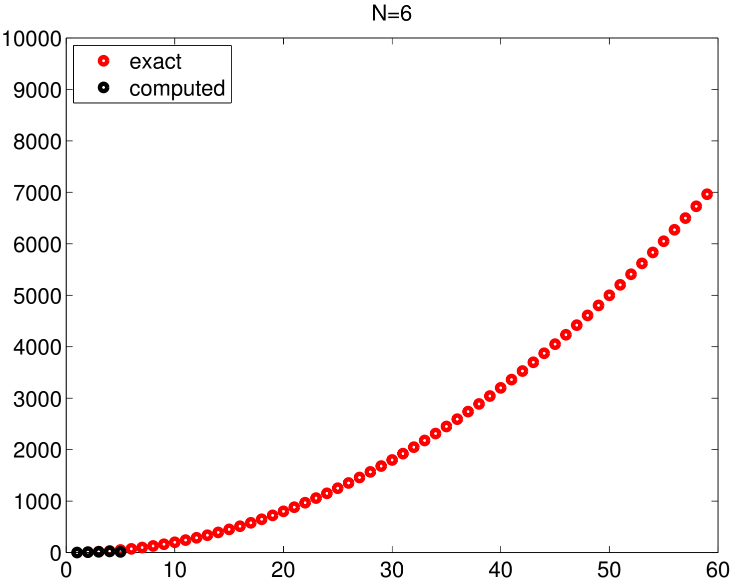
Pointwise vs. uniform convergence



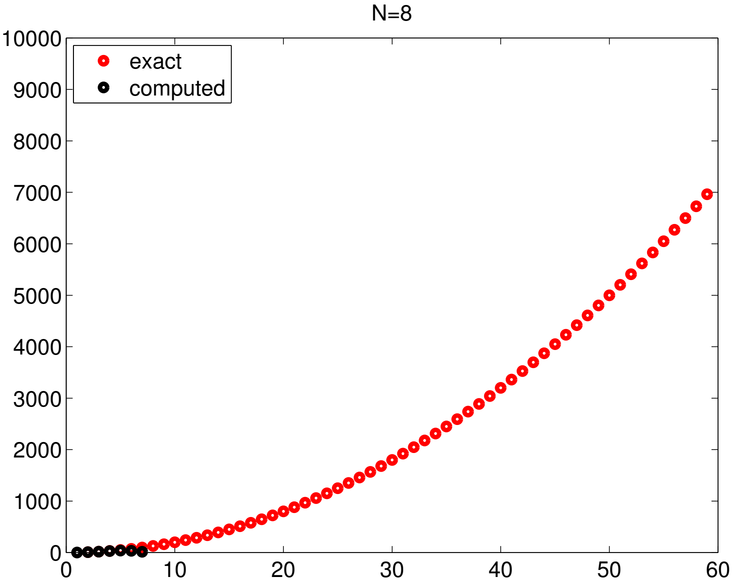
Pointwise vs. uniform convergence



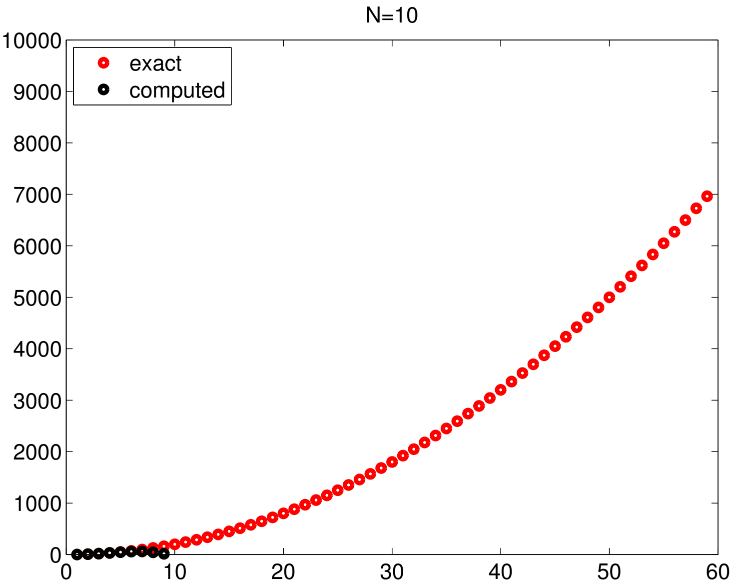
Pointwise vs. uniform convergence



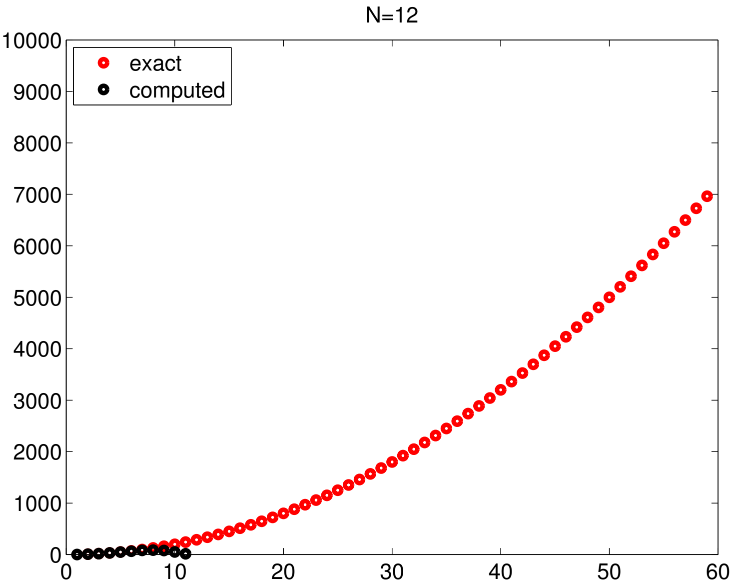
Pointwise vs. uniform convergence



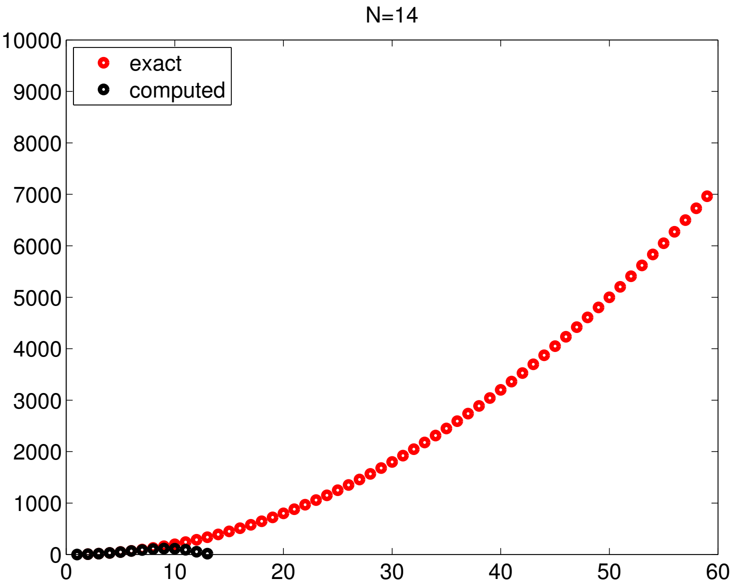
Pointwise vs. uniform convergence



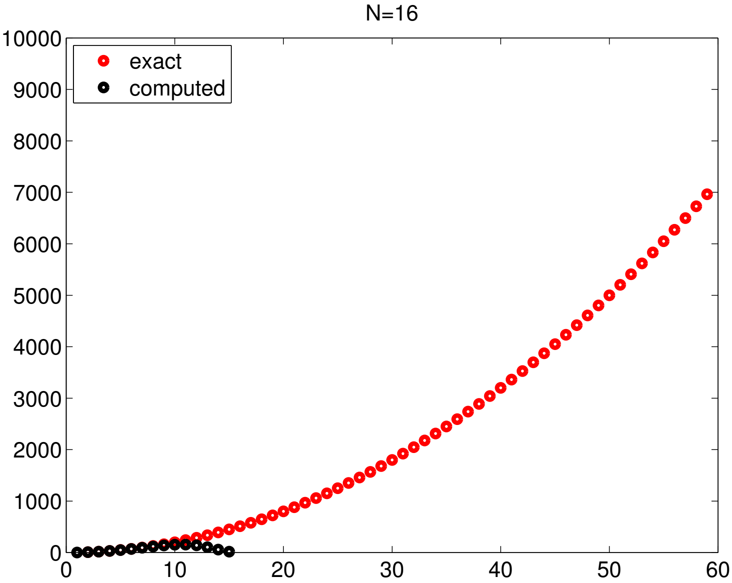
Pointwise vs. uniform convergence



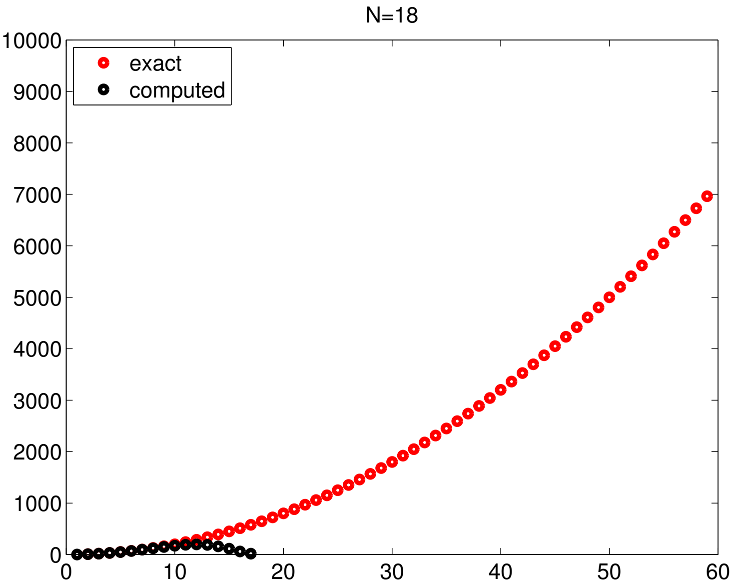
Pointwise vs. uniform convergence



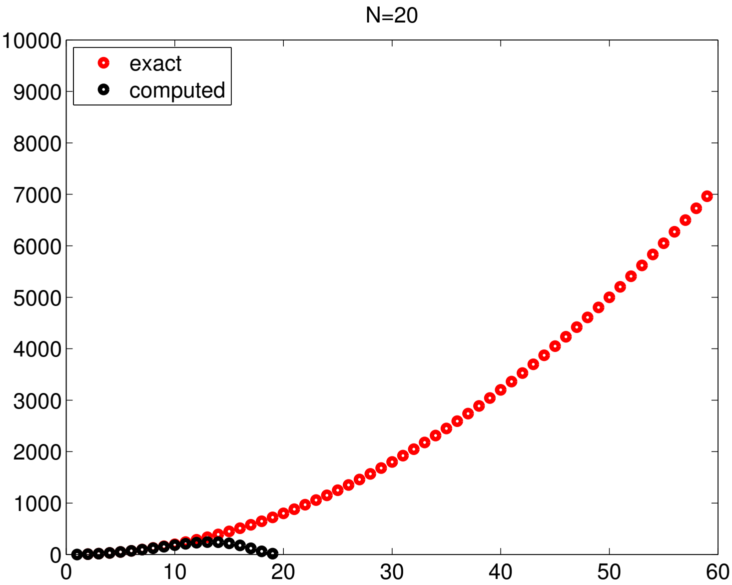
Pointwise vs. uniform convergence



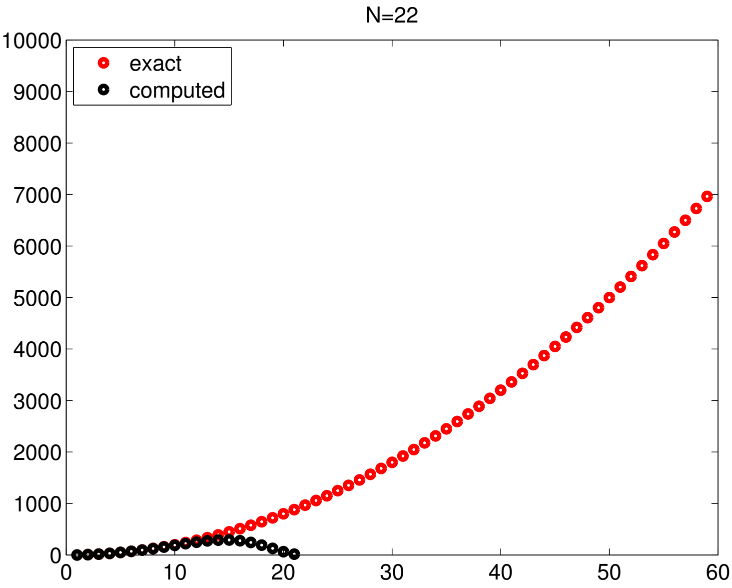
Pointwise vs. uniform convergence



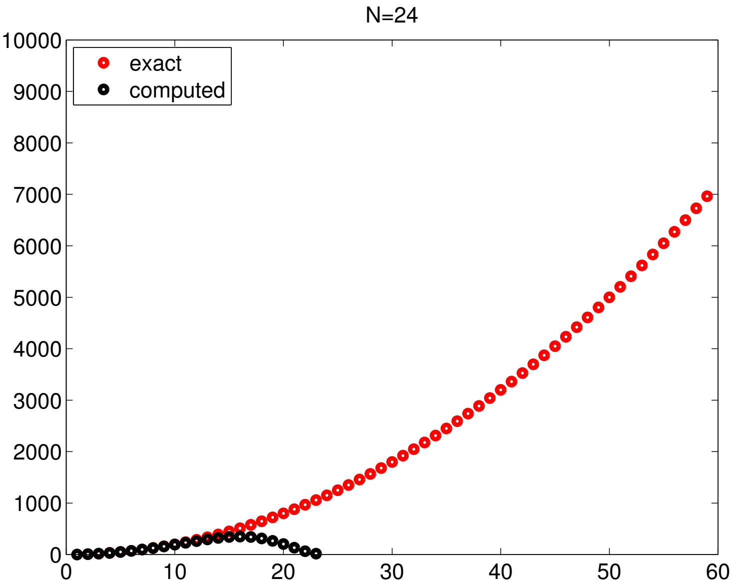
Pointwise vs. uniform convergence



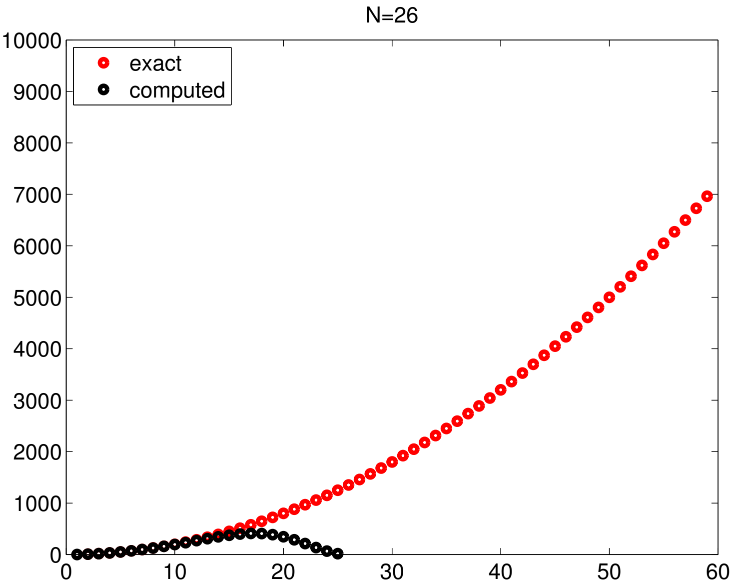
Pointwise vs. uniform convergence



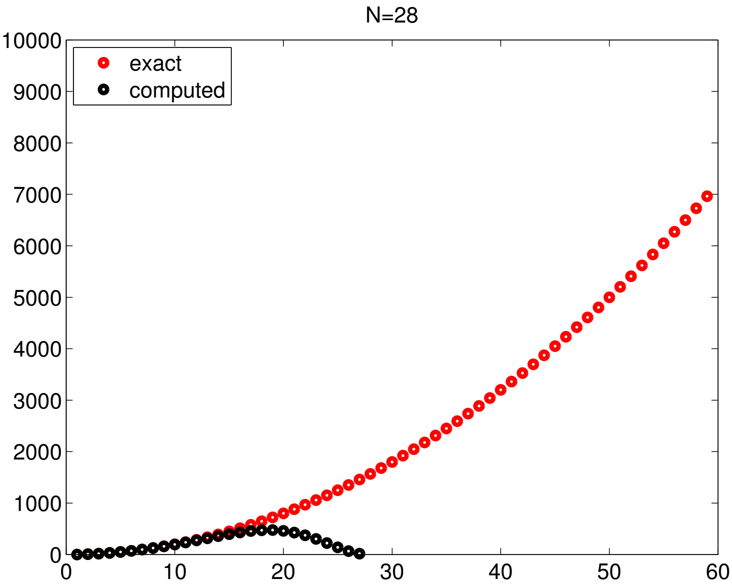
Pointwise vs. uniform convergence



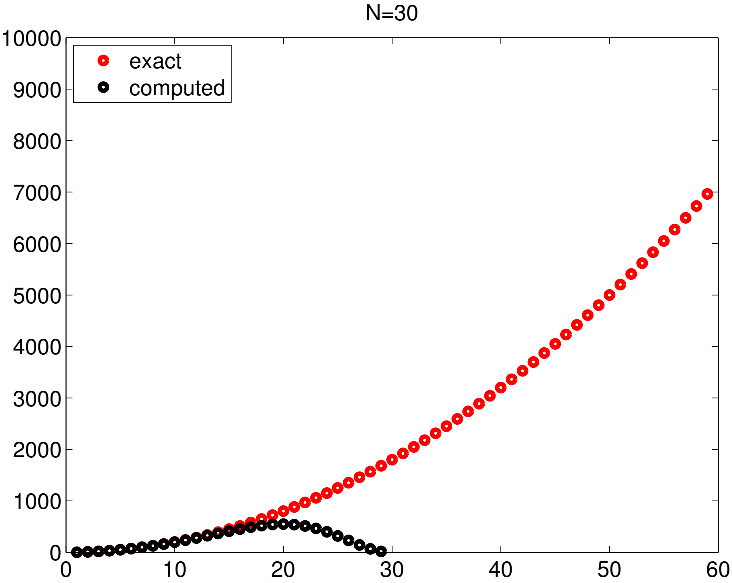
Pointwise vs. uniform convergence



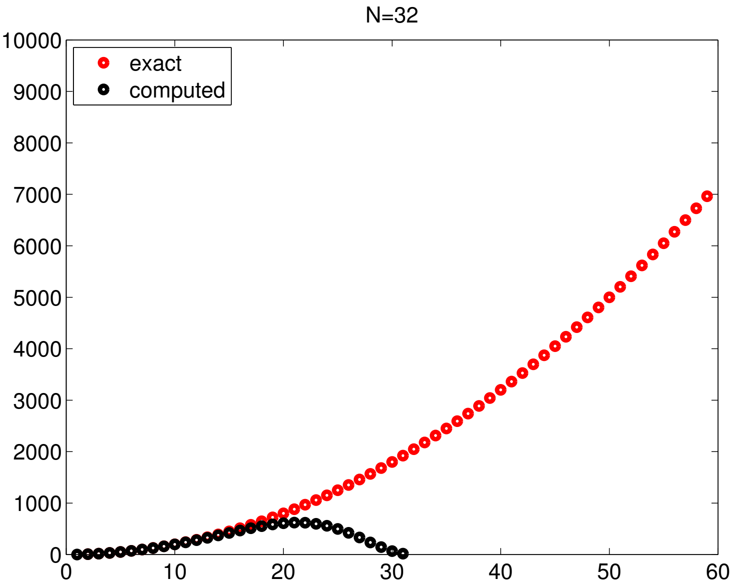
Pointwise vs. uniform convergence



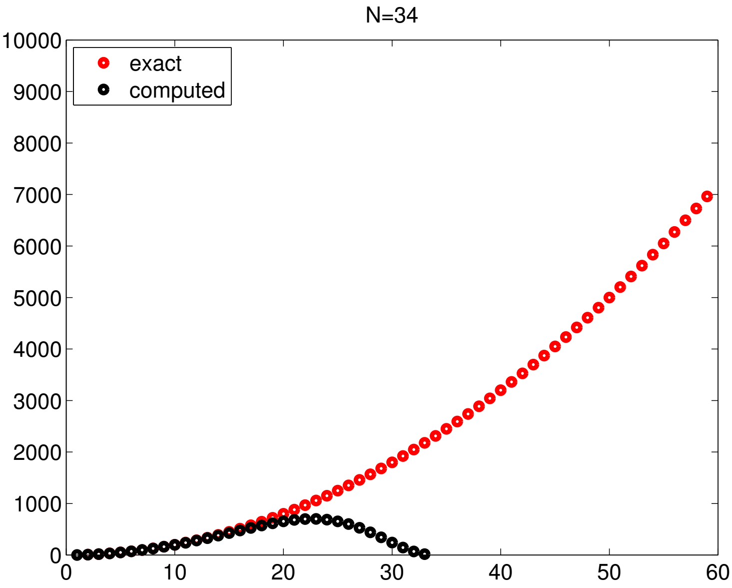
Pointwise vs. uniform convergence



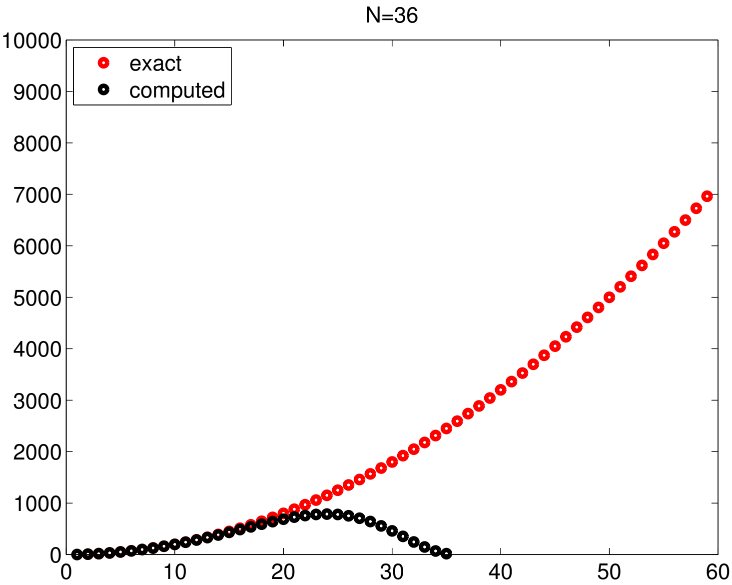
Pointwise vs. uniform convergence



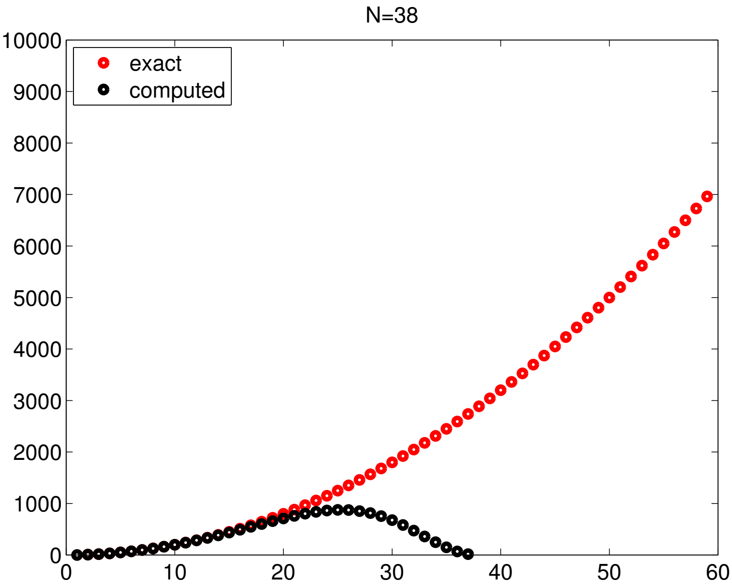
Pointwise vs. uniform convergence



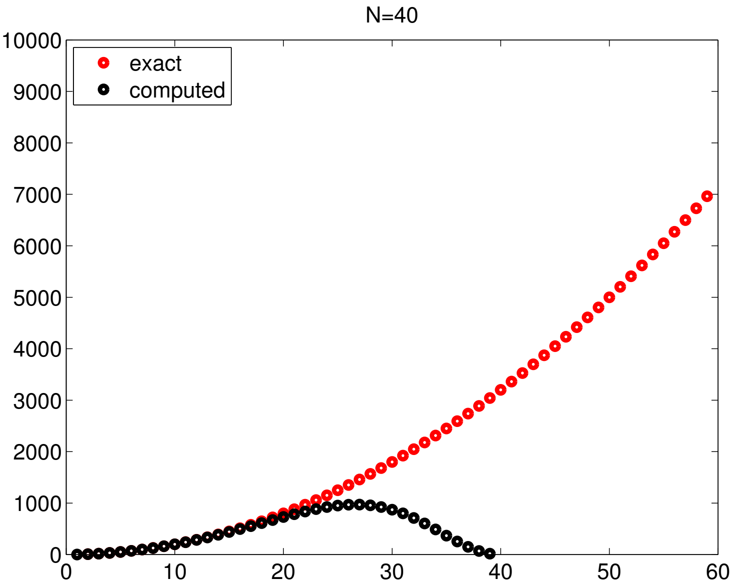
Pointwise vs. uniform convergence



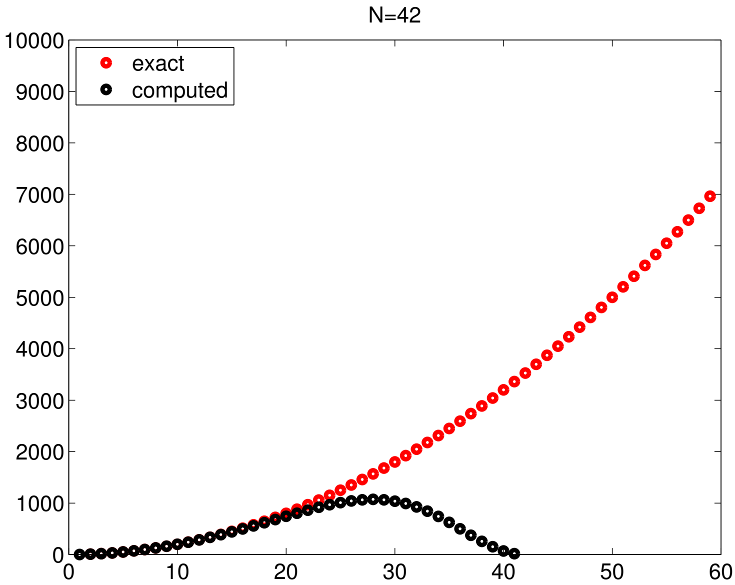
Pointwise vs. uniform convergence



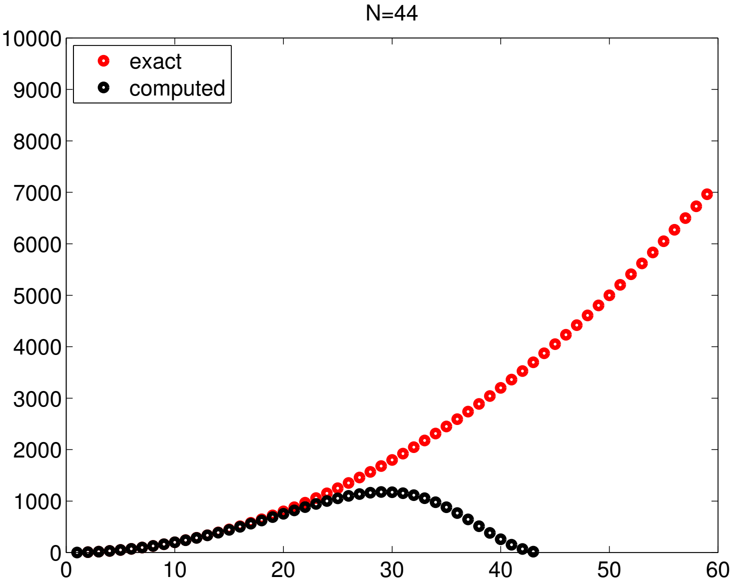
Pointwise vs. uniform convergence



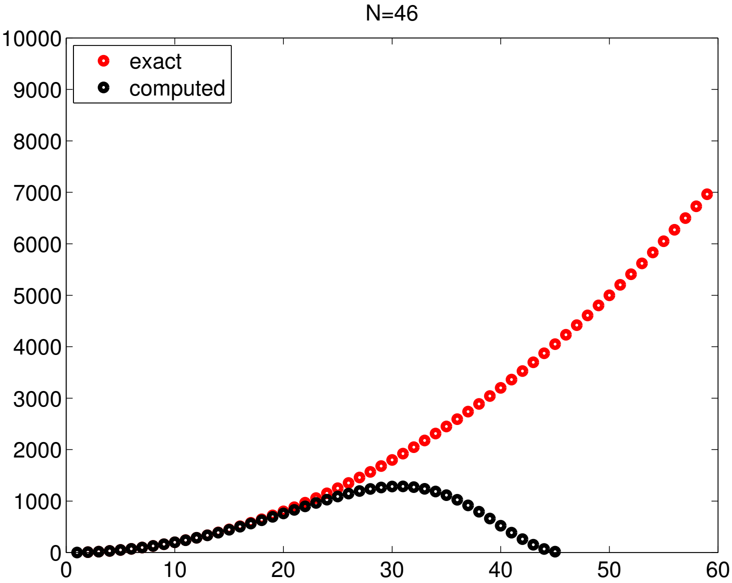
Pointwise vs. uniform convergence



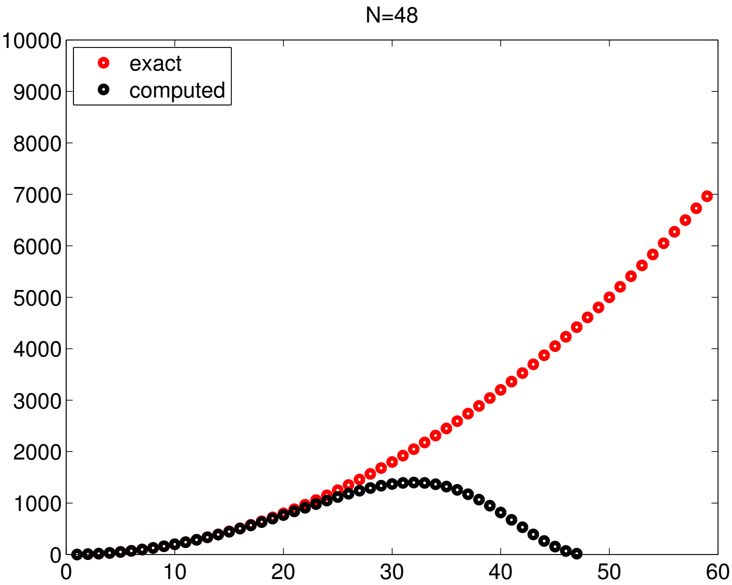
Pointwise vs. uniform convergence



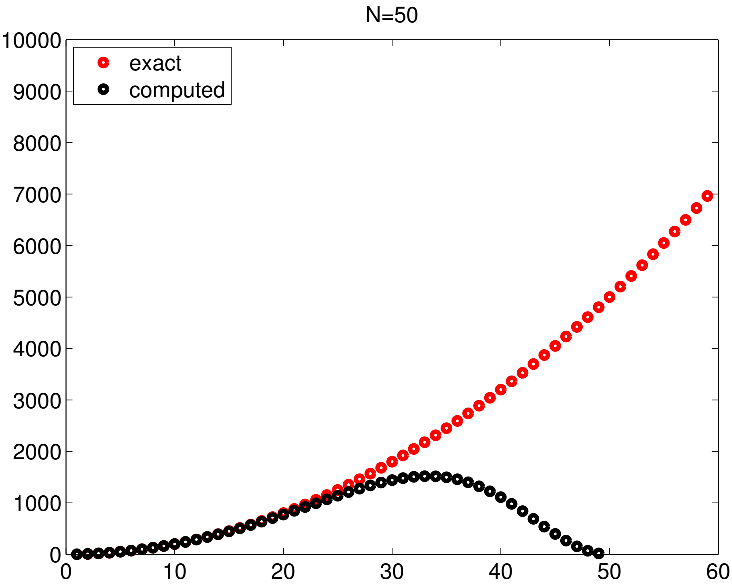
Pointwise vs. uniform convergence



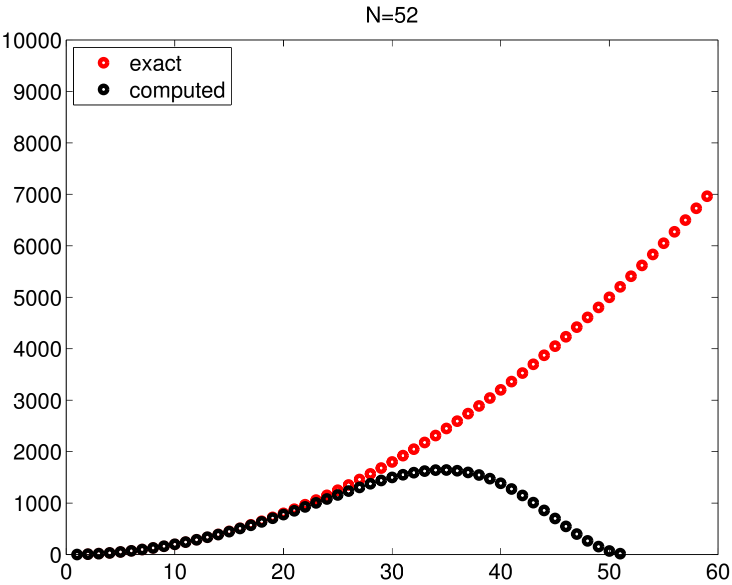
Pointwise vs. uniform convergence



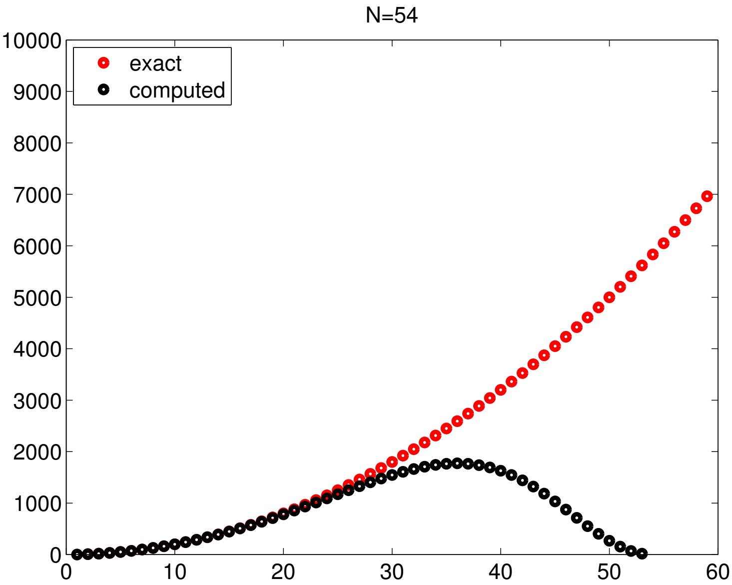
Pointwise vs. uniform convergence



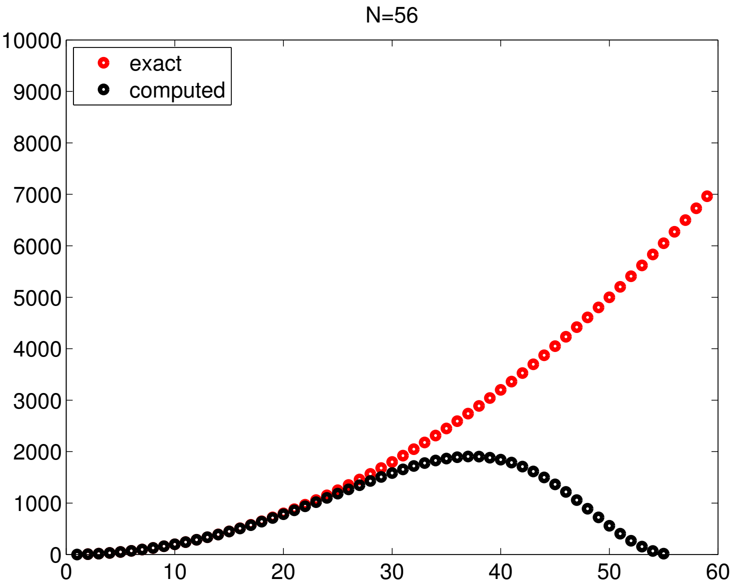
Pointwise vs. uniform convergence



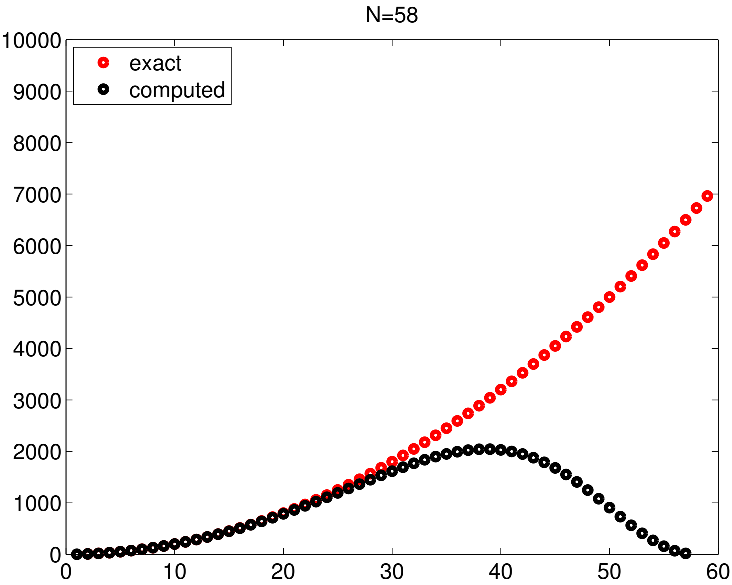
Pointwise vs. uniform convergence



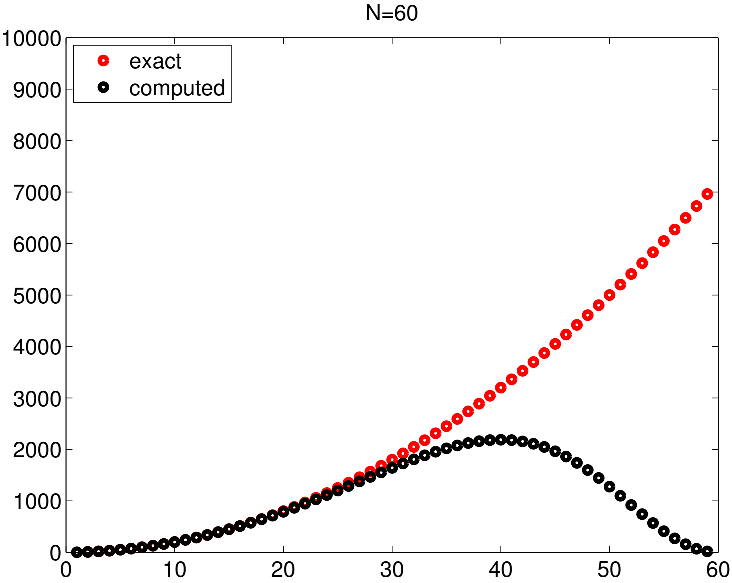
Pointwise vs. uniform convergence



Pointwise vs. uniform convergence



Pointwise vs. uniform convergence



Raviart–Thomas element

Unstructured mesh

	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.0138	1.9989 (3.6)	1.9997 (1.7)	1.9999 (2.7)	2.0000 (2.8)
5	4.8696	4.9920 (4.0)	5.0000 (8.0)	4.9999 (-2.1)	5.0000 (3.7)
5	4.8868	4.9952 (4.5)	5.0006 (3.0)	5.0000 (5.8)	5.0000 (2.6)
8	8.6905	7.9962 (7.5)	7.9974 (0.6)	7.9995 (2.5)	7.9999 (2.2)
10	9.7590	9.9725 (3.1)	9.9980 (3.8)	9.9992 (1.3)	9.9999 (3.2)
10	11.4906	9.9911 (7.4)	10.0007 (3.7)	10.0005 (0.4)	10.0001 (2.4)
13	11.9051	12.9250 (3.9)	12.9917 (3.2)	12.9998 (5.4)	12.9999 (1.8)
13	12.7210	12.9631 (2.9)	12.9950 (2.9)	13.0000 (7.5)	13.0000 (1.1)
17	13.5604	16.8450 (4.5)	16.9848 (3.4)	16.9992 (4.3)	16.9999 (2.5)
17	14.1813	16.9659 (6.4)	16.9946 (2.7)	17.0009 (2.6)	17.0000 (5.5)
#	32	142	576	2338	9400

Raviart–Thomas element (cont'd)

Uniform mesh

	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.1048	2.0258 (2.0)	2.0064 (2.0)	2.0016 (2.0)	2.0004 (2.0)
5	5.9158	5.2225 (2.0)	5.0549 (2.0)	5.0137 (2.0)	5.0034 (2.0)
5	5.9158	5.2225 (2.0)	5.0549 (2.0)	5.0137 (2.0)	5.0034 (2.0)
8	9.7268	8.4191 (2.0)	8.1033 (2.0)	8.0257 (2.0)	8.0064 (2.0)
10	13.8955	11.0932 (1.8)	10.2663 (2.0)	10.0660 (2.0)	10.0165 (2.0)
10	13.8955	11.0932 (1.8)	10.2663 (2.0)	10.0660 (2.0)	10.0165 (2.0)
13	17.7065	14.2898 (1.9)	13.3148 (2.0)	13.0781 (2.0)	13.0195 (2.0)
13	17.7065	14.2898 (1.9)	13.3148 (2.0)	13.0781 (2.0)	13.0195 (2.0)
17	20.5061	20.1606 (0.1)	17.8414 (1.9)	17.2075 (2.0)	17.0517 (2.0)
17	20.5061	20.4666 (0.0)	17.8414 (2.0)	17.2075 (2.0)	17.0517 (2.0)
#	16	64	256	1024	4096

Commuting diagram property (de Rham complex)

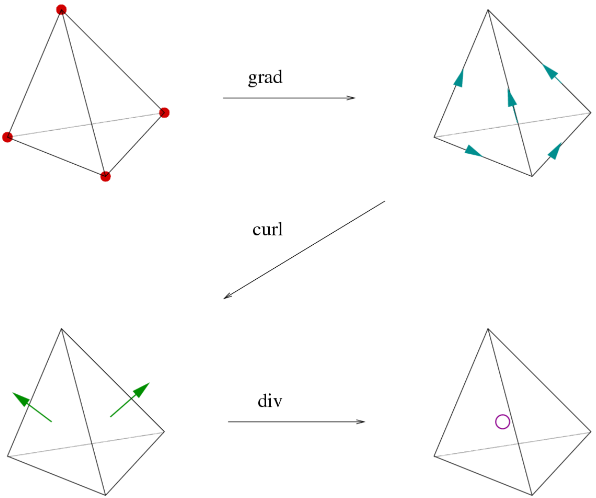
⟨Douglas–Roberts ’82⟩
⟨Bossavit ’88⟩
⟨Arnold ’02⟩
⟨Arnold–Falk–Winther ’10⟩

$Q \subset H^1_0,\ V \subset \mathbf{H}_0(\mathbf{curl}),\ U \subset \mathbf{H}_0(\mathrm{div}),\ S \subset L^2/\mathbb{R}$

$$\begin{array}{ccccccccccc}
0 \to & Q & \xrightarrow{\nabla} & V & \xrightarrow{\mathbf{curl}} & U & \xrightarrow{\mathrm{div}} & S & \to 0 \\
& \downarrow \Pi_k^Q & & \downarrow \Pi_k^V & & \downarrow \Pi_k^U & & \downarrow \Pi_k^S & \\
0 \to & Q_k & \xrightarrow{\nabla} & V_k & \xrightarrow{\mathbf{curl}} & U_k & \xrightarrow{\mathrm{div}} & S_k & \to 0
\end{array}$$

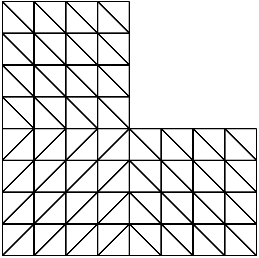
- Kikuchi formulation uses Q and V
- Alternative formulation uses V and U
- U and S are used for Darcy flow or mixed Laplacian

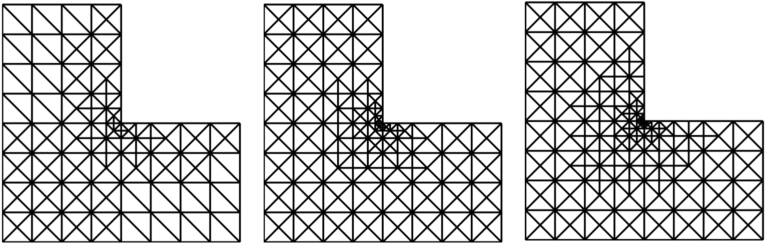
Lowest order finite elements



Some comments on adaptive schemes

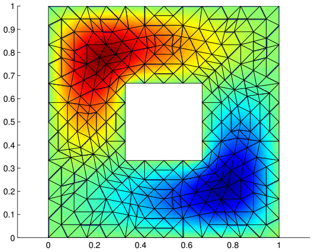
- A posteriori error analysis
- Convergence study for adaptive schemes



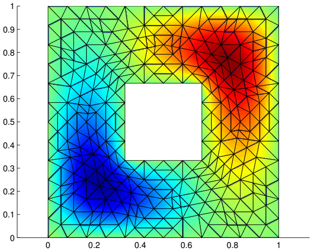


Multiple eigenvalues: the square ring

λ_2



$\lambda_3 = \lambda_2$



Question

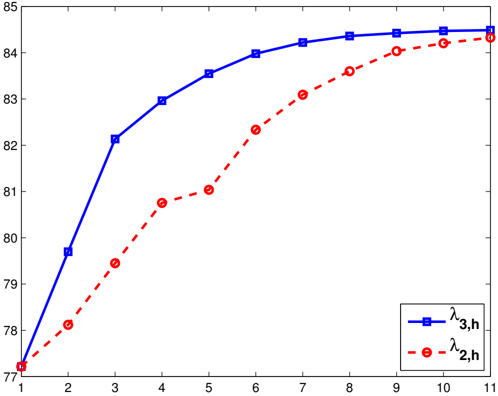
What is the best adaptive strategy for the approximation of the multiple eigenvalue?

1. Indicator based on $(\lambda_{h,2}, u_{h,2})$
2. Indicator based on $(\lambda_{h,3}, u_{h,3})$
3. Indicator based on both $(\lambda_{h,2}, u_{h,2})$ and $(\lambda_{h,3}, u_{h,3})$

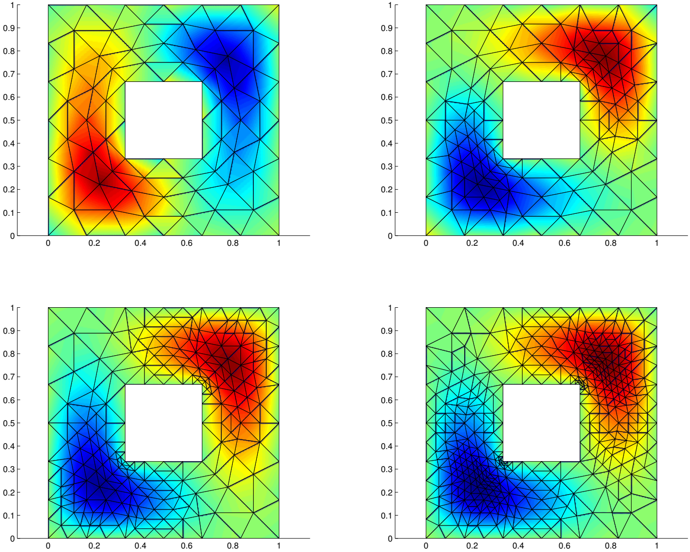
Refinement based on $\lambda_{h,3}$

⟨B.–Durán–Gardini–Gastaldi 2015⟩

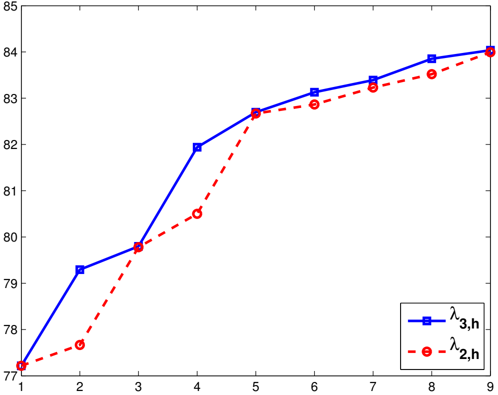
Remark: here we are using a nonconforming discretization which provides eigenvalue approximation from below



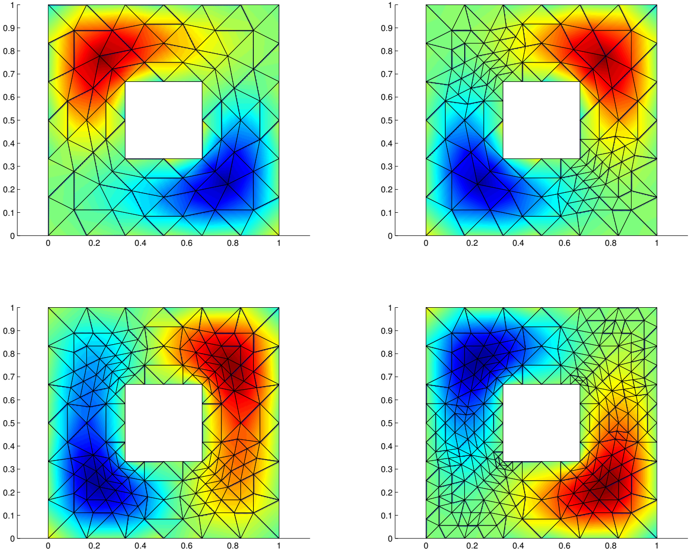
Refinement based on $\lambda_{h,3}$ (eigenfunction $u_{h,3}$)



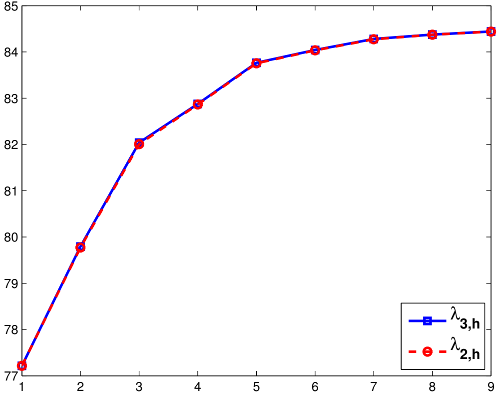
Refinement based on $\lambda_{h,2}$



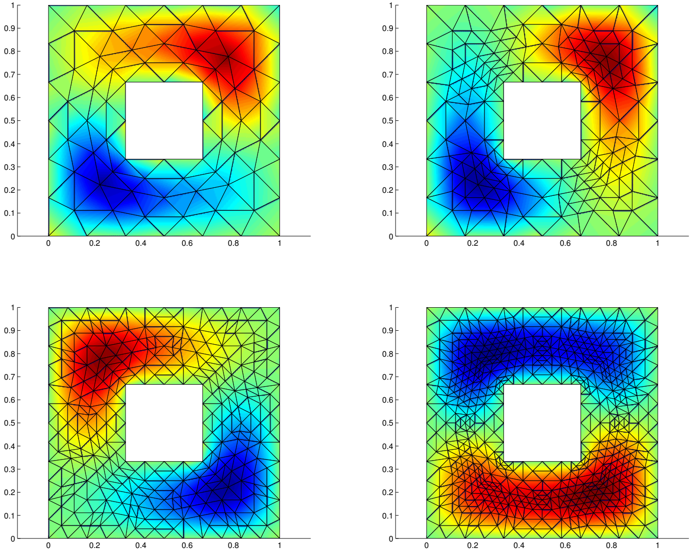
Refinement based on $\lambda_{h,2}$ (eigenfunction $u_{h,2}$)



Refinement based on $\lambda_{h,2}$ and $\lambda_{h,3}$ (eigenvalues)



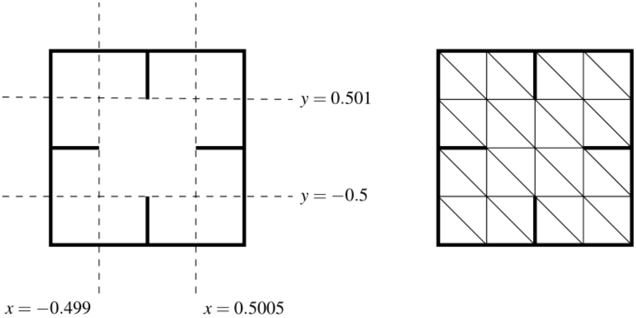
Refinement based on $\lambda_{h,2}$ and $\lambda_{h,3}$ (eigenfunction $u_{h,2}$)



Cluster of eigenvalues

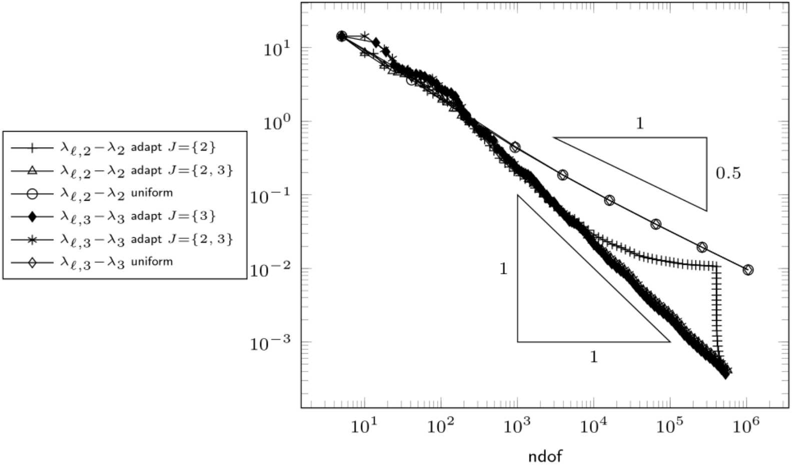
⟨Gallistl '14⟩

A slightly non-symmetric domain



Now $\lambda_2 < \lambda_3$ but they are very close to each other

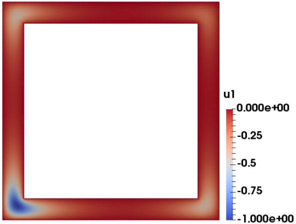
Non-symmetric slit domain



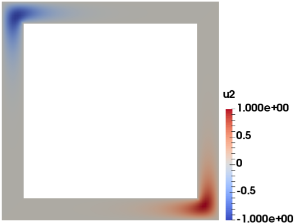
A slightly non-symmetric domain

A square ring for which the first four modes are the following ones (computed on an adapted mesh with 4,122,416 dof's)

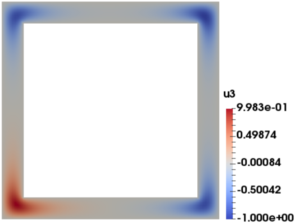
916.745



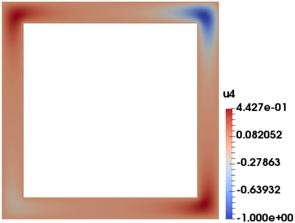
917.008



917.010

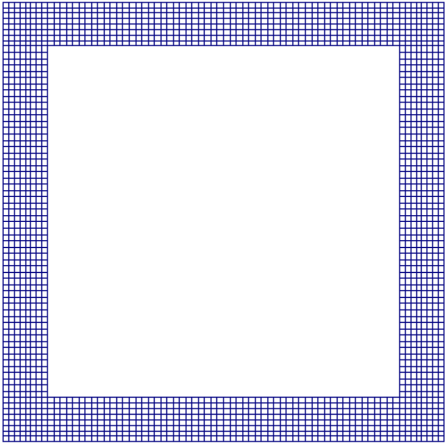


917.274



Approximation of first frequency

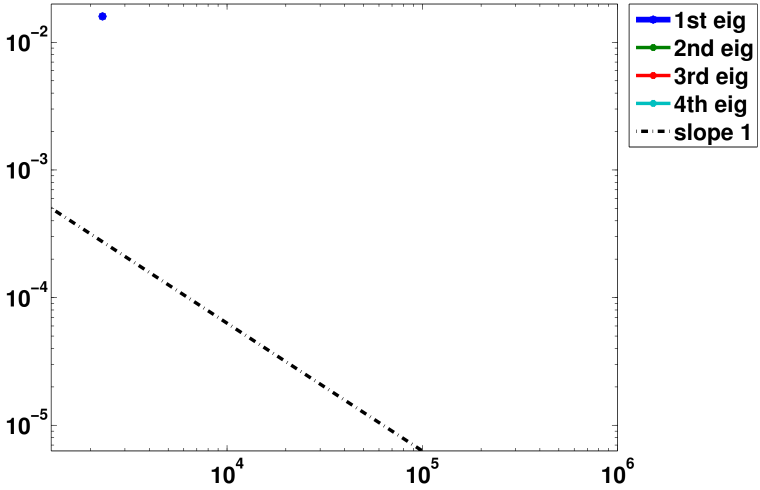
Initial mesh



At each refinement level we compute the first four eigenmodes and drive the adaptive strategy according to the error indicator related to the first eigenmode

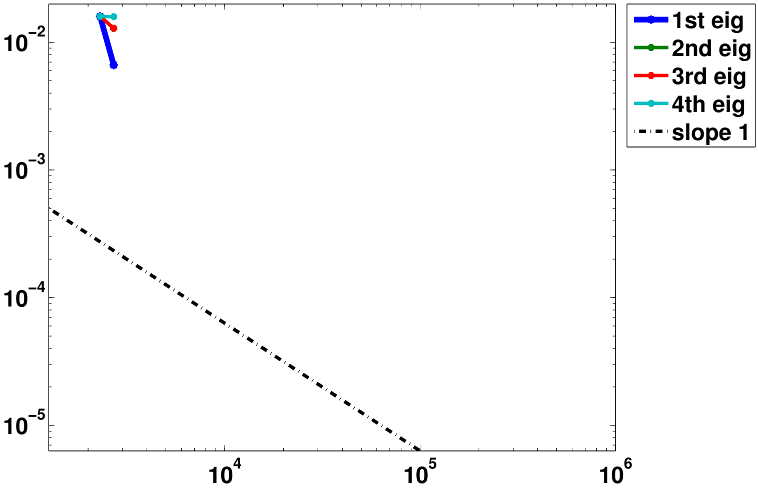
Approximation of first frequency

Bulk parameter=0.3, Refinement level=0 (initial mesh)



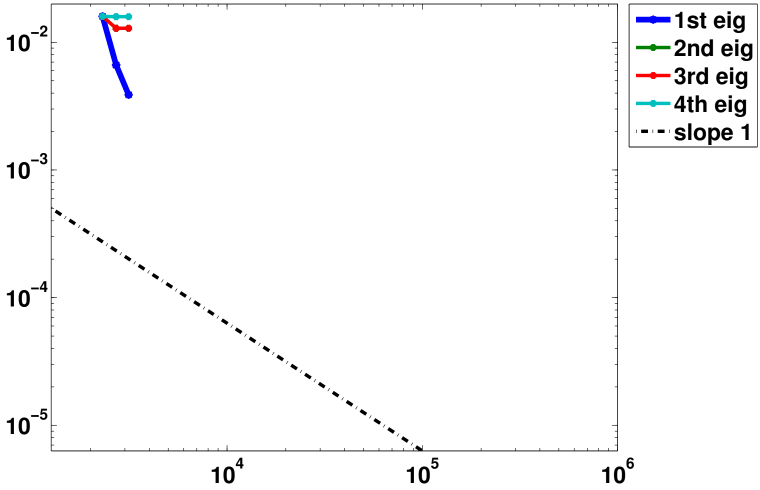
Approximation of first frequency

Bulk parameter=0.3, Refinement level=1



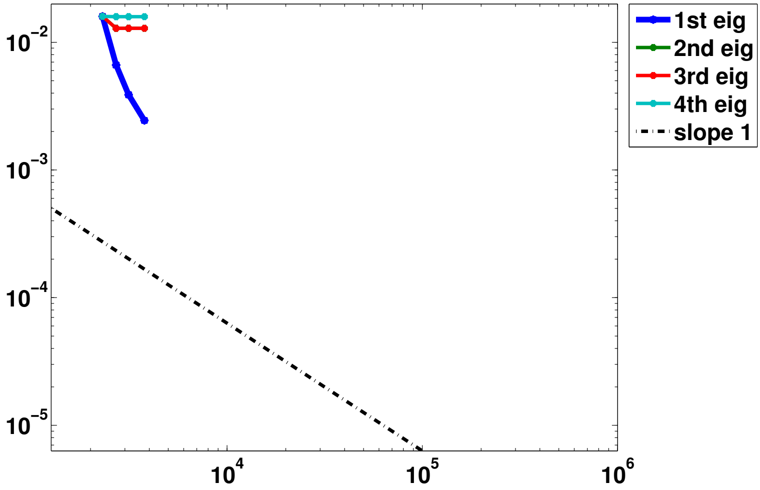
Approximation of first frequency

Bulk parameter=0.3, Refinement level=2



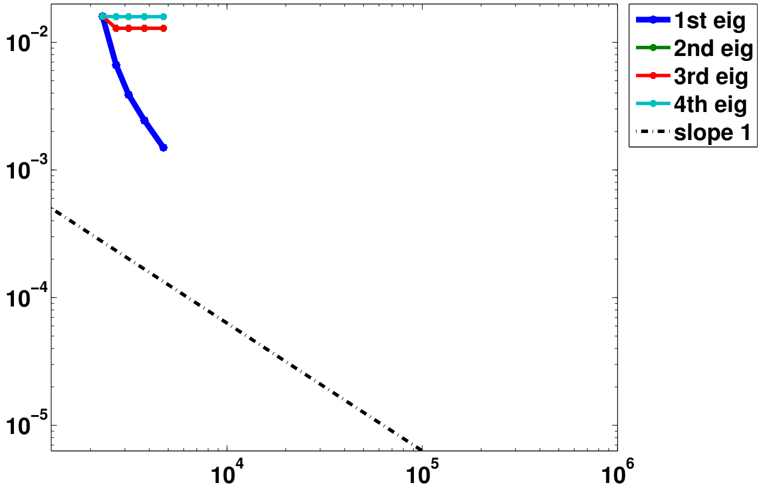
Approximation of first frequency

Bulk parameter=0.3, Refinement level=3



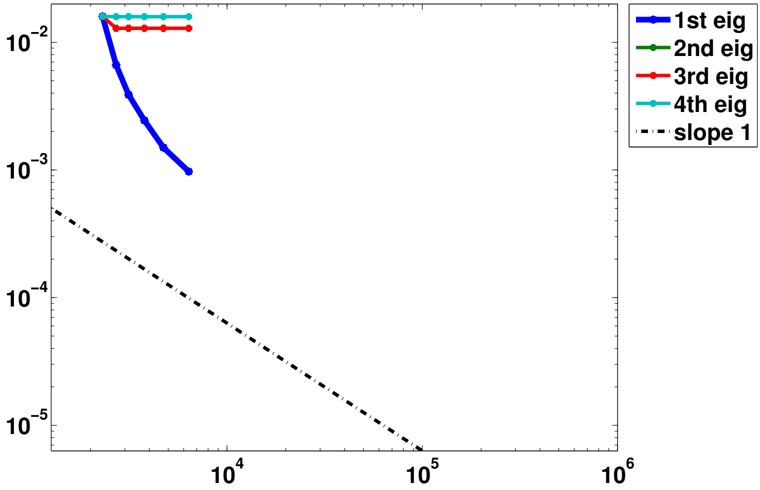
Approximation of first frequency

Bulk parameter=0.3, Refinement level=4



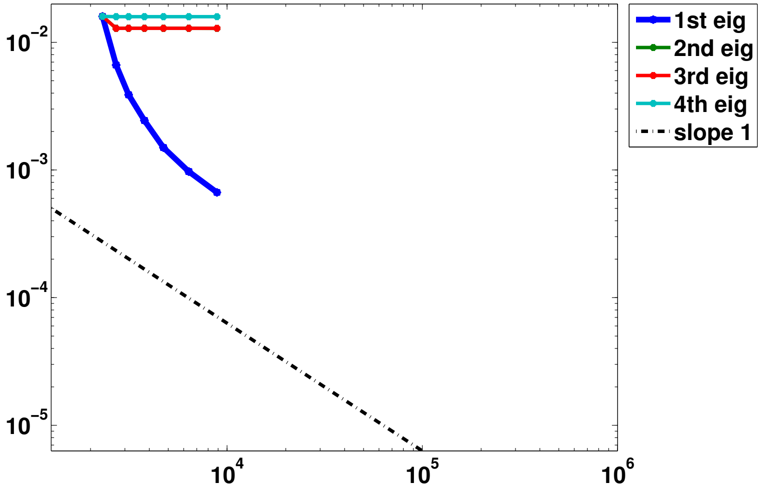
Approximation of first frequency

Bulk parameter=0.3, Refinement level=5



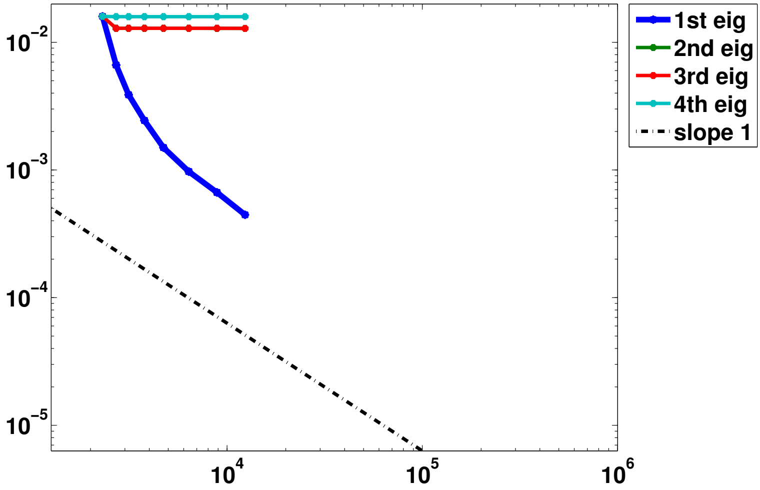
Approximation of first frequency

Bulk parameter=0.3, Refinement level=6



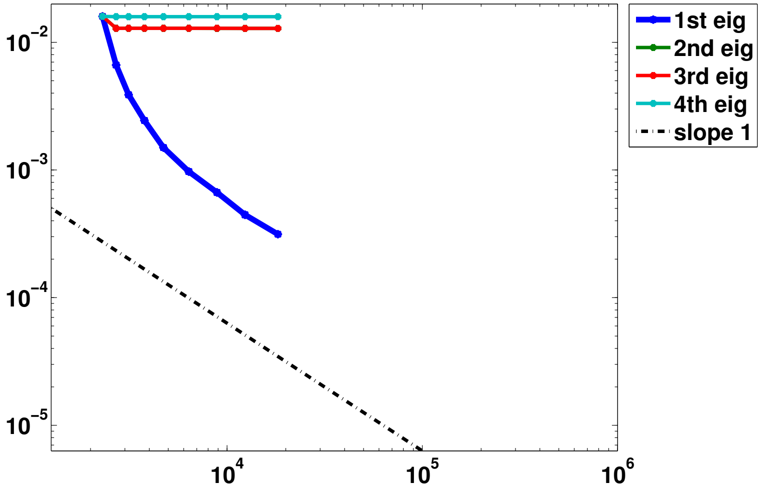
Approximation of first frequency

Bulk parameter=0.3, Refinement level=7



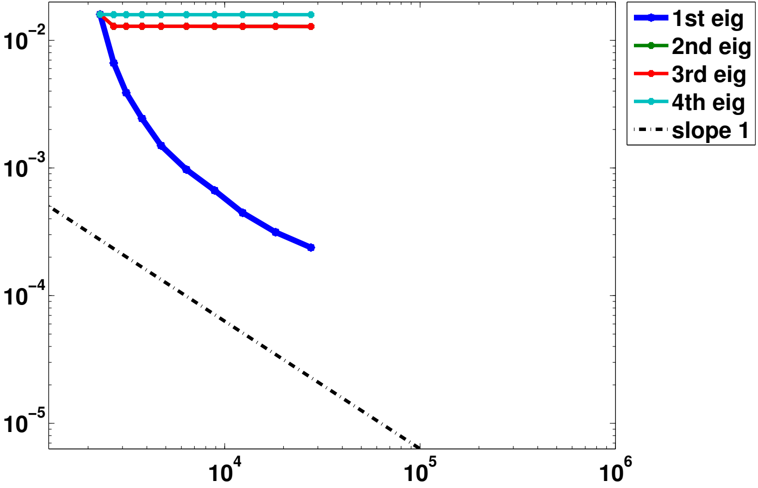
Approximation of first frequency

Bulk parameter=0.3, Refinement level=8



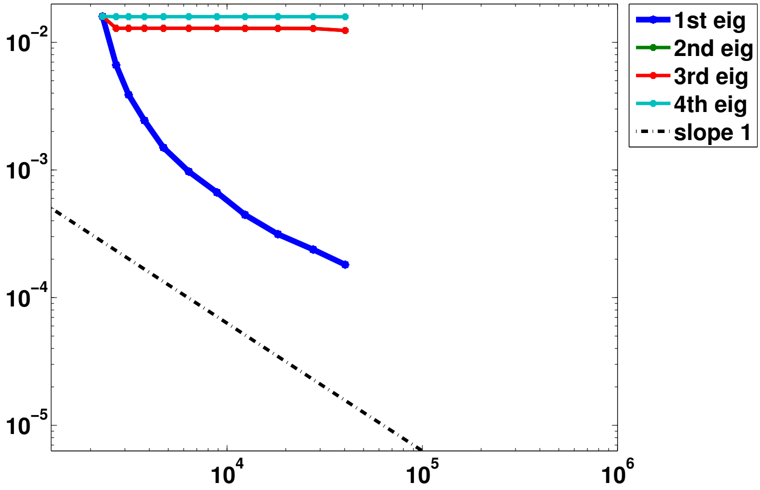
Approximation of first frequency

Bulk parameter=0.3, Refinement level=9



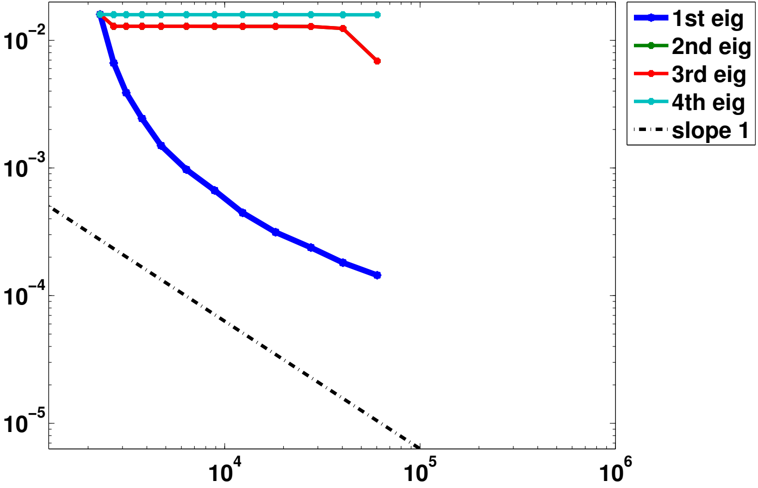
Approximation of first frequency

Bulk parameter=0.3, Refinement level=10



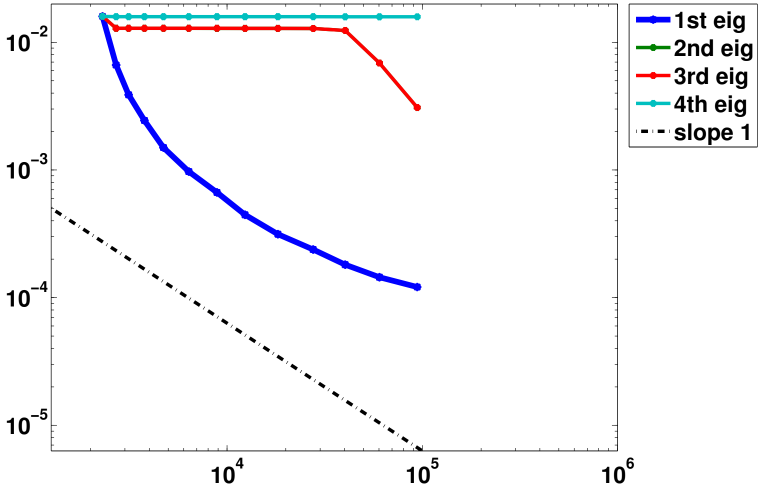
Approximation of first frequency

Bulk parameter=0.3, Refinement level=11



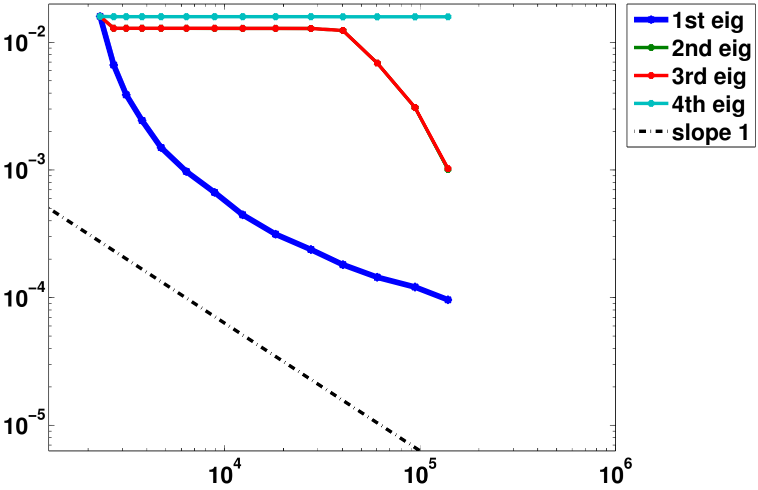
Approximation of first frequency

Bulk parameter=0.3, Refinement level=12



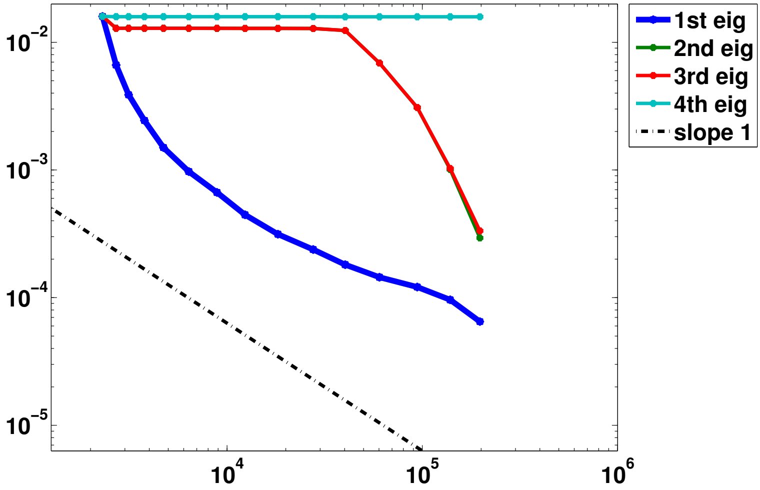
Approximation of first frequency

Bulk parameter=0.3, Refinement level=13



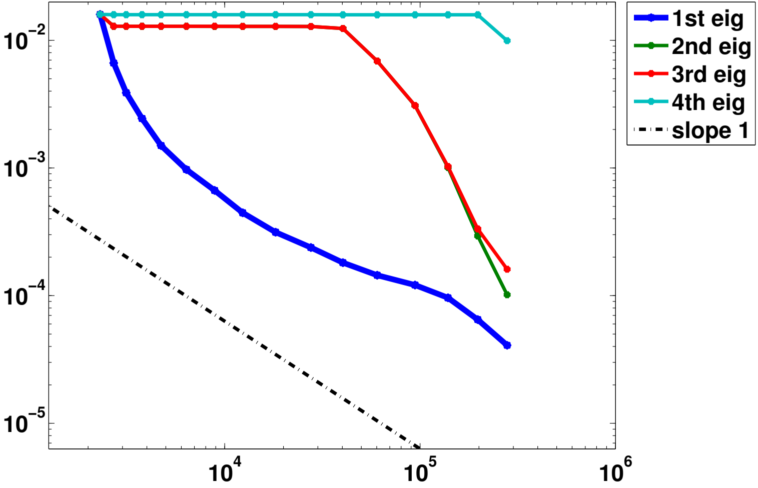
Approximation of first frequency

Bulk parameter=0.3, Refinement level=14



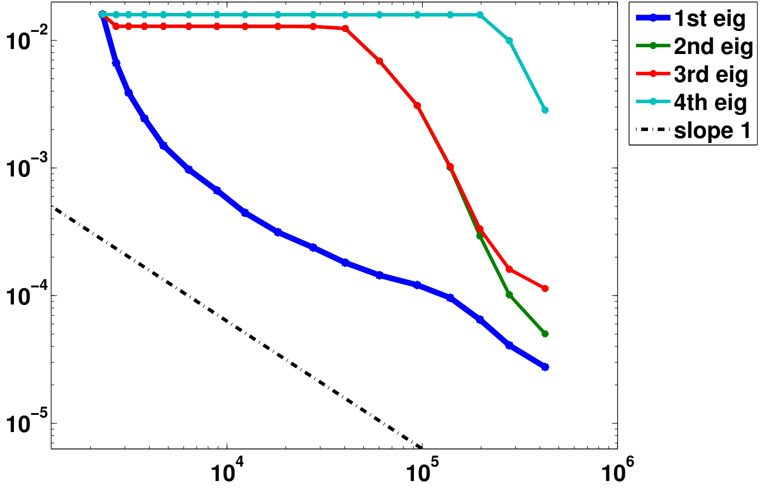
Approximation of first frequency

Bulk parameter=0.3, Refinement level=15



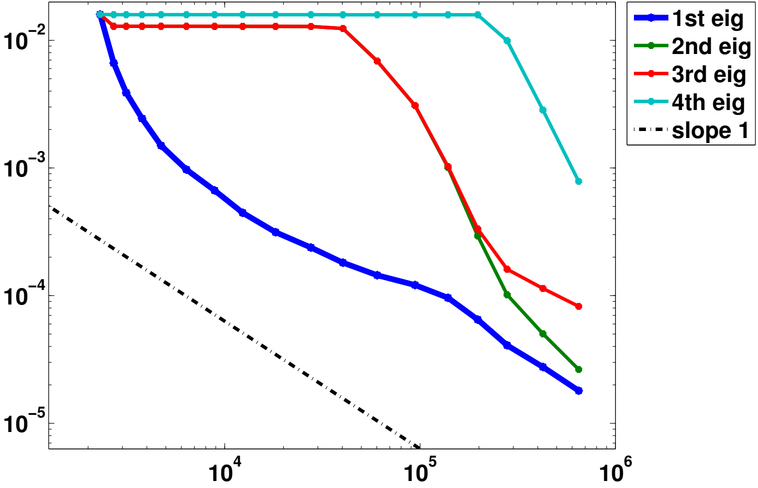
Approximation of first frequency

Bulk parameter=0.3, Refinement level=16



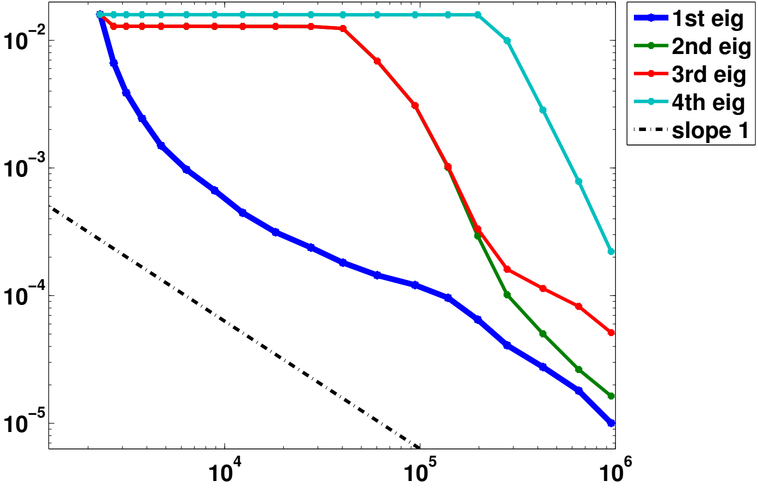
Approximation of first frequency

Bulk parameter=0.3, Refinement level=17



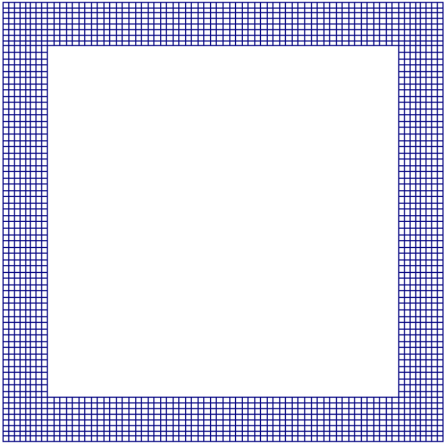
Approximation of first frequency

Bulk parameter=0.3, Refinement level=18



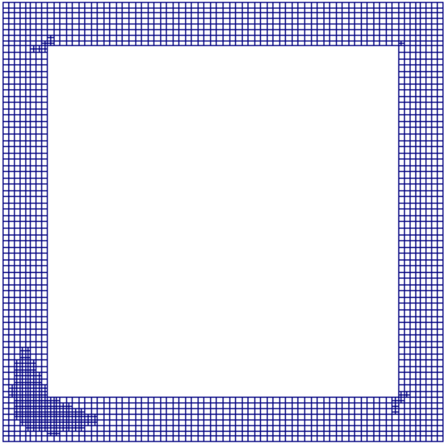
Approximation of first frequency: underlying mesh

Initial mesh



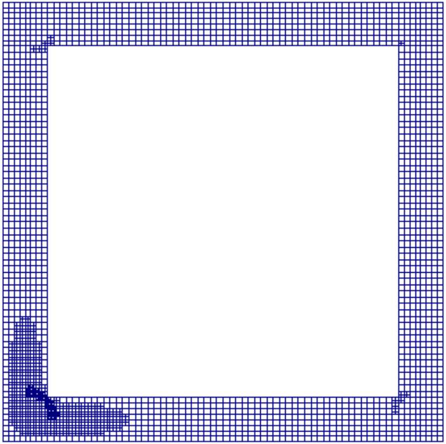
Approximation of first frequency: underlying mesh

Refinement level=1



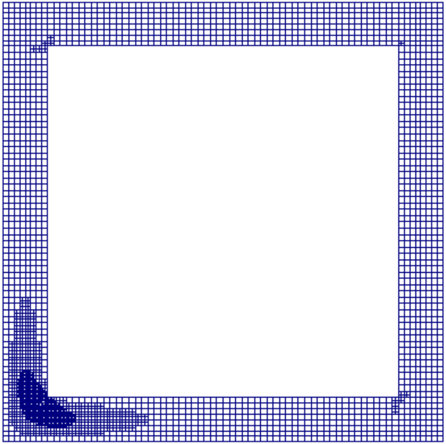
Approximation of first frequency: underlying mesh

Refinement level=2



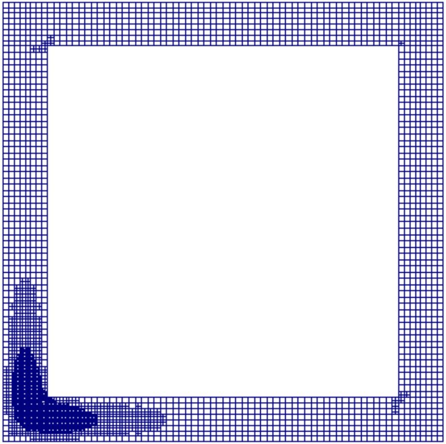
Approximation of first frequency: underlying mesh

Refinement level=3



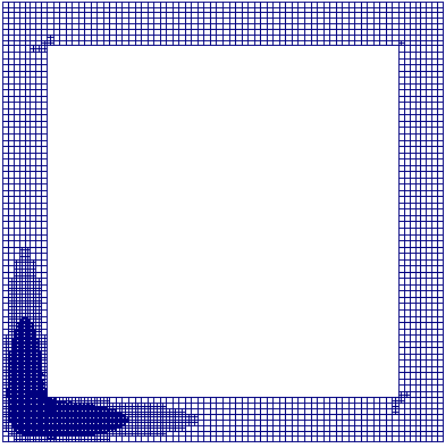
Approximation of first frequency: underlying mesh

Refinement level=4



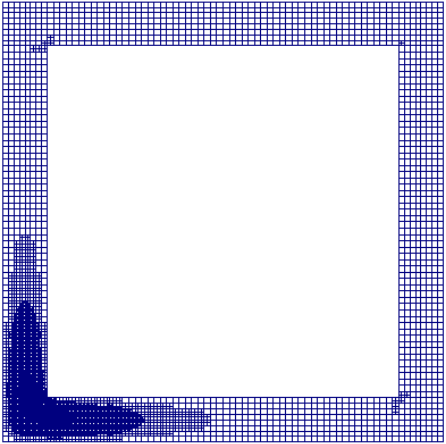
Approximation of first frequency: underlying mesh

Refinement level=5



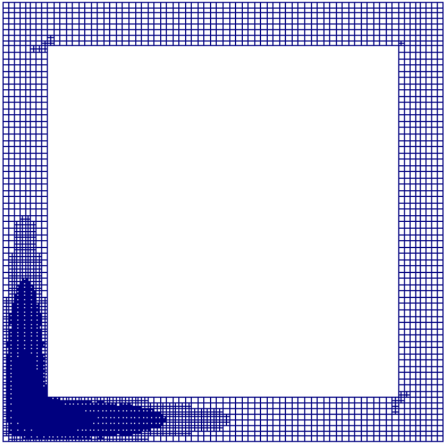
Approximation of first frequency: underlying mesh

Refinement level=6



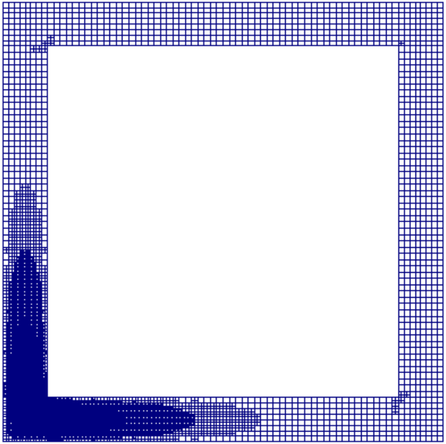
Approximation of first frequency: underlying mesh

Refinement level=7



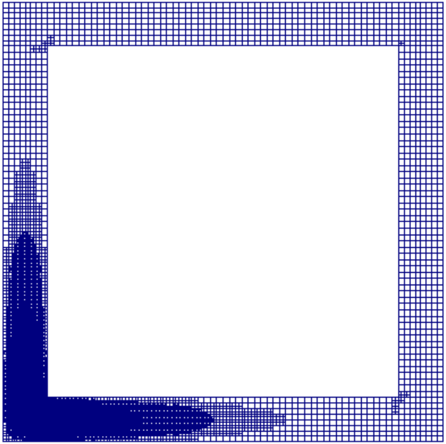
Approximation of first frequency: underlying mesh

Refinement level=8



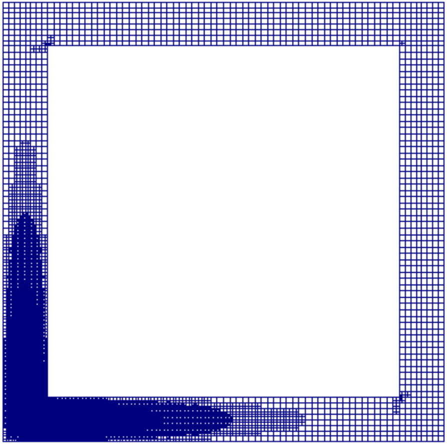
Approximation of first frequency: underlying mesh

Refinement level=9



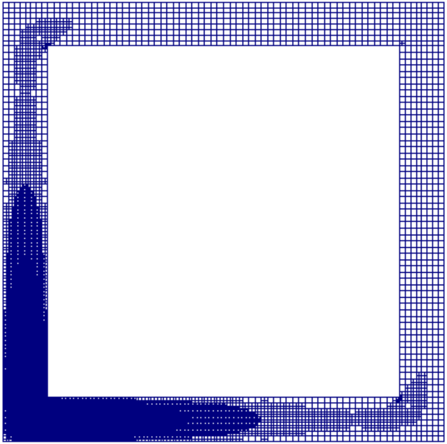
Approximation of first frequency: underlying mesh

Refinement level=10



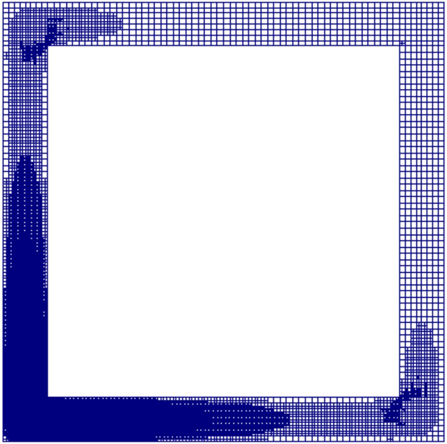
Approximation of first frequency: underlying mesh

Refinement level=11



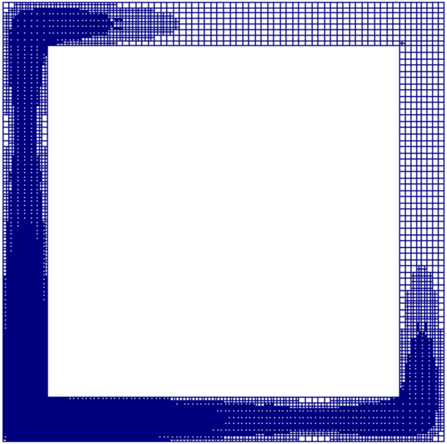
Approximation of first frequency: underlying mesh

Refinement level=12



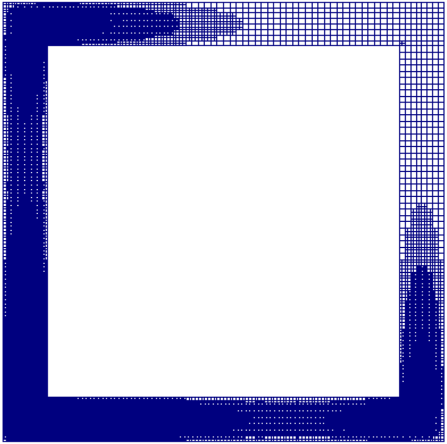
Approximation of first frequency: underlying mesh

Refinement level=13



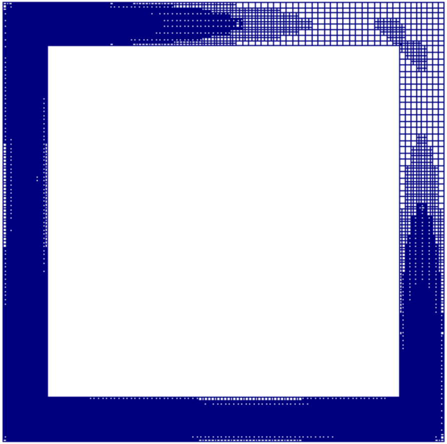
Approximation of first frequency: underlying mesh

Refinement level=14



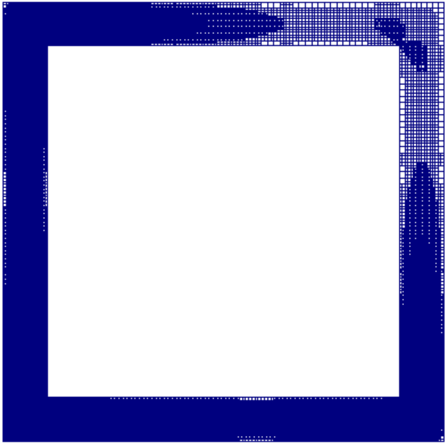
Approximation of first frequency: underlying mesh

Refinement level=15



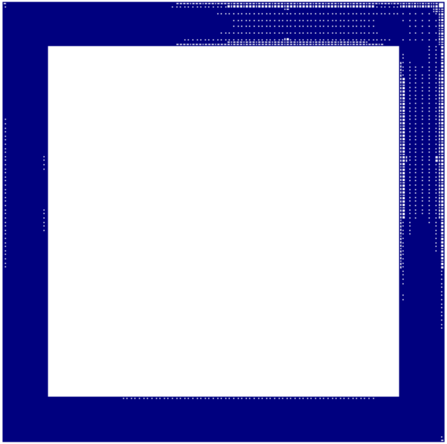
Approximation of first frequency: underlying mesh

Refinement level=16



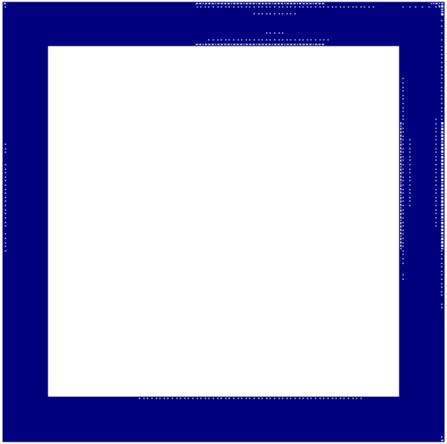
Approximation of first frequency: underlying mesh

Refinement level=17



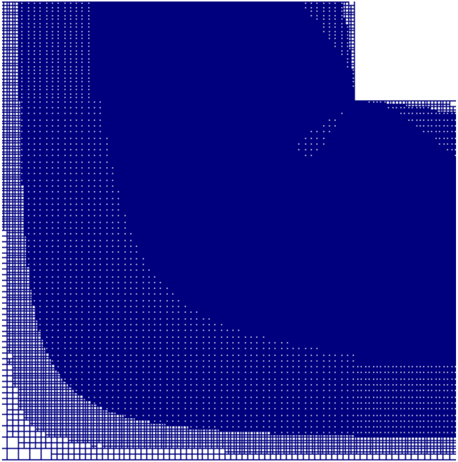
Approximation of first frequency: underlying mesh

Refinement level=18

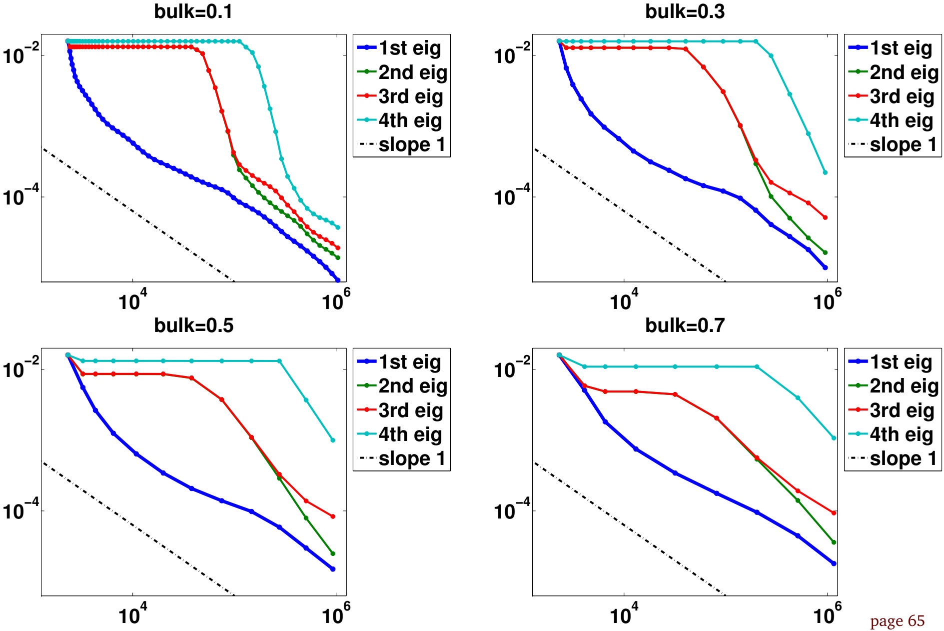


Approximation of first frequency: underlying mesh

Detail of the last mesh

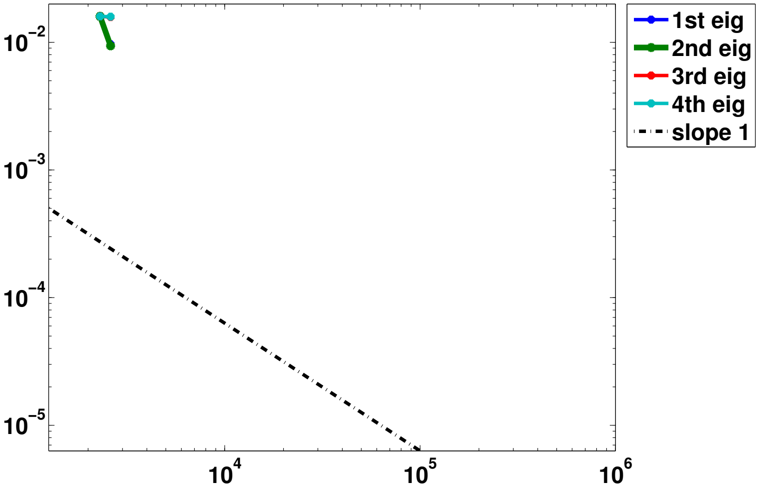


Changing the bulk parameter doesn't help



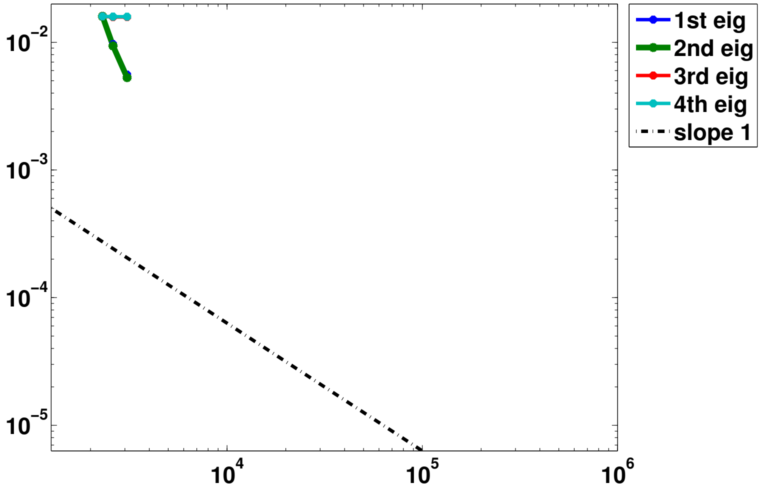
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=1



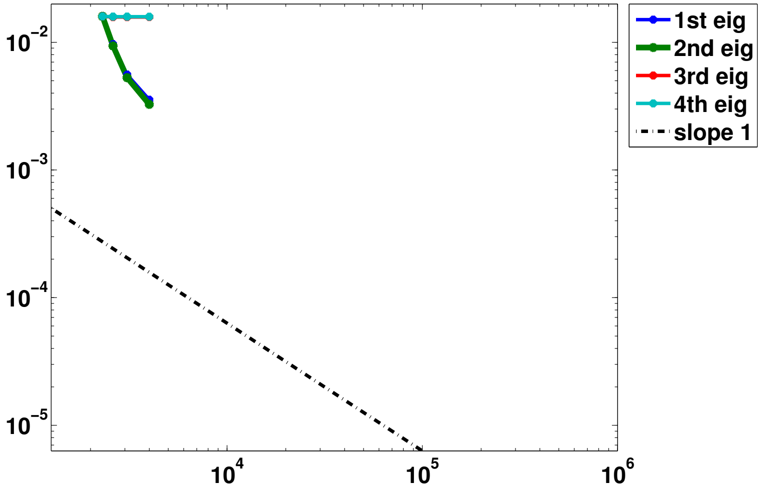
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=2



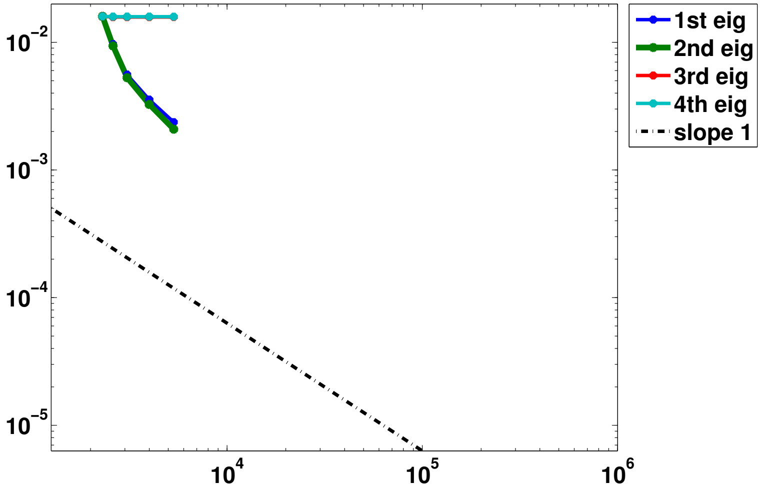
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=3



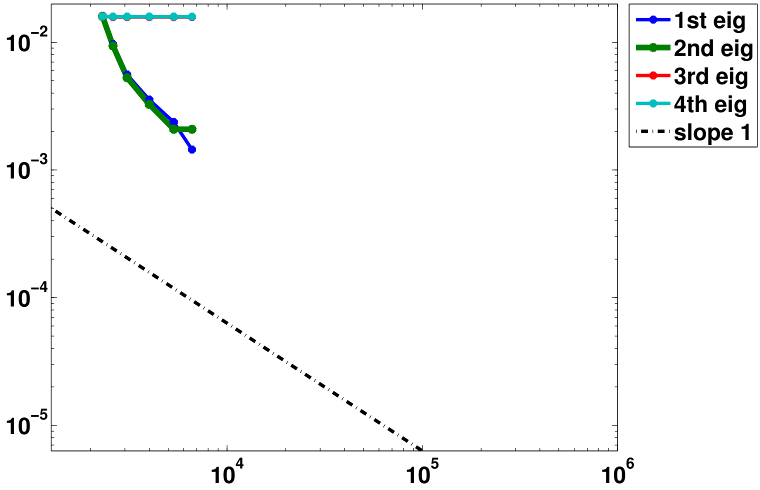
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=4



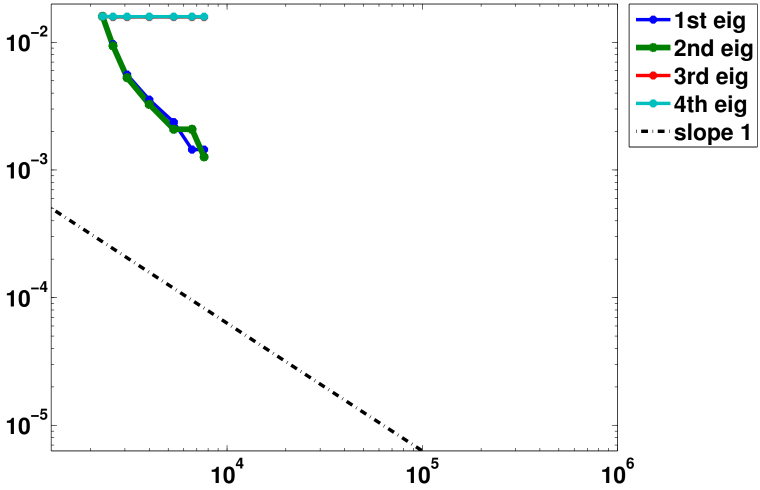
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=5



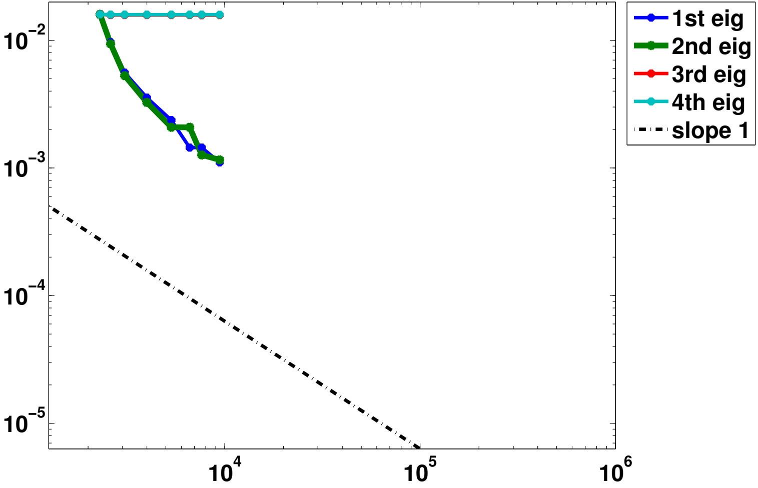
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=6



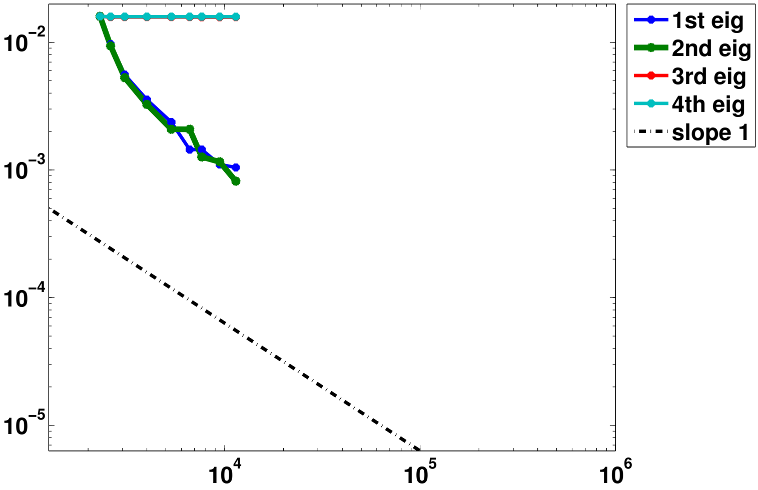
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=7



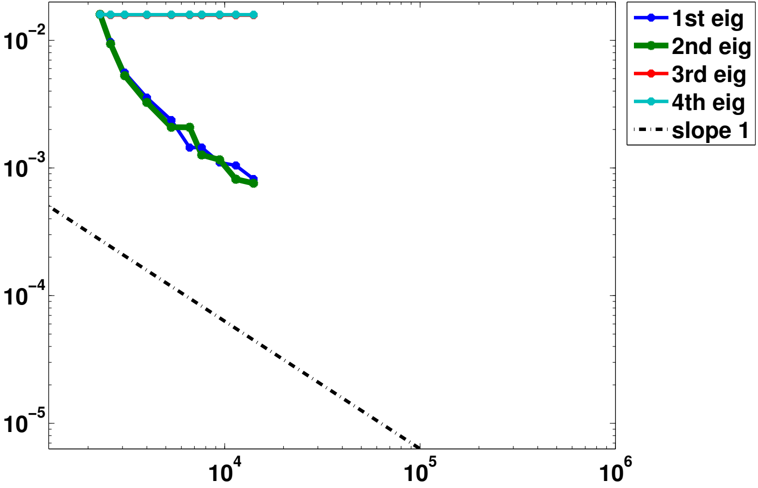
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=8



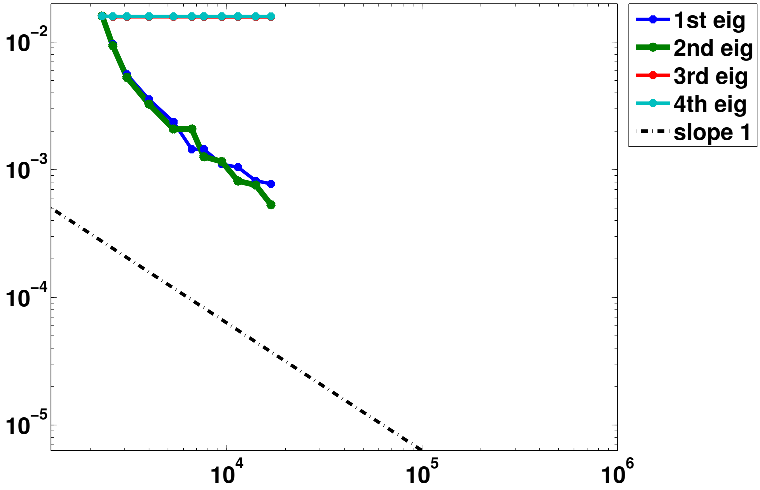
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=9



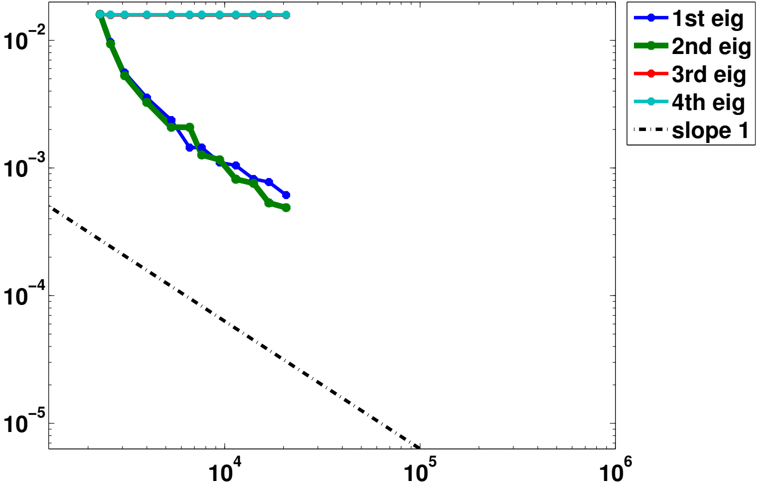
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=10



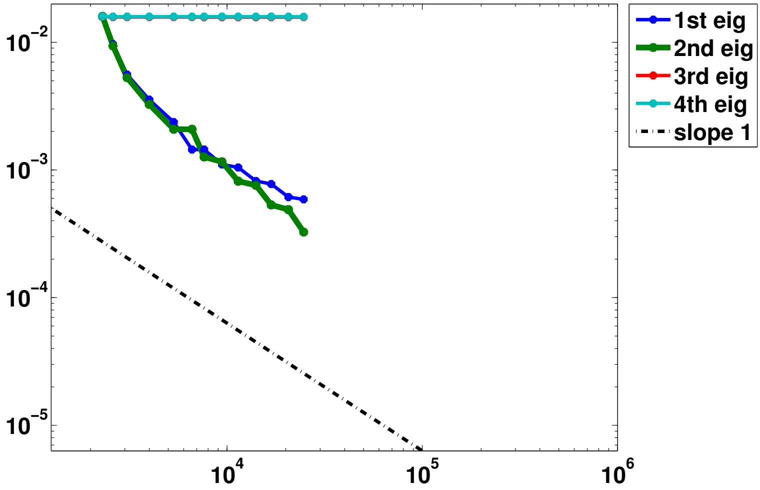
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=11



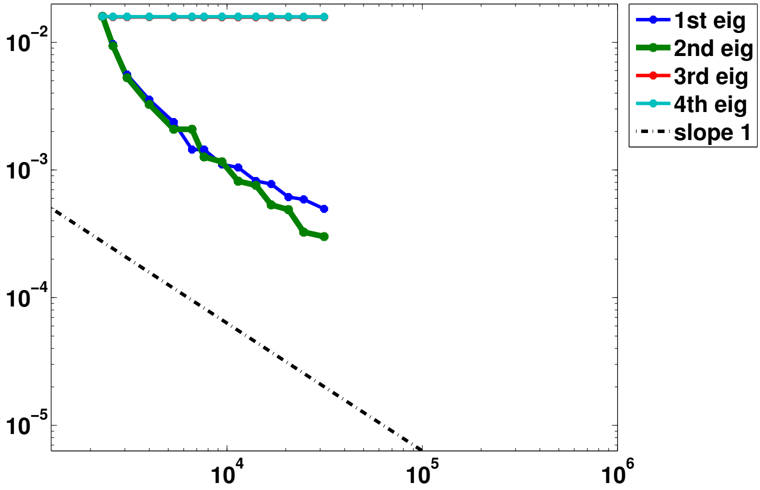
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=12



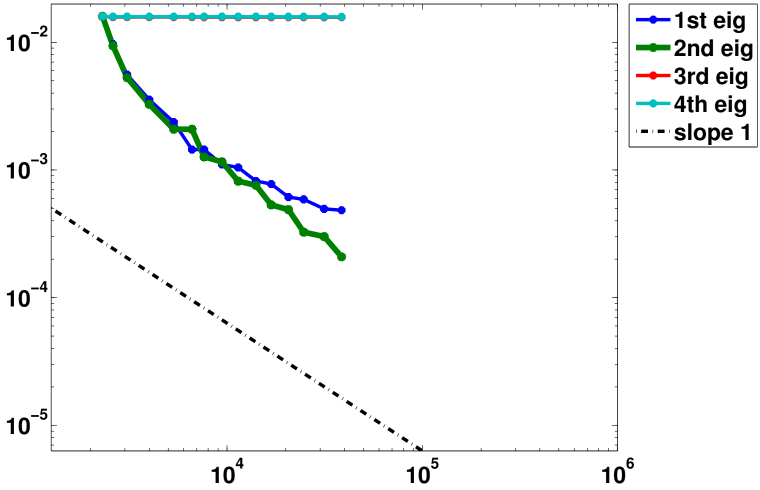
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=13



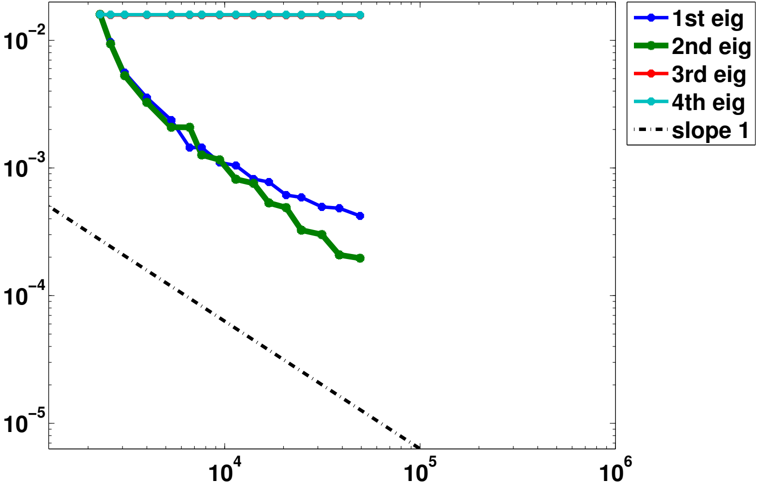
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=14



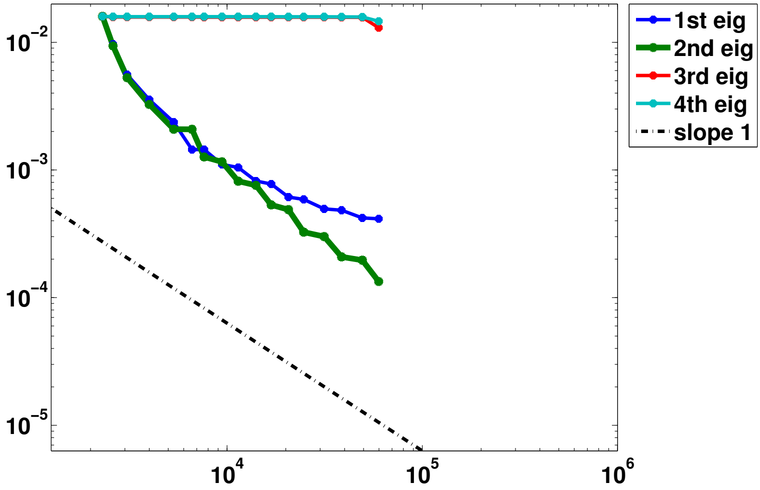
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=15



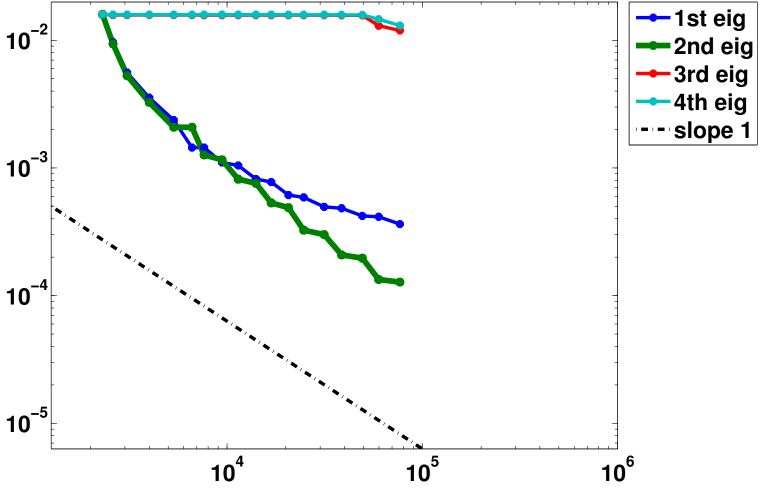
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=16



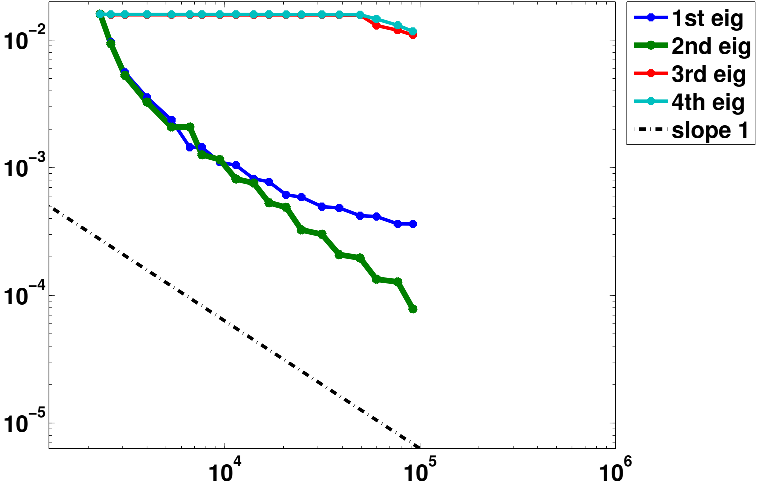
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=17



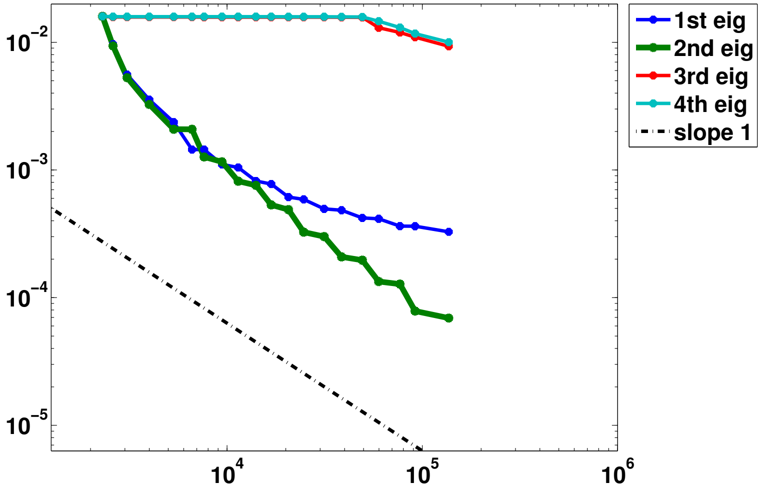
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=18



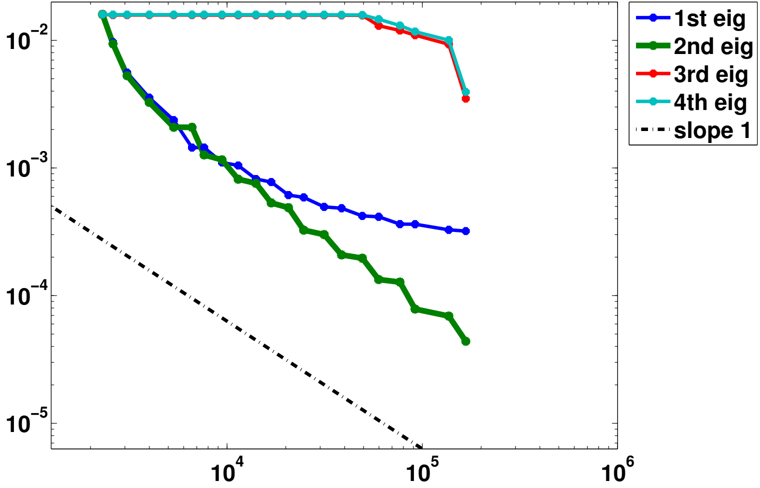
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=19



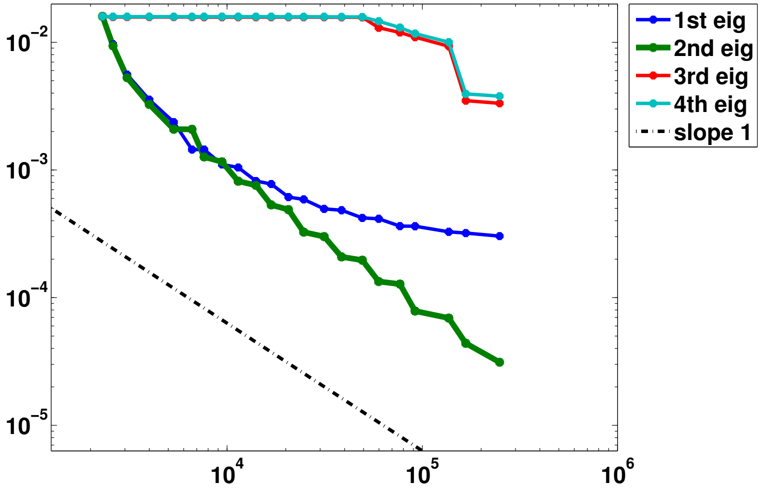
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=20



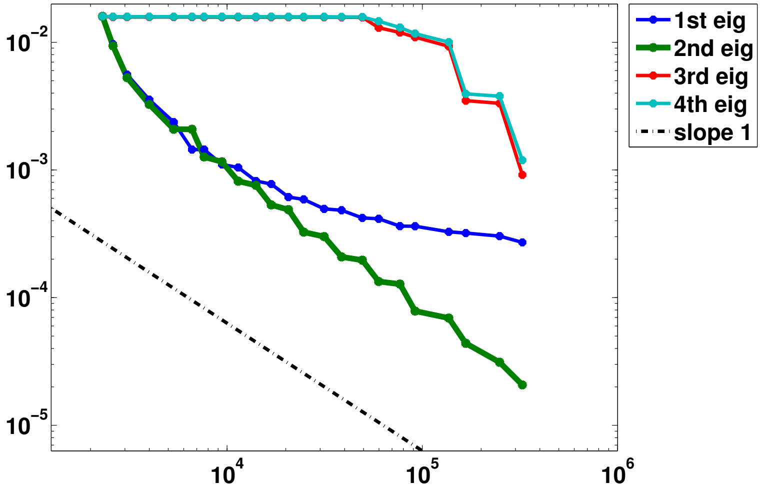
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=21



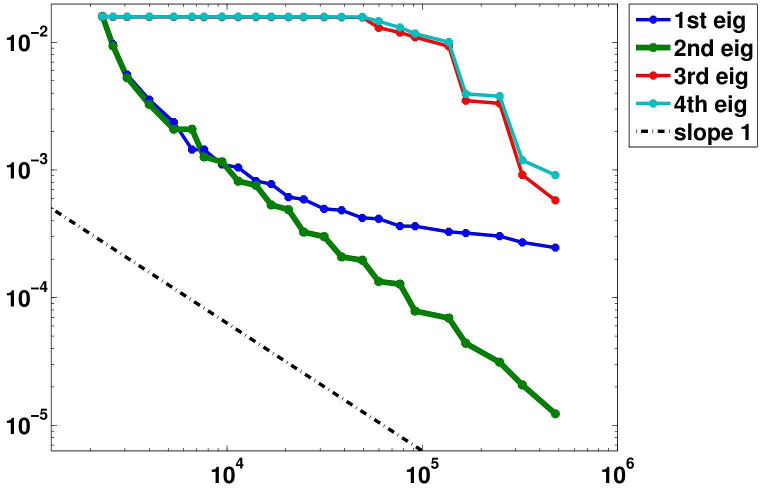
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=22



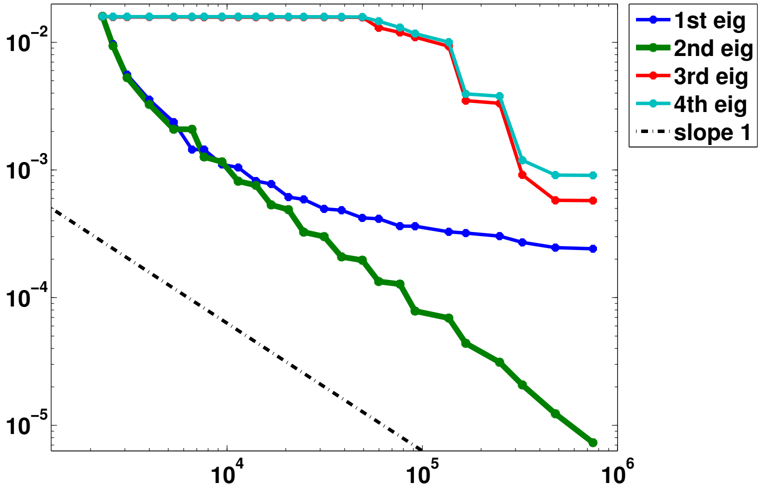
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=23



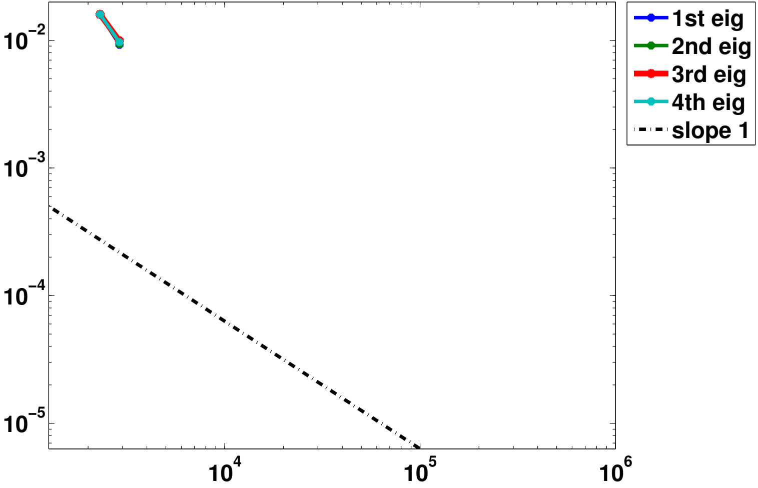
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=24



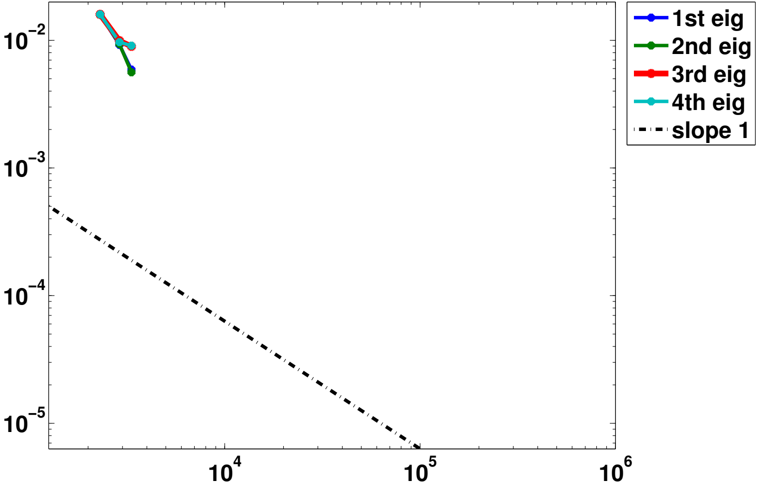
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=1



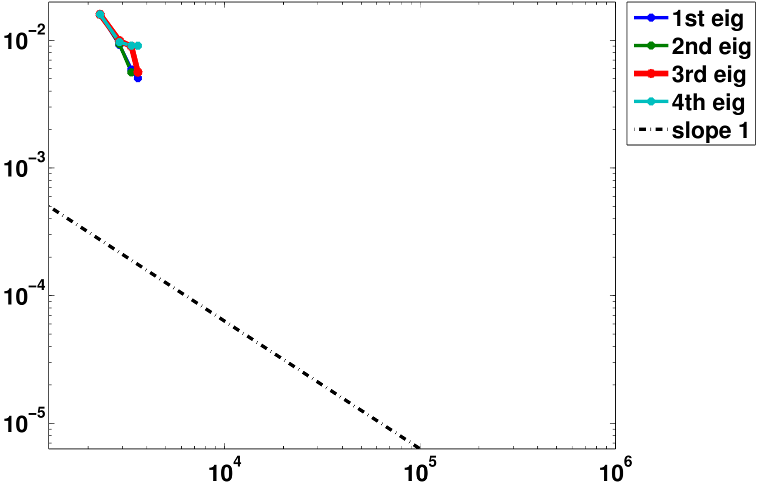
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=2



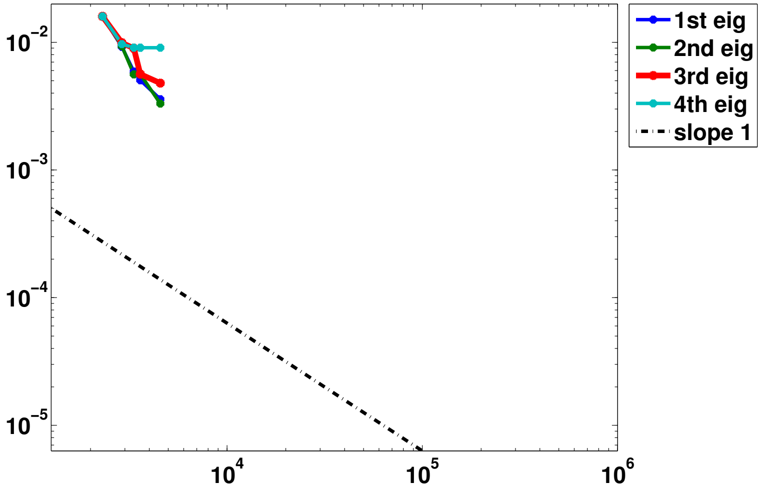
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=3



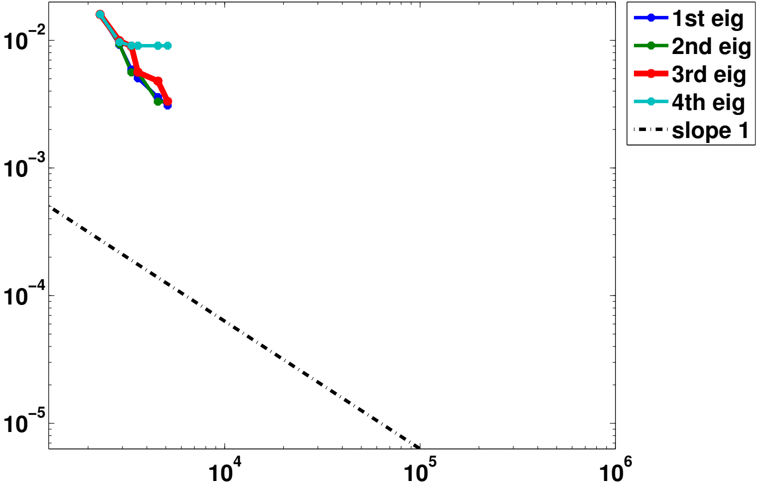
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=4



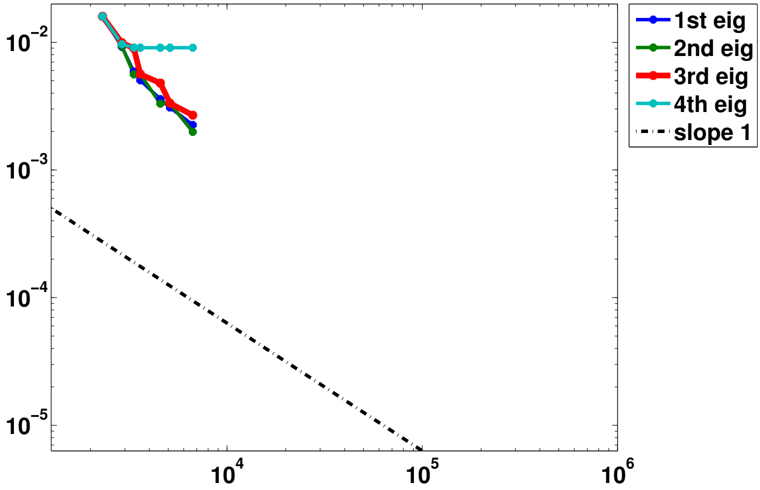
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=5



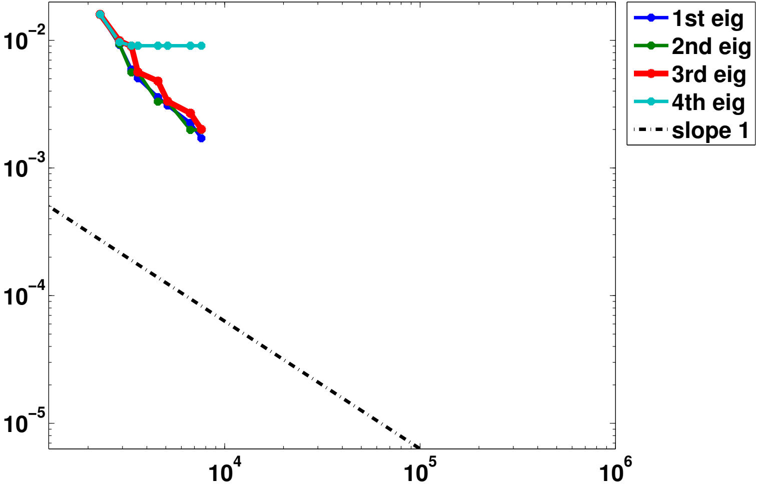
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=6



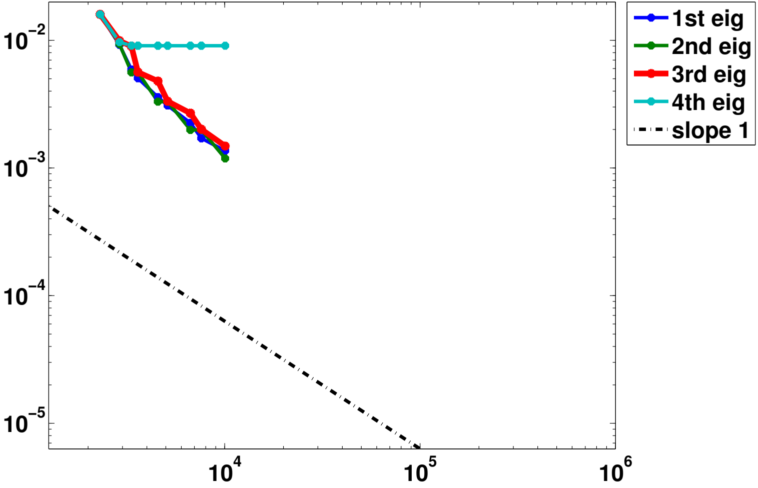
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=7



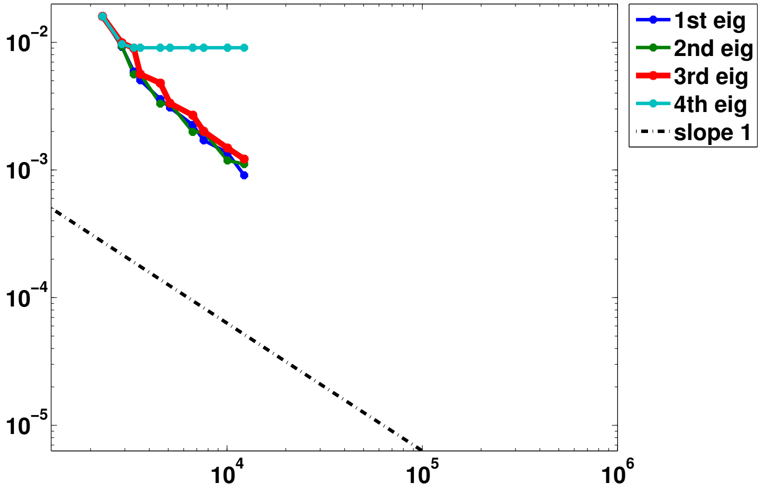
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=8



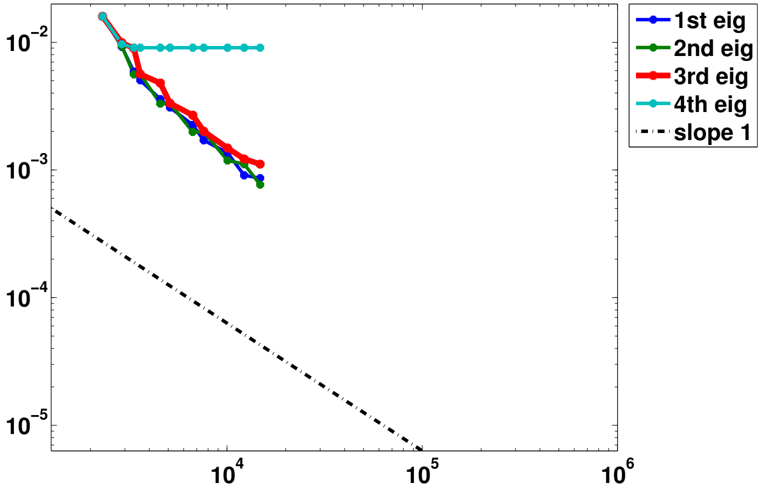
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=9



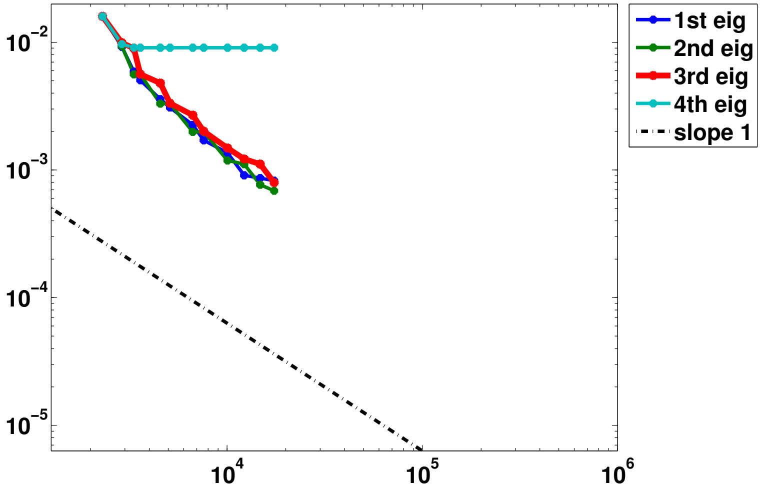
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=10



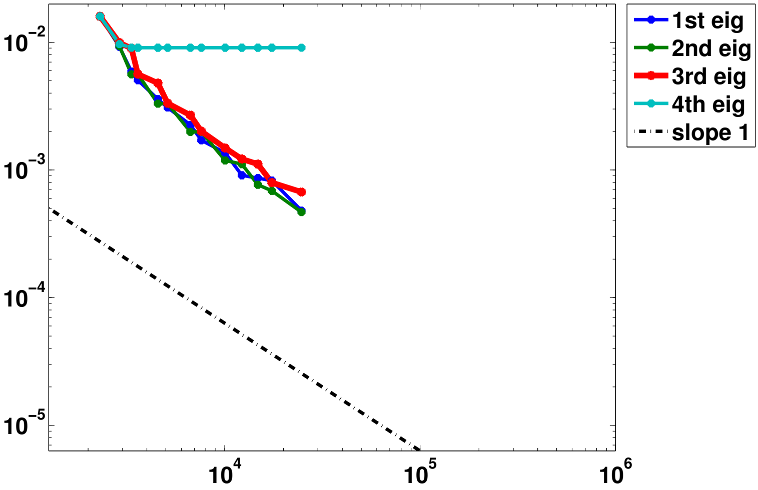
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=11



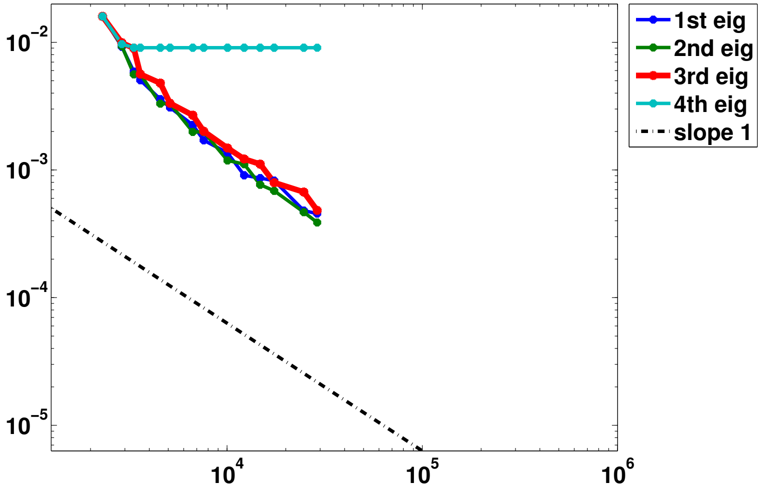
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=12



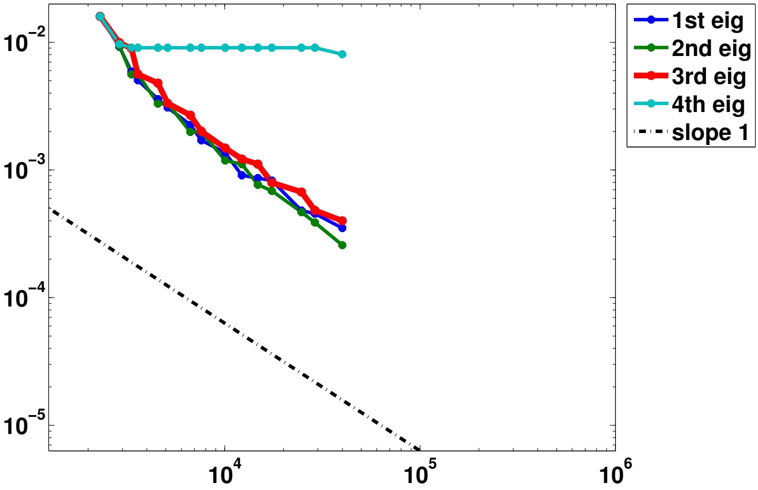
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=13



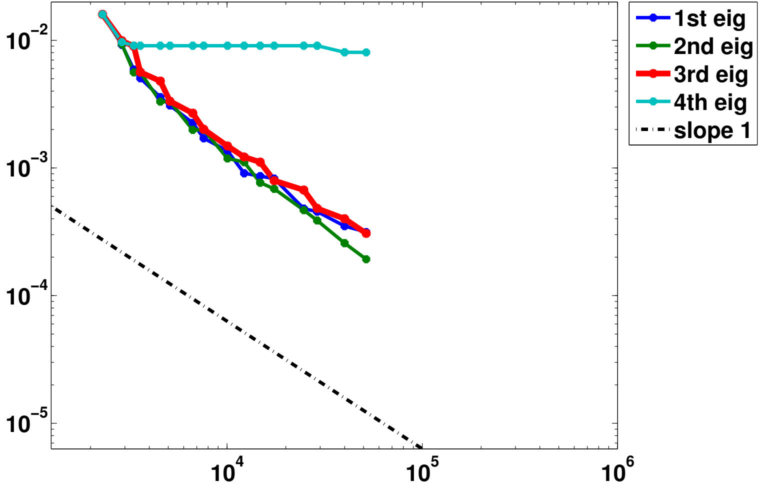
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=14



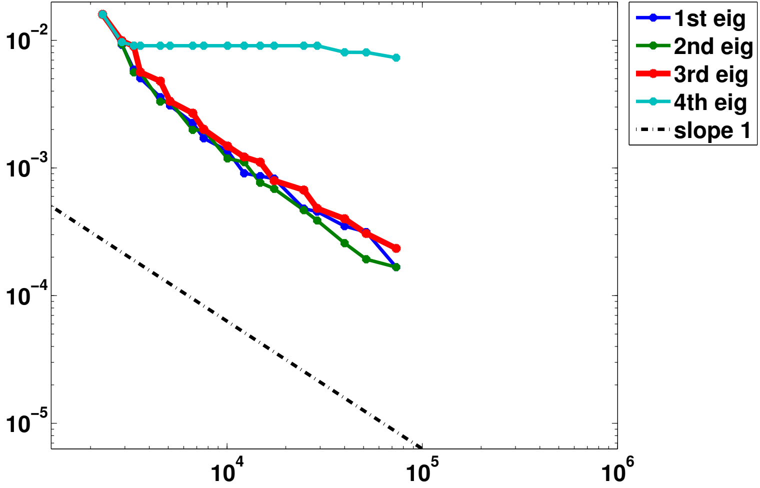
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=15



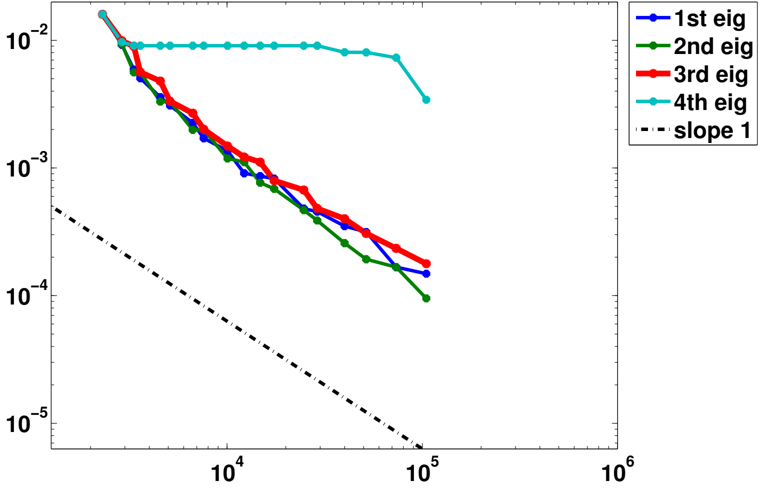
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=16



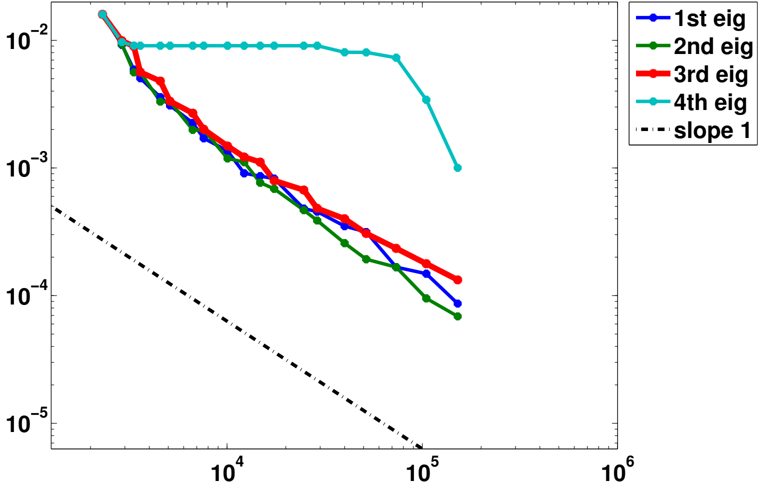
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=17



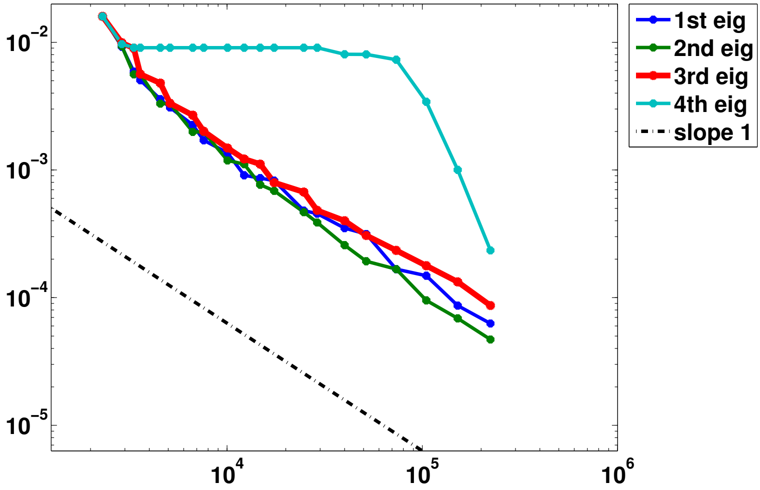
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=18



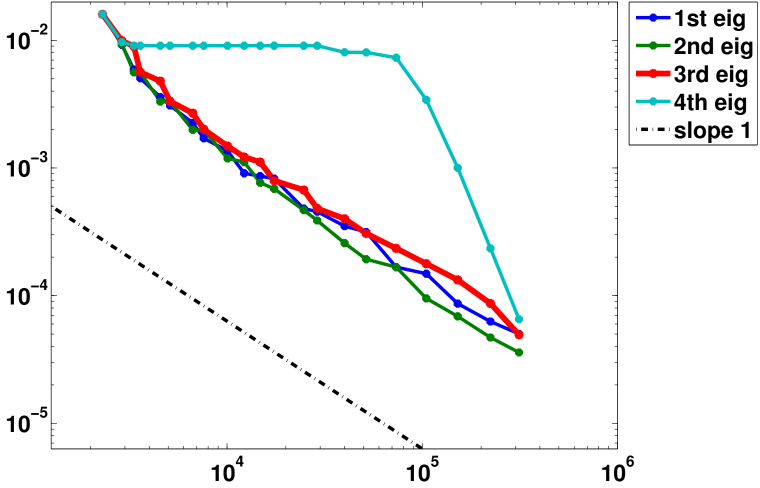
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=19



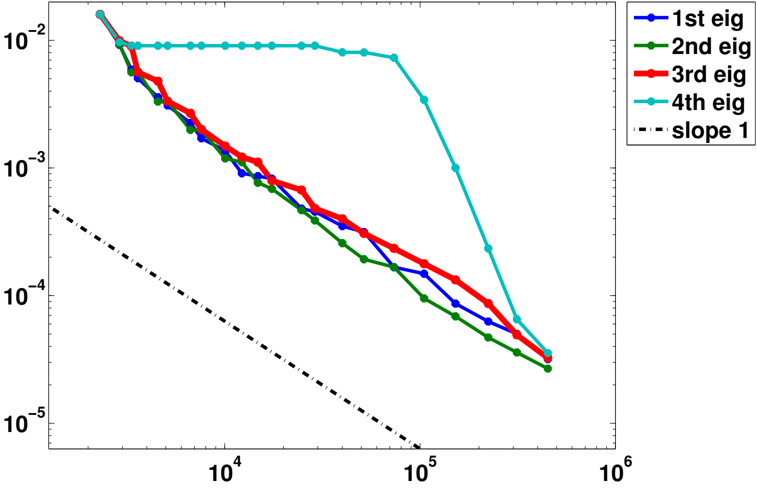
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=20



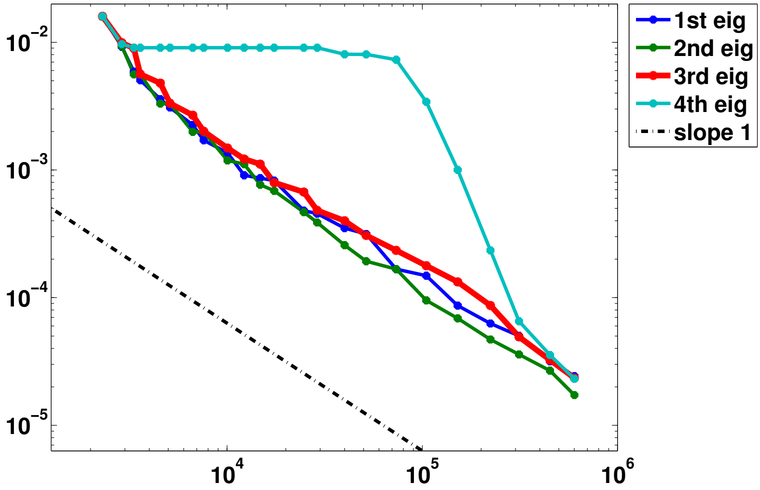
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=21



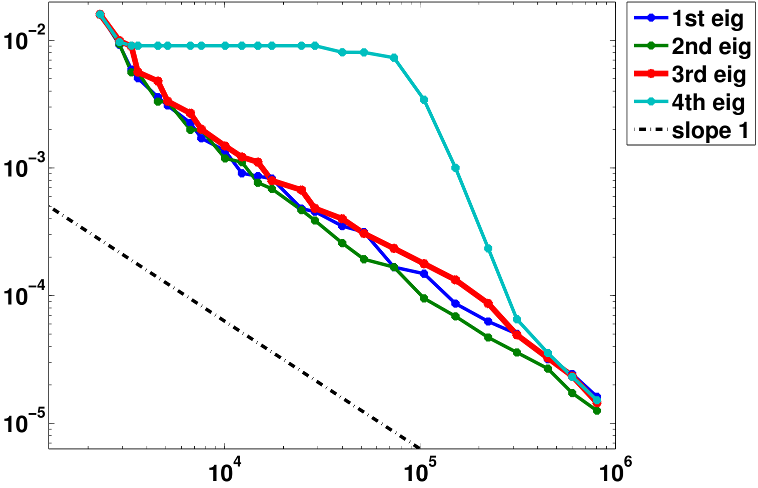
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=22



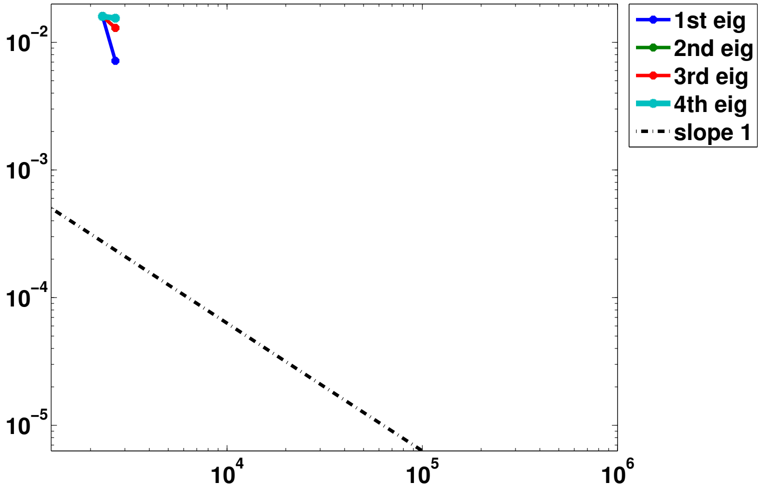
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=23



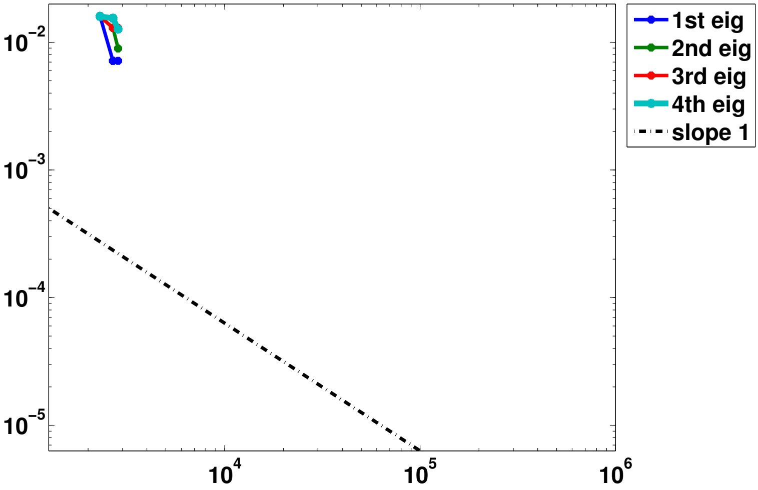
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=1



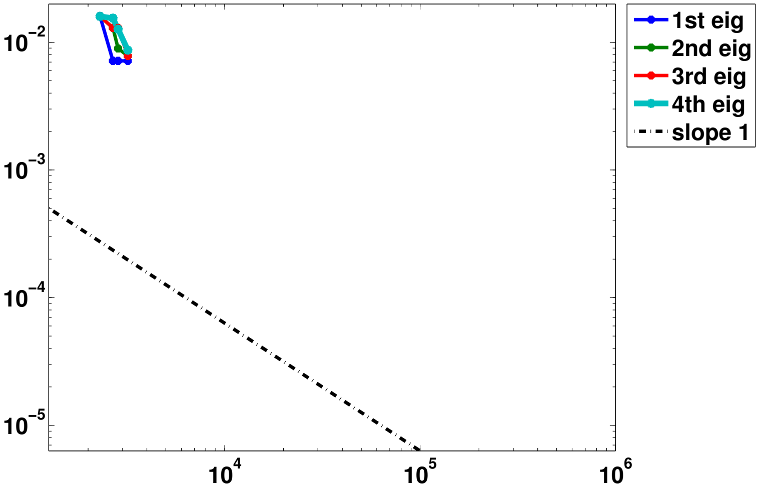
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=2



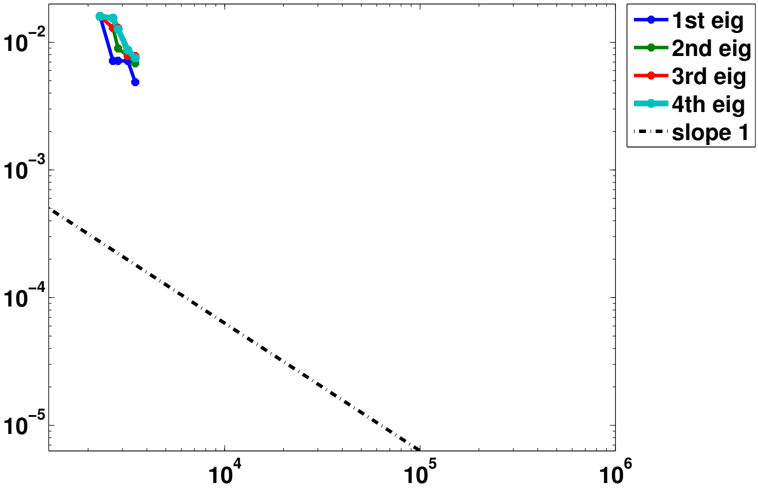
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=3



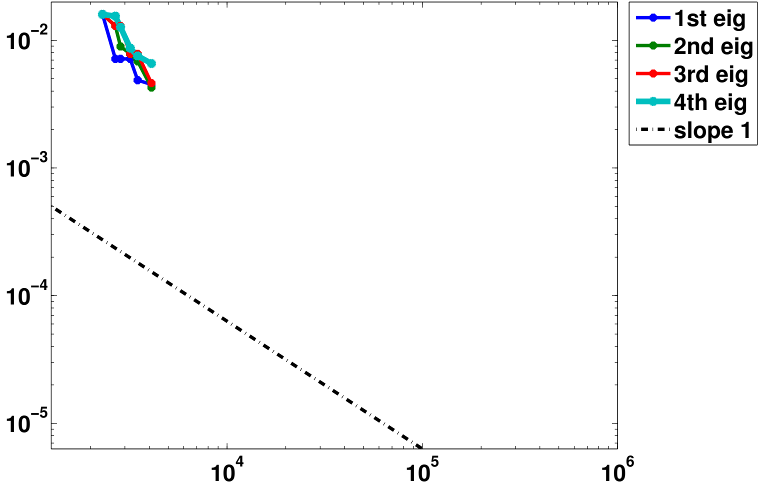
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=4



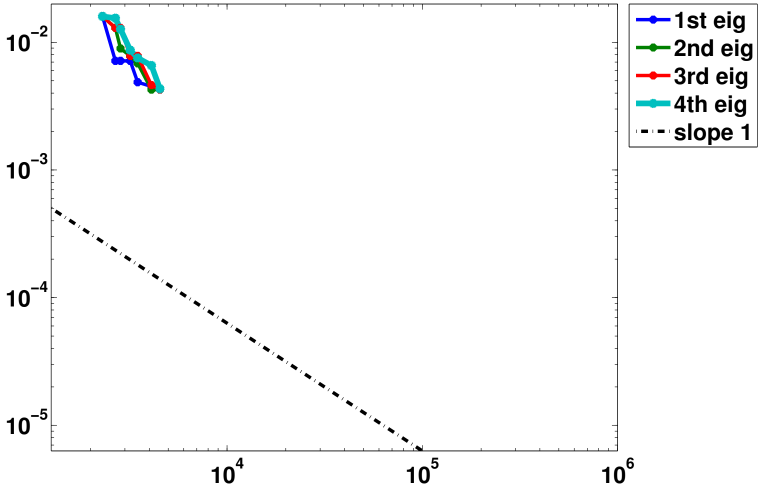
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=5



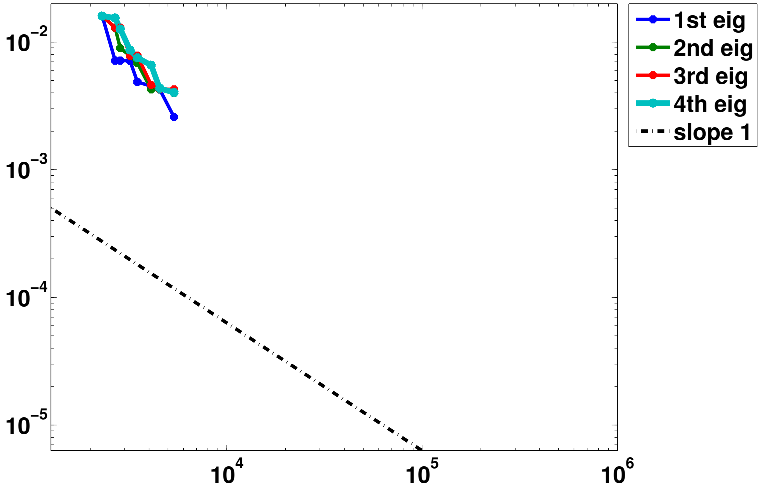
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=6



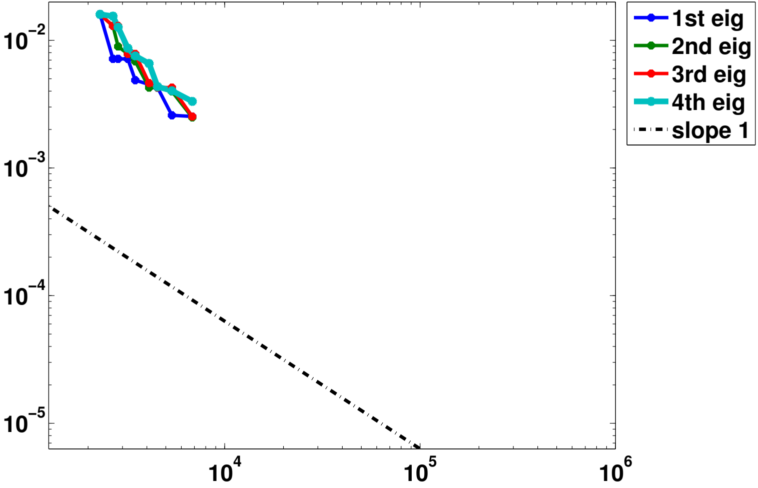
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=7



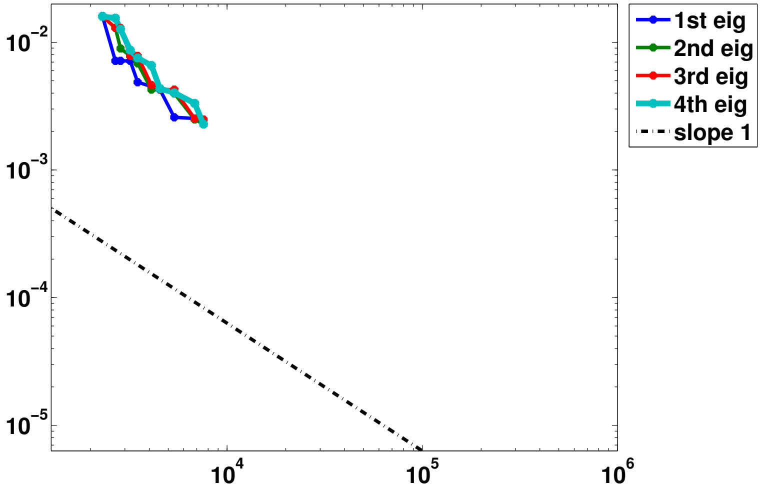
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=8



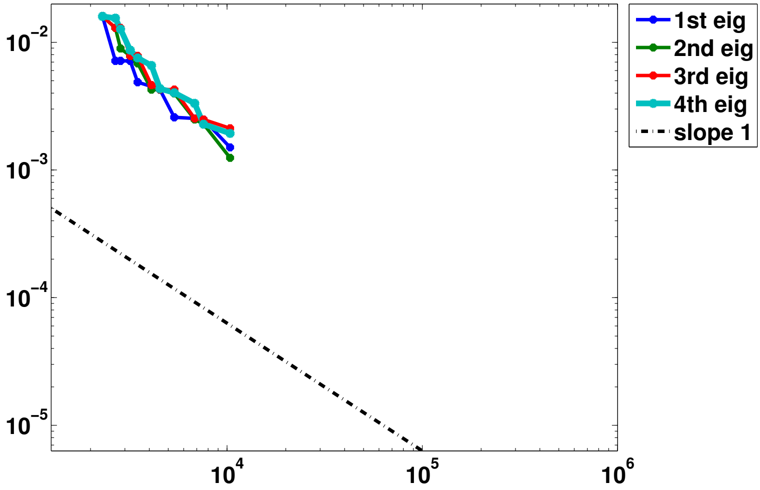
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=9



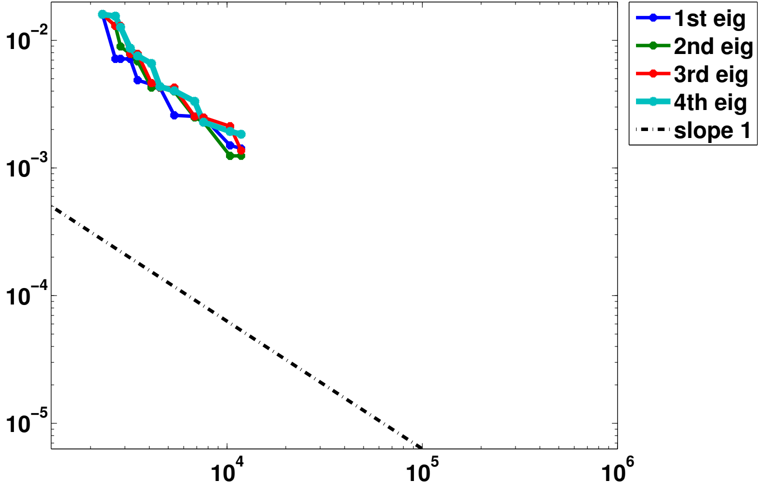
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=10



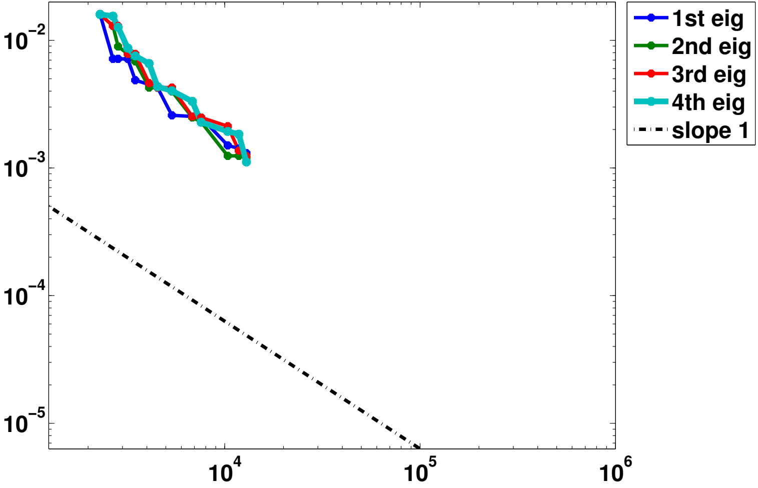
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=11



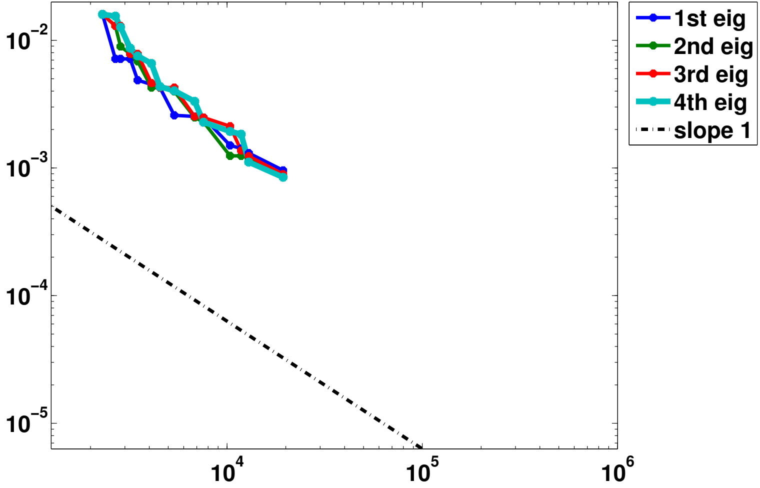
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=12



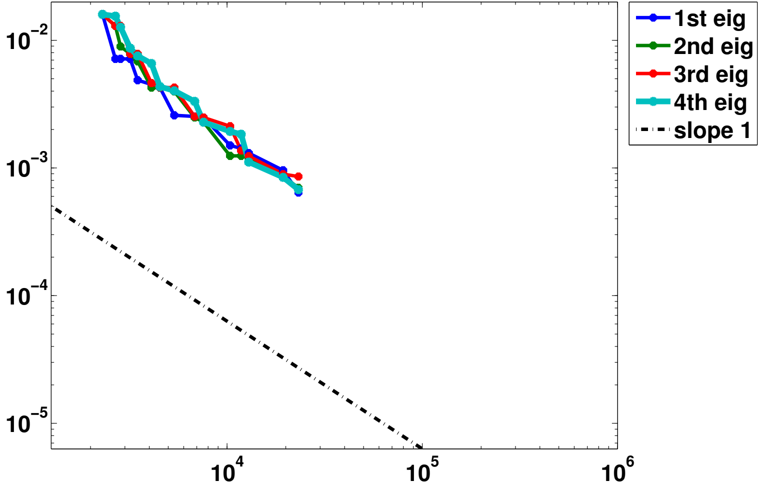
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=13



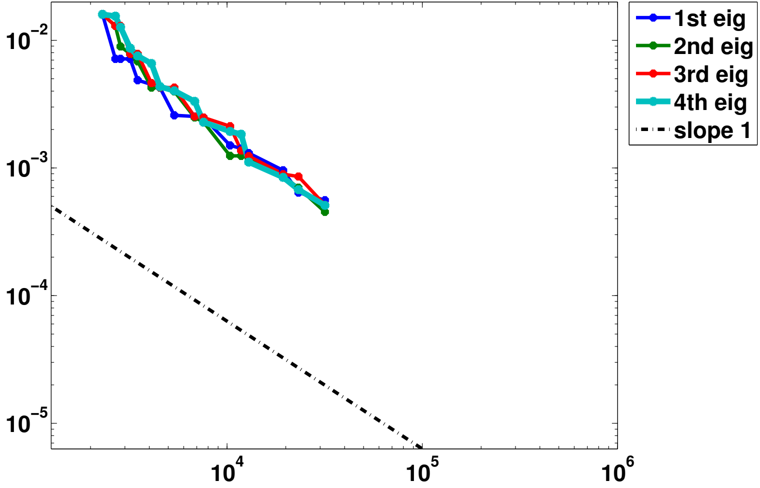
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=14



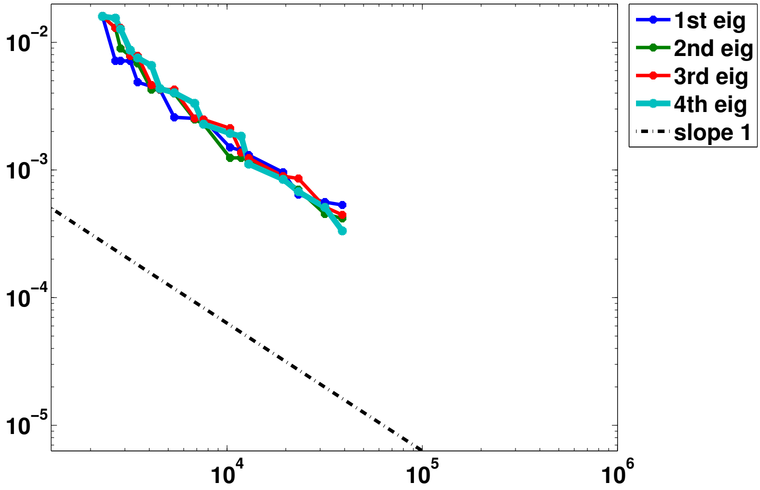
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=15



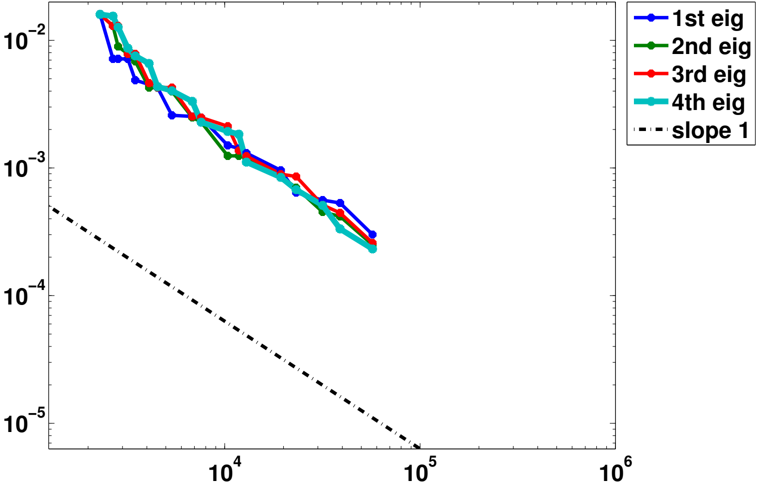
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=16



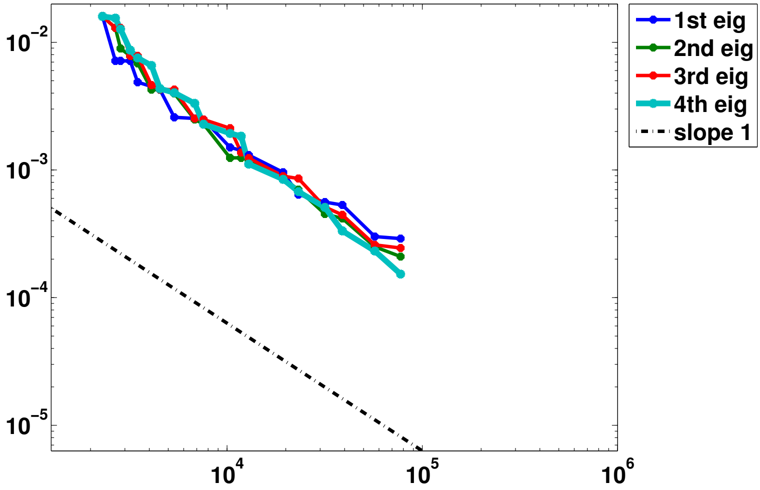
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=17



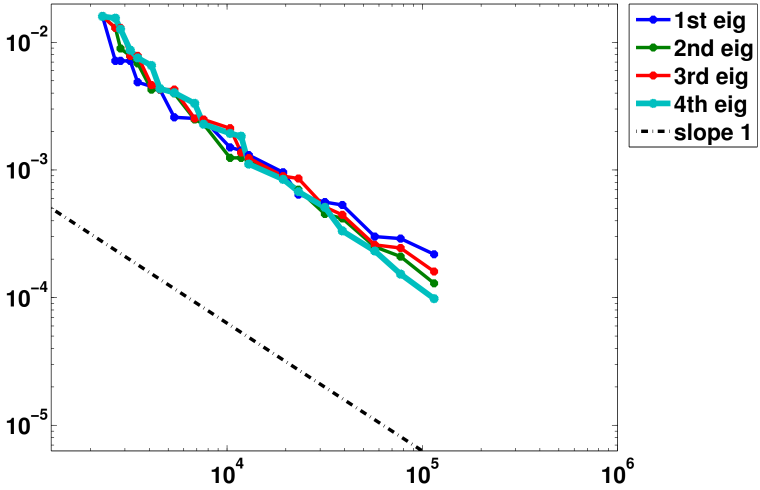
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=18



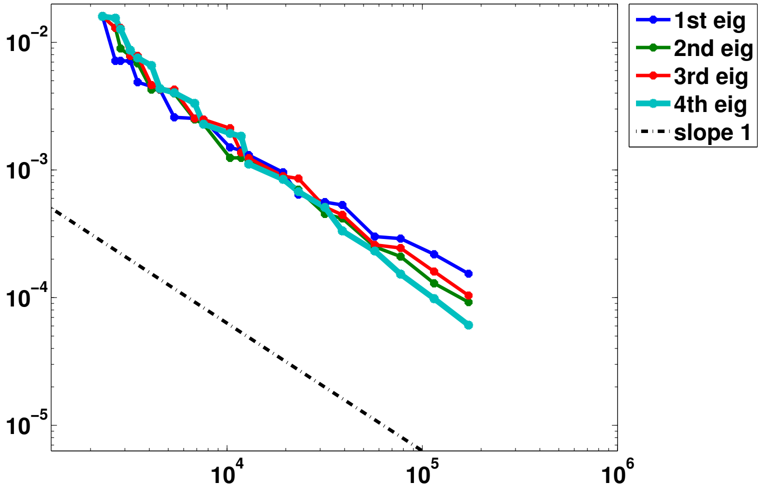
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=19



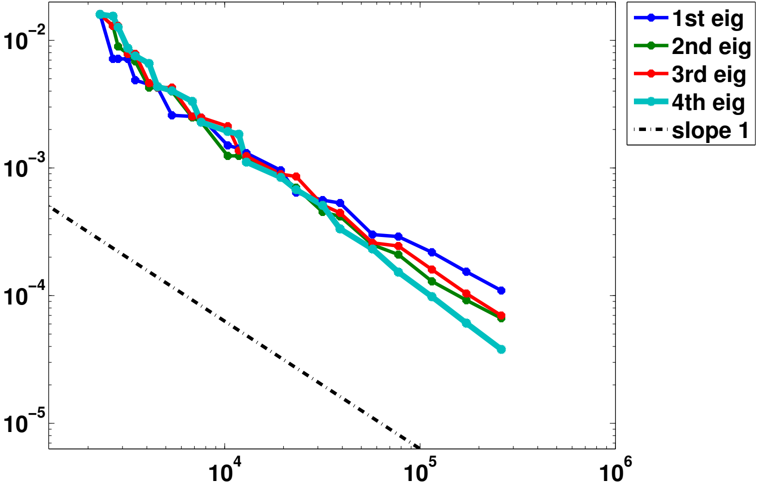
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=20



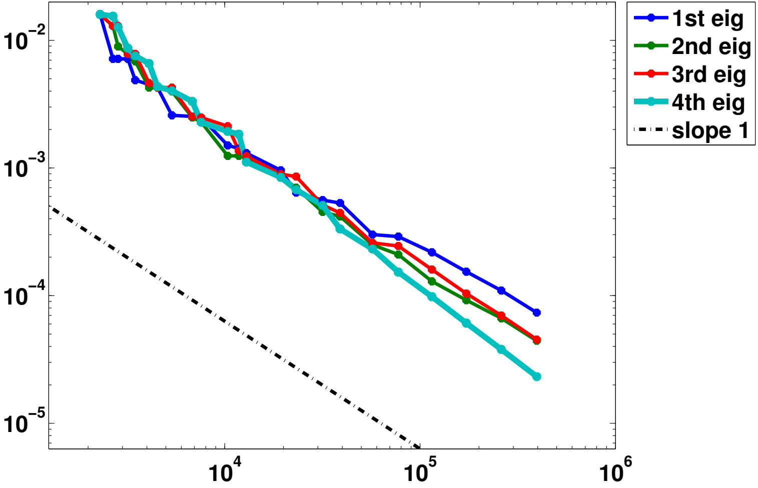
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=21



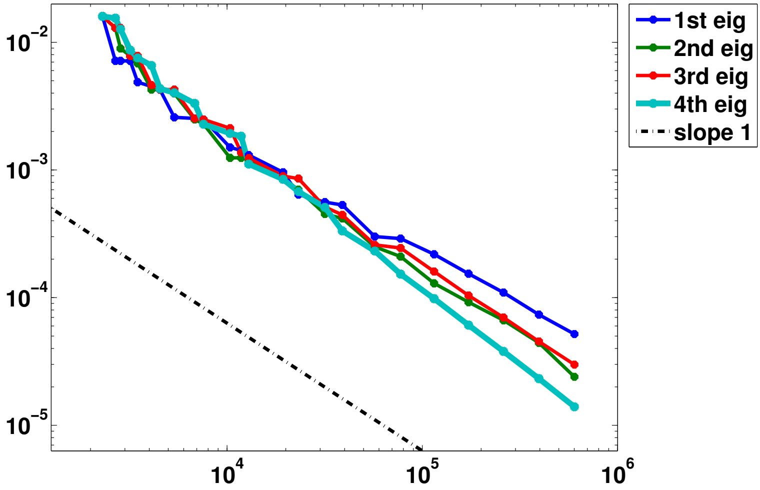
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=22



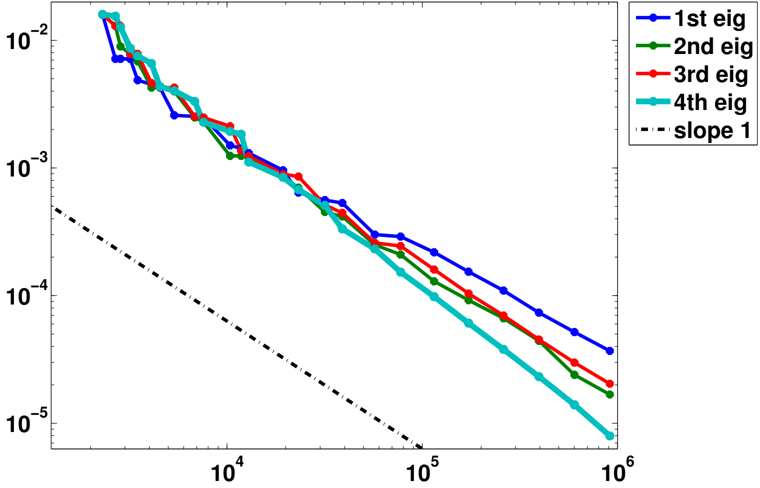
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=23



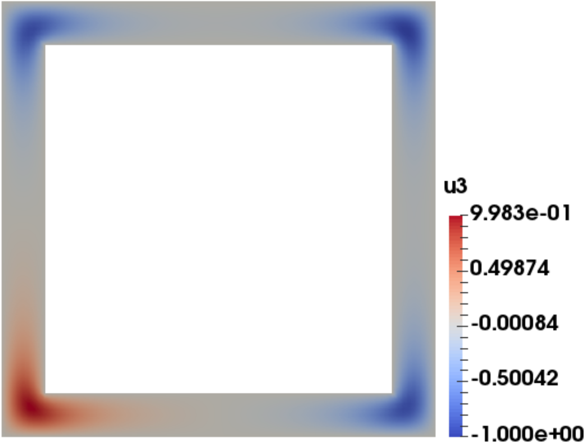
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=24



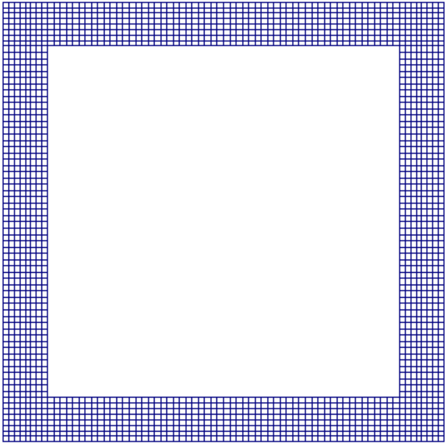
What is going on?

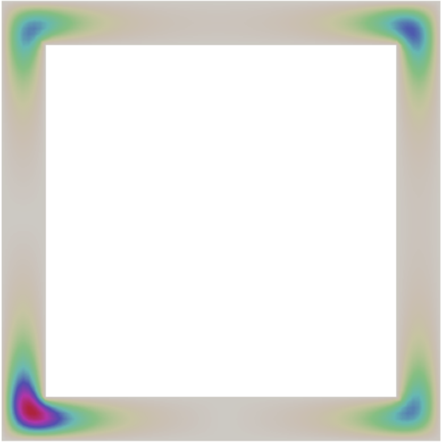
Let’s see the mesh sequence and the computed eigenfunctions, for instance, in the case of the third frequency



What is going on?

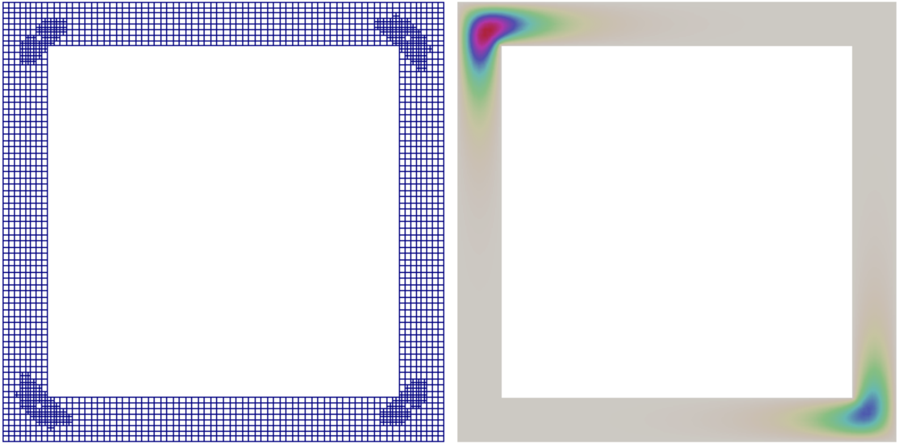
Initial mesh





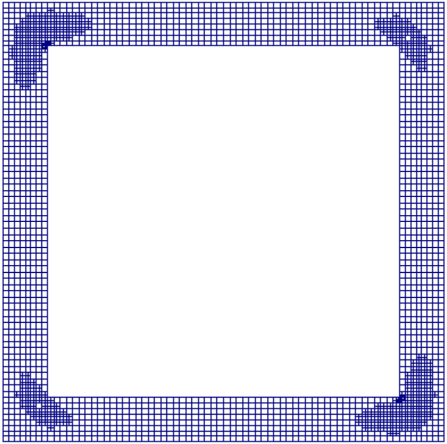
What is going on?

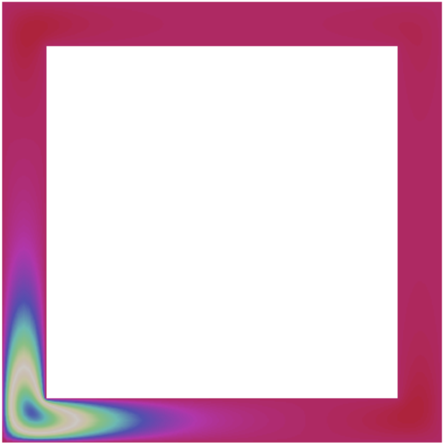
Refinement level=1



What is going on?

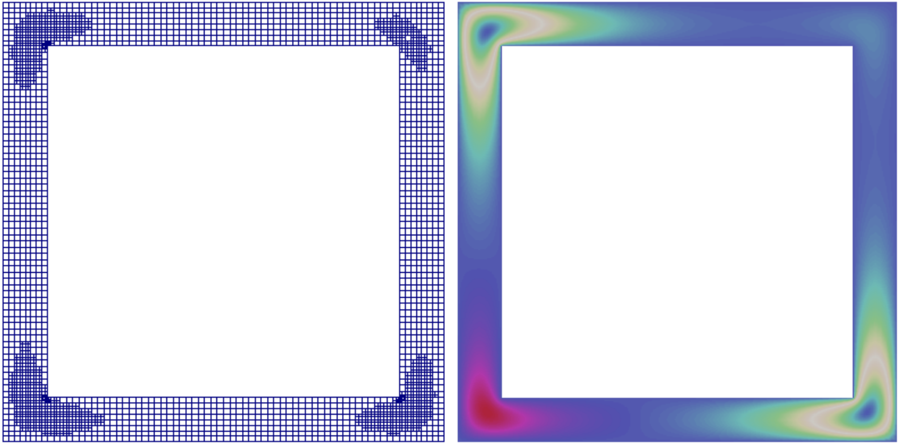
Refinement level=2





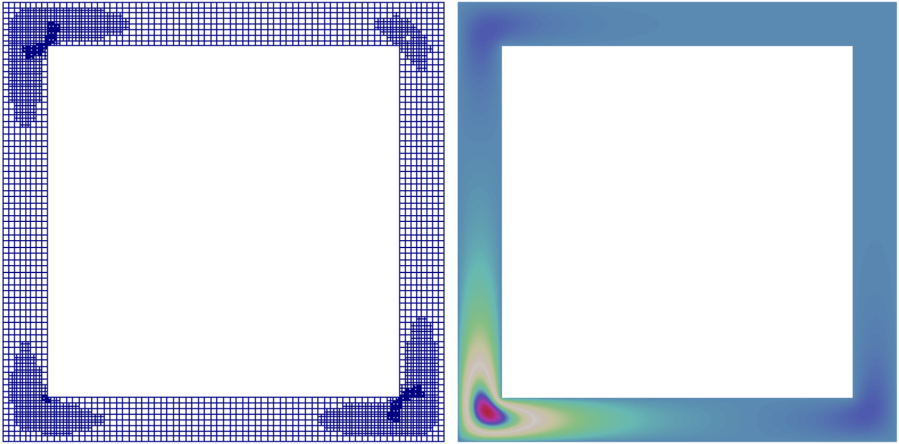
What is going on?

Refinement level=3



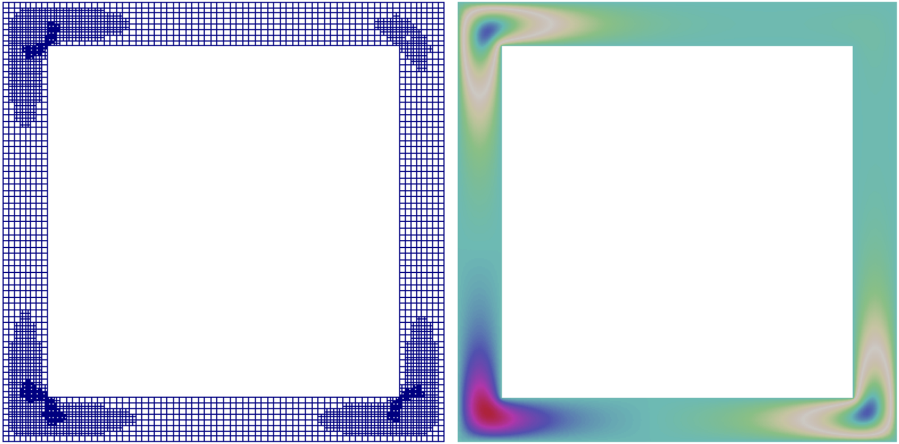
What is going on?

Refinement level=4



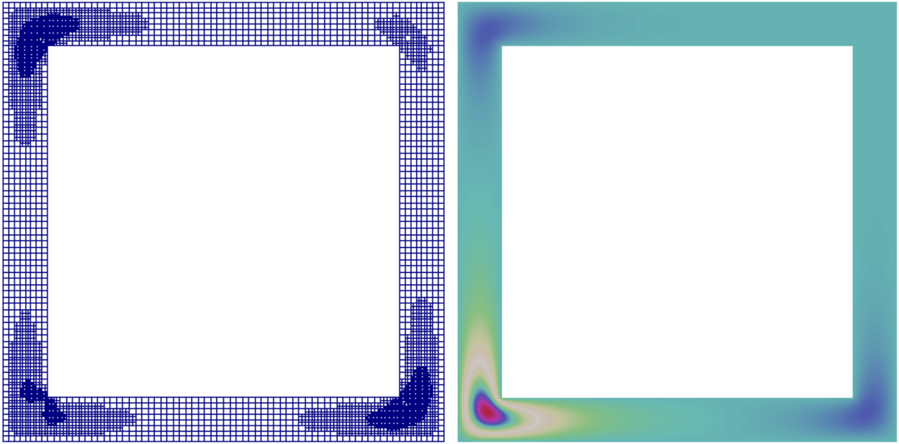
What is going on?

Refinement level=5



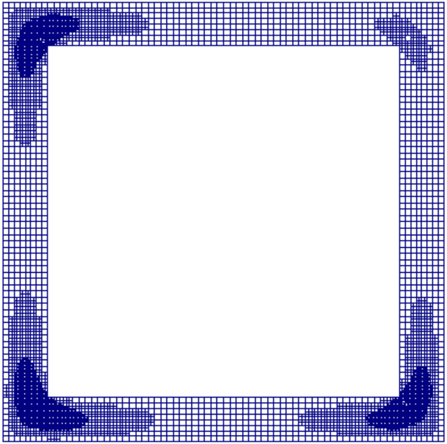
What is going on?

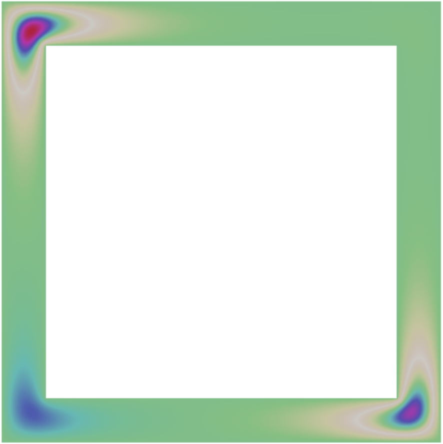
Refinement level=6



What is going on?

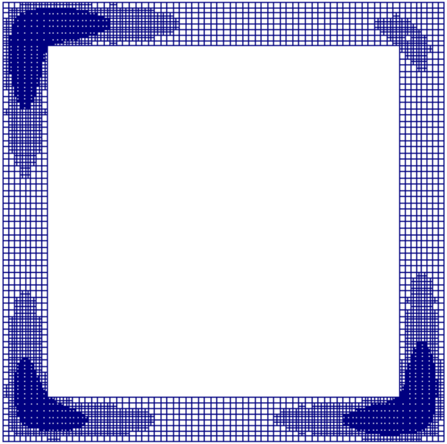
Refinement level=7

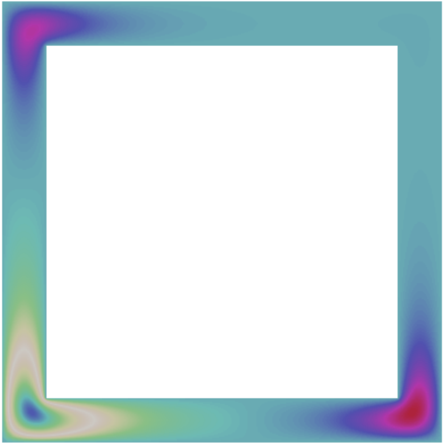




What is going on?

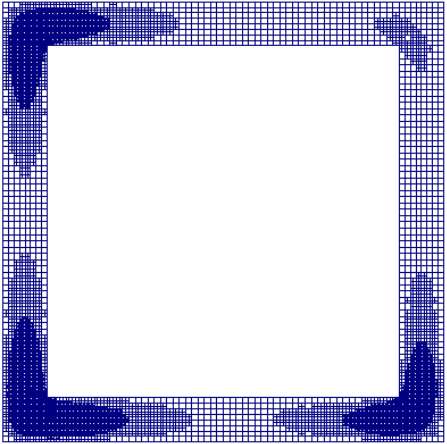
Refinement level=8

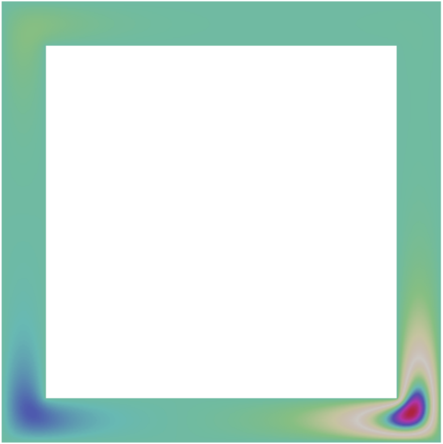




What is going on?

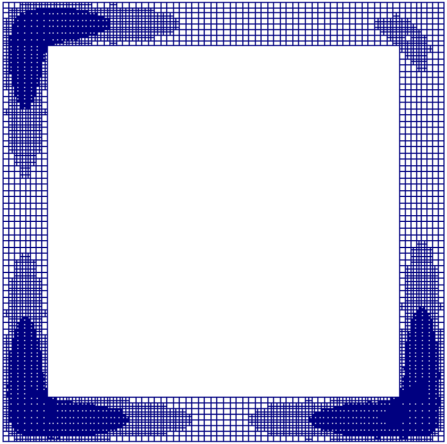
Refinement level=9

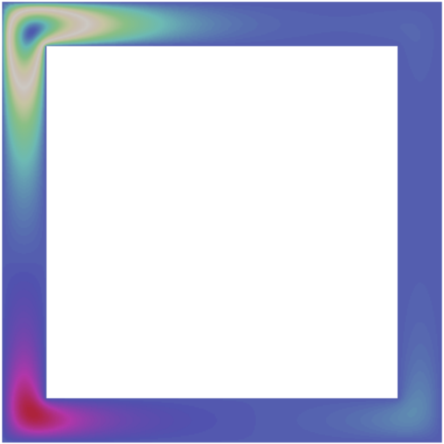




What is going on?

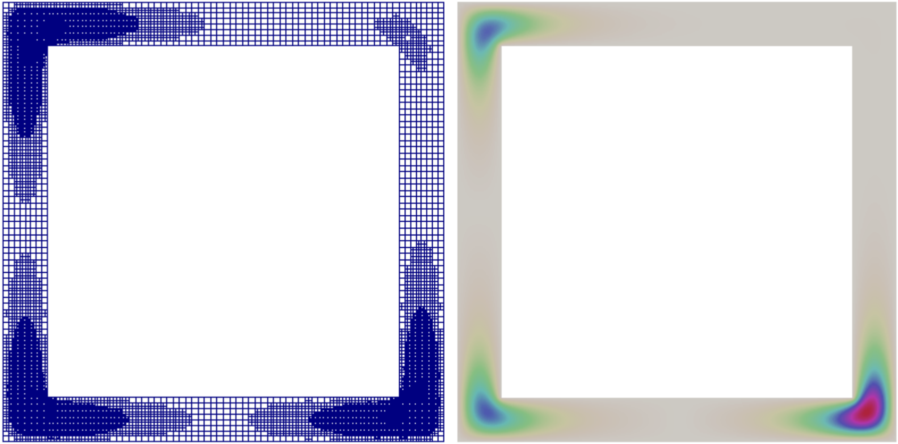
Refinement level=10





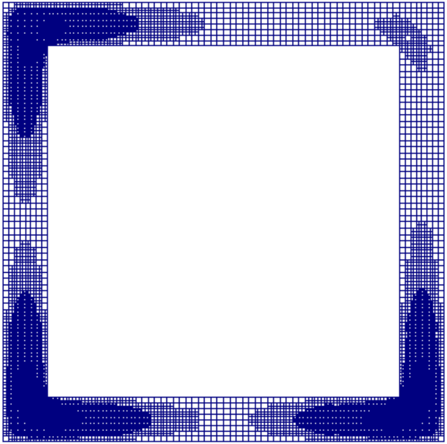
What is going on?

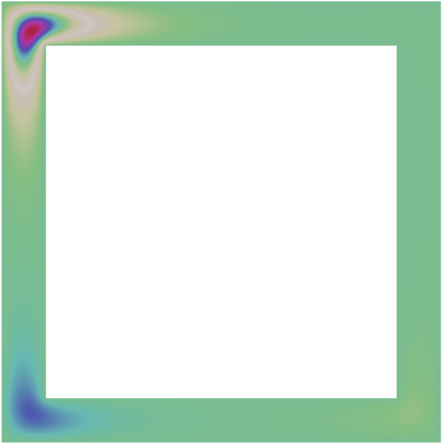
Refinement level=11



What is going on?

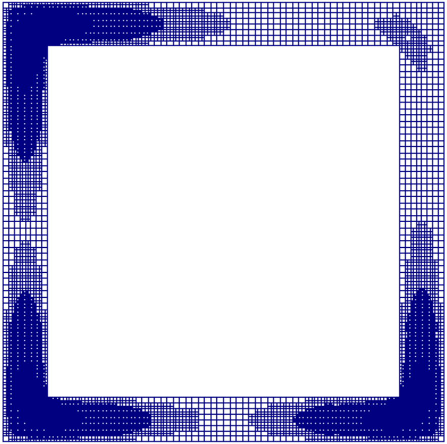
Refinement level=12

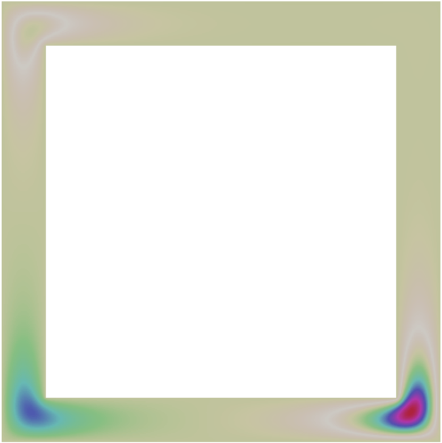




What is going on?

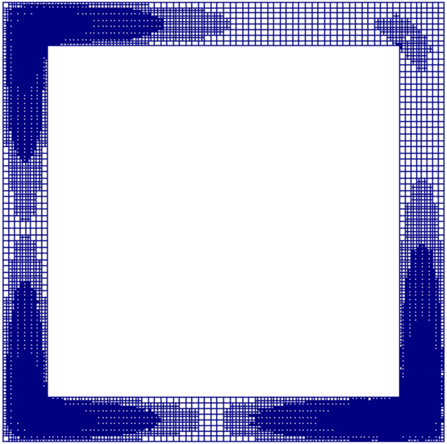
Refinement level=13

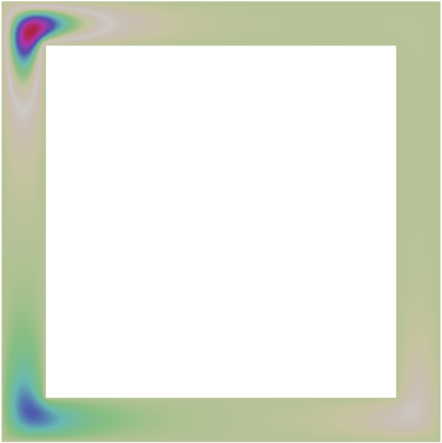




What is going on?

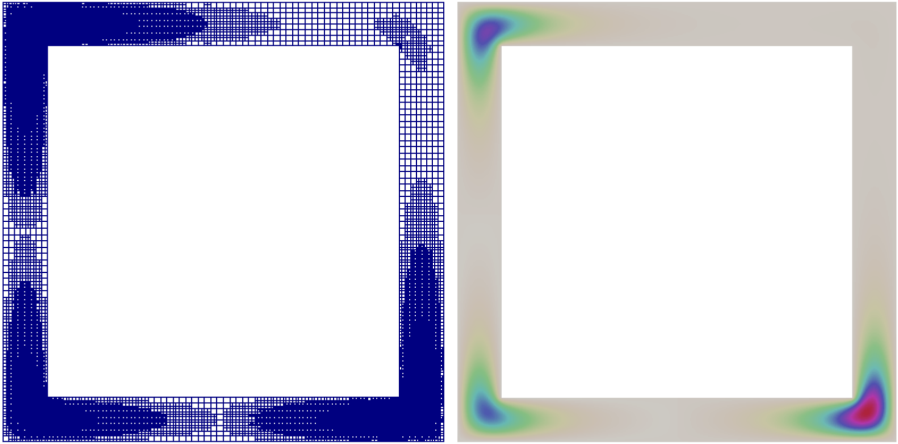
Refinement level=14





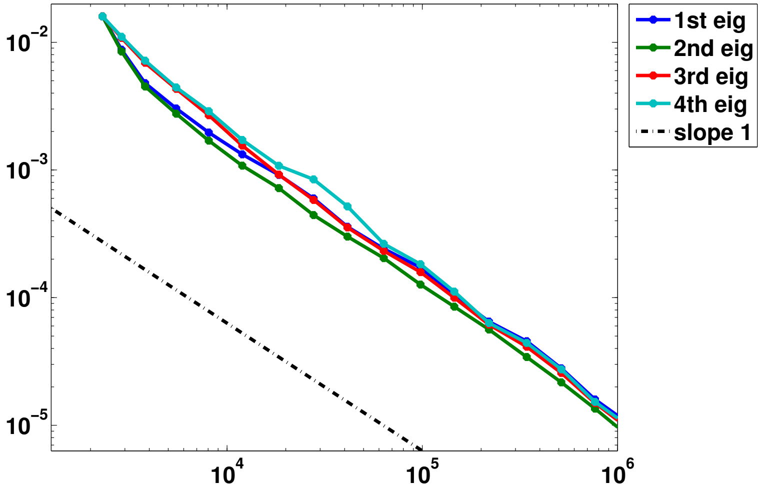
What is going on?

Refinement level=15

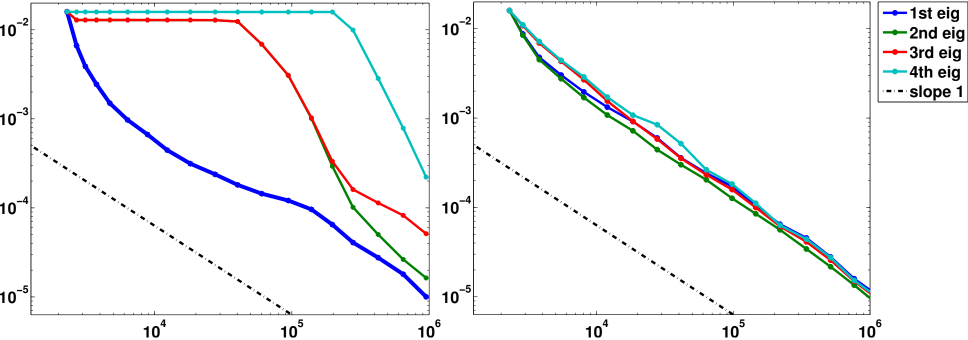


Approximation of the first four frequencies altogether

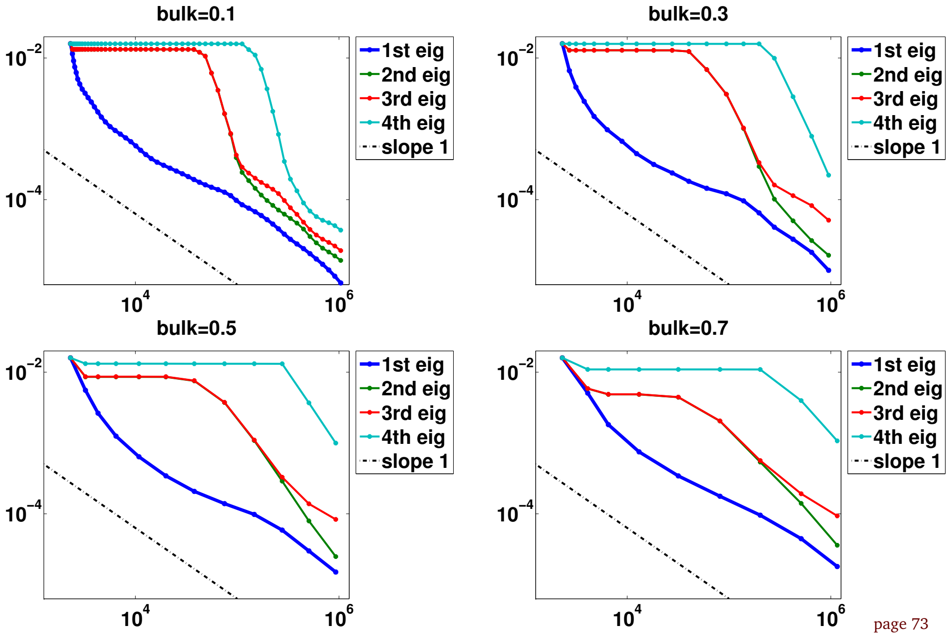
Bulk parameter=0.3, Refinement level=16



Convergence rate vs. computational cost



Influence of the bulk parameter



A tricky example

Solve the Laplace eigenvalue problem on this domain

First four exact values: 9.636. . . , 9.638. . . , 15.17. . . , 15.18. . .
(computed on an adaptively refined mesh with 8,913,989 dofs)

The computational framework

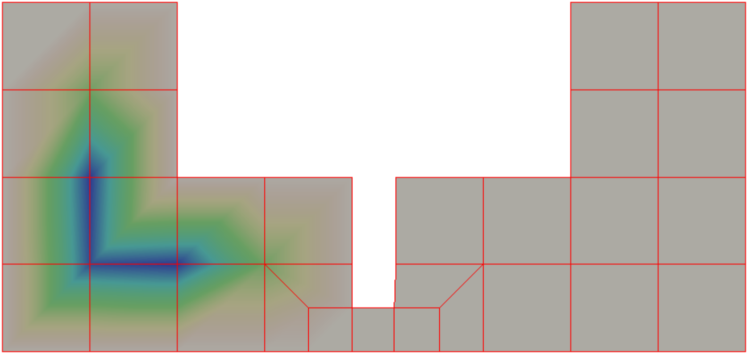
All computations performed on my relatively old
desktop computer
(2 Intel Xeon CPU's @ 3.60GHz, 2 cores each, 16Gb RAM)

AFEM algorithm implemented within the deal.II library ⟨Heltai⟩

Tensor product mesh
Finite element space: continuous bilinear space Q_1
Solution of algebraic eigenvalue problem by SLEPc
`http://slepc.upv.es` using `hypre` multigrid solver
SLEPc is built on top of PETSc for the parallelization

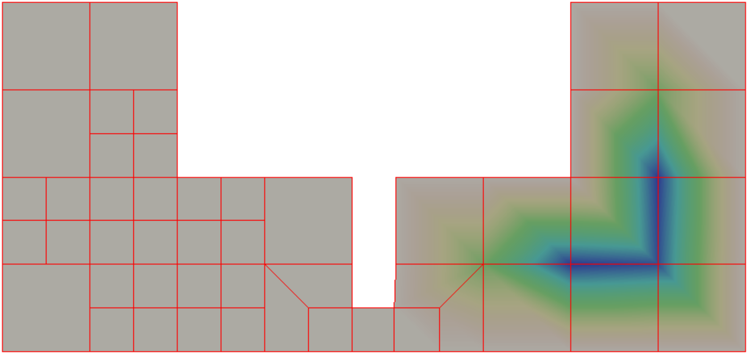
Approximation of the second frequency

Initial mesh: 48 dofs



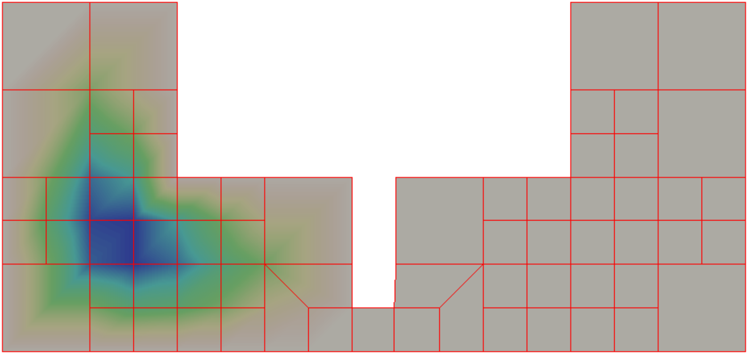
Approximation of the second frequency

Refinement # 1: 72 dofs



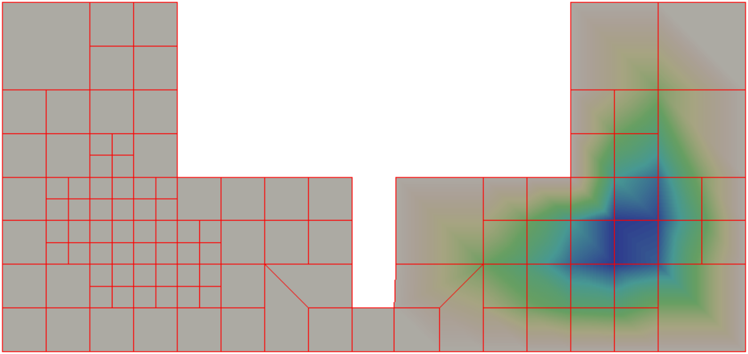
Approximation of the second frequency

Refinement # 2: 96 dofs



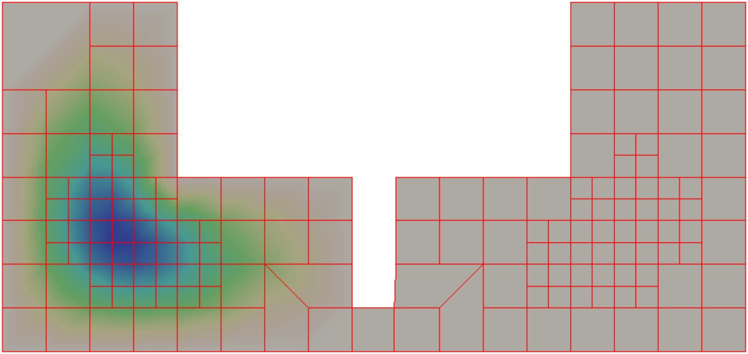
Approximation of the second frequency

Refinement # 3: 151 dofs



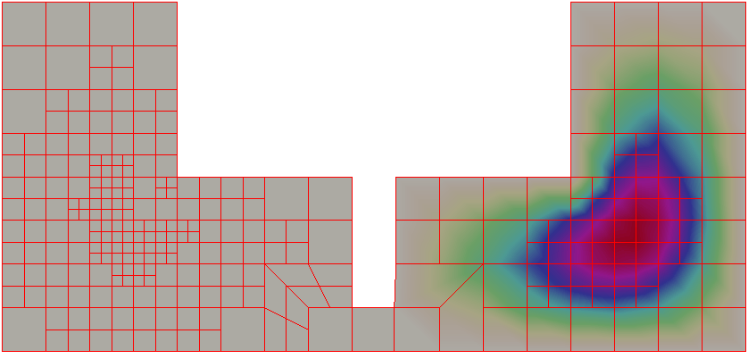
Approximation of the second frequency

Refinement # 4: 209 dofs



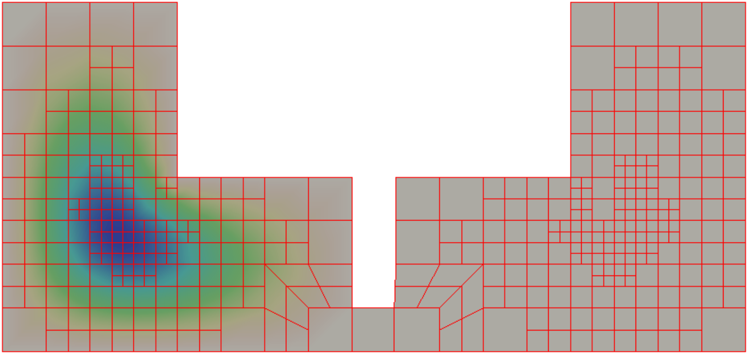
Approximation of the second frequency

Refinement # 5: 356 dofs



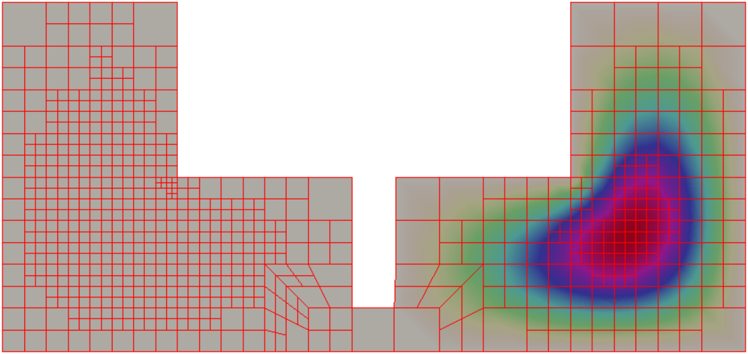
Approximation of the second frequency

Refinement # 6: 512 dofs



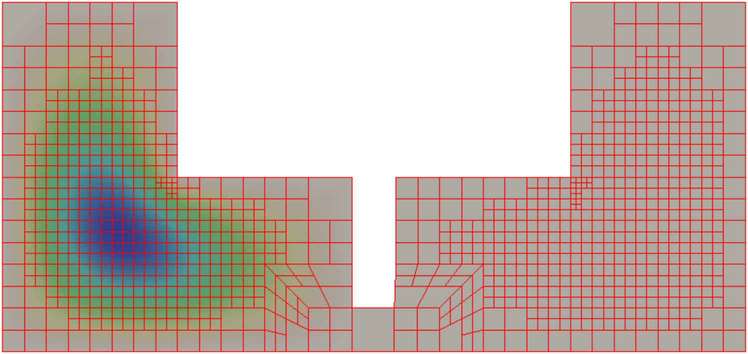
Approximation of the second frequency

Refinement # 7: 822 dofs



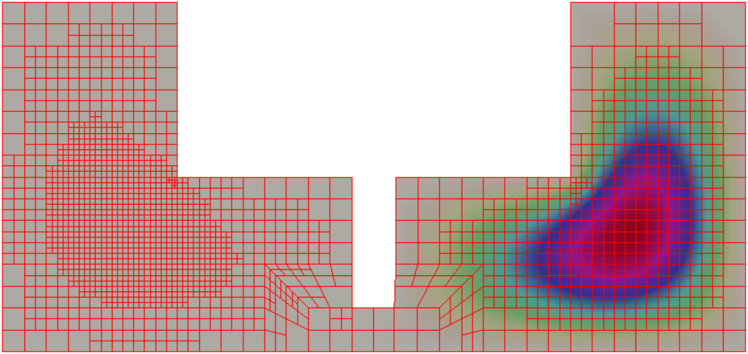
Approximation of the second frequency

Refinement # 8: 1,161 dofs



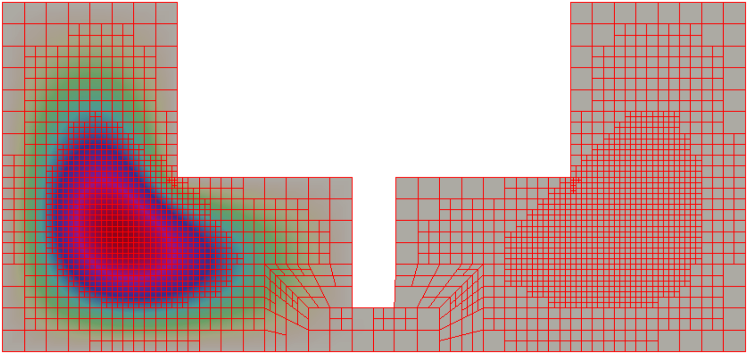
Approximation of the second frequency

Refinement # 9: 2,014 dofs



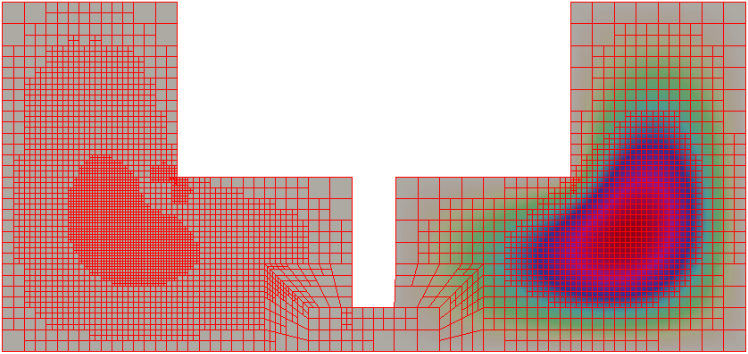
Approximation of the second frequency

Refinement # 10: 2,906 dofs



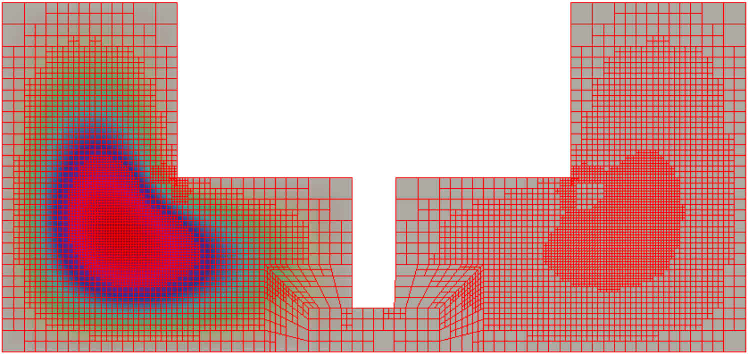
Approximation of the second frequency

Refinement # 11: 5,255 dofs



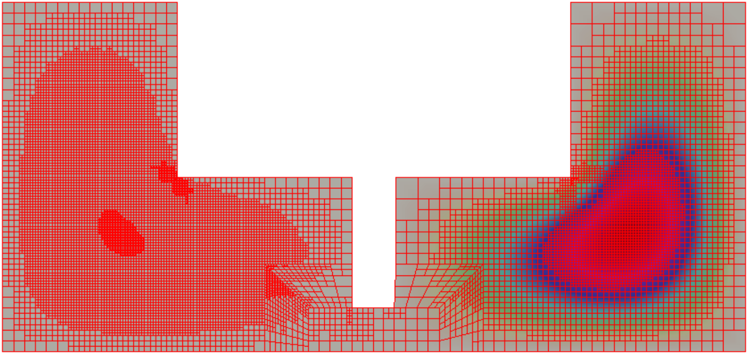
Approximation of the second frequency

Refinement # 12: 7,740 dofs



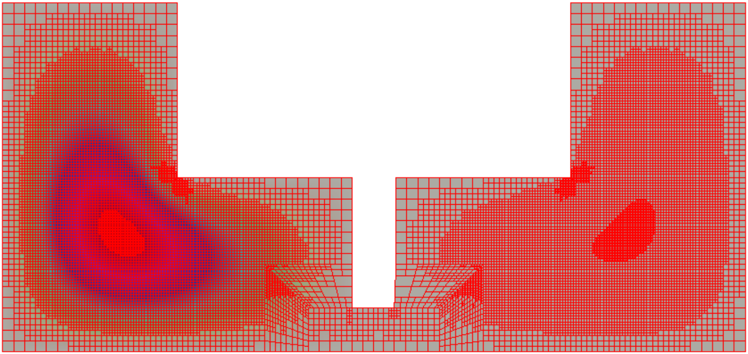
Approximation of the second frequency

Refinement # 13: 13,701 dofs



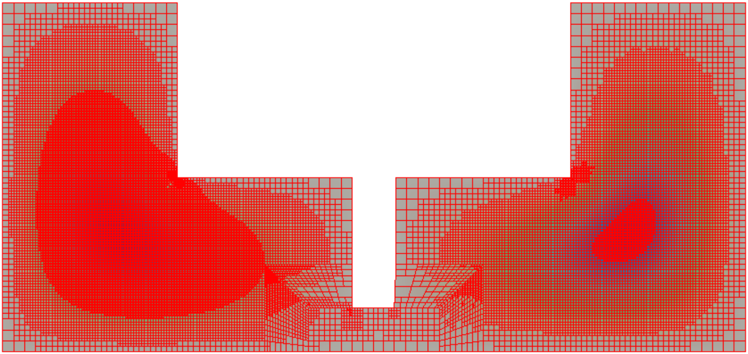
Approximation of the second frequency

Refinement # 14: 20,033 dofs



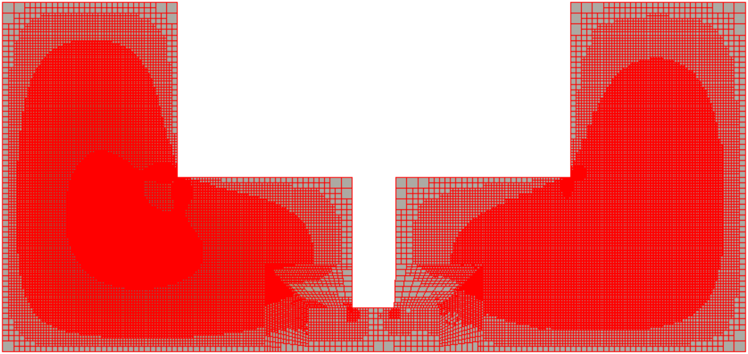
Approximation of the second frequency

Refinement # 15: 35,787 dofs



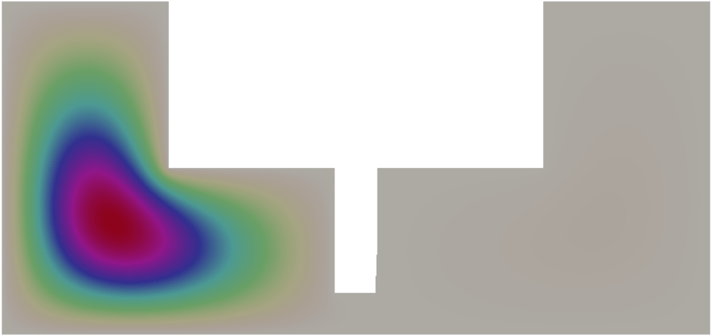
Approximation of the second frequency

Refinement # 16: 92,676 dofs



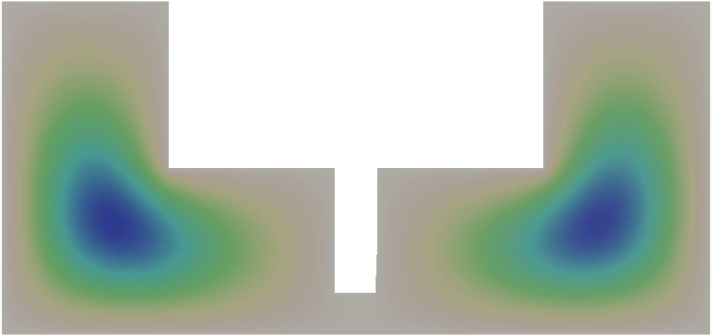
Better view

Refinement # 14: 20,033 dofs



Better view

Refinement # 15: 35,787 dofs



Better view

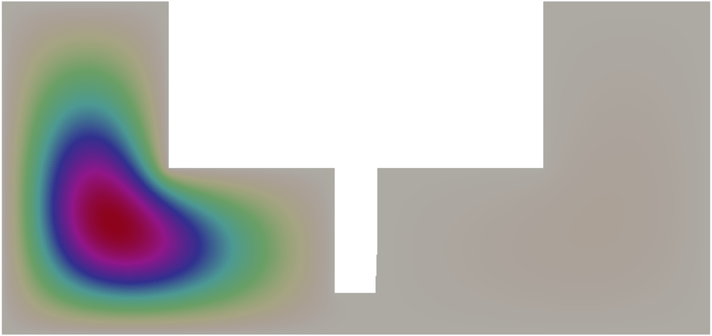
Refinement # 16: 92,676 dofs

Better view

Refinement # 17: 199,818 dofs

Better view

Refinement # 18: 485,503 dofs



Better view

Refinement # 19: 1,312,781 dofs

How much does it cost?

Global steps (Except Output)	CPU Time	
Solve	738,00	45,97%
Assemble System	337,00	20,99%
Estimate Error	239,00	14,89%
Mark and Refine	226,00	14,08%
Setup Dofs	38,00	2,37%
Create snapshot	24,90	1,55%
Setup System	2,34	0,15%
	1605,24	**100,00%**

Some comments on the parallel solver

Cycle	Dofs	Total CPU Time per cycle	
0	48	0,20	0,007%
1	72	0,35	0,011%
2	96	0,32	0,010%
3	151	0,41	0,013%
4	209	0,37	0,012%
5	356	0,40	0,013%
6	512	0,38	0,012%
7	822	0,47	0,015%
8	1,161	0,58	0,019%
9	2,014	0,70	0,023%
10	2,906	0,92	0,030%
11	5,255	1,07	0,034%
12	7,740	1,66	0,053%
13	13,701	2,67	0,086%
14	20,033	3,87	0,125%
15	35,787	6,51	0,210%
16	92,676	16,00	0,516%
17	199,818	36,20	1,166%
18	485,503	85,40	2,752%
19	1,312,781	239,00	7,701%
20	3,558,963	686,00	22,104%
21	8,913,989	2020,00	65,088%
Total	**14,654,593**	**3103,48**	100.000%

More detailed comments on the parallel solver

Cycle	Dofs	CPU time x dof	Ideal Wall Clock	Wall Clock time x dof	Efficiency
0	48	42.50	10.63	412.50	0.03
1	72	47.92	11.98	20.28	0.59
2	96	32.92	8.23	10.42	0.79
3	151	27.09	6.77	8.87	0.76
4	209	17.80	4.45	5.60	0.79
5	356	11.26	2.82	3.54	0.80
6	512	7.40	1.85	2.29	0.81
7	822	5.72	1.43	1.93	0.74
8	1,161	5.01	1.25	1.52	0.83
9	2,014	3.50	0.87	1.04	0.84
10	2,906	3.17	0.79	0.92	0.86
11	5,255	2.04	0.51	0.54	0.94
12	7,740	2.14	0.54	0.57	0.94
13	13,701	1.95	0.49	0.51	0.95
14	20,033	1.93	0.48	0.50	0.97
15	35,787	1.82	0.45	0.46	0.98
16	92,676	1.73	0.43	0.44	0.99
17	199,818	1.81	0.45	0.46	0.99
18	485,503	1.76	0.44	0.44	0.99
19	1,312,781	1.82	0.46	0.46	0.99
20	3,558,963	1.93	0.48	0.49	0.99
21	8,913,989	2.27	0.57	1.10	0.51
Total	**14,654,593**	**2.12**		**0.86**	***.1 ms**

AFEM for clusters of eigenvalues

Cluster of length N

$\lambda_{n+1}, \dots, \lambda_{n+\mathsf{N}}$

$J = \{n+1, \dots, n+\mathsf{N}\}$

Corresponding combination of eigenspaces

$W = \mathrm{span}\{u_j \mid j \in J\}$

$W_{\mathcal{T}_h} = W_h = \mathrm{span}\{u_{h,j} \mid j \in J\}$

How to implement the AFEM scheme

Consider contribution of all elements in W_ℓ simultaneously

$$\theta \sum_{j\in J} \eta_{\ell,j}(\mathcal{T}_\ell)^2 \le \sum_{j\in J} \eta_{\ell,j}(\mathcal{M}_\ell)^2$$

Approximation of the second frequency (cluster of two)

Initial mesh: 48 dofs

Approximation of the second frequency (cluster of two)

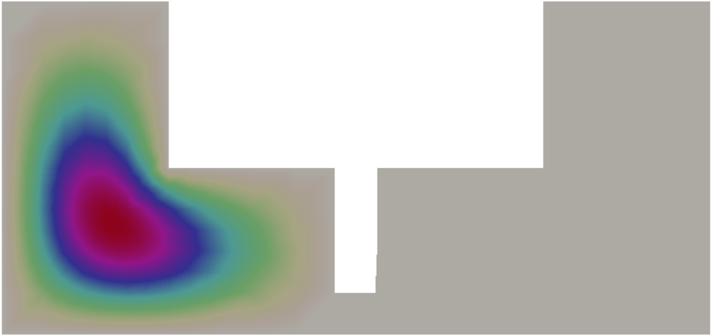
Refinement # 1: 99 dofs

Approximation of the second frequency (cluster of two)

Refinement # 2: 221 dofs

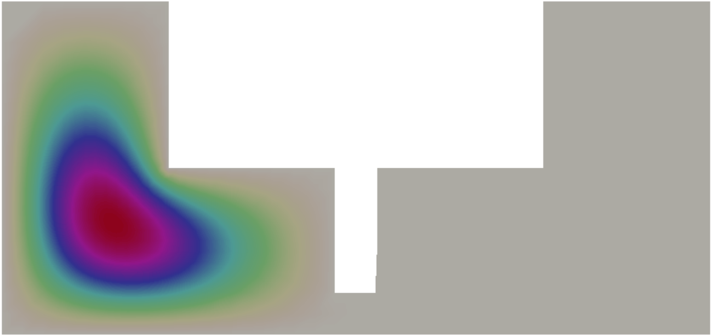
Approximation of the second frequency (cluster of two)

Refinement # 3: 543 dofs



Approximation of the second frequency (cluster of two)

Refinement # 4: 1,295 dofs



Approximation of the second frequency (cluster of two)

Refinement # 5: 3,141 dofs

Approximation of the second frequency (cluster of two)

Refinement # 6: 8,324 dofs

Approximation of the second frequency (cluster of two)

Refinement # 7: 21,419 dofs

Approximation of the second frequency (cluster of two)

Refinement # 8: 52,783 dofs

Approximation of the second frequency (cluster of two)

Refinement # 9: 143,641 dofs

Approximation of the second frequency (cluster of two)

Refinement # 10: 384,389 dofs

Approximation of the second frequency (cluster of two)

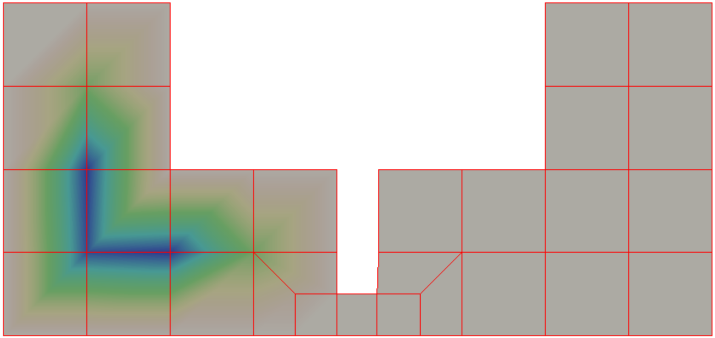
Refinement # 11: 949,442 dofs

Approximation of the second frequency (cluster of two)

Refinement # 12: 2,559,242 dofs

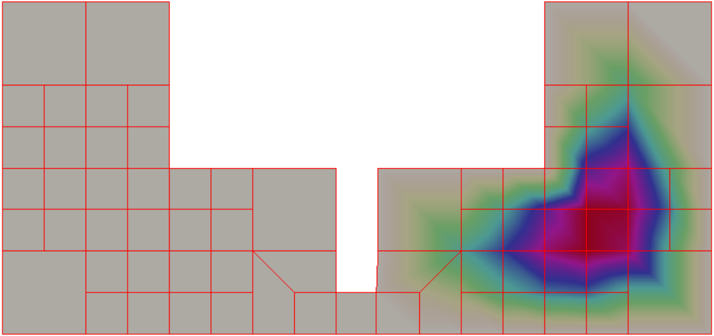
Cluster of two: underlying mesh

Initial mesh: 48 dofs



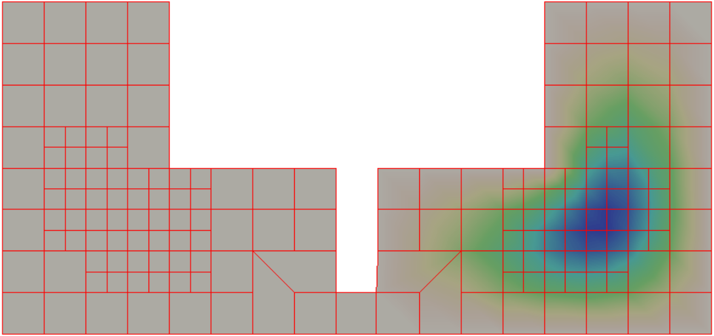
Cluster of two: underlying mesh

Refinement # 1: 99 dofs



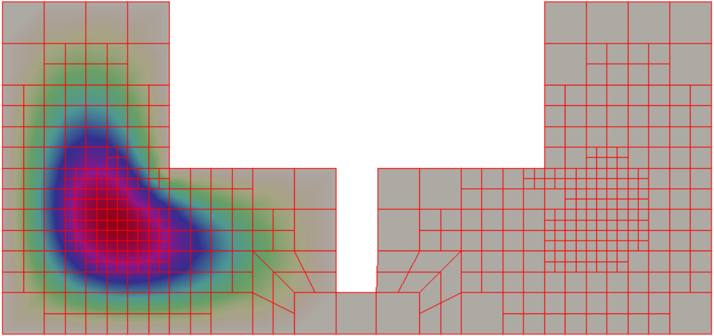
Cluster of two: underlying mesh

Refinement # 2: 221 dofs



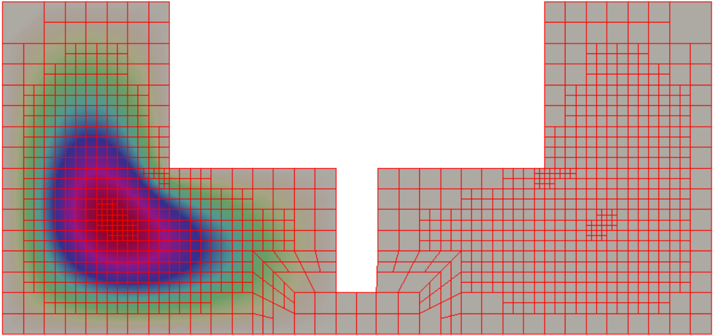
Cluster of two: underlying mesh

Refinement # 3: 543 dofs



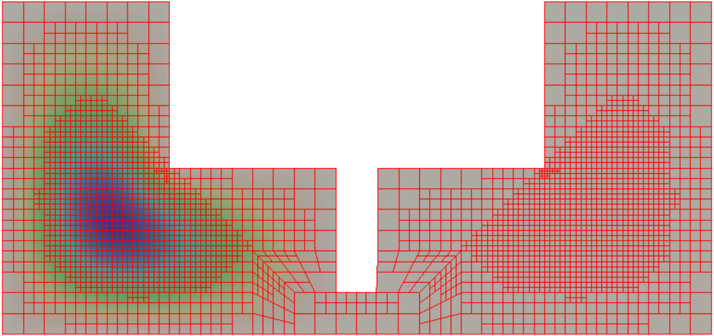
Cluster of two: underlying mesh

Refinement # 4: 1,295 dofs



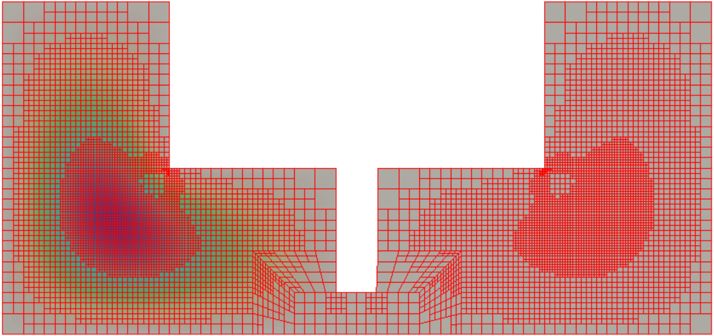
Cluster of two: underlying mesh

Refinement # 5: 3,141 dofs



Cluster of two: underlying mesh

Refinement # 6: 8,324 dofs



Some theoretical framework

⟨Babuška–Osborn '91⟩
⟨B. '00⟩

Variationally posed eigenproblem (Laplace operator)

Abstract framework

$H \quad (= L^2(\Omega)) \qquad V \quad (= H^1_0(\Omega)) \qquad \subset H$

Hilbert spaces, V compactly embedded in H

$a(u,v) \quad (= \int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x}) \qquad V \times V \to \mathbb{R}$

bilinear, continuous, symmetric, coercive

$b(u,v) \quad (= (u,v)) \qquad H \times H \to \mathbb{R}$

bilinear, continuous, symmetric

Eigenvalue problem

Find $\lambda \in \mathbb{R}$ such that for some $u \in V$ with $u \not\equiv 0$ it holds

$a(u,v) = \lambda b(u,v) \quad \forall v \in V$

Rayleigh quotient

$$\lambda^{(1)} \le \lambda^{(2)} \le \cdots \le \lambda^{(k)} \le \cdots$$

$$a(u^{(m)}, u^{(n)}) = b(u^{(m)}, u^{(n)}) = 0 \quad \text{if } m \neq n$$

$$\lambda^{(1)} = \min_{v \in V} \frac{a(v,v)}{b(v,v)} \qquad u^{(1)} = \arg\min_{v \in V} \frac{a(v,v)}{b(v,v)}$$

$$\lambda^{(k)} = \min_{v \in \left(\bigoplus_{i=1}^{k-1} E^{(i)} \right)^{\perp}} \frac{a(v,v)}{b(v,v)} \qquad u^{(k)} = \arg\min_{v \in \left(\bigoplus_{i=1}^{k-1} E^{(i)} \right)^{\perp}} \frac{a(v,v)}{b(v,v)}$$

Rayleigh quotient (discrete)

$$\lambda_h^{(1)} \le \lambda_h^{(2)} \le \cdots \le \lambda_h^{(k)} \le \lambda_h^{(N(h))}$$

$$a(u_h^{(m)}, u_h^{(n)}) = b(u_h^{(m)}, u_h^{(n)}) = 0 \quad \text{if } m \ne n$$

$$\lambda_h^{(1)} = \min_{v \in V_h} \frac{a(v,v)}{b(v,v)} \qquad u_h^{(1)} = \arg\min_{v \in V_h} \frac{a(v,v)}{b(v,v)}$$

$$\lambda_h^{(k)} = \min_{v \in \left(\bigoplus_{i=1}^{k-1} E_h^{(i)}\right)^\perp} \frac{a(v,v)}{b(v,v)} \qquad u_h^{(k)} = \arg\min_{v \in \left(\bigoplus_{i=1}^{k-1} E_h^{(i)}\right)^\perp} \frac{a(v,v)}{b(v,v)}$$

$$\boxed{\Rightarrow \lambda^{(1)} \le \lambda_h^{(1)}}$$

Minmax characterization

The k-th eigenvalue $\lambda^{(k)}$ satisfies

$$\lambda^{(k)} = \min_{E \in V^{(k)}} \max_{v \in E} \frac{a(v,v)}{b(v,v)}$$

$V^{(k)}$ set of all subspaces of V with $\dim(E) = k$

Proof.

$\boxed{\lambda^{(k)} \geq \min\max}$: take $E = \bigoplus_{i=1}^{k} E^{(i)}$, so that $v = \sum_{i=1}^{k} \alpha_i u^{(i)}$

Then $a(v,v)/b(v,v) \leq \lambda^{(k)}$ thanks to orthogonalities

$\boxed{\lambda^{(k)} \leq \min\max}$: minimum $\lambda^{(k)}$ attained for $E = \bigoplus_{i=1}^{k} E^{(i)}$ and the choice $v = u^{(k)}$. Otherwise, there exists $v \in E$ orthogonal to $u^{(i)}$ for all $i \leq k$ and hence $a(v,v)/b(v,v) \geq \lambda^{(k)}$

□

Minmax characterization (continuous and discrete)

$$\lambda^{(k)} = \min_{E \in V^{(k)}} \max_{v \in E} \frac{a(v,v)}{b(v,v)}$$

$V^{(k)}$ set of all subspaces of V with $\dim(E) = k$

$$\lambda_h^{(k)} = \min_{E_h \in V_h^{(k)}} \max_{v \in E_h} \frac{a(v,v)}{b(v,v)}$$

$V_h^{(k)}$ set of all subspaces of V_h with $\dim(E) = k$

$$\boxed{\Rightarrow \lambda^{(k)} \le \lambda_h^{(k)} \quad \forall k}$$

Laplace's eigenvalues

We need the upper bound

$$\lambda_h^{(k)} \le \lambda^{(k)} + \varepsilon(h)$$

with $\varepsilon(h)$ tending to zero as h goes to zero

We are going to use

$$E_h = \Pi_h \mathcal{E}^{(k)}$$

in the minmax characterization of the discrete eigenvalues, where

$$\mathcal{E}^{(k)} = \bigoplus_{i=1}^{k} E^{(i)}$$

and $\Pi_h : V \to V_h$ denotes the elliptic projection

$$(\nabla(u - \Pi_h u), \nabla v_h) = 0 \quad \forall v_h \in V_h$$

We need to check that the dimension of E_h is equal to k

Take h small enough so that
$\|v - \Pi_h v\|_{L^2(\Omega)} \le \frac{1}{2}\|v\|_{L^2(\Omega)} \quad \forall v \in \mathcal{E}^{(k)}$

Then $\|\Pi_h v\|_{L^2(\Omega)} \ge \|v\|_{L^2(\Omega)} - \|v - \Pi_h v\|_{L^2(\Omega)} \quad \forall v \in V$ implies that Π_h is injective from $\mathcal{E}^{(k)}$ to E_h

Taking E_h in the discrete minmax equation gives

$$\begin{aligned}
\lambda_h^{(k)} &\le \max_{w\in E_h} \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\|w\|^2_{L^2(\Omega)}} = \max_{v\in\mathcal{E}^{(k)}} \frac{\|\nabla(\Pi_h v)\|^2_{L^2(\Omega)}}{\|\Pi_h v\|^2_{L^2(\Omega)}} \\
&\le \max_{v\in\mathcal{E}^{(k)}} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|\Pi_h v\|^2_{L^2(\Omega)}} = \max_{v\in\mathcal{E}^{(k)}} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^2(\Omega)}} \frac{\|v\|^2_{L^2(\Omega)}}{\|\Pi_h v\|^2_{L^2(\Omega)}} \\
&\le \lambda^{(k)} \max_{v\in\mathcal{E}^{(k)}} \frac{\|v\|^2_{L^2(\Omega)}}{\|\Pi_h v\|^2_{L^2(\Omega)}}.
\end{aligned}$$

Take Ω convex (for simplicity). Then

$$\begin{aligned}
\|v-\Pi_h v\|_{L^2(\Omega)} &\le Ch^2\|\Delta v\|_{L^2(\Omega)} \le C\lambda^{(k)}h^2\|v\|_{L^2(\Omega)} \\
&= C(k)h^2\|v\|_{L^2(\Omega)}
\end{aligned}$$

Hence

$$\|\Pi_h v\|_{L^2(\Omega)} \ge \|v\|_{L^2(\Omega)}(1-C(k)h^2)$$

Eigenvalue estimate

Finally

$$\lambda_h^{(k)} \le \lambda^{(k)} \left(\frac{1}{1 - C(k)h^2} \right)^2 \simeq \lambda^{(k)} \left(1 + C(k)h^2\right)^2$$
$$\simeq \lambda^{(k)} (1 + 2C(k)h^2)$$

In general

$$\lambda_h^{(k)} \le \lambda^{(k)} \Big(1 + C(k) \sup_{\substack{v \in \mathcal{E}^{(k)} \\ \|v\|=1}} \|v - \Pi_h v\|^2_{H^1(\Omega)} \Big)$$

Estimate for the eigenfunctions

Let's start with a simple eigenvalue $\lambda^{(k)}$

$$\rho_h^{(k)} = \max_{i \neq k} \frac{\lambda^{(k)}}{|\lambda^{(k)} - \lambda_h^{(i)}|},$$

$$w_h^{(k)} = (\Pi_h u^{(k)}, u_h^{(k)}) u_h^{(k)}$$

$$\begin{aligned} \|u^{(k)} - u_h^{(k)}\|_{L^2(\Omega)} &\leq \|u^{(k)} - \Pi_h u^{(k)}\| \\ &+ \|\Pi_h u^{(k)} - w_h^{(k)}\| \\ &+ \|w_h^{(k)} - u_h^{(k)}\| \end{aligned}$$

Second term

$$\Pi_h u^{(k)} - w_h^{(k)} = \sum_{i\ne k} (\Pi_h u^{(k)}, u_h^{(i)}) u_h^{(i)}$$

$$\|\Pi_h u^{(k)} - w_h^{(k)}\|^2 = \sum_{i\ne k} (\Pi_h u^{(k)}, u_h^{(i)})^2$$

$$\begin{aligned}(\Pi_h u^{(k)}, u_h^{(i)}) &= \frac{1}{\lambda_h^{(i)}} (\nabla(\Pi_h u^{(k)}), \nabla u_h^{(i)}) \\ &= \frac{1}{\lambda_h^{(i)}} (\nabla u^{(k)}, \nabla u_h^{(i)}) = \frac{\lambda^{(k)}}{\lambda_h^{(i)}} (u^{(k)}, u_h^{(i)})\end{aligned}$$

$$\boxed{\lambda_h^{(i)} (\Pi_h u^{(k)}, u_h^{(i)}) = \lambda^{(k)} (u^{(k)}, u_h^{(i)})}$$

$$(\lambda_h^{(i)} - \lambda^{(k)})(\Pi_h u^{(k)}, u_h^{(i)}) = \lambda^{(k)}(u^{(k)} - \Pi_h u^{(k)}, u_h^{(i)})$$

$$|(\Pi_h u^{(k)}, u_h^{(i)})| \leq \rho_h^{(k)} |(u^{(k)} - \Pi_h u^{(k)}, u_h^{(i)})|$$

$$\begin{aligned}\|\Pi_h u^{(k)} - w_h^{(k)}\|^2 &\leq \left(\rho_h^{(k)}\right)^2 \sum_{i \neq k} (u^{(k)} - \Pi_h u^{(k)}, u_h^{(i)})^2 \\ &\leq \left(\rho_h^{(k)}\right)^2 \|u^{(k)} - \Pi_h u^{(k)}\|^2\end{aligned}$$

Third term

We are going to show that

$$\|u_h^{(k)} - w_h^{(k)}\| \le \|u^{(k)} - w_h^{(k)}\|$$

so that

$$\|u_h^{(k)} - w_h^{(k)}\| \le \|u^{(k)} - \Pi_h u^{(k)}\| + \|\Pi_h u^{(k)} - w_h^{(k)}\|$$

$$u_h^{(k)} - w_h^{(k)} = u_h^{(k)}(1 - (\Pi_h u^{(k)}, u_h^{(k)})).$$

$$\|u^{(k)}\| - \|u^{(k)} - w_h^{(k)}\| \le \|w_h^{(k)}\| \le \|u^{(k)}\| + \|u^{(k)} - w_h^{(k)}\|$$

$$1 - \|u^{(k)} - w_h^{(k)}\| \le |(\Pi_h u^{(k)}, u_h^{(k)})| \le 1 + \|u^{(k)} - w_h^{(k)}\|,$$

$$\boxed{\left| |(\Pi_h u^{(k)}, u_h^{(k)})| - 1 \right| \le \|u^{(k)} - w_h^{(k)}\|}$$

Simple eigenfunction estimate

Sign choice for $u_h^{(k)}$ such that

$$(\Pi_h u^{(k)}, u_h^{(k)}) \geq 0$$

Then $\left| |(\Pi_h u^{(k)}, u_h^{(k)})| - 1 \right| = \|w_h^{(k)} - u_h^{(k)}\|$

Final estimate

$$\|u^{(k)} - u_h^{(k)}\|_{L^2(\Omega)} \leq 2(1 + \rho_h^{(k)})\|u^{(k)} - \Pi_h u^{(k)}\|_{L^2(\Omega)}$$

Eigenfunction estimate in H^1

$$
\begin{aligned}
C\|u^{(k)} - u_h^{(k)}\|^2_{H^1(\Omega)} &\le \|\nabla(u^{(k)} - u_h^{(k)})\|^2_{L^2(\Omega)} \\
&= \|\nabla u^{(k)}\|^2 - 2(\nabla u^{(k)}, \nabla u_h^{(k)}) + \|\nabla u_h^{(k)}\|^2 \\
&= \lambda^{(k)} - 2\lambda^{(k)}(u^{(k)}, u_h^{(k)}) + \lambda_h^{(k)} \\
&= \lambda^{(k)} - 2\lambda^{(k)}(u^{(k)}, u_h^{(k)}) + \lambda^{(k)} - (\lambda^{(k)} - \lambda_h^{(k)}) \\
&= \lambda^{(k)}\|u^{(k)} - u_h^{(k)}\|^2_{L^2(\Omega)} - (\lambda^{(k)} - \lambda_h^{(k)})
\end{aligned}
$$

$$
\boxed{\|u^{(k)} - u_h^{(k)}\|_{H^1(\Omega)} \le C(k) \sup_{\substack{v \in \mathcal{E}^{(k)} \\ \|v\|=1}} \|v - \Pi_h v\|_{H^1(\Omega)}}
$$

Multiple eigenfunctions

$\lambda^{(k)} = \lambda^{(k+1)}$
$\lambda^{(i)} \neq \lambda^{(k)}$ for $i \neq k, k+1$

$$\rho_h^{(k)} = \max_{i \neq k,k+1} \frac{\lambda^{(k)}}{|\lambda^{(k)} - \lambda_h^{(i)}|},$$

$$w_h^{(k)} = \alpha_h u_h^{(k)} + \beta_h u_h^{(k+1)}$$

$$\alpha_h = (\Pi_h u^{(k)}, u_h^{(k)}), \quad \beta_h = (\Pi_h u^{(k)}, u_h^{(k+1)})$$

$$\|u^{(k)} - w_h^{(k)}\|_{L^2(\Omega)} \leq (1 + \rho_h^{(k)}) \|u^{(k)} - \Pi_h u^{(k)}\|_{L^2(\Omega)}$$

$$\|u^{(k)} - w_h^{(k)}\|_{H^1(\Omega)} \leq C(k) \sup_{\substack{v \in V^{(k+1)} \\ \|v\|=1}} \|v - \Pi_h v\|_{H^1(\Omega)}$$

Standard Laplace eigenvalue problem

Strong form

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Weak form

$\lambda \in \mathbb{R}$, $u \in V$, $u \not\equiv 0$:

$a(u,v) = \lambda b(u,v) \quad \forall v \in V$

Solution operator

$T : H \to H$, $\qquad T(H) \subset V$ implies T is compact

$a(Tf, v) = b(f, v) \quad \forall v \in V$

$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_i \le \cdots$

$E_i = \operatorname{span}(u_i)$, normalization $b(u_i, u_i) = 1$

$V = \bigoplus_{i=1}^{\infty} E_i$

Galerkin approximation

Discrete problem

$V_h \subset V$, $\dim V_h = N(h)$

Find $\lambda_h \in \mathbb{R}$ such that for some $u_h \in V_h$ with $u_h \not\equiv 0$ it holds
$a(u_h, v) = \lambda_h b(u_h, v) \quad \forall v \in V_h$

Discrete (compact) solution operator

$T_h : H \to H$
$a(T_h f, v) = b(f, v) \quad \forall v \in V_h$

$\lambda_{1,h} \le \lambda_{2,h} \le \cdots \le \lambda_{i,h} \le \cdots \le \lambda_{N(h),h}$
$E_{i,h} = \operatorname{span}(u_{i,h})$, normalization $b(u_{i,h}, u_{i,h}) = 1$
$V_h = \bigoplus_{i=1}^{N(h)} E_{i,h}$

Definition of convergence

Absence of spurious modes

For any compact set $K \subset \rho(T)$ there exists $h_0 > 0$ such that for all $h < h_0$ it holds $K \subset \rho(T_h)$

Convergence

If μ is a nonzero eigenvalue of T with algebraic multiplicity equal to m, then there are exactly m eigenvalues $\mu_{1,h}, \mu_{2,h}, \dots, \mu_{m,h}$ of T_h, repeated according to their algebraic multiplicities, such that $\mu_{i,h} \to \mu$ for all i
Moreover, the gap between the direct sum of the generalized eigenspaces associated with $\mu_{1,h}, \mu_{2,h}, \dots, \mu_{m,h}$ and the generalized eigenspace associated with μ tends to zero

Gap

$\hat{\delta}(E,F) = \max(\delta(E,F), \delta(F,E))$, where E, F subspaces of H

$$\delta(E,F) = \sup_{u \in E,\ ||u||_H = 1} \inf_{v \in F} ||u - v||_H$$

Uniform convergence

Convergence in norm

$$||T - T_h||_{\mathcal{L}(H,H)} \to 0$$

Theorem

If T is selfadjoint and compact

Uniform convergence $\Longleftrightarrow$ *Eigenmodes convergence*

Strategy

1) prove uniform convergence,
2) estimate the order of convergence

Galerkin approximation of compact operators

⟨Bramble–Osborn ’73⟩
⟨Osborn ’75⟩
⟨Kolata ’78⟩

Céa’s Lemma

$T_h = P_h T$, with P_h projection w.r.t. bilinear form a

$$T - T_h = (I - P_h)T$$

Consequence of $a(Tf - T_h f, v_h) = 0 \quad \forall v_h \in V_h$

If $I - P_h$ converges to zero *pointwise* and T is *compact*, then $T - T_h$ converges to zero *uniformly* (consequence of Banach–Steinhaus uniform boundedness principle)

Crucial proof

First we show that $\{\|I - P_h\|_{\mathcal{L}(V,H)}\}$ is bounded

Define $c(h,u)$ by $\|(I - P_h)u\|_H = c(h,u)\|u\|_V$
For each u we have $c(h,u) \to 0$ (pointwise convergence)

$M(u) = \max_h c(h,u) < \infty$ implies $\|I - P_h\|_{\mathcal{L}(V,H)} \le C$ uniformly

Take $\{f_h\}$ s.t. $\|f_h\|_H = 1$ and $\|T - T_h\|_{\mathcal{L}(H)} = \|(T - T_h)f_h\|_H$

Extract subsequence with $Tf_h \to w$ in V

$$\begin{aligned} \|(I - P_h)Tf_h\|_H &\le \|(I - P_h)(Tf_h - w)\|_H + \|(I - P_h)w\|_H \\ &\le C\|Tf_h - w\|_V + \|(I - P_h)w\|_H \le \varepsilon \end{aligned}$$

Comment on the norms

1. $T : H \to V$ compact + p/w convergence $V \to H$ $\qquad \mathcal{L}(H)$
2. $T : V \to V$ compact + p/w convergence $V \to V$ $\qquad \mathcal{L}(V)$

Standard Galerkin formulation are OK for eigenvalues

Important conclusion

Standard Galerkin formulation: all finite element schemes providing good approximation to the source problem can be successfully applied to the corresponding eigenvalue problem

Is the same true for eigenvalue problems in mixed form?

Laplace eigenproblem in mixed form

⟨Mercier–Osborn–Rappaz–Raviart '81⟩
⟨B.–Brezzi–Gastaldi '97-'00⟩

Find $\lambda \in \mathbb{R}$ and $u \in U$ with $u \not\equiv 0$ such that for some $\sigma \in \Sigma$

$$\begin{cases} (\sigma,\tau) + (\operatorname{div}\tau, u) = 0 & \forall \tau \in \Sigma \qquad \boldsymbol{\sigma} = \nabla u \\ (\operatorname{div}\sigma, v) = -\lambda(u,v) & \forall v \in U \qquad \operatorname{div}\boldsymbol{\sigma} = -\lambda u \end{cases}$$

Matrix form ($\Sigma_h \subset \Sigma$, $U_h \subset U$)

$$\begin{bmatrix} A & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\lambda \begin{bmatrix} 0 & 0 \\ 0 & M_U \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Similarly, one could deal with problems of the type

$$\begin{bmatrix} A & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} M_\Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Definition of the solution operator

Source problem

$$\begin{cases} (\sigma,\tau)+(\operatorname{div}\tau,u)=0 & \forall\tau\in\Sigma \\ (\operatorname{div}\sigma,v)=-(g,v) & \forall v\in U \end{cases} \qquad \begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\nabla} u \\ -\operatorname{div}\boldsymbol{\sigma} &= g \end{aligned}$$

A first natural (but wrong) definition

$$T_1 : U \to \Sigma\times U, \qquad T_1(g)=(\sigma,u)$$

One would like to compute eigenvalues...

$T_2 : (\Sigma\times U)' \to \Sigma\times U$
$T_2(f,g)=(\sigma,u)$ with

$$\begin{cases} (\sigma,\tau)+(\operatorname{div}\tau,u)=\langle f,v\rangle & \forall\tau\in\Sigma \\ (\operatorname{div}\sigma,v)=-(g,v) & \forall v\in U \end{cases}$$

$$T_{\Sigma U} \quad \boxed{\begin{matrix} (f,g)\overset{\text{cutoff}}{\longmapsto}(0,g)\overset{T_2}{\longmapsto}(\sigma,u) \\ L^2\times L^2 \longrightarrow L^2\times L^2 \end{matrix}} \quad \text{is compact}$$

Uniform convergence?

Let's try to follow Kolata's argument

$T_{\Sigma U} - T_{\Sigma U,h} = (I - Q_h)T_{\Sigma U}$

✓ $||(I - Q_h)(\sigma, u)||_{\Sigma \times U} \to 0$ for all $(\sigma, u) \in \Sigma \times U$

✗ $T_{\Sigma U} : L^2 \times L^2 \to \Sigma \times U$ is not compact

✗ $T_{\Sigma U} : \Sigma \times U \to \Sigma \times U$ is not compact either

Standard mixed estimates don't help

$$||\sigma - \sigma_h||_\Sigma + ||u - u_h||_U \le C \inf_{\tau_h, v_h} \Big(\underbrace{||\sigma - \tau_h||_\Sigma}_{O(1)} + \underbrace{||u - v_h||_U}_{O(h)}\Big)$$

$$\inf_{\tau_h} \|\sigma - \tau_h\|_{H(\mathrm{div})} \le Ch^s(\|\sigma\|_{H^s} + \|\operatorname{div}\sigma\|_{H^s})$$

Better definition of the solution operator

⟨B.–Brezzi–Gastaldi '97⟩

$T_U : U \to U$

$\sigma \in \Sigma$, $T_U g \in U$ such that

$$\begin{cases} (\sigma,\tau) + (\operatorname{div}\tau, T_U g) = 0 & \forall\tau\in\Sigma \\ (\operatorname{div}\sigma, v) = -(g, v) & \forall v\in U \end{cases}$$

Operator is now compact, but standard mixed estimates don't help again

$$||\sigma-\sigma_h||_\Sigma + ||u-u_h||_U \le C \inf_{\tau_h, v_h} (\underbrace{||\sigma-\tau_h||_\Sigma}_{O(1)} + \underbrace{||u-v_h||_U}_{O(h)})$$

Fundamental comment

We need an estimate for u_h which does not involve $\operatorname{div}\sigma$

Uniform convergence $||T_U - T_{U,h}|| \to 0$

- Ellipticity in the kernel

$$||\tau_h||^2_{L^2} \geq \alpha ||\tau_h||^2_{\Sigma}$$

for all $\tau_h \in \Sigma_h$ s.t. $\{(\operatorname{div} \tau_h, v) = 0,\ \forall v \in U_h\}$

- Fortin operator $\Pi_h : \Sigma^+ \to \Sigma_h$ s.t.

$$(\operatorname{div}(\sigma - \Pi_h \sigma), v) = 0 \quad \forall v \in U_h$$
$$||\Pi_h \sigma||_{\Sigma} \leq C ||\sigma||_{\Sigma^+}$$

Theorem

$$||\sigma - \sigma_h||_{L^2} \leq C \left(||\sigma - \Pi_h \sigma||_{L^2} + (1/\sqrt{\alpha}) \inf_{v_h \in U_h} ||u - v_h||_U \right)$$

$$||u - u_h||_U \leq C \left(\inf_{v_h \in U_h} ||u - v_h||_U + ||\sigma - \sigma_h||_{L^2} \right)$$

Proof

$P = L^2$-projection onto U_h

$$\begin{aligned}
||\Pi_h\sigma - \sigma_h||_{L^2}^2 &= (\Pi_h\sigma - \sigma, \Pi_h\sigma - \sigma_h) + (\sigma - \sigma_h, \Pi_h\sigma - \sigma_h) \\
&= (\Pi_h\sigma - \sigma, \Pi_h\sigma - \sigma_h) - (\operatorname{div}(\Pi_h\sigma - \sigma_h), u - Pu) \\
&\leq ||\Pi_h\sigma - \sigma||_{L^2}||\Pi_h\sigma - \sigma_h||_{L^2} + ||\operatorname{div}(\Pi_h\sigma - \sigma_h)||_{L^2}||u - Pu||_U \\
&\leq ||\Pi_h\sigma - \sigma_h||_{L^2}\left(||\Pi_h\sigma - \sigma||_{L^2} + (1/\sqrt{\alpha})||u - Pu||_U\right)
\end{aligned}$$

$$\begin{aligned}
||Pu - u_h||_U &\leq C\sup_{\tau_h}\frac{(Pu - u_h, \operatorname{div}\tau_h)}{||\tau_h||_\Sigma} \\
&\leq C\sup_{\tau_h}\frac{(Pu - u, \operatorname{div}\tau_h) + (u - u_h, \operatorname{div}\tau_h)}{||\tau_h||_\Sigma} \\
&\leq C\left(||Pu - u||_U + \sup_{\tau_h}\frac{-(\sigma - \sigma_h, \tau_h)}{||\tau_h||_\Sigma}\right) \\
&\leq C\left(||Pu - u||_U + ||\sigma - \sigma_h||_{L^2}\right)
\end{aligned}$$

Fortid condition

Definition

The spaces Σ_h, U_h satisfy the Fortid condition if there exists a Fortin operator which converges strongly to the identity operator, namely

$\Pi_h : \Sigma^+ \to \Sigma_h$ s.t.

$$(\mathrm{div}(\sigma - \Pi_h\sigma), v) = 0 \quad \forall v \in U_h$$
$$||\Pi_h\sigma||_\Sigma \le C||\sigma||_{\Sigma^+}$$

$$\boxed{||I - \Pi_h||_{\mathcal{L}(\Sigma^+, L^2)} \to 0}$$

Final convergence result

Theorem

Assume ellipticity in the kernel and Fortid condition
For any $N \in \mathbb{N}$ *define* $\rho_N(h) :]0,1] \to \mathbb{R}$ *as*

$$\rho_N(h) = \sup_{u \in \bigoplus_{i=1}^{m(N)} E_i} \left(\inf_{v_h} ||u - v_h||_U + ||\nabla u - \Pi_h \nabla u||_{L^2} \right)$$

Then $||T_U - T_{U,h}||_{\mathcal{L}(U,U)} \to 0$ *and the following estimates hold true*

$$\sum_{i=1}^{m(N)} |\lambda_i - \lambda_{i,h}| \le C(\rho_N(h))^2$$

$$\hat{\delta}\left(\bigoplus_{i=1}^{m(N)} E_i, \bigoplus_{i=1}^{m(N)} E_{i,h} \right) \le C\rho_N(h)$$

Back to the criss-cross (counter)-example

Crisscross mesh

$\Sigma_h = \{\text{continuous p/w linears (componentwise)}\}$

$U_h = \operatorname{div} \Sigma_h \subset \{\text{p/w constants}\}$

Theorem

With the above choice of spaces, there exists a sequence $\{g_h\} \subset U$ *with* $\|g_h\|_0 = 1$ *s.t.*

$$\boxed{||u - u_h||_U \not\to 0}$$

that is $\|T_U - T_{U,h}\|_{\mathcal{L}(U,U)} \not\to 0$

Proof.

Estimate by Qin '94 based on idea of Johnson–Pitkäranta '82 □

Raviart–Thomas scheme

General mesh (triangles, parallelograms, tetrahedrons, parallelepipeds)

Σ_h : Raviart–Thomas space of order k
U_h : $\mathcal{P}_{k-1}$ or tensor product polynomials $\mathcal{Q}_{k-1}$

Fortid

The interpolant is a Fortin operator

See also Falk–Osborn ’80

Convergence: $O(h^{2k})$ eigenvalues, $O(h^k)$ eigenfunctions