Periodic homogenization of the Prandtl-Reuss model with hardening

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Abstract

We study the n-dimensional wave equation with an elasto-plastic nonlinear stress-strain relation. We investigate the case of heterogeneous materials, i.e. x-dependent parameters that are periodic at the scale $\eta > 0$. We study the limit $\eta \to 0$ and derive the plasticity equations for the homogenized material. We prove the well-posedness for the original and the effective system with a finite-element approximation. The approximate solutions are also used in the homogenization proof which is based on oscillating test function and an adapted version of the div-curl Lemma.

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1 Introduction

We are interested in the description of deformation waves in plastic materials and in the derivation of effective (or homogenized) models. For the description of the problem we use a polygonal reference volume $\Omega \subset \mathbb{R}^n$ occupied by the plastic material, a time interval $(0,T)$ and $\Omega_T := \Omega \times (0,T)$. The dependent variables are the deformation $u : \Omega_T \to \mathbb{R}^n$ with the symmetric gradient $\nabla^s u = \frac{1}{2}(\nabla u + (\nabla u)^t) : \Omega_T \to \mathbb{R}^{n\times n}$ (strain), which we decompose into its elastic part and its plastic part $\nabla^s u = \varepsilon_e + \varepsilon_p$, and the stress tensor $\sigma : \Omega_T \to \mathbb{R}^{n\times n}$. The wave equation with density ϱ and volume load f then reads

$$\varrho \partial_t^2 u - \nabla \cdot \sigma = f \tag{1.1}$$

on Ω_T, it expresses conservation of momentum and is valid for elastic and plastic materials. Elastic materials are characterized by a linear dependence between strain $\nabla^s u$ and stress σ. Plastic materials show an elastic response for deformations $\nabla^s u$ within a subset of $\mathbb{R}^{n\times n}$, beyond the boundary of this set (the flow surface), even small forces can result in large deformations. We use the common model where the stress in the material depends only on the elastic part of the strain, and the relation is linearly determined by the compliance tensor C

$$C\sigma = \nabla^s u - \varepsilon_p. \tag{1.2}$$

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We investigate the Prandtl-Reuss model with linear hardening (see [1, 15]), which reads

$$\partial_t \varepsilon_p \in \partial\chi(\sigma - b\varepsilon_p). \tag{1.3}$$

Here $b \in \mathbb{R}_+$ is a hardening parameter, and the nonlinear function $\chi : \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{+\infty\}$ is an indicator function which is discontinuous at the flow surface.

One of our results is an existence result for the above problem, where we use a solution concept that is adapted to energy estimates. Our method of proof relies on a regularization of the equations and a finite element scheme to construct approximate solutions. Uniform estimates for the approximate solutions allow to take limits, the limit functions turn out to be strong variational solutions. Theorem 1.1 provides an existence result for the original system with variable parameters and, additionally, an existence result for the homogenized system.

Our main interest lies in the homogenization of the above equations (1.1)–(1.3). We assume that the material characteristics C, b, ϱ, and χ depend on $x \in \Omega$ in a periodic way. More precisely, for the unit cube $Y := [0, 1[^n$ with periodic identifications of the boundaries, we assume that with $C : Y \to \mathbb{R}^{n\times n}$ the first material parameter is $C_\eta(x) = C(x/\eta)$, and similarly for the other parameters. The vector of unknowns in the η-problem is

$$(u^\eta(x,t), \sigma^\eta(x,t), \varepsilon_p^\eta(x,t)) = (\text{displacement vector, stress tensor, plastic strain tensor}).$$

Our interest is to find a homogenized model that allows to calculate directly the weak limits $(u, \sigma, \varepsilon_p)$ of the above solutions for $\eta \to 0$.

The *homogenized* model has the form of a two-scale model [6], i.e. some quantities depend not only on the coarse scale parameters x and t, but additionally on the fine scale parameter $y \in Y$. The main unknowns in the homogenized problem are

$$(u(x,t), z(x,t,y), w(x,t,y)) = (\text{displacement vector, stress tensor, plastic strain tensor}).$$

The equations for the η-problem and the homogenized system are given as problems (P^η) and (P) in Subsection 1.5. With our main result, Theorem 1.2, we show that solutions of problem (P^η) converge to solutions of problem (P).

The method of proof is to start from a solution to problem (P), to construct from these functions a family of approximate solutions to problem (P^η), and to use them as test-functions in problem (P^η). An important technical problem is that system (P) does not provide the regularity that is needed to make this method rigorous. This problem was solved in [5, 24, 25] with a regularization procedure. Our approach is to use the finite element approximate solution to problem (P) for the construction of test-functions. as it was used in [28, 29].

Literature. As a general reference for plasticity equations we mention the books [1, 15], a more general treatment of hysteresis equations is given in [8, 30]. Existence results for plasticity equations can be found in all these books, additionally e.g. in [4] and in all the references below concerning homogenization. The reference [1] covers very general laws, classified as “constitutive equations of monotone type”. This class includes our problem (P^η), but not the limit problem (P).

The existence result for problem (P^η) stated in Theorem 1.1 is basically a special case of existence results proved in [17, 1, 10]. However, Theorem 1.1 is a slight improvement over these results, since the volume force f need only have one time derivative, whereas in the cited results two time derivatives of f are required.

In the case of the quasistatic problem, existence of solutions for the constitutive equations (1.1)-(1.3) has first been proven in [14, 16]. Reference [4] concerns an existence theory for the quasistatic approximation of viscoelasticity with nonlinear kinematic hardening (or no hardening).

Homogenization of the (spatially) one-dimensional case was studied in [11, 13, 29]. The one-dimensional case is much more accessible than the general case, since with the divergence of the stress all derivatives of the stress are controlled.

Homogenization of plasticity equations in higher space dimension has been treated with different techniques. The two-scale convergence method [6, 26] was employed for the quasi-static (no $\partial_t^2 u$-term)† visco-elasticity ($\chi \in C^0$) in [33], for the Kelvin-Voigt model ($\chi \in C^0$) in [32], for Maxwell and for Prandtl-Reuss without strain hardening ($b = 0$) in [34]. We note that in some publications the name "Prager model" is used instead of "Prandtl-Reuss with hardening" (e.g. in [29, 31]), and in [32], which does not contain the rigorous homogenization of that model. Gamma-convergence served the investigation of rate-independent systems in [21]. The tools of Steklov regularization and phase-shift convergence were adopted in the homogenization of quasi-static monotone constitutive equations [5, 24, 25]. We refer additionally to [2, 3] for the development of these tools.

Homogenization is also a recent subject in investigations from the engineering point of view, e.g. in [18, 19, 9]. The only references for rigorous homogenization in the stochastic case seem to be [11, 29], which are restricted to the one-dimensional case. We emphasize that, as in the contribution [28] regarding hysteresis in porous media, the homogenized system is transformed, in the one-dimensional case, into a system of simpler structure than in the case of higher dimensional plasticity equations: it is not a general two-scale system with doubled spatial variables x and y, but a system in only x and t with an averaged hysteresis operator of Prandtl-Ishlinskii type.

Our results are closest to [5] and [34], but we treat the wave equation and $b \neq 0$ (and an oscillatory density). The main distinction is that we use a different method, namely the very direct and general (compare [23]) tool of oscillating test functions developed by Tartar (e.g. appendix of [27], see also [7]). The main advantage of this method is that, in principle, it can be used also in the stochastic case, see [29].

Plan of the paper. In Subsection 1.1 we present the rheological model, discussing the structural assumptions and reviewing some classical results of convex analysis. In Subsection 1.2 we state the homogenization problem (P$^\eta$)and the limit problem (P). Subsections 1.3 and 1.4 contain the choice of boundary data and the definition of strong solutions, respectively. The main results are stated in Subsection 1.5.

In Section 2 we prove the existence and uniqueness of strong solutions to problems (P$^\eta$) and (P), as stated in Theorem 1.1. The proof of the homogenization Theorem 1.2 is given in Section 3.

1.1 The rheological model

The aim of this Section is to explain in detail the plasticity model introduced in (1.3). Under the hypothesis of small displacements, the stress tensor depends only on the linear strain, i.e. on the symmetrized gradient of the displacement $\nabla^s u := \frac{1}{2}\left(\nabla u + \nabla u^T\right)$. The linear operator C maps stress tensors to strain tensors and we therefore introduce some notation regarding tensors.

†in the description of other models of the literature we indicate in brackets the main distinction to our model.

We denote the space of second order tensors by $\mathcal{T}^2 = \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, and symmetric tensors by $\mathcal{T}^2_s$. As a scalar product on the space $\mathcal{T}^2$ we adopt the standard product $\sigma : \varepsilon = \sum_{i,j=1}^n \sigma_{ij}\varepsilon_{ij}$, with the induced norm $|\sigma|^2 = \sigma : \sigma = \sum_{i,j=1}^n \sigma_{ij}^2$. The space of fourth order tensors is denoted by $\mathcal{T}^4 := \mathcal{L}(\mathcal{T}^2, \mathcal{T}^2)$. With indices we write $C = \{c_{ij}^{kl}\} \in \mathcal{T}^4$ and, for $\xi = \{\xi_{ij}\} \in \mathcal{T}^2$, $(C\xi)_{ij} = \sum_{k,l} c_{ij}^{kl}\xi_{kl}$.

Review of convex analysis

We review some basic facts of convex analysis. We give the statements in the case when X is a separable Hilbert space, with scalar product "$\cdot$", having in mind the application $X = \mathcal{T}^2_s$. Let

$$\varphi : X \to \mathbb{R} \cup \{+\infty\}, \text{ \textit{convex} and \textit{lower-semicontinuous}, with } \varphi \not\equiv +\infty. \tag{1.4}$$

The domain of φ is

$$dom(\varphi) := \{\sigma \in X : \varphi(\sigma) < +\infty\}.$$

The Legendre-Fenchel conjugate φ^* is given by

$$\varphi^* : X \to \mathbb{R} \cup \{+\infty\}, \quad \varepsilon \mapsto \sup_{\sigma \in X}\{\varepsilon \cdot \sigma - \varphi(\sigma)\}.$$

The subdifferential $\partial\varphi : dom(\varphi) \to \mathcal{P}(X)$ is the set

$$\partial\varphi(\sigma) = \{\varepsilon \in X \text{ such that } \varphi(\xi) \geq \varphi(\sigma) + \varepsilon \cdot (\xi - \sigma) \quad \forall\, \xi \in X\}.$$

A multivalued operator $f : dom(f) \subset X \to \mathcal{P}(X)$ is said to be *monotone* if

$$(\sigma_1 - \sigma_2) \cdot (\varepsilon_1 - \varepsilon_2) \geq 0, \quad \forall\, \varepsilon_i \in dom(f), \quad \forall\, \sigma_i \in f(\varepsilon_i),\ (i = 1, 2).$$

Some useful properties of convex functions are summarized in the following lemma, for a proof we refer to [12].

Lemma 1.1. *For every φ as in* (1.4) *holds*

(i) φ^* *is convex, lower-semicontinuous, and* $dom(\varphi^*) \neq \emptyset$,

(ii) $\partial\varphi$, $\partial\varphi^*$ *are monotone operators,*

(iii) $\varphi(\sigma) + \varphi^*(\varepsilon) \geq \sigma \cdot \varepsilon, \qquad \forall\, \sigma, \varepsilon \in X$.

(iv) $\sigma \in dom(\varphi)$ *and* $\varepsilon \in \partial\varphi(\sigma) \;\Leftrightarrow\; \varepsilon \in dom(\varphi^*)$ *and* $\sigma \in \partial\varphi^*(\varepsilon)$.

(v) $\varepsilon \in dom(\varphi^*)$ *and* $\sigma \in \partial\varphi^*(\varepsilon) \;\Leftrightarrow\; \varphi(\sigma) + \varphi^*(\varepsilon) = \sigma \cdot \varepsilon$.

The equality in (v) is known as Fenchel's equality, while (iii) is referred to as Fenchel's in*equality.*

Decomposition into spheric and deviatoric part

In our setting of plasticity, the plastic response is given by a flow surface in $\mathcal{T}^2_s$, which defines the convex function χ, and by a parameter b of linear hardening. In order to describe real materials, it is important to note that the flow surface is contained in a lower dimensional subspace of $\mathcal{T}^2_s$, the deviatoric tensors. This reflects the fact that the material responds in an elastic way to deformations that correspond to purely volumetric changes. We therefore use an orthogonal decomposition of $\mathcal{T}^2_s$ into spheric and deviatoric components [15],

$$\mathcal{T}^2_s = \mathcal{S}^2 + \mathcal{D}^2, \quad \sigma = \sigma^S + \sigma^D,$$

the first component associated to volumetric changes, the latter associated to other deformations. For $\sigma \in \mathcal{T}_s^2$, the projections onto spheric part σ^S and deviatoric part σ^D are given by

$$\sigma^S := P^S(\sigma) := \frac{1}{n} tr(\sigma) I_{\mathcal{T}^2}, \qquad \sigma^D := P^D(\sigma) := \sigma - \sigma^S,$$

where $I_{\mathcal{T}^2}$ is the identity tensor in $\mathcal{T}^2$ and n is the space dimension. We note that the subspaces $\mathcal{S}^2 = P^S(\mathcal{T}_s^2)$ and $\mathcal{D}^2 = P^D(\mathcal{T}_s^2)$ are orthogonal, i.e., $\sigma^S : \sigma^D = 0$ for all $\sigma^S \in \mathcal{S}^2$ and $\sigma^D \in \mathcal{D}^2$.

The indicator function χ (including von-Mises and Tresca models)

We represent the yield criterion of the studied material by a set ω and a function χ_ω

$$\begin{aligned} &\omega \subset \mathcal{D}^2 \text{ is a bounded closed convex set}, \quad 0 \in \omega, \\ &\chi_\omega : \mathcal{T}^2 \to \mathbb{R} \cup \{+\infty\} \qquad \chi_\omega(\sigma) := \begin{cases} 0 & \text{if } \sigma^D \in \omega, \\ +\infty & \text{if } \sigma^D \notin \omega. \end{cases} \end{aligned} \tag{1.5}$$

We want to allow ω to depend on the spatial variable x. In the homogenization process we will assume that the spatial dependence is highly oscillatory, therefore the set ω is assumed to depend on the fine-scale variable y.

As a special case of our setting, we discuss here the von-Mises yield criterion with a variable positive radius $\gamma \in C(Y;\mathbb{R})$. It reads

$$\omega_M(y) := \left\{ \sigma \in \mathcal{D}^2 \text{ such that } |\sigma| \leq \gamma(y) \right\}.$$

In this model, we can compute explicitly the conjugate function and the subdifferentials, the latter are always understood as subdifferential for fixed value of $y \in Y$.

$$\begin{aligned} \partial\chi_{\omega_M}(\sigma; y) &= \left\{ \varepsilon \in \mathcal{T}_s^2 \text{ such that } \varepsilon^S = 0,\ \varepsilon : (\xi - \sigma) \leq 0, \quad \forall \xi \in \mathcal{T}_s^2 \text{ s. th. } \xi^D \in \omega_M(y) \right\} \\ &= \begin{cases} \emptyset & \text{if } \sigma^D \notin \omega_M, \\ 0 & \text{if } \sigma^D \in \omega_M \text{ and } |\sigma^D| < \gamma(y), \\ \{\lambda \sigma^D,\ \lambda \geq 0\} & \text{if } \sigma^D \in \omega_M \text{ and } |\sigma^D| = \gamma(y), \end{cases} \\ \chi^*_{\omega_M}(\varepsilon; y) &= \max_{\sigma^D \in \omega_M(y)} \{\varepsilon : \sigma\} = \begin{cases} +\infty & \text{if } \varepsilon^S \neq 0, \\ \gamma(y)|\varepsilon^D| & \text{if } \varepsilon^S = 0, \end{cases} \\ \partial\chi^*_{\omega_M}(\varepsilon; y) &= \begin{cases} \emptyset & \text{if } \varepsilon^S \neq 0, \\ \omega_M(y) & \text{if } \varepsilon^D = 0, \\ \gamma(y)\varepsilon^D/|\varepsilon^D| & \text{if } |\varepsilon^D| > 0, \end{cases} \end{aligned}$$

where we identified the one-point set $\{\gamma(y)\varepsilon^D/|\varepsilon^D|\}$ of the last line with its element.

Another frequently used model is given by the Tresca criterion

$$\omega_{\mathrm{Tr}}(y) := \left\{ \sigma \in \mathcal{D}^2 : \max_{i,j=1,\dots,n} |\lambda_i(\sigma) - \lambda_j(\sigma)| \leq \gamma(y) \right\}, \tag{1.6}$$

where $\lambda_i(\sigma)$ are the eigenvalues of σ.

Remark 1.2. An indicator function $\chi = \chi_\omega$, as in (1.5), has an important feature: even though $\partial\chi^*$ is a multivalued operator, the map $\varepsilon \mapsto \partial\chi^*(\varepsilon) : \varepsilon$ is single valued and nonnegative. In fact, owing to Lemma 1.1–(v),

$$\sigma : \varepsilon = \chi^*(\varepsilon) + \chi(\sigma) = \chi^*(\varepsilon) \geq 0, \qquad \forall\, \varepsilon \in dom(\chi^*),\ \forall\, \sigma \in \partial\chi^*(\varepsilon).$$

We note that the boundedness of ω implies that $dom(\chi^*) = \mathcal{D}^2$, and that $0 \in \omega$ implies $\chi^* \geq 0$.

We emphasize that the subdifferential inclusion of (1.3)

$$\partial_t \varepsilon_p \in \partial\chi(\sigma - b\varepsilon_p),$$

always demands, in particular, that $\sigma - b\varepsilon_p \in dom(\chi)$. It additionally demands the following. (i) for $\sigma^D - b\varepsilon_p$ inside $\omega \subset \mathcal{D}^2$, the material response is $\partial_t\varepsilon_p = 0$. (ii) on the yield surface, i.e. for $\sigma^D - b\varepsilon_p \in \partial\omega \subset \mathcal{D}^2$, the tensor $\xi := \sigma^D - b\varepsilon_p$ determines the affine material response. Loosely speaking, the flow of the plastic strain ε_p occurs in direction ξ.

Structural assumptions, positivity and continuity

In order to model highly heterogeneous media, we assume that the material parameters are given by maps $C : Y \to \mathcal{T}^4$, $b, \varrho : Y \to \mathbb{R}_+$, where $Y := [0,1[^n$ is the periodic unit cell and $\mathcal{T}_s^2$ is the space of fourth order tensors, i.e. linear maps $\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$. We assume that

$$\begin{aligned} &C \in C^0(Y;\mathcal{T}^4),\ b \in C^0(Y;\mathbb{R}_+) \text{ satisfy, for some } \alpha,\beta > 0 : \\ &C(y)\sigma : \sigma \geq \alpha|\sigma|^2, \quad b(y) \geq \beta \qquad \text{for all } y \in Y,\ \text{for all } \sigma \in \mathbb{R}^{n\times n}. \end{aligned} \tag{1.7}$$

We also demand $C_{ij}^{kl} = C_{kl}^{ij}$ in order to have $\partial_t(\sigma : C\sigma) = 2\sigma : C\partial_t\sigma$.

We assume that the convex function $\chi(\,\cdot\,; y)$ is given by

$$\begin{aligned} &\chi(\sigma\,;\,y) := \chi_{\omega(y)}(\sigma) \ \text{ with } \omega(y) \subset \mathcal{D}^2 \text{ as in (1.5)}, \\ &|\chi^*(\sigma\,;\,y_1) - \chi^*(\sigma\,;\,y_2)| \leq m(|y_2 - y_1|)|\sigma| \qquad \forall\, y_1, y_2 \in Y, \end{aligned} \tag{1.8}$$

where the continuous function m, with $m(0) = 0$, is an upper bound for the continuity modulus of $\chi^*(\sigma\,;\cdot)$. This model includes the von-Mises and the Tresca criterion for continuous $\gamma(y)$. Finally, we denote by $\varrho \in L^\infty(Y;\mathbb{R})$ the density of the material. We assume that there exists $\varrho_m > 0$ such that

$$\varrho(y) \geq \varrho_m \quad \text{a.e. in } Y. \tag{1.9}$$

We note that the positivity of all coefficients as above is crucial for our method. The positivity assumption on b restricts the method to models with kinematic hardening and excludes e.g. the case of perfect plasticity.

Regularized model

The proofs of Theorem 1.1 and of Theorem 1.2 are based on the existence of a regularization of χ. We assume that there exists a family of convex functions $\{\chi_\delta\}_\delta$, depending on $\delta \in (0,1)$ and on the parameter $y \in Y$, which satisfies the following requirements.

$$\chi_\delta : \mathcal{T}_s^2 \to \mathbb{R} \quad \text{is convex} \tag{1.10a}$$

$$\lim_{\delta\to 0} \chi_\delta(\sigma\,;\,y) = \chi(\sigma\,;\,y), \quad \forall\,\sigma \in \mathcal{T}_s^2, \tag{1.10b}$$

for all $y \in Y$. We additionally ask that $\chi_\delta \geq -c\delta$ on $\mathcal{T}_s^2$, in order to fix the rate of convergence and give an explicit bound in the a priori estimates. For example, it is possible to choose χ_δ as the Yosida transform of χ, also referred to as the *inf-convolution* of χ and $|\cdot|^2/(2\delta)$

$$\chi_\delta(\sigma) := \inf_{\xi\in\mathcal{T}_s^2} \left\{ \chi(\xi) + \frac{|\xi - \sigma|^2}{2\delta} \right\}.$$

1.2 Statement of the homogenization problems

Denoting by $\eta > 0$ the small length scale of the periodicity cells, we consider

$$C_\eta(x) = C\left(\frac{x}{\eta}\right), \quad b_\eta(x) = b\left(\frac{x}{\eta}\right), \quad \varrho_\eta(x) = \varrho\left(\frac{x}{\eta}\right), \quad \chi_\eta(\,\cdot\,;x) = \chi\left(\,\cdot\,;\frac{x}{\eta}\right). \tag{1.11}$$

We now formulate (1.1)–(1.3) for oscillatory coefficients.

Problem (P$^\eta$). Find $u^\eta : \Omega_T \to \mathbb{R}^n$ and $\sigma^\eta, \varepsilon_p^\eta : \Omega_T \to \mathcal{T}_s^2$ solving

$$\varrho_\eta \partial_t^2 u^\eta - \nabla \cdot \sigma^\eta = f, \tag{1.12a}$$
$$C_\eta \sigma^\eta = \nabla^s u^\eta - \varepsilon_p^\eta, \tag{1.12b}$$
$$\partial_t \varepsilon_p^\eta \in \partial\chi_\eta(\sigma^\eta - b_\eta \varepsilon_p^\eta). \tag{1.12c}$$

We next state the homogenized two-scale problem, which contains $y \in Y$ as an additional independent variable. An interesting aspect of the homogenized problem (P) is the additional unknown v in (1.13b), which models local variations of the displacement vector. It plays the role of a Lagrange parameter for the local incompressibility constraint (1.13d).

Limit Problem (P). For given $\bar\varrho > 0$, find $u : \Omega_T \to \mathbb{R}^n$, $v : \Omega_T \times Y \to \mathbb{R}^n$, and $w, z : \Omega_T \times Y \to \mathcal{T}_s^2$, solving

$$\bar\varrho \partial_t^2 u - \nabla \cdot \left(\int_Y z\, dy\right) = f, \tag{1.13a}$$
$$Cz = \nabla_x^s u + \nabla_y^s v - w, \tag{1.13b}$$
$$\partial_t w \in \partial\chi(z - bw\,; y), \tag{1.13c}$$
$$\mathrm{div}_y z = 0. \tag{1.13d}$$

In these equations, w is a plastic strain variable and z is a stress variable.

1.3 Initial and boundary conditions

Special care must be taken of compatibility requirements on the initial data.

Compatible initial data for problem (P$^\eta$). Initial values for problem (1.12) are given by $u_0, u_1 \in H^1(\Omega, \mathbb{R}^n)$ and $\sigma_0 \in L^2(\Omega, \mathcal{T}_s^2)$ with $\mathrm{div}\,\sigma_0 \in L^2(\Omega, \mathbb{R}^n)$, and we impose the initial and boundary conditions

$$u^\eta(\cdot, 0) = u_0, \quad \partial_t u^\eta(\cdot, 0) = u_1, \quad \sigma^\eta(\cdot, 0) = \sigma_0 \quad \text{in } \Omega \tag{1.14}$$
$$\sigma^\eta \cdot \nu = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1.15}$$

where ν is the outward-directed unit normal vector on $\partial\Omega$. In order to satisfy (1.12c) we demand that the plastic strain, according to (1.12b) initially given by

$$\varepsilon_{p,0}^\eta := \nabla^s u_0 - C_\eta \sigma_0, \tag{1.16}$$

satisfies the compatibility conditions

$$\sigma_0 - b_\eta \varepsilon_{p,0}^\eta \in dom(\chi_\eta), \qquad (\varepsilon_{p,0}^\eta)^S = 0 \qquad \text{in } \Omega. \tag{1.17}$$

It would also be possible to allow for oscillating initial displacements u_0^η. We restrict to non-oscillating for an easier form of the compatibility condition below.

Compatible initial data for problem (P). For the homogenized problem (1.13), initial and boundary conditions are given by $u_0^*, u_1^* \in H^1(\Omega, \mathbb{R}^n)$, $v_0 \in L^2(\Omega; H^1(Y, \mathbb{R}^n))$, and $z_0 \in L^2(\Omega \times Y, \mathcal{T}_s^2)$ through

$$u(\cdot, 0) = u_0^*, \quad \partial_t u(\cdot, 0) = u_1^*, \quad v(\cdot, 0) = v_0, \quad z(\cdot, 0) = z_0 \quad \text{in } \Omega \tag{1.18}$$

$$\overline{z} \cdot \nu = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1.19}$$

where we used $\bar{z}(x) = \int_Y z(x, .)$ for the Y-average. We demand $\mathrm{div}_y z_0 = 0$ and $\mathrm{div}\, \bar{z}_0 \in L^2(\Omega, \mathbb{R}^n)$ for the Y-average $\bar{z}_0(x) = \int_Y z_0(x, .)$.

With the help of (1.13b) we can extract from the above data also the initial condition for w as

$$w_0(x, y) := \nabla^s u_0^*(x) + \nabla_y^s v_0(x, y) - C(y) z_0(x, y). \tag{1.20}$$

We assume the compatibility conditions

$$z_0 - b w_0 \in dom(\chi), \qquad (w_0)^S = 0 \qquad \text{in } \Omega \times Y. \tag{1.21}$$

Relation of the initial values for original and limit problem. We demand the relations

$$u_0^* = u_0, \quad u_1^* = u_1, \quad w_0(x, y) = \nabla^s u_0(x) - C(y)\sigma_0(x). \tag{1.22}$$

Loosely speaking, we calculate w_0 from the initial data u_0 and σ_0 of the η-problem and demand that the initial data v_0 and z_0 satisfy (1.20) as in a Hopf decomposition. We emphasize that the choice of trivial initial conditions (for position, velocity, and stress) is compatible for both problems and satisfies (1.22).

1.4 Solution concepts.

We can now define a concept of strong solution for our problems.

Definition 1.3 (Strong solutions)**.** *Let $f \in H^1(0, T; L^2(\Omega))$ be given. A vector $(u^\eta, \varepsilon_p^\eta, \sigma^\eta) \in L^2(\Omega_T; \mathbb{R}^n) \times L^2(\Omega_T; \mathcal{T}_s^2)^2$ is called a* strong solution *to Problem (P$^\eta$) if the distributional derivatives satisfy*

$$\partial_t^2 u^\eta,\ \partial_t \nabla u^\eta,\ \partial_t \varepsilon_p^\eta,\ \partial_t \sigma^\eta \in L^\infty(0, T; L^2(\Omega)), \tag{1.23}$$

equations (1.12a)*,* (1.12b) *are satisfied in the sense of distributions and relation* (1.12c) *is satisfied almost everywhere in Ω_T.*

A vector $(u, v, w, z) \in L^2(\Omega_T; \mathbb{R}^n) \times L^2(\Omega_T \times Y; \mathbb{R}^n) \times L^2(\Omega_T \times Y; \mathcal{T}_s^2)^2$ is called a strong solution *to Problem (P) if the distributional derivatives satisfy*

$$\partial_t^2 u,\ \partial_t \nabla u \in L^\infty(0, T; L^2(\Omega)), \tag{1.24}$$

$$\partial_t \nabla_y v,\ \partial_t w,\ \partial_t z \in L^\infty(0, T; L^2(\Omega \times Y)), \tag{1.25}$$

equations (1.13a)*,* (1.13b)*, and* (1.13d) *are satisfied in the sense of distributions and equation* (1.13c) *is satisfied almost everywhere in $\Omega_T \times Y$.*

In both problems we additionally demand that the initial conditions are satisfied in the sense of traces, the boundary condition through the weak formulation of equations (1.12a) *and* (1.13a)*.*

Remark 1.4. Due to the regularity of strong solutions, equations (1.12b) and (1.13b) hold pointwise almost everywhere. The weak formulation of (1.12a), which contains the boundary condition of a vanishing normal component of the stress, is

$$-\int_{\Omega_T} \varrho_\eta \partial_t u^\eta \cdot \partial_t \psi + \int_{\Omega_T} \sigma^\eta : \nabla\psi = \int_{\Omega_T} f \cdot \psi, \qquad \forall \psi \in C_c^\infty((0,T)\times\overline{\Omega}).$$

The weak formulation of (1.13a) is analogous.

It will be useful to introduce a second concept of solutions. This concept of variational solutions, which was already exploited in [29], uses an energy inequality in the characterization of solutions. It will actually turn out to be equivalent to the first concept, but it can be verified more easily for weak limits. We emphasize that this use of an energy inequality was extended into a theory of energetic solutions in [20], and it was used in the analysis of elastoplasticity problems, e.g., in [31, 32, 22, 33, 34] in order to characterize *weak* variational solutions.

Definition 1.5 (Strong variational solutions)**.** *Let $f \in H^1(0,T;L^2(\Omega))$ be given. A vector $(u^\eta, \varepsilon_p^\eta, \sigma^\eta) \in L^2(\Omega_T;\mathbb{R}^n) \times L^2(\Omega_T;\mathcal{T}_s^2)^2$ is called a* strong variational solution *to Problem (P^η) if*

$(i)_\eta$ *it has the regularity of a strong solution* (1.23)*,*

$(ii)_\eta$ *it solves equations* (1.12a) *and* (1.12b) *in the distributional sense in Ω_T,*

$(iii)_\eta$ $\sigma^\eta - b_\eta \varepsilon_p^\eta \in dom(\chi_\eta)$ *a.e. in Ω_T,*

$(iv)_\eta$ *it satisfies the energy inequality*

$$\frac{1}{2}\int_\Omega \varrho_\eta |\partial_t u^\eta|^2 + b_\eta|\varepsilon_p^\eta|^2 + \sigma^\eta : C_\eta \sigma^\eta \Big|_0^t + \int_{\Omega_t} \chi_\eta^*(\partial_t \varepsilon_p^\eta) \le \int_{\Omega_t} f \cdot \partial_t u^\eta \tag{1.26}$$

for almost every $t \in (0,T)$.

A vector $(u, v, w, z) \in L^2(\Omega_T;\mathbb{R}^n) \times L^2(\Omega_T \times Y;\mathbb{R}^n) \times L^2(\Omega_T \times Y;\mathcal{T}_s^2)^2$ is called a strong variational solution *to Problem (P) if*

(i) *it has the regularity of a strong solution* (1.24)*,* (1.25)*,*

(ii) *it solves equations* (1.13a)*,* (1.13b)*, and* (1.13d) *in the distributional sense in $\Omega_T \times Y$,*

(iii) $z - bw \in dom(\chi)$*, a.e. in $\Omega_T \times Y$,*

(iv) *it satisfies the energy inequality*

$$\frac{1}{2}\left(\int_\Omega \bar{\varrho}|\partial_t u|^2 + \int_{\Omega\times Y} b|w|^2 + z : Cz\right)\Big|_0^t + \int_{\Omega_t \times Y} \chi^*(\partial_t w) \le \int_{\Omega_t} f \cdot \partial_t u. \tag{1.27}$$

for almost every $t \in (0,T)$.

Remark 1.6. Inequality (1.26) is a mathematical formulation of the *principle of maximal dissipation.* Consider the standard free energy (see, e.g., [1])

$$\psi(\nabla^s u, \varepsilon_p) := \frac{1}{2}(\nabla^s u - \varepsilon_p) : C^{-1}(\nabla^s u - \varepsilon_p) + \frac{1}{2} b\varepsilon_p : \varepsilon_p,$$

and the dissipation rate

$$\mathscr{D}(\nabla^s u, \varepsilon_p) := -\nabla_{\varepsilon_p}\psi : \partial_t \varepsilon_p = (\sigma - b\varepsilon_p) : \partial_t \varepsilon_p.$$

By Fenchel's inequality Lemma 1.1-(iii), using $\chi(\sigma - b\varepsilon_p) = 0$, there always holds

$$\mathscr{D}(\nabla^s u, \varepsilon_p) \leq \chi^*(\partial_t \varepsilon_p). \tag{1.28}$$

In this sense, $\chi^*(\partial_t \varepsilon_p)$ is the *maximal* dissipation rate. It is easy to see that the equality in (1.28) is achieved by strong solutions of equations (1.1)–(1.3), indeed (1.3) implies, by Fenchel's equality in Lemma 1.1-(v),

$$(\sigma - b\varepsilon_p) : \partial_t \varepsilon_p = \chi(\sigma - b\varepsilon_p) + \chi^*(\partial_t \varepsilon_p).$$

Since by (1.3) there holds $\sigma - b\varepsilon_p \in dom(\chi)$, and thus $\chi(\sigma - b\varepsilon_p) = 0$, we conclude

$$\mathscr{D}(\nabla^s u, \varepsilon_p) = (\sigma - b\varepsilon_p) : \partial_t \varepsilon_p = \chi^*(\partial_t \varepsilon_p).$$

Similarly, one can show for strong variational solutions, which satisfy inequality (1.26), that

$$\int_{\Omega_T} \mathscr{D}(\nabla^s u, \varepsilon) = \int_{\Omega_T} f \cdot \partial_t u - \partial_t \left(\frac{1}{2} \varrho |\partial_t u|^2 + \psi(\nabla^s u, \varepsilon_p) \right) \geq \int_{\Omega_T} \chi^*(\partial_t \varepsilon_p).$$

We conclude from (1.28) that the dissipation rate must be maximal at all times.

Lemma 1.7. *Every strong variational solution according to Definition 1.5 is also a strong solution in the sense of Definition 1.3. Vice versa, every strong solution is a strong variational solution.*

Proof. We show the result for problem (P^η), the proof for the limit problem (P) is analogous. It is easy to check that every strong solution of problem (P^η) satisfies (1.26) as an equality. We skip this calculation, which is also contained in the proof of Lemma 2.1. The regularity assumed in (1.23) is sufficient to perform all the computations rigorously. With this observation it is shown that every strong solution is a strong variational solution.

Let us prove the converse implication: for a strong variational solution $(u^\eta, \varepsilon_p^\eta, \sigma^\eta)$ of problem (P^η) we need to show that equation (1.12c) is satisfied a.e. in Ω_T. We start from the energy inequality (1.26) to calculate

$$\begin{aligned}
\int_{\Omega_t} \chi_\eta^*(\partial_t \varepsilon_p^\eta) &\leq \int_{\Omega_t} f \cdot \partial_t u^\eta - \frac{1}{2} \int_\Omega \varrho_\eta |\partial_t u^\eta|^2 + b|\varepsilon_p^\eta|^2 + \sigma^\eta : C_\eta \sigma^\eta \Big|_0^t \\
&= \int_{\Omega_t} f \cdot \partial_t u^\eta - \int_{\Omega_t} \varrho_\eta \partial_t^2 u^\eta \cdot \partial_t u^\eta + \partial_t \varepsilon_p^\eta : b_\eta \varepsilon_p^\eta + \sigma^\eta : C_\eta \partial_t \sigma^\eta \\
&\overset{(1.12a)}{=} \int_{\Omega_t} \sigma^\eta : \partial_t \nabla u^\eta - \int_{\Omega_t} \partial_t \varepsilon_p^\eta : b_\eta \varepsilon_p^\eta + \sigma^\eta : C_\eta \partial_t \sigma^\eta \\
&\overset{(1.12b)}{=} \int_{\Omega_t} \sigma^\eta : \partial_t (C_\eta \sigma^\eta + \varepsilon_p^\eta) - \int_{\Omega_t} \partial_t \varepsilon_p^\eta : b_\eta \varepsilon_p^\eta + \sigma^\eta : C_\eta \partial_t \sigma^\eta \\
&= \int_{\Omega_t} \partial_t \varepsilon_p^\eta : \left[\sigma^\eta - b_\eta \varepsilon_p^\eta \right].
\end{aligned} \tag{1.29}$$

The last integrand can be estimated by Fenchel's inequality of Lemma 1.1-(iii) as

$$\partial_t \varepsilon_p^\eta : \left[\sigma^\eta - b_\eta \varepsilon_p^\eta \right] \leq \chi_\eta^*(\partial_t \varepsilon_p^\eta) + \chi_\eta(\sigma^\eta - b_\eta \varepsilon_p^\eta), \quad \text{a.e. in } \Omega. \tag{1.30}$$

Due to property $(\mathrm{iii})_\eta$ of Definition 1.5 we have $\chi_\eta(\sigma^\eta - b_\eta \varepsilon_p^\eta) = 0$. We can therefore estimate the integrand of the right hand side of (1.29) by the integrand of the left hand side. We obtain the equality

$$\partial_t \varepsilon_p^\eta : \left[\sigma^\eta - b_\eta \varepsilon_p^\eta \right] = \chi_\eta^*(\partial_t \varepsilon_p^\eta) + \chi_\eta(\sigma^\eta - b_\eta \varepsilon_p^\eta), \quad \text{a.e. in } \Omega.$$

Properties (iv) and (v) of Lemma 1.1 imply

$$\partial_t \varepsilon_p^\eta \in \partial\chi_\eta(\sigma^\eta - b_\eta \varepsilon_p^\eta).$$

This shows (1.12c) and hence the equivalence of the solution concepts. □

1.5 Main results

We can state now the main theorems of this paper.

Theorem 1.1. *Let $f \in H^1(0,T;L^2(\Omega;\mathbb{R}^n))$ be given, and let initial data and boundary conditions be as in* (1.14)–(1.21)*. Then, with a constant c that depends on C, b, χ, ϱ, as defined in* (1.7)–(1.10)*, and on Ω_T, but not on η, we have the following existence result with uniform estimates.*

For every $\eta > 0$ there exists a unique strong solution $(u^\eta, \varepsilon_p^\eta, \sigma^\eta)$ of problem (P^η) with

$$\begin{aligned}\|u^\eta\|_{W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^n))} &+ \|\partial_t^2 u^\eta\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} \\ &+ \|\varepsilon_p^\eta\|_{W^{1,\infty}(0,T;L^2(\Omega;\mathcal{T}_s^2))} + \|\sigma^\eta\|_{W^{1,\infty}(0,T;L^2(\Omega;\mathcal{T}_s^2))} \le c\Big(\|u_0\|_{H^1(\Omega;\mathbb{R}^n)} \\ &+ \|u_1\|_{H^1(\Omega;\mathbb{R}^n)} + \|\sigma_0\|_{L^2(\Omega;\mathcal{T}_s^2)} + \|\operatorname{div}\sigma_0\|_{L^2} + \|f\|_{H^1(0,T;L^2(\Omega))}\Big).\end{aligned}$$

There exists a unique strong solution (u, v, w, z) of the homogenized problem (P) with

$$\begin{aligned}\|u\|_{W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^n))} &+ \|\partial_t^2 u\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\nabla_y v\|_{W^{1,\infty}(0,T;L^2(\Omega\times Y;\mathcal{T}_s^2))} \\ &+ \|w\|_{W^{1,\infty}(0,T;L^2(\Omega\times Y;\mathcal{T}_s^2))} + \|z\|_{W^{1,\infty}(0,T;L^2(\Omega\times Y;\mathcal{T}_s^2))} \\ &\le c\Big(\|u_0^*\|_{H^1(\Omega;\mathbb{R}^n)} + \|u_1^*\|_{H^1(\Omega;\mathbb{R}^n)} + \|v_0\|_{L^2(\Omega;H^1(Y;\mathbb{R}^n))} \\ &\quad + \|\operatorname{div}\bar z_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega\times Y;\mathcal{T}_s^2)} + \|f\|_{H^1(0,T;L^2(\Omega))}\Big).\end{aligned}$$

Theorem 1.2. *Let $(u^\eta, \varepsilon_p^\eta, \sigma^\eta)$ be a sequence of strong solutions of problems (P^η) and let (u, v, w, z) be a strong solution of problem (P), as in Theorem 1.1. Set $\bar\varrho = \int_Y \varrho$. Let the initial data be compatible in the sense of* (1.17) *and* (1.21)*, and let them be in relation* (1.22) *to each other. Then, as $\eta \to 0$,*

$$\begin{aligned} \partial_t u^\eta \to \partial_t u \qquad & \text{strongly in } L^2(\Omega_T), \\ \sigma^\eta \rightharpoonup \int_Y z\,dy, \quad \varepsilon_p^\eta \rightharpoonup \int_Y w\,dy \qquad & \text{weakly in } L^2(\Omega_T). \end{aligned}$$

2 Existence

In this section we prove Theorem 1.1 with the help of approximate equations. We replace the nonlinear function χ (which has flat parts and infinite slopes) by a regularized function χ_δ satisfying hypothesis (1.10a) and (1.10b). Furthermore, we will discretize the equations in space with a small parameter h, such that the existence to the approximate system will be a consequence of the Picard-Lindelöf theorem for ordinary differential equations. The derivation of uniform estimates for the approximate problems allows to perform the limits $\delta \to 0$ and $h \to 0$ and to find solutions to the original problems together with the same estimates.

For ease of notation we perform all calculations in this section under the hypothesis $\varrho \equiv 1$, i.e. $\varrho_\eta = \bar\varrho = 1$. In Section 3 we allow again the y-dependent density.

2.1 Regularization and a priori estimates

Let $\chi_{\delta,\eta}(\,\cdot\,) := \chi_\delta(\,\cdot\,;\ x/\eta)$, $\chi_\delta(\,\cdot\,) := \chi_\delta(\,\cdot\,;\ y)$, and denote

$$U := \Omega \times Y, \qquad U_T := \Omega \times Y \times (0,T).$$

In the statement of the regularized problems we use the dual formulation of the subdifferential inclusion, which is equivalent to the primal formulation by Lemma 1.1 (iv).

Regularized Problem ($\mathbf{P}^\eta_\delta$). Find $u^\eta_\delta : \Omega_T \to \mathbb{R}^n$ and $\sigma^\eta_\delta, \varepsilon^\eta_{p,\delta} : \Omega_T \to \mathcal{T}^2_s$ solving

$$\begin{aligned}
&\partial_t^2 u^\eta_\delta - \nabla \cdot \sigma^\eta_\delta = f\\
&C_\eta \sigma^\eta_\delta = \nabla^s u^\eta_\delta - \varepsilon^\eta_{p,\delta}.\\
&\sigma^\eta_\delta = b_\eta \varepsilon^\eta_{p,\delta} + \nabla\chi^*_{\delta,\eta}\left(\partial_t \varepsilon^\eta_{p,\delta}\right)
\end{aligned}$$

Regularized Limit Problem ($\mathbf{P}_\delta$). Find $u_\delta : \Omega_T \to \mathbb{R}^n$, $v_\delta : U_T \to \mathbb{R}^n$, and $w_\delta, z_\delta : U_T \to \mathcal{T}^2_s$, solving

$$\partial_t^2 u_\delta - \nabla \cdot \left(\int_Y z_\delta \, dy\right) = f \tag{2.1a}$$

$$C z_\delta = \nabla^s u_\delta + \nabla^s_y v_\delta - w_\delta \tag{2.1b}$$

$$z_\delta = b w_\delta + \nabla\chi^*_\delta(\partial_t b w_\delta) \tag{2.1c}$$

$$\mathrm{div}_y z_\delta = 0. \tag{2.1d}$$

The following lemma collects the energy estimates for strong solutions. It helps to identify useful function spaces for the construction of solutions. For our methods, it will be necessary to improve the estimates by one order in time, which is done in Lemma 2.2.

Lemma 2.1 (Energy estimates for the δ-regularized problem)**.** *There exists a constant $c > 0$, independent of δ and η, such that strong solutions $(u^\eta_\delta, \sigma^\eta_\delta, \varepsilon^\eta_{p,\delta})$ of (P^η_δ) satisfy*

$$\begin{aligned}
&\|\varepsilon^\eta_{p,\delta}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\sigma^\eta_\delta\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\chi^*_{\delta,\eta}(\partial_t\varepsilon^\eta_{p,\delta})\|_{L^1(\Omega_T)}\\
&\qquad + \|\chi_{\delta,\eta}(\sigma^\eta_\delta - b_\eta\varepsilon^\eta_{p,\delta})\|_{L^1(\Omega_T)} + \|\partial_t u^\eta_\delta\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u^\eta_\delta\|^2_{L^\infty(0,T;H^1(\Omega))}\\
&\qquad\qquad \le c\left(\|u_0\|^2_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)} + \|\sigma_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(U_T)} + \delta\right),
\end{aligned}$$

and solutions $(u_\delta, v_\delta, w_\delta, z_\delta)$ of (P_δ) satisfy

$$\begin{aligned}
&\|w_\delta\|^2_{L^\infty(0,T;L^2(U))} + \|z_\delta\|^2_{L^\infty(0,T;L^2(U))} + \|\chi^*_\delta(\partial_t w_\delta)\|_{L^1(U_T)} + \|\chi_\delta(z_\delta - b w_\delta)\|_{L^1(U_T)}\\
&\qquad + \|\partial_t u_\delta\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_\delta\|^2_{L^\infty(0,T;H^1(\Omega))} + \|\nabla_y v_\delta\|^2_{L^2(U_T)}\\
&\le c\left(\|u^*_0\|^2_{H^1(\Omega)} + \|u^*_1\|^2_{L^2(\Omega)} + \|v_0\|^2_{L^2(\Omega;H^1(Y))} + \|z_0\|^2_{L^2(U)} + \|f\|^2_{L^2(U_T)} + \delta\right).
\end{aligned} \tag{2.2}$$

Proof. We multiply equation (2.1a) with $\partial_t u_\delta$ and integrate over Ω to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_t u_\delta|^2 - \int_\Omega f \cdot \partial_t u_\delta &= -\int_\Omega \left(\int_Y z_\delta\, dy\right) : \partial_t \nabla^s u_\delta\\
&= -\int_U z_\delta : \partial_t \nabla^s u_\delta \overset{(2.1b)}{=} -\int_U z_\delta : \partial_t(w_\delta + C z_\delta - \nabla^s_y v_\delta)\\
&= -\int_U (z_\delta : \partial_t w_\delta) - \int_U (z_\delta : B\partial_t z_\delta) + \int_U (z_\delta : \partial_t \nabla^s_y v_\delta)\\
&=: -I_1 - I_2 + I_3.
\end{aligned} \tag{2.3}$$

Using relation (2.1c) and Fenchel's equality, we compute

$$\begin{aligned}
I_1 &= \int_U (z_\delta : \partial_t w_\delta) = \int_U \big[b w_\delta + z_\delta - b w_\delta \big] : \partial_t w_\delta \\
&= \int_U \partial_t w_\delta : b w_\delta + \int_U \big(\chi_\delta (z_\delta - b w_\delta) + \chi_\delta^*(\partial_t w_\delta) \big) \\
&= \frac{1}{2}\frac{d}{dt} \int_U w_\delta : b w_\delta + \int_U \chi_\delta(z_\delta - b w_\delta) + \int_U \chi_\delta^*(\partial_t w_\delta) \\
I_2 &= \frac{1}{2}\frac{d}{dt} \int_U z_\delta : C z_\delta \\
I_3 &= \int_\Omega \int_Y z_\delta : \partial_t \nabla_y v_\delta = \int_\Omega \int_Y \operatorname{div}_y z_\delta \cdot \partial_t v_\delta \overset{(2.1d)}{=} 0.
\end{aligned}$$

Inserting into (2.3) and integrating in time from 0 to s we find

$$\begin{aligned}
\frac{1}{2} \Big(\int_\Omega |\partial_t u_\delta|^2 + \int_U b |w_\delta|^2 + \int_U z_\delta : C z_\delta \Big) \Big|_{t=0}^{t=s} & \\
+ \int_U \chi_\delta(z_\delta - b w_\delta) + \int_U \chi_\delta^*(\partial_t w_\delta) &= \int_{\Omega_s} f \cdot \partial_t u_\delta .
\end{aligned}$$

By positivity of C and b, we can write this as an estimate in function spaces with the help of the Cauchy-Schwarz inequality.

$$\begin{aligned}
&\|\partial_t u_\delta\|^2_{L^\infty(0,T;L^2(\Omega))} + \|w_\delta\|^2_{L^\infty(0,T;L^2(U))} + \|z_\delta\|^2_{L^\infty(0,T;L^2(U))} + \|\chi_\delta^*(\partial_t w_\delta)\|_{L^1(U_T)} \\
&+ \|\chi_\delta(z_\delta - b w_\delta)\|_{L^1(U_T)} \le c \Big(\|u_1^*\|^2_{L^2(\Omega)} + \|u_0^*\|^2_{H^1(\Omega)} + \|z_0\|^2_{L^2(U)} + \|f\|^2_{L^2(\Omega_T)} + \delta \Big).
\end{aligned}$$

It remains to conclude the $H^1(\Omega)$-type estimate for u_δ and the $L^2(U)$-type estimate for $\nabla_y v_\delta$ in (2.2). We exploit relation (2.1b),

$$w_\delta(x,y) + C(y) z_\delta(x,y) = \nabla^s u_\delta(x) + \nabla^s_y v_\delta(x,y).$$

Integration over Y yields, because of Y-periodicity of v_δ and y-independence of u_δ,

$$\int_Y [w_\delta(x,y) + C(y) z_\delta(x,y)]\, dy = \nabla^s u_\delta(x),$$

such that Korn's inequality implies the $H^1(\Omega)$-bound for u_δ. This, in turn, provides the estimate for $\nabla_y v_\delta$ from relation (2.1b). $\square$

We can obtain higher order estimates by differentiating the equation with respect to time and testing with $\partial_t^2 u_\delta$. We will state and motivate the estimates here and provide the rigorous proof with a spatial discretization in the next subsection.

Lemma 2.2 (Higher order estimates for the δ-regularized problem)**.** *There exists a constant $c > 0$, independent of δ and η, such that solutions $(u_\delta^\eta, \sigma_\delta^\eta, \varepsilon_{p,\delta}^\eta)$ of (P_δ^η) satisfy*

$$\begin{aligned}
&\|\partial_t u_\delta^\eta\|^2_{L^\infty(0,T;H^1(\Omega))} + \|\partial_t^2 u_\delta^\eta\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t \varepsilon_{p,\delta}^\eta\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\quad + \|\partial_t \sigma_\delta^\eta\|^2_{L^\infty(0,T;L^2(\Omega))} \le c \left(\|\operatorname{div} \sigma_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{H^1(\Omega)} + \|f\|^2_{H^1(0,T;L^2(\Omega))} \right),
\end{aligned}$$

and solutions $(u_\delta, v_\delta, w_\delta, z_\delta)$ of (P_δ) satisfy

$$\begin{aligned}
&\|\partial_t w_\delta\|^2_{L^\infty(0,T;L^2(U))} + \|\partial_t z_\delta\|^2_{L^\infty(0,T;L^2(U))} + \|\partial_t^2 u_\delta\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&+ \|\partial_t u_\delta\|^2_{L^\infty(0,T;H^1(\Omega))} + \|\partial_t \nabla_y v_\delta\|^2_{L^\infty(0,T;L^2(U))} \\
&\quad \le c \left(\|\operatorname{div} \bar{z}_0\|^2_{L^2(\Omega)} + \|u_1^*\|^2_{H^1(\Omega)} + \|f\|^2_{H^1(0,T;L^2(\Omega))} \right).
\end{aligned}$$

For simplicity, we assume in this calculation $\chi_\delta^* \in C^2(\mathcal{D}^2)$, and note that the assumption can be relaxed to $\chi_\delta^* \in C^{1,1}(\mathcal{D}^2)$ as in (1.10a), by an argument with finite differences. We differentiate equation (2.1a) with respect to time. The resulting equation is multiplied with $\partial_t^2 u_\delta$ and integrated. We set $g := \partial_t f$ to find

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_t^2 u_\delta|^2 - \int_\Omega g\cdot \partial_t^2 u_\delta &= -\int_U \partial_t z_\delta : \partial_t^2 \nabla^s u_\delta \\
&\overset{(2.1b)}{=} -\int_U \partial_t z_\delta : \partial_t^2 (w_\delta + C z_\delta - \nabla_y^s v_\delta) \\
&= -\int_U (\partial_t z_\delta : \partial_t^2 w_\delta) - \int_U (\partial_t z_\delta : C\partial_t^2 z_\delta) + \int_U (\partial_t z_\delta : \partial_t^2 \nabla_y^s v_\delta) \\
&\overset{(2.1c)}{=} -\int_U \partial_t \big[b w_\delta + \nabla \chi_\delta^*(\partial_t w_\delta)\big] : \partial_t^2 w_\delta \\
&\quad - \frac{1}{2}\frac{d}{dt}\int_U \partial_t z_\delta : C(\partial_t z_\delta) - \int_\Omega\int_Y \partial_t(\mathrm{div}_y z_\delta)\cdot \partial_t^2 v_\delta \\
&= -\frac{1}{2}\frac{d}{dt}\int_U \partial_t w_\delta : b\partial_t w_\delta - \int_U \big[\nabla^2\chi_\delta^*(\partial_t w_\delta)\partial_t^2 w_\delta\big] : \partial_t^2 w_\delta \\
&\quad - \frac{1}{2}\frac{d}{dt}\int_U \partial_t z_\delta : C(\partial_t z_\delta).
\end{aligned}$$

By convexity of χ_δ^* (1.10a), we can estimate

$$\left(\int_\Omega |\partial_t^2 u_\delta|^2 + \int_U b|\partial_t w_\delta|^2 + \int_U \partial_t z_\delta : C(\partial_t z_\delta)\right)\bigg|_{t=0}^{t=s} \le 2\int_{\Omega_s} g\cdot \partial_t u_\delta. \tag{2.4}$$

Note that the compatibility condition (1.21) and property (1.10b) of the regularization imply that $\partial_t w_\delta|_{t=0}$ is bounded in $L^\infty(U)$, independently of δ. By equation (1.13a) we get

$$\partial_t^2 u_\delta|_{t=0} = \mathrm{div}\int_Y z_0\,dy + f|_{t=0} \in L^2(\Omega).$$

Differentiating (1.13b) with respect to time and multiplication with $\partial_t z$ provides, exploiting (1.13d),

$$\int_{\Omega\times Y} C\partial_t z : \partial_t z\,dx\,dy = \int_\Omega \nabla\partial_t u : \int_Y \partial_t z\,dy\,dx - \int_{\Omega\times Y} \partial_t w : \partial_t z\,dy\,dx,$$

and hence

$$\|\partial_t z_\delta(0)\|^2_{L^2(U)} \le c\left(\|u_1^*\|^2_{H^1(\Omega)} + \|\partial_t w_\delta(0)\|^2_{L^2(U)}\right).$$

Estimate (2.4) then yields

$$\begin{aligned}
\|\partial_t^2 u_\delta\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t w_\delta\|^2_{L^\infty(0,T;L^2(U))} + \|\partial_t z_\delta\|^2_{L^\infty(0,T;L^2(U))} \\
\le c\Big(\|\mathrm{div}\,\bar z_0\|^2_{L^2(\Omega)} + \|u_1^*\|^2_{H^1(\Omega)} + \|f\|^2_{H^1(0,T;L^2(\Omega))}\Big).
\end{aligned}$$

Treating the remaining two quantities as in the energy estimate, we find the result.

2.2 Discretization and rigorous estimates

We introduce a space-discretization of the limit system (P). Let Ω be polygonal and let

$$\mathscr{T}_h^\Omega := \{K_q\}_{q\in\Lambda_h^\Omega} \quad \text{be a subdivision of } \Omega,$$

where K_q are simplices such that $\max\{\text{diam}(K_q),\ q \in \Lambda_h^\Omega\} = h$ and Λ_h^Ω is a finite set of indexes. In the same way, we choose a triangular mesh $\mathscr{T}_\tau^Y$ of Y with maximal diameter τ. Let $\mathscr{P}_k(K)$ be the space of polynomials of degree at most $k \geq 0$ on K. Moreover, for every $q \in \Lambda_h^\Omega$, $p \in \Lambda_\tau^Y$ we can choose a point $x_q \in K_q^\circ$ (the internal part of the triangle K_q), and a point $y_p \in K_p^\circ$, for example the baricenters. We can then use the projections P_h^Ω, P_τ^Y, defined almost everywhere,

$$\begin{aligned} P_h^\Omega(x) &:= x_q \quad \text{if } x \in K_q^\circ \\ P_\tau^Y(y) &:= y_p \quad \text{if } y \in K_p^\circ \end{aligned}$$

to discretize the tensors and functions as

$$C_\tau(y) := C(P_\tau^Y(y)), \quad b_\tau(y) := b(P_\tau^Y(y)),$$

Similarly, the discretization of the regularized function χ_δ is

$$\chi_{\delta,\tau}(\xi;\, y) := \chi_\delta(\xi;\, P_\tau^Y(y)), \tag{2.5}$$

which implies $\chi_{\delta,\tau}^*(\xi;\, y) := \chi_\delta^*(\xi;\, P_\tau^Y(y))$. We define spaces of piecewise linear and piecewise constant functions as

$$PL_\tau(Y;\mathbb{R}^n) := \left\{ v \in H^1(Y;\mathbb{R}^n) : v_{|K} \in \mathscr{P}_1(K;\mathbb{R}^n),\ \forall\, K \in \mathscr{T}_\tau^Y \right\}$$

$$PC_h(\Omega;\mathcal{T}_s^2) := \{ f : L^2(\Omega;\mathcal{T}_s^2) : f_{|K} \in \mathscr{P}_0(K;\mathcal{T}_s^2),\ \forall\, K \in \mathscr{T}_h^\Omega \},$$

and will search for the approximate solution in the finite-dimensional spaces

$$\begin{aligned} U_h &:= PL_h(\Omega;\mathbb{R}^n) := \left\{ u \in H^1(\Omega;\mathbb{R}^n) : u_{|K} \in \mathscr{P}_1(K;\mathbb{R}^n)\, \forall\, K \in \mathscr{T}_h^\Omega \right\}, \\ V_{h,\tau} &:= \left\{ v \in L^2(\Omega; H^1(Y;\mathbb{R}^n)) : v_{|K} \in \mathscr{P}_0(K; PL_\tau(Y;\mathbb{R}^n))\, \forall\, K \in \mathscr{T}_h^\Omega \right\}, \\ W_{h,\tau} &:= \left\{ w : L^2(\Omega \times Y;\mathbb{R}^{n\times n}) : w_{|K_q \times K_p} \in \mathscr{P}_0(K_q \times K_p)\, \forall\, K_q \in \mathscr{T}_h^\Omega,\ K_p \in \mathscr{T}_\tau^Y \right\}. \end{aligned}$$

The discretized problems and local existence

We now define approximated problems which constitute the core of the proof of existence for the original problems. The notation is unavoidably involved, so we first summarize the employed symbols. Starting from the approximation of the homogenized problem (P), the solutions depend on three parameters.

h is the size of the mesh on Ω,
τ is the size of the mesh on Y,
δ is the regularization parameter for χ.

For every $t \in (0, T]$, the unknowns of the problem are then

$$u_{h,\tau,\delta}(t) \in U_h, \quad v_{h,\tau,\delta}(t) \in V_{h,\tau}, \quad w_{h,\tau,\delta}(t),\, z_{h,\tau,\delta}(t) \in W_{h,\tau}.$$

With the combined function space $X_{h,\tau} := U_h \times V_{h,\tau} \times W_{h,\tau} \times W_{h,\tau}$, we can now define the space-discretized approximation corresponding to problem (P). Find

$$(u_{h,\tau,\delta}, v_{h,\tau,\delta}, w_{h,\tau,\delta}, z_{h,\tau,\delta}) : [0,T] \to X_{h,\tau},$$

such that for a.e. $t \in (0,T)$, $\partial_t u_{h,\tau,\delta}, \partial_t^2 u_{h,\tau,\delta} \in U_h$, $\partial_t w_{h,\tau,\delta} \in W_{h,\tau}$, and the following system $(\mathrm{P}_{h,\tau,\delta})$ of equations is satisfied

$$\int_\Omega \partial_t^2 u_{h,\tau,\delta} \cdot \psi \, dx + \int_\Omega \left(\int_Y z_{h,\tau,\delta} \, dy \right) : \nabla\psi \, dx = \int_\Omega f \cdot \psi \, dx \quad \forall\, \psi \in U_h, \tag{2.6a}$$

$$\partial_t w_{h,\tau,\delta} = (\nabla\chi_{\delta,\tau})(z_{h,\tau,\delta} - b_\tau w_{h,\tau,\delta};\, y), \qquad \text{a.e. } (x,y) \in \Omega\times Y \tag{2.6b}$$

$$C_\tau z_{h,\tau,\delta} = \nabla^s u_{h,\tau,\delta} + \nabla_y^s v_{h,\tau,\delta} - w_{h,\tau,\delta}, \qquad \text{a.e. } (x,y) \in \Omega\times Y \tag{2.6c}$$

$$\int_\Omega \int_Y z_{h,\tau,\delta} : \nabla_y \xi \, dy \, dx = 0 \qquad \forall\, \xi \in V_{h,\tau}. \tag{2.6d}$$

We give some remarks on the above discrete system. Relation (2.6a) consists in $\dim(U_h)$-equations, and can hence be understood as an evolution equation for the vector $u_{h,\tau,\delta}$. All the functions appearing in (2.6b) and (2.6c) are in $W_{h,\tau}$, i.e. piecewise constant in x and y. The nonlinear function $\nabla\chi_{\delta,\tau}$ is the y-discretized gradient (in the matrix entry) of the regularized function of (2.5). Finally, from the number of equations, the side condition (2.6d) can determine $v_{h,\tau,\delta}(t) \in V_{h,\tau}$.

To pose the initial conditions, we denote by $\mathcal{P}(x;X)$ the orthogonal projection of x onto the space X. We ask that solutions of (2.6) satisfy the initial data

$$u_{h,\tau,\delta}(0) = \mathcal{P}(u_0^*; U_h), \qquad \partial_t u_{h,\tau,\delta}(0) = \mathcal{P}(u_1^*; U_h), \tag{2.7}$$

$$v_{h,\tau,\delta}(0) = \mathcal{P}(v_0; V_{h,\tau}), \qquad z_{h,\tau,\delta}(0) = \mathcal{P}(z_0; Z_{h,\tau}), \tag{2.8}$$

where

$$Z_{h,\tau} := \{ z \in W_{h,\tau} : z \text{ satisfies (2.6d)} \}\,.$$

As in the original problem, the initial value for $w_{h,\tau,\delta}$ is determined, through equation (2.6c), by

$$w_{h,\tau,\delta}(0) = \nabla^s u_{h,\tau,\delta}(0) + \nabla_y^s v_{h,\tau,\delta}(0) - C_\tau z_{h,\tau,\delta}(0). \tag{2.9}$$

Note that, as $h \to 0$, $P(u_0^*; U_h) \to u_0^* \in H^1(\Omega)$ (and similarly for the other initial data). Furthermore, the norms of the discrete initial data are bounded by the norms of the original initial data.

Lemma 2.3. *The space-discrete problem ($P_{h,\tau,\delta}$) of* (2.6a)-(2.6d) *can be written as a system of ordinary differential equations with Lipschitz continuous right hand side in the unknowns $u_{h,\tau,\delta} \in U_h$ and $w_{h,\tau,\delta} \in W_{h,\tau}$. It admits a unique solution satisfying*

$$\begin{aligned}&(u_{h,\tau,\delta}, \partial_t u_{h,\tau,\delta}, v_{h,\tau,\delta}, w_{h,\tau,\delta}, z_{h,\tau,\delta}) \\ &\qquad \in C^1\left([0,T]; U_h \times U_h \times PC_h(\Omega, V_\tau^Y) \times W_{h,\tau} \times W_{h,\tau}\right)\end{aligned}$$

to the initial data (2.7)-(2.9).

Proof. The evolution system is given by (2.6a) for $u_{h,\tau,\delta}$ and (2.6b) for $w_{h,\tau,\delta}$ when we insert the solutions $z_{h,\tau,\delta}$ and $v_{h,\tau,\delta}$ of the other two equations.

The first step of this procedure is to invert C_τ in order to find an explicit expression for $z_{h,\tau,\delta}$ in terms of $(u_{h,\tau,\delta}, v_{h,\tau,\delta},\, w_{h,\tau,\delta})$. Note that C_τ is invertible by positivity of C. After this first modification the system reads

$$\begin{aligned}\int_\Omega \partial_t^2 u_{h,\tau,\delta} \cdot \psi \, dx = &- \int_\Omega \left(\int_Y C_\tau^{-1} (\nabla^s u_{h,\tau,\delta} + \nabla_y^s v_{h,\tau,\delta}) \, dy \right) : \nabla\psi \, dx \\ &+ \int_\Omega \left[f \cdot \psi + \int_Y C_\tau^{-1} w_{h,\tau,\delta} \, dy : \nabla\psi \right] dx \quad \forall\, \psi \in U_h,\end{aligned} \tag{2.10}$$

$$\begin{aligned}\partial_t w_{h,\tau,\delta} = (\nabla\chi_{\delta,\tau})(C_\tau^{-1}(\nabla^s u_{h,\tau,\delta} + \nabla_y^s v_{h,\tau,\delta}) - (C_\tau^{-1} + b)w_{h,\tau,\delta};\ y)& \\ \text{for a.e. } (x,y) \in \Omega \times Y,&\end{aligned} \tag{2.11}$$

$$\begin{aligned}\int_Y C_\tau^{-1}\nabla_y^s v_{h,\tau,\delta} : \nabla_y\xi\, dy = -\int_Y C_\tau^{-1}\nabla^s u_{h,\tau,\delta} : \nabla_y\xi\, dy& \\ + \int_Y C_\tau^{-1} w_{h,\tau,\delta} : \nabla_y \xi\, dy \quad \forall\, \xi \in V_{h,\tau},\ \text{a.e. } x \in \Omega.&\end{aligned} \tag{2.12}$$

By positivity of C_τ^{-1}, relation (2.12) is, for every $x = x_q$, $q \in \Lambda_h^\Omega$, (the discretized version of) an elliptic equation on Y for the piecewise linear function $v_{h,\tau,\delta}(x,.)$. Normalizing the solution, e.g. with the condition of a vanishing average, by Lax-Milgram's Lemma the system admits a unique solution $v_{h,\tau,\delta}(t) = \mathscr{F}_v(u_{h,\tau,\delta}(t), w_{h,\tau,\delta}(t)) \in V_{h,\tau}$, with a linear solution operator $\mathscr{F}_v : U_h \times W_{h,\tau} \to V_{h,\tau}$. In the existence argument it is crucial that the construction provides functions C_τ, $\nabla^s u_{h,\tau,\delta}$, and $w_{h,\tau,\delta}$ that are piecewise constant on the triangulation of Ω. For later use we note that the norm of $\mathscr{F}_v$ only depends on the lower bound for C_τ^{-1},

$$\|\nabla_y v_{h,\tau,\delta}(t)\|_{L^2(\Omega\times Y)} \le c\left(\|\nabla^s u_{h,\tau,\delta}(t)\|_{L^2(\Omega)} + \|w_{h,\tau,\delta}(t)\|_{L^2(\Omega\times Y)}\right),$$

with c independent of the parameters h, τ, δ.

We can now insert $v_{h,\tau,\delta}(t) = \mathscr{F}_v(u_{h,\tau,\delta}(t), w_{h,\tau,\delta}(t))$ in equations (2.10) and (2.11). The result is the desired ordinary differential equation with a Lipschitz continuous right-hand side, since by (1.10a) the function $\nabla\chi_{\delta,\tau} = ((\nabla_\sigma\chi_\delta)(\,\cdot\,; y))_{h,\tau}$ is Lipschitz-continuous $\mathcal{T}_s^2 \to \mathcal{T}_s^2$. Equation (2.10) can be solved for the finitely many unknowns of $\partial_t^2 u_{h,\tau,\delta}$, since the mass matrix to piecewise linear elements on Ω is invertible.

In order to conclude the proof of the Lemma, it remains to show that the local solution can be extended to the whole interval $[0,T]$. This fact is a consequence of the time-independent L^∞ estimates on solutions, which are provided in Lemma 2.5 below. □

After having described the spatial discretization of the homogenized system, we also briefly describe the discretization of the (regularized) η-problem. Since no dependence on the y-variable appears, the discretization is much simpler. For every $t \in (0,T]$, the unknowns of the problem are

$$u_{h,\delta}^\eta(t) \in U_h, \quad \varepsilon_{p,h,\delta}^\eta(t), \sigma_{h,\delta}^\eta(t) \in PC_h(\Omega; \mathcal{T}_s^2).$$

In perfect analogy with the homogenized problem, we define the space-discretized approximation of problem (P_δ^η) which we label $(\mathrm{P}_{h,\delta}^\eta)$,

$$\int_\Omega \partial_t^2 u_{h,\delta}\cdot\psi\, dx = -\int_\Omega \sigma_{h,\delta}^\eta : \nabla\psi\, dx + \int_\Omega f\cdot\psi\, dx \qquad \forall\, \psi \in U_h, \tag{2.13a}$$

$$\partial_t \varepsilon_{p,h,\delta}^\eta = \nabla\chi_{\eta,h,\delta}\left(\sigma_{h,\delta}^\eta - b_{\eta,h}\varepsilon_{p,h,\delta}^\eta(t);\ x\right) \qquad \text{for a.e. } x \in \Omega, \tag{2.13b}$$

$$C_{\eta,h}\sigma_{h,\delta}^\eta = \nabla^s u_{h,\delta}^\eta - \varepsilon_{p,h,\delta}^\eta(t) \qquad \text{for a.e. } x \in \Omega, \tag{2.13c}$$

supplied with suitable initial data. Equation (2.13c) contains only piecewise constant functions on Ω, by positivity of C it can be solved for $\sigma_{h,\delta}^\eta$. At this point we note that the coefficient functions are obtained as the piecewise constant discretizations of the functions, e.g. $x \mapsto C(x/\eta)$, hence

$$C_{\eta,h}(x) := C_\eta(P_h^\Omega(x)) = C\left(\frac{P_h^\Omega(x)}{\eta}\right),$$

and similarly for $b_{\eta,h}$, $\chi_{\eta,h,\delta}$.

Inserting the solution $\sigma^\eta_{h,\delta}$ of (2.13c) into equations (2.13a) and (2.13b), the latter transform into an ordinary differential equation with Lipschitz continuous right hand side. Up to the uniform bounds for solutions which are provided below, we find the following result for the η-problem.

Lemma 2.4. *There exists a unique solution of problem $(P^\eta_{h,\delta})$,*

$$(u^\eta_{h,\delta}, \partial_t u^\eta_{h,\delta}, \varepsilon^\eta_{p,h,\delta}, \sigma^\eta_{h,\delta}) \in C^1([0,T]; U_h \times U_h \times PC_h(\Omega;\mathcal{T}^2_s) \times PC_h(\Omega;\mathcal{T}^2_s)).$$

Uniform estimates for the discretized problems

In order to abbreviate the statement of the uniform estimates, we set $E := W^{1,\infty}(0,T; L^2(\Omega \times Y);\mathcal{T}^2)$, and extend ∇u as constant functions to $\Omega \times Y$ by setting $\nabla u_{h,\tau,\delta}(x,y) := \nabla u_{h,\tau,\delta}(x)$.

Lemma 2.5. *There exists a constant $c > 0$, independent of T and independent of h, τ, δ, such that every solution of system $(P_{h,\tau,\delta})$ as in Lemma 2.3 satisfies*

$$\begin{aligned} \|\partial_t^2 u_{h,\tau,\delta}\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u_{h,\tau,\delta}\|_E + \|\nabla_y v_{h,\tau,\delta}\|_E + \|w_{h,\tau,\delta}\|_E + \|z_{h,\tau,\delta}\|_E \\ \le c\Big(\|u_0^*\|_{H^1(\Omega)} + \|u_1^*\|_{H^1(\Omega)} + \|v_0\|_{L^2(\Omega;H^1(Y))} \\ + \|\operatorname{div} \bar z_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega\times Y)} + \|f\|_{H^1(0,T;L^2(\Omega))} + \delta\Big). \end{aligned} \tag{2.14}$$

We recall that initial data are given by u_0^*, u_1^*, v_0, z_0 through (2.7)–(2.9)*, the right hand side be given by f.*

Proof. We note that $t \mapsto \partial_t u_{h,\tau,\delta}(t) \in U_h$ and $t \mapsto \partial_t^2 u_{h,\tau,\delta}(t) \in U_h$ have sufficient regularity in order to be used as test functions in equation (2.6a). We can therefore follow the calculations of Lemma 2.1 and of Lemma 2.2. Using $\psi = \partial_t u_{h,\tau,\delta}$ in (2.6a) we find

$$\int_\Omega \partial_t^2 u_{h,\tau,\delta} \cdot \partial_t u_{h,\tau,\delta} + \int_\Omega \int_Y z_{h,\tau,\delta}\, dy : \partial_t \nabla^s u_{h,\tau,\delta} = \int_\Omega f \cdot \partial_t u_{h,\tau,\delta}.$$

The functions $z_{h,\tau,\delta}, w_{h,\tau,\delta}$ and $\nabla u_{h,\tau,\delta}$ are piecewise constant functions on every $K_q \times K_p \in \mathscr{T}^\Omega_h \times \mathscr{T}^Y_\tau$. Therefore, the integrals below are in fact simply weighted sums.

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_t u_{h,\tau,\delta}|^2 - \int_\Omega f \cdot \partial_t u_{h,\tau,\delta} &= -\int_\Omega\int_Y z_{h,\tau,\delta} : \partial_t \nabla^s u_{h,\tau,\delta} \\ &= -\int_\Omega\int_Y z_{h,\tau,\delta} : \partial_t \Big[w_{h,\tau,\delta} + C_\tau z_{h,\tau,\delta} - \nabla^s_y v_{h,\tau,\delta}\Big] \\ &= -I_1 - I_2 + I_3. \end{aligned}$$

With the help of (2.6b) we replace $z_{h,\tau,\delta}$ in I_1 and compute with Lemma 1.1–(v)

$$\begin{aligned} I_1 &= \int_\Omega\int_Y b_\tau w_{h,\tau,\delta} : \partial_t w_{h,\tau,\delta} + \big(z_{h,\tau,\delta} - b_\tau w_{h,\tau,\delta}\big) : \partial_t w_{h,\tau,\delta} \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega\int_Y b_\tau |w_{h,\tau,\delta}|^2 + \int_\Omega\int_Y \chi_{\delta,\tau}(z_{h,\tau,\delta} - b_\tau w_{h,\tau,\delta}) + \chi^*_{\delta,\tau}(\partial_t w_{h,\tau,\delta}), \\ I_2 &= \frac{1}{2}\frac{d}{dt}\int_\Omega\int_Y z_{h,\tau,\delta} : C_\tau z_{h,\tau,\delta}\,. \end{aligned}$$

The function $\xi = \partial_t v_{h,\tau,\delta}$ is admissible in equation (2.6d) and therefore

$$I_3 = \int_\Omega \int_Y z_{h,\tau,\delta} : \nabla_y \partial_t v_{h,\tau,\delta} = 0.$$

For arbitrary $s \in (0,T)$ in the interval of existence of the discrete solution we integrate over $t \in (0,s)$ to find

$$\frac{1}{2}\Big(\int_\Omega \int_Y |\partial_t u_{h,\tau,\delta}|^2 + b_\tau |w_{h,\tau,\delta}|^2 + z_{h,\tau,\delta} : C_\tau z_{h,\tau,\delta} \Big)\Big|_{t=0}^{t=s} \\ + \int_0^s \int_\Omega \int_Y \chi_{\delta,\tau}(z_{h,\tau,\delta} - b_\tau w_{h,\tau,\delta}) + \chi^*_{\delta,\tau}(\partial_t w_{h,\tau,\delta}) = \int_{\Omega_s} f \cdot \partial_t u_{h,\tau,\delta}. \quad (2.15)$$

With Gronwall's inequality we conclude the uniform bound

$$\sup_{s \in (0,\epsilon)} \left\{ \int_\Omega \int_Y |\partial_t u_{h,\tau,\delta}(s)|^2 + |w_{h,\tau,\delta}(s)|^2 + z_{h,\tau,\delta}(s) : C_\tau z_{h,\tau,\delta}(s) \right\} \\ + \int_0^\epsilon \int_\Omega \int_Y \chi_{\delta,\tau}(z_{h,\tau,\delta} - b_\tau w_{h,\tau,\delta}) + \chi^*_{\delta,\tau}(\partial_t w_{h,\tau,\delta}) \\ \leq c \left(\|u_0^*\|^2_{H^1(\Omega)} + \|u_1^*\|^2_{L^2(\Omega)} + \|v_0\|^2_{L^2(\Omega;H^1(Y))} + \|z_0\|^2_{L^2(U)} + \|f\|^2_{L^2(\Omega_T)} \right).$$

This uniform bound is the discrete analogue of the energy estimate of Lemma 2.1. It provides, in particular, the existence of the solution to the ordinary differential equation of Lemma 2.3 on the whole interval $[0,T]$. Moreover, following the calculations of the higher order estimates of Lemma 2.2 and proceeding as above, we obtain (2.14). □

2.3 Proof of the existence theorem

We are now in the position to give the rigorous proof of the main existence theorem stated in the introduction. We will obtain the solution as a weak limit of the approximate discrete solutions. We note already here that the estimates of Lemma 2.5 carry over to the limit solution, thus providing the proof of the estimates in Lemma 2.2.

Proof of Theorem 1.1. We will show existence and uniqueness of strong solutions (u,v,w,z) of the homogenized problem (P). Once more, the proof for problem (P$^\eta$) is completely analogous, actually slightly simpler.

Existence. The approach to the existence result is very direct. We use the solution $(u_{h,\tau,\delta}, v_{h,\tau,\delta}, w_{h,\tau,\delta}, z_{h,\tau,\delta})$ of problem (P$_{h,\tau,\delta}$), which exists on $[0,T]$ by Lemma 2.3. Owing to the a priori estimates in Lemma 2.5 we find a subsequence $\{h_k, \tau_k, \delta_k\}_{k\in\mathbb{N}}$, which we relabel h, τ, δ, and a limit vector $(\nabla\bar{u}, \nabla_y\bar{v}, \bar{w}, \bar{z}) \in E^4$ such that, for all $p \in [1,\infty)$, as $(h,\tau,\delta) \to (0,0,0)$,

$$(\nabla u_{h,\tau,\delta}, \nabla_y v_{h,\tau,\delta}, w_{h,\tau,\delta}, z_{h,\tau,\delta}) \rightharpoonup (\nabla\bar{u}, \nabla_y\bar{v}, \bar{w}, \bar{z}),$$

weakly in $W^{1,p}(0,T; L^2(\Omega \times Y))$. Our aim is to show that the vector $(\bar{u}, \bar{v}, \bar{w}, \bar{z})$ is a strong variational solution of problem (P), (see Definition 1.5). Lemma 1.7 then guarantees that $(\bar{u}, \bar{v}, \bar{w}, \bar{z})$ is a strong solution.

Step 1. Properties (i) and (ii) of Definition 1.5. The estimates of Lemma 2.5 coincide exactly with the regularity requirement (i) of Definition 1.5.

Regarding the solution properties we note that we can pass directly to the limit in equation (2.6a) and find (1.13a). Similarly, taking limits in equation (2.6c) yields that $(\bar{u}, \bar{v}, \bar{w}, \bar{z})$ satisfies equation (1.13b) for a.e. $(x, y) \in \Omega \times Y$ and in the distributional sense. Analogously, (1.13d) follows from (2.6d).

Step 2. Properties (iii) and (iv) of Definition 1.5. We pass to the limit in inequality (2.15). For every $y \in Y$, $\forall \tau > 0$ the maps

$$\xi \mapsto |\xi|^2, \quad \xi \mapsto b_\tau |\xi|^2, \quad \xi \mapsto \xi : C_\tau \xi,$$

are convex, and therefore lower-semicontinuous w.r.t. weak convergence in $L^2(\Omega \times Y)$. Since (1.7) implies that $b_\tau \to b$, and $C_\tau \to C$, uniformly in Y as $\tau \to 0$, we deduce that for a.e. $s \in (0, T)$

$$\begin{aligned} \liminf_{h,\tau,\delta \to 0} \Big(\int_\Omega \int_Y |\partial_t u_{h,\tau,\delta}|^2 + b_\tau |w_{h,\tau,\delta}|^2 + z_{h,\tau,\delta} : C_\tau z_{h,\tau,\delta} \Big) \Big|_{t=0}^{t=s} \\ \geq \Big(\int_\Omega \int_Y |\partial_t \bar{u}|^2 + b |\bar{w}|^2 + \bar{z} : C \bar{z} \; dx \, dy \Big) \Big|_{t=0}^{t=s}. \end{aligned} \tag{2.16}$$

Regarding the convergence of $\chi^*_{\delta,\tau}$, we start by state the following lemma.

Lemma 2.6 (Lower-semicontinuity of χ^*_δ)**.** *Let $U_s := \Omega \times Y \times (0, s)$. Let $u_\delta, u \in L^2(U_s)^n$, such that $u_\delta \rightharpoonup u$ weakly in $L^2(U_s)^n$, as $\delta \to 0$. Assume that χ_δ, χ satisfy assumptions* (1.10a) *and* (1.10b)*. Then*

$$\liminf_{\delta \to 0} \int_{U_s} \chi^*_\delta(u_\delta) \geq \int_{U_s} \chi^*(u).$$

Proof. First let us define $(\chi^*)^m$ as the maximum of finitely many affine functions:

$$(\chi^*)^m(p) = \max \left\{ p : \sigma^i - \chi(\sigma^i), \quad \sigma^i \in dom(\chi), \quad i = 1, \dots, m \right\} \qquad \text{for all } p \in \mathcal{T}^2_s.$$

Define

$$E_i := \left\{ (x, y, t) \in U_s : \quad (\chi^*)^m(u(x, y, t)) = u : \sigma^i - \chi(\sigma^i) \right\}.$$

Then, since $u_\delta \rightharpoonup u$ by hypothesis, $\chi_\delta(\sigma^i) \to \chi(\sigma^i)$ by (1.10b), and by the definition of Legendre transform

$$\liminf_{\delta \to 0} \int_{U_s} \chi^*_\delta(u_\delta) \geq \liminf_{\delta \to 0} \sum_{i=1}^m \int_{E_i} u_\delta : \sigma^i - \chi_\delta(\sigma^i) = \sum_{i=1}^m \int_{E_i} u : \sigma^i - \chi(\sigma^i) = \int_{U_s} (\chi^*)^m(u).$$

Finally we can write $\chi^*(p) = \lim_{m \to \infty} (\chi^*)^m(p)$, and pass to the limit by the Monotone Convergence theorem. □

Recalling hypothesis (1.8) we compute

$$\begin{aligned} &\int_{U_s} \chi^*_{\delta,\tau}(\partial_t w_{h,\tau,\delta}) \\ &\quad = \int_{U_s} \Big[\chi^*_{\delta,\tau}(\partial_t w_{h,\tau,\delta}) - \chi^*_\tau(\partial_t w_{h,\tau,\delta}) + \chi^*_\tau(\partial_t w_{h,\tau,\delta}) - \chi^*(\partial_t w_{h,\tau,\delta}) + \chi^*(\partial_t w_{h,\tau,\delta}) \Big] \\ &\quad \geq \int_{U_s} \Big[\chi^*_{\delta,\tau}(\partial_t w_{h,\tau,\delta}) - \chi^*_\tau(\partial_t w_{h,\tau,\delta}) \Big] - c m(\tau) \int_{U_s} |\partial_t w_{h,\tau,\delta}| + \int_{U_s} \chi^*(\partial_t w_{h,\tau,\delta}), \end{aligned}$$

where $U_s = \Omega \times Y \times (0, s)$. Applying Lemma 2.6 to $\chi^*_{\delta,\tau}(\partial_t w_{h,\tau,\delta})$, and using estimate (2.14) and convexity of χ^* we conclude

$$\liminf_{h,\tau,\delta\to 0} \int_{U_s} \chi^*_{\delta,\tau}(\partial_t w_{h,\tau,\delta}) \geq \int_{U_s} \chi^*(\partial_t \bar{w}). \tag{2.17}$$

Collecting inequalities (2.16), (2.17), and taking the liminf in (2.15), we get

$$\frac{1}{2}\Big(\int_\Omega \int_Y |\partial_t \bar{u}|^2 + b|\bar{w}|^2 + \bar{z} : C\bar{z}\Big)\Big|_{t=0}^{t=s} + \int_{U_s} \chi^*(\partial_t \bar{w}) \leq \int_{\Omega_s} f \cdot \partial_t \bar{u}$$

which is the energy inequality (1.27). Thus property (iv) is satisfied. Let $\chi_k := \chi_{\delta_k,\tau_k}$, $g_k := z_{h_k,\tau_k,\delta_k} - b_{\tau_k} w_{h_k,\tau_k,\delta_k}$, and let $g := \bar{z} - b\bar{w}$ be the weak limit of g_k in $L^2(U_s)$. Let $B_\rho(y) := \{\sigma \in \mathcal{D}^2 : d(\sigma, \omega(y)) \leq \rho\}$ and let $\psi^\rho_y : \mathcal{T}^2_s \to \mathbb{R}$ be given by $\psi^\rho_y(\sigma) := d(B_\rho(y), \sigma^D)$. By (1.10b) we can find a monotone function $\lambda : \mathbb{R} \to \mathbb{R}^+$:

$$\lim_{k\to\infty} \lambda(k) = +\infty \qquad \text{and} \qquad \lambda(k)\psi^\rho_y(\sigma) \leq \chi_k(\sigma\,;\, y) \quad \forall\, \sigma \in \mathcal{T}^2_s,\ y \in Y. \tag{2.18}$$

In order to prove property (iii) in Definition 1.5 we show that

$$\int_{U_s} \chi_k(g_k) \leq c \quad \forall\, k \in \mathbb{N} \ \Rightarrow\ g(x, y, t) \in dom(\chi(y)) \text{ for a.e. } (x, y, t) \in U_s,$$

by proving that $\psi^\rho_y(g(x, y, t)) = 0$ for a.e. $(x, y, t) \in U_s$, $\forall\, \rho > 0$. By convexity of $\psi^\rho_y(\cdot)$, and continuity of $y \mapsto \psi^\rho_y$ we obtain

$$\int_{U_s} \psi^\rho_y(g) \leq \liminf_{k\to\infty} \int_{U_s} \psi^\rho_y(g_k) \overset{(2.18)}{\leq} \liminf_{k\to\infty} \frac{1}{\lambda(k)} \int_{U_s} \chi_k(g_k\,;\, y) \overset{(2.15)}{\leq} \liminf_{k\to\infty} \frac{c}{\lambda(k)} = 0.$$

By arbitrariety of $\rho > 0$, we conclude that $g \in dom(\chi(y))$ a.e. in U_s.

Uniqueness. Let (u_i, v_i, w_i, z_i), $i = 1, 2$, be two strong solutions of Problem (P), with the same initial data, and let $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z}) := (u_1, v_1, w_1, z_1) - (u_2, v_2, w_2, z_2)$ be their difference. By equation (1.13a), orthogonality of $\tilde{z}$ and $\nabla_y \tilde{v}$, and (1.13c), we calculate

$$\begin{aligned}
\int_\Omega \partial^2_t \tilde{u} \cdot \partial_t \tilde{u} &= -\int_\Omega \left(\int_Y \tilde{z}\right) : \nabla^s \partial_t \tilde{u} = -\int_{\Omega\times Y} \tilde{z} : \nabla^s \partial_t \tilde{u} \\
&= -\int_{\Omega\times Y} \tilde{z} : \partial_t\big(\tilde{w} + C\tilde{z} - \nabla_y \tilde{v}\big) \in -\int_{\Omega\times Y} \tilde{z} : C\partial_t \tilde{z} \\
&\quad - \int_{\Omega\times Y} (\partial\chi^*(\partial_t w_1) - \partial\chi^*(\partial_t w_2)) : \partial_t(w_1 - w_2) - \int_{\Omega\times Y} b\tilde{w} : \partial_t \tilde{w}.
\end{aligned}$$

Monotonicity of $\partial\chi^*$ implies $\partial_t \tilde{u} = \tilde{w} = \tilde{z} = 0$, which also provides $\tilde{v} = 0$. □

Partial limit of the discretized solutions as $\tau, \delta \to 0$

In order to prove existence for solutions of problem (P), we performed the limit as all the parameters $h, \tau, \delta \to 0$ simultaneously. For the purpose of homogenization we need a test function which

- is regular enough to be evaluated in $(x, y) = (x, x/\eta)$;
- is *exactly* div_y-free (in the component approximating the stress);

- solves a suitable approximation of problem (P).

We choose to take the limit as $\tau \to 0$ and $\delta \to 0$, leaving h as a positive parameter related to the spatial discretization on Ω. Let $(u_{h,\tau,\delta}, v_{h,\tau,\delta}, w_{h,\tau,\delta}, z_{h,\tau,\delta})$ be the solution of problem $(\mathrm{P}^{\eta}_{h,\tau,\delta})$ found in Lemma 2.3. By compactness we can find a subsequence $(\tau_k, \delta_k) \to 0$ and a limit $(u_h, v_h, w_h, z_h) \equiv (u_{h,0,0}, v_{h,0,0}, w_{h,0,0}, z_{h,0,0})$ such that

$$(\nabla u_{h,\tau_k,\delta_k}, v_{h,\tau_k,\delta_k}, w_{h,\tau_k,\delta_k}, z_{h,\tau_k,\delta_k}) \rightharpoonup (\nabla u_h, v_h, w_h, z_h)$$

weakly in the topology of $W^{1,\infty}(0,T; L^2(\Omega \times Y))^4$. with the function spaces

$$\begin{aligned} V_h &:= \left\{ v \in L^2(\Omega; H^1(Y;\mathbb{R}^n)) : v_{|K} \in \mathscr{P}_0(K; H^1(Y;\mathbb{R}^n)) \ \forall K \in \mathscr{T}_h^{\Omega} \right\}, \\ W_h &:= \left\{ w : L^2(\Omega; L^2(Y;\mathbb{R}^{n\times n}) : w_{|K} \in \mathscr{P}_0(K; L^2(Y;\mathbb{R}^{n\times n})) \ \forall K \in \mathscr{T}_h^{\Omega} \right\}. \end{aligned}$$

we find the following existence result.

Lemma 2.7. *Every weak limit (u_h, v_h, w_h, z_h) solves the following system (P_h)*

$$\int_\Omega \left(\int_Y z_h \, dy \right) : \nabla\psi = \int_\Omega (f - \partial_t^2 u_h) \cdot \psi \qquad \forall\, \psi \in U_h \tag{2.19a}$$

$$\partial_t w_h \in \partial\chi(z_h - b w_h;\, y) \qquad \textit{for a.e.}\, (x,y) \in \Omega \times Y \tag{2.19b}$$

$$C z_h = \nabla^s u_h + \nabla^s_y v_h - w_h \qquad \textit{for a.e.}\, (x,y) \in \Omega \times Y \tag{2.19c}$$

$$\int_Y z_h : \nabla\xi = 0, \qquad \forall\, \xi \in H^1(Y),\ \textit{for a.e.}\, x \in \Omega, \tag{2.19d}$$

with initial conditions

$$u_h(0) = \mathcal{P}(u_0^*; U_h),\ \partial_t u_h(0) = \mathcal{P}(u_1^*; U_h),\ v_h(0) = \mathcal{P}(v_0; V_h),\ z_h(0) = \mathcal{P}(z_0; W_h). \tag{2.20}$$

It satisfies an a priori estimate as in (2.14)*.*

As above, equation (2.19c) allows to extract the initial condition

$$w_{h,0}(x,y) := \nabla^s \mathcal{P}(u_0^*; U_h)(x) + \nabla^s_y \mathcal{P}(v_0; V_h)(x,y) - C(y)\mathcal{P}(z_0; W_h)(x,y). \tag{2.21}$$

Proof. We only sketch the proof which is similar to that of Theorem 1.1. The a priori estimate (2.14) allows to select a weakly convergent subsequence, with the weak limit (u_h, v_h, w_h, z_h) satisfying the same estimates. It is straightforward to conclude from equations (2.6a), (2.6c), and (2.6d) for the h, τ, δ-solutions equations (2.19a), (2.19c), and (2.19d) for (u_h, v_h, w_h, z_h).

Passing to the liminf as $\tau, \delta \to 0$ in inequality (2.15) we obtain

$$\frac{1}{2}\Big(\int_{\Omega\times Y} |\partial_t u_h|^2 + b|w_h|^2 + z_h : C z_h \Big)\Big|_{t=0}^{t=s} + \int_0^s \int_{\Omega\times Y} \chi(z_h - b w_h) + \chi^*(\partial_t w_h) \le \int_{\Omega_s} f \cdot \partial_t u_h.$$

We can once more argue by the equivalence of strong solutions and strong variational solutions to conclude that $\partial_t w_h \in \partial\chi(z_h - b w_h;\, y)$, a.e. in $\Omega \times Y \times (0,T)$. We recall that in this formula all functions are piecewise constant in Ω. $\square$

3 Homogenization

This Section is devoted to the proof of Theorem 1.2, which provides, in particular, the convergence $u^\eta \to u$. We state below with Proposition 3.1 an intermediate result which compares u^η with the solution u_h of the discretized problem. The theorem is an immediate consequence of the proposition.

In the formulation and in the proof of the proposition, the fundamental tool is to construct from multi-scale solutions such as $w(x,y,t)$ oscillating functions on Ω_T. We denote the resulting oscillatory function with a lower index η. To be precise, for any function g defined on $\Omega \times Y \times (0,T)$, we set

$$g_\eta = g_\eta(x,t) := g(x, \tfrac{x}{\eta}, t).$$

Proposition 3.1. *Let $(u^\eta, \varepsilon_p^\eta, \sigma^\eta)$ be a sequence of strong solutions to problem (P^η) in* (1.12)*. Let (u_h, v_h, w_h, z_h) be a sequence of semi-discrete solutions to problem (P_h) with parameter $h > 0$ in* (2.19)*. Then*

$$\lim_{h\to 0}\lim_{\eta\to 0} \|\partial_t u^\eta - \partial_t u_h\|_{L^\infty(0,T;L^2(\Omega))} = 0,$$
$$\lim_{h\to 0}\lim_{\eta\to 0} \|\varepsilon_p^\eta - w_{h,\eta}\|_{L^\infty(0,T;L^2(\Omega))} = 0,$$
$$\lim_{h\to 0}\lim_{\eta\to 0} \|\sigma^\eta - z_{h,\eta}\|_{L^\infty(0,T;L^2(\Omega))} = 0.$$

The proof of the proposition is given in Subsection 3.3.

We note here that Theorem 1.2 follows easily from Proposition 3.1. We use the triangle inequality

$$\|\partial_t u^\eta - \partial_t u\|_{L^2(\Omega_T)} \le \|\partial_t u^\eta - \partial_t u_h\|_{L^2(\Omega_T)} + \|\partial_t u_h - \partial_t u\|_{L^2(\Omega_T)},$$

and notice that the first term of the right-hand side vanishes for $\eta \to 0$ and $h \to 0$ by Proposition 3.1, and that the last term vanishes for $h \to 0$ due to estimates (2.14) and the compact embedding $H^1(\Omega_T) \subset L^2(\Omega_T)$.

Similarly, concerning the weak convergence of the strains, we write for an arbitrary $\phi \in L^2(\Omega_T)$

$$\left| \int_{\Omega_T} \left(\varepsilon_p^\eta - \int_Y w \right) \phi \right| \le \left| \int_{\Omega_T} \left(\varepsilon_p^\eta - w_{h,\eta} \right) \phi \right| + \left| \int_{\Omega_T} \left(w_{h,\eta} - \int_Y w \right) \phi \right|. \tag{3.1}$$

The convergence to 0 of the first term of the right hand side is stated in the proposition. Concerning the second term we recall that w_h is piecewise constant in x, the values in the grid points are functions in $W^{1,\infty}(0,T;L^2(Y))$. The regularity in x allows to calculate the weak limits of the corresponding oscillating functions in the classical way as averages. Furthermore, averages of w_h converge to averages of w.

$$w_{h,\eta} \rightharpoonup \int_Y w_h \, dy \text{ in } L^2(\Omega_T) \text{ for } \eta \to 0, \quad \int_Y w_h \, dy \rightharpoonup \int_Y w \, dy \text{ in } L^2(\Omega_T) \text{ for } h \to 0.$$

Therefore, with respect to the weak topology of $L^2(\Omega_T)$, we may write

$$\lim_{h\to 0} \left(\lim_{\eta \to 0} w_{h,\eta} \right) = \int_Y w \, dy.$$

This provides the convergence of the last term in (3.1). The weak convergence of σ^η to its Y-average is calculated in exactly the same way.

Before giving the proof of Proposition 3.1, we provide a short computation, with the intent of showing the core of the homogenization procedure.

3.1 Homogenization with appropriate test-functions

In the following remark we present the estimate that is the key of the homogenization result. It is only formal, since expressions such as

$$(\partial_t w)_\eta(x) = \partial_t w\left(x, \frac{x}{\eta}, t\right)$$

are used. Since $\partial_t w$ is only of the quality $L^2(\Omega_T \times Y)$, this function need not even be measurable (see e.g. [6]). In that sense, the result of the remark is only reflecting a formal calculation.

Remark 3.2. Let $(u^\eta, \varepsilon_p^\eta, \sigma^\eta)$ be a solution of Problem (P $^\eta$) in (1.12) and let (u, v, w, z) be a solution of Problem (P) in (1.13). Denote

$$\varrho_\eta = \varrho\left(\frac{x}{\eta}\right), \quad w_\eta = w\left(x, \frac{x}{\eta}, t\right), \quad z_\eta = z\left(x, \frac{x}{\eta}, t\right), \quad (\nabla_y v)_\eta = (\nabla_y v)\left(x, \frac{x}{\eta}, t\right).$$

Then, assuming that all involved functions exist in appropriate L^2 spaces, there holds

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left\{\varrho_\eta|\partial_t u^\eta - \partial_t u|^2 + (\sigma^\eta - z_\eta) : C_\eta(\sigma^\eta - z_\eta) + b|\varepsilon_p^\eta - w_\eta|^2\right\} \\
&\quad \le -\int_\Omega \left[z_\eta - \int_Y z\right] \cdot \partial_t \nabla^s(u^\eta - u) - \int_\Omega (\sigma^\eta - z_\eta) : \partial_t(\nabla_y v)_\eta \\
&\qquad + \int_\Omega \left[\varrho_\eta - \int_Y \varrho\right] \partial_t^2 u \cdot (\partial_t u^\eta - \partial_t u).
\end{aligned} \tag{3.2}$$

Proof. Let $\bar\varrho = \int_Y \varrho\, dy$. We examine the following expression.

$$E_\eta := \int_\Omega (\varrho_\eta \partial_t^2 u^\eta - \bar\varrho \partial_t^2 u) \cdot \partial_t(u^\eta - u) + \int_\Omega (\sigma^\eta - z_\eta) : \partial_t(\nabla^s u^\eta - \nabla^s u - (\nabla_y v)_\eta).$$

Using equation (1.12b) to replace $\nabla^s u^\eta$ and (1.13b) to replace $(\nabla^s u + (\nabla_y v)_\eta)$, we compute

$$\begin{aligned}
E_\eta &= \frac{1}{2}\frac{d}{dt}\int_\Omega \varrho_\eta|\partial_t u^\eta - \partial_t u|^2 + \int_\Omega (\varrho_\eta - \bar\varrho)\partial_t^2 u \cdot (\partial_t u^\eta - \partial_t u) \\
&\quad + \int_\Omega (\sigma^\eta - z_\eta) : \partial_t\left[\varepsilon_p^\eta + C_\eta \sigma^\eta - (w_\eta + C_\eta z_\eta)\right] \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega \varrho_\eta|\partial_t u^\eta - \partial_t u|^2 + \int_\Omega (\varrho_\eta - \bar\varrho)\partial_t^2 u \cdot (\partial_t u^\eta - \partial_t u) \\
&\quad + \int_\Omega (\sigma^\eta - z_\eta) : \partial_t\left(\varepsilon_p^\eta - w_\eta\right) + \int_\Omega (\sigma^\eta - z_\eta) : C_\eta \partial_t(\sigma^\eta - z_\eta).
\end{aligned}$$

Denoting the integrals on the right-hand side by I_1 to I_4, we note that I_1 and I_4 are time derivatives of positive quantities. In order to treat I_3 we recall that the inverse relations to (1.13c) and (1.12c) are

$$z_\eta \in b_\eta w_\eta + \partial\chi_\eta^*(\partial_t w_\eta), \qquad \sigma^\eta \in b_\eta \varepsilon_p^\eta + \partial\chi_\eta^*(\partial_t \varepsilon_p^\eta),$$

so that I_3 becomes

$$\begin{aligned}
\int_\Omega (\sigma^\eta - z_\eta) : \partial_t(\varepsilon_p^\eta - w_\eta) &\in \int_\Omega b_\eta(\varepsilon_p^\eta - w_\eta) : \partial_t(\varepsilon_p^\eta - w_\eta) \\
&\quad + \int_\Omega (\partial\chi_\eta^*(\partial_t\varepsilon_p^\eta) - \partial\chi_\eta^*(\partial_t w_\eta)) : \partial_t(\varepsilon_p^\eta - w_\eta).
\end{aligned}$$

This expression is the time derivative of a positive quantity plus a non-negative term, owing to monotonicity of $\partial\chi_\eta^*$. The integral I_2 remains as an error term on the right hand side of (3.2).

We note that E_η is constructed as a difference of similar equations, tested by the time derivative of a solution difference, with the addition of a term containing $\nabla_y v$. In fact, we have, by (1.12a) and (1.13a),

$$\begin{aligned} E_\eta &= \int_\Omega \Big[\big(\varrho_\eta \partial_t^2 u^\eta - \operatorname{div}\sigma^\eta\big) - \big(\bar\varrho \partial_t^2 u - \operatorname{div} z_\eta\big) \Big] \cdot \partial_t(u^\eta - u) - \int_\Omega (\sigma^\eta - z_\eta) : \partial_t(\nabla_y^s v)_\eta \\ &= \int_\Omega \left[f - f - \operatorname{div} \int_Y z + \operatorname{div} z_\eta \right] \cdot \partial_t(u^\eta - u) - \int_\Omega (\sigma^\eta - z_\eta) : \partial_t(\nabla_y^s v)_\eta \\ &= - \int_\Omega \left[z_\eta - \int_Y z \right] \cdot \partial_t \nabla^s (u^\eta - u) - \int_\Omega (\sigma^\eta - z_\eta) : \partial_t(\nabla_y^s v)_\eta . \end{aligned}$$

Collecting the various terms implies (3.2). □

In order to conclude the homogenization result from a result as in Remark 3.2, we must derive the smallness of the right hand side of (3.2). Concerning the first integral we note that — if z has some regularity —

$$z_\eta \rightharpoonup \int_Y z(y)\, dy \quad \text{weakly in } L^2(\Omega_T). \tag{3.3}$$

Also the second factor in the first integral has some weak limit. In order to deal with the product of two weakly convergent sequences, the idea is to use the *div-curl* Lemma in order to pass to the two weak limits under the integral. If an appropriate *div-curl* Lemma can be applied, we find a vanishing limit of the integral in the limit $\eta \to 0$.

Concerning the second integral we note that — if $\nabla_y v$ is regular enough —

$$(\nabla_y v)_\eta \rightharpoonup \int_Y \nabla_y v(y)\, dy = 0 \quad \text{weakly in } L^2(\Omega_T). \tag{3.4}$$

Since the first factor of the second integral has a weak limit, an application of a suitable *div-curl* Lemma can show that also second integral vanishes for $\eta \to 0$.

The vanishing limit of the third term is immediate, since $\varrho \in C^0$ ensures $\varrho_\eta \rightharpoonup \bar\varrho$, weakly in any $L^p(\Omega_T)$, while the second factor converges strongly.

This concludes the homogenization limit, since the left hand side of (3.2) controls differences between η-solutions and homogenized solutions.

Obstacles to the rigorous justification. We have to overcome the following difficulties. We need

i) some regularity of $z(\cdot, t, y)$, in the calculation of Remark 3.2 and for (3.3)

ii) some regularity of $\nabla_y v(\cdot, t, y)$, in order for (3.4) to hold.

iii) to analyze the divergence of $z_\eta - \int_Y z$

iv) to analyze the curl of $\nabla_y v$

v) a div-curl lemma with boundary

We solve problems i) and ii) by analyzing a discretized problem and using the test functions z_h and v_h and, correspondingly, the oscillating functions $z_{h,\eta}$ and $(\nabla_y v_h)_\eta$. In order to exploit (2.19d) which provides a relation for test functions in U_h, we will introduce projections.

Problem v) corresponds to the fact that the weak convergence of the integrand to 0 does not imply the convergence of the integral to 0, since concentration effects may occur along the boundary. We solve problems iii) and v) simultaneously with the *div-curl* type Lemma 3.3. Problem iv) is solved by a variant of that lemma, formulated as Lemma 3.4. The core of both lemmata is that concentration effects are ruled out by the periodicity of z_η and that of $(\nabla_y v)_\eta$. In the application of Lemma 3.3 we insert $z_h(x,t,.)$ for the function u, while in the application of Lemma 3.4 we use $(\partial_t v_h)_\eta$ for χ_η.

3.2 The div-curl lemma with boundary

Lemma 3.3 (div-curl Lemma with boundary)**.** *Let $T \subset \mathbb{R}^m$ be an open and bounded set, with Lipschitz boundary ∂T, and let $Y := [0,1[^m$ denote the flat torus. Assume*

$$u \in L^2(Y;\mathbb{R}^m) \quad \textit{with} \quad \operatorname{div} u = 0 \ \textit{in}\ \mathcal{D}'(Y), \tag{3.5}$$

$$\varphi^\eta,\, \varphi \in H^1(T;\mathbb{R}) \quad \textit{with} \quad \varphi^\eta \rightharpoonup \varphi \ \textit{in}\ H^1(T;\mathbb{R}). \tag{3.6}$$

Then

$$\lim_{\eta\to 0} \int_T u\big(\tfrac{x}{\eta}\big)\cdot \nabla\varphi^\eta(x)\,dx = \int_T \bar u \cdot \nabla\varphi(x)\,dx,$$

where $\bar u := \displaystyle\int_Y u(y)\,dy$*.*

Proof. The result is clearly true for constant functions u. By linearity of the expressions it is therefore sufficient to show the result for functions with vanishing average. In the following we hence analyze oscillatory functions $u_\eta(x) := u(\frac{x}{\eta})$ with $\bar u = 0$.

Step 1. Boundary layer. Let $\delta > 0$ be small. We consider a tubular δ-neighborhood V_δ of the boundary ∂T,

$$V_\delta := B_\delta(\partial T)\cap T := \{x \in T : d(x,\partial T) < \delta\}.$$

We divide $\mathbb{R}^m$ into d-cubes of size η

$$Y_k^\eta := \eta(Y+k), \qquad \mathcal{L}^m(Y_k^\eta) = \eta^m, \quad \forall\, k \in \mathbb{Z}^m.$$

It is useful to define a suitable covering of V_δ by cubes. Precisely, let us define the set of indices $I_\delta^\eta := \{k \in \mathbb{Z}^m \ :\ Y_k^\eta \cap V_\delta \neq \emptyset\}$, then

$$V_\delta \subset \bigcup_{k\in I_\delta^\eta} Y_k^\eta =: V_\delta^\eta.$$

Denoting by $\mathcal{L}^m$ the m-dimensional Lebesgue measure, by Lipschitz regularity of ∂T, we have

$$\mathcal{L}^m(V_\delta) \le \mathcal{L}^m(V_\delta^\eta) \le C(\delta+\eta). \tag{3.7}$$

The small volume of V_δ^η together with the periodicity of u_η implies that there exists $C > 0$, independent of δ and η, such that

$$\|u_\eta\|^2_{L^2(V_\delta)} \le C\big(\delta+\eta\big), \quad \forall\,\delta>0,\ \forall\,\eta>0. \tag{3.8}$$

Indeed,

$$\int_{V_\delta} |u_\eta(x)|^2\,dx \le \sum_{k\in I_\delta^\eta} \int_{Y_k^\eta} \left|u\left(\tfrac{x}{\eta}\right)\right|^2 dx = \eta^m \sum_{k\in I_\delta^\eta} \int_Y |u(y)|^2\,dy$$
$$= \|u\|_{L^2(Y)}^2 \sum_{k\in I_\delta^\eta} \mathcal{L}^m(Y_k^\eta) = \|u\|_{L^2(Y)}^2\, \mathcal{L}^m(V_\delta^\eta).$$

Step 2. Convergence. With the Euclidean distance d in $\mathbb{R}^m$ we define the Lipschitz-continuous cut-off function $\psi : T \to \mathbb{R}$

$$\psi(x) := \min\left\{\frac{1}{\delta} d(x, \partial T), 1\right\}.$$

We start our convergence calculation with the Gauß theorem. Using $\psi(x) \equiv 0$ on ∂T and (3.5) yields

$$0 = \int_T \operatorname{div}(u_\eta \varphi^\eta \psi)\,dx = \int_T u_\eta \cdot \nabla\varphi^\eta \psi\,dx + \int_T u_\eta \cdot \nabla\psi\varphi^\eta\,dx.$$

The last integral converges to 0 for $\eta \to 0$, since φ^η converges strongly in L^2 and u_η converges weakly to its average 0. We conclude that also the first integral on the right hand side vanishes in the limit $\eta \to 0$.

With this information we write the expression of interest as

$$\left|\int_T u_\eta \cdot \nabla\varphi^\eta\right| \le \left|\int_T u_\eta \cdot \nabla\varphi^\eta\psi\right| + \left|\int_T u_\eta \cdot \nabla\varphi^\eta(1-\psi)\right|.$$

For the first integral on the right hand side we already know convergence to 0. In the second integral we can replace the integral over T by an integral over V_δ and calculate using (3.6) and (3.8),

$$\left|\int_{V_\delta} u_\eta(x) \cdot \nabla\varphi^\eta(x)(1-\psi(x))\,dx\right| \le \|u_\eta\|_{L^2(V_\delta)} \|\nabla\varphi^\eta\|_{L^2(V_\delta)} \le C(\delta+\eta)^{1/2}.$$

Since $\delta > 0$ can be chosen arbitrarily small, this provides the convergence result. □

Lemma 3.4 (Variant of the div-curl Lemma with boundary)**.** *Let $T \subset \mathbb{R}^m$ be an open and bounded set with Lipschitz boundary ∂T, and let $Y := [0,1[^m$ denote the flat torus. Assume $u^\eta \rightharpoonup u$ in $L^2(T, \mathbb{R}^m)$ with $\operatorname{div} u^\eta \rightharpoonup u$ in $L^2(T)$. Let, for $\varphi \in H^1(Y)$, φ_η be the η-periodic function $\varphi_\eta(x) = \varphi(x/\eta)$. Then*

$$\lim_{\eta\to 0} \int_T u^\eta \cdot \nabla(\eta\varphi_\eta)(x)\,dx = 0.$$

Proof. We use the notation of the last proof. We start once more with the Gauß theorem. Since the cut-off function satisfies $\psi(x) \equiv 0$ on ∂T, there holds

$$0 = \int_T \operatorname{div}(u^\eta \eta\varphi_\eta\psi) = \int_T (\operatorname{div} u^\eta)\eta\varphi_\eta\psi + \int_T u^\eta \cdot \nabla(\eta\varphi_\eta)\psi + \int_T u^\eta \cdot \nabla\psi\eta\varphi_\eta.$$

The first and the last integral converge to 0 for $\eta \to 0$ due to the explicit η-factor. The second integral coincides with the integral of the claim except for the error

$$e_\eta^\delta = \left|\int_T u^\eta \cdot \nabla(\eta\varphi_\eta)(1-\psi)\right|.$$

Smallness of e^δ_η for small $\delta > 0$ (independent of η) follows as in the last line of the last proof from the boundedness of $u^\eta \in L^2(T)$ and the boundedness and periodicity of $\nabla(\eta\varphi_\eta) \in L^2(T)$. $\square$

We note that a very short proof of the div-curl lemma with boundary can be given with the theory of two-scale convergence. One only has to exploit that one of the two factors under the integral converges weakly in two scales, the other factor converges strongly in two scales. We are grateful to A. Visintin for pointing out this alternative proof. We include here the more elementary proof since it is independent of two-scale convergence methods.

3.3 Rigorous homogenization estimate

We can now give the proof of the homogenization result of Proposition 3.1. As announced, we construct test-functions from the semi-discrete approximate solutions of Lemma 2.7. We have to be careful about the fact that, in equation (2.19a), the test-function must be piece-wise linear. We therefore introduce $P_h : L^2(\Omega) \to U_h$, the orthogonal projection onto the space U_h of piece-wise linear functions on the mesh $\mathscr{T}_h^\Omega$, and let I be the identity operator in $L^2(\Omega)$. We first state and prove the rigorous version of Remark 3.2 in the next lemma. We will afterwards show that the error terms on the right-hand side vanish in the limit.

Lemma 3.5. *Let $(u^\eta, \varepsilon_p^\eta, \sigma^\eta)$ be a solution of Problem (P^η) in* (1.12) *and let (u_h, v_h, w_h, z_h) be a solution of Problem (P_h) in* (2.19). *The functions w_h, z_h, and v_h are piece-wise constant in x and we introduce the measurable functions*

$$w_{h,\eta}(x,t) = w_h\left(x, \frac{x}{\eta}, t\right),\ z_{h,\eta}(x,t) = z_h\left(x, \frac{x}{\eta}, t\right),\ (\nabla_y v_h)_\eta(x,t) = (\nabla_y v_h)\left(x, \frac{x}{\eta}, t\right).$$

Then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \varrho_\eta|\partial_t u^\eta - \partial_t u_h|^2 + (\sigma^\eta - z_{h,\eta}) : C_\eta(\sigma^\eta - z_{h,\eta}) + b_\eta|\varepsilon_p^\eta - w_{h,\eta}|^2 \\
&\quad \le \int_\Omega \left[\int_Y z_h - z_{h,\eta}\right] : \nabla^s P_h(\partial_t u^\eta - \partial_t u_h) - \int_\Omega (\sigma^\eta - z_{h,\eta}) : (\nabla^s_y \partial_t v_h)_\eta \\
&\qquad + \int_\Omega (\varrho_\eta - \bar\varrho)\partial_t^2 u_h \cdot (\partial_t u^\eta - \partial_t u_h) + \int_\Omega (\varrho_\eta \partial_t^2 u^\eta - \bar\varrho\partial_t^2 u_h)\cdot(I - P_h)(\partial_t u^\eta - \partial_t u_h) \\
&\qquad + \int_\Omega (\sigma^\eta - z_{h,\eta}) : \nabla^s (I - P_h)(\partial_t u^\eta - \partial_t u_h).
\end{aligned} \tag{3.9}$$

Proof. We examine the following expression.

$$\begin{aligned}
E_{h,\eta} \ := \ & \int_\Omega (\varrho_\eta\partial_t^2 u^\eta - \bar\varrho\partial_t^2 u_h)\cdot\partial_t(u^\eta - u_h) \\
& + \int_\Omega (\sigma^\eta - z_{h,\eta}) : \partial_t(\nabla^s u^\eta - \nabla^s u_h - (\nabla^s_y v_h)_\eta).
\end{aligned}$$

As in the proof of Remark 3.2, using equation (1.12b) and (2.19c), we compute

$$\begin{aligned}
E_\eta = {} & \frac{1}{2}\frac{d}{dt}\int_\Omega \varrho_\eta|\partial_t u^\eta - \partial_t u_h|^2 + \int_\Omega (\varrho_\eta - \bar\varrho)\partial_t^2 u_h\cdot(\partial_t u^\eta - \partial_t u_h) \\
& + \int_\Omega (\sigma^\eta - z_{h,\eta}) : \partial_t\left(\varepsilon_p^\eta - w_{h,\eta}\right) + \int_\Omega (\sigma^\eta - z_{h,\eta}) : C_\eta \partial_t(\sigma^\eta - z_{h,\eta}).
\end{aligned}$$

Denoting the integrals on the right by I_1 to I_4, we note that I_1 and I_4 are time derivatives of positive quantities. In order to treat I_3, we use the inverse relations to (2.19b) and (1.12c) so that

$$\begin{aligned}\int_\Omega (\sigma^\eta - z_{h,\eta}) : \partial_t(\varepsilon_p^\eta - w_{h,\eta}) \in \int_\Omega b_\eta(\varepsilon_p^\eta - w_{h,\eta}) : \partial_t(\varepsilon_p^\eta - w_{h,\eta})& \\ + \int_\Omega (\partial\chi_\eta^*(\partial_t\varepsilon_p^\eta) - \partial\chi_\eta^*(\partial_t w_{h,\eta})) : \partial_t(\varepsilon_p^\eta - w_{h,\eta}).&\end{aligned}$$

This expression is the time derivative of a positive quantity plus a non-negative term, owing to monotonicity of $\partial\chi_\eta^*$.

It remains to evaluate $E_{h,\eta}$. We expect the smallness of the expression due to the conservation laws for u^η and u_h. To find this result, we re-write $E_{h,\eta}$ in such a way that (1.12a) and (2.19a) appear as the first two integrals. We use the abbreviation $\psi_{h,\eta} := (\partial_t u^\eta - \partial_t u_h)$ and blow up the expression $E_{h,\eta}$ by writing (terms 2,4 and 6 can be added and terms 1 and 3 can be added)

$$\begin{aligned}E_{h,\eta} = &\int_\Omega \left[\varrho_\eta \partial_t^2 u^\eta - \bar\varrho\partial_t^2 u_h\right] \cdot P_h(\psi_{h,\eta}) + \int_\Omega \left[\sigma^\eta - \int_Y z_h\right] : \nabla^s P_h(\psi_{h,\eta}) \\ &+ \int_\Omega (\varrho_\eta \partial_t^2 u^\eta - \bar\varrho\partial_t^2 u_h) \cdot (I - P_h)(\psi_{h,\eta}) - \int_\Omega \left[z_{h,\eta} - \int_Y z_h\right] : \nabla^s P_h(\psi_{h,\eta}) \\ &- \int_\Omega (\sigma^\eta - z_{h,\eta}) : (\nabla_y^s \partial_t v_h)_\eta + \int_\Omega [\sigma^\eta - z_{h,\eta}] : \nabla^s(I - P_h)(\psi_{h,\eta}).\end{aligned}$$

The integrals in the first line vanish by the conservation laws. Collecting the other terms yields (3.9). □

With this estimate we can now conclude the proof of Proposition 3.1 and thus the homogenization result of Theorem 1.2.

Proof of Proposition 3.1. It remains to show that the right-hand side of estimate (3.9) vanishes, in the limit as $\eta \to 0$ and then $h \to 0$. Note that by estimate (2.14), up to subsequences, $\{\partial_t u_h\}_h$ and $\{\partial_t u^\eta\}_\eta$ converge weakly in $H^1(\Omega_T)$ and strongly in $L^2(\Omega_T)$. We denote the limits of u^η and u_h by u^* and u_*, respectively.

The first integral is

$$I_1 := \int_\Omega \left[\int_Y z_h - z_{h,\eta}\right] : \nabla^s P_h(\psi_{h,\eta}) = \sum_{T\in\mathscr{T}_h^\Omega} \int_T \left[\int_Y z_h - z_{h,\eta}\right] : \nabla^s P_h(\psi_{h,\eta}).$$

Using Lemma 3.3 with $u = z_h$ on each triangle T, we find $\lim_{\eta\to 0} I_1 = 0$.

Similarly, we treat the second integral. We use Lemma 3.3 on each triangle T with factors z_h and $\nabla(\eta\partial_t v_{h,\eta})$. The latter converges weakly to 0 in $L^2(T)$. For the contribution of σ^η we use Lemma 3.4.

$$I_2 := \int_\Omega (z_{h,\eta} - \sigma^\eta) : (\nabla_y \partial_t v_h)_\eta = \sum_{T\in\mathscr{T}_h^\Omega} \int_T (z_{h,\eta} - \sigma^\eta) : \nabla_y(\eta\partial_t v_h) \to 0.$$

Concerning the third integral, we recall that $\varrho_\eta \rightharpoonup \bar\varrho$, weakly in $L^2(\Omega_T)$. By strong convergence of $\partial_t u^\eta$ in $L^2(\Omega_T)$ we obtain for $\eta \to 0$, recalling that u_h maps into a finite dimensional space,

$$I_3 := \int_\Omega (\varrho_\eta - \bar\varrho)\partial_t^2 u_h \cdot (\partial_t u^\eta - \partial_t u_h) \to 0.$$

In the fourth integral we exploit that u_h maps into the right function space such that it coincides with its projection. The fourth integral therefore reads

$$I_4 := \int_\Omega (\varrho_\eta \partial_t^2 u^\eta - \bar\varrho \partial_t^2 u_h) \cdot (I - P_h)(\psi_{h,\eta}) = \int_\Omega (\varrho_\eta \partial_t^2 u^\eta - \bar\varrho \partial_t^2 u_h) \cdot (I - P_h)(\partial_t u^\eta).$$

Let ξ be the $L^2(\Omega_T)$-weak limit of $\varrho_\eta \partial_t^2 u^\eta$ (if necessary, along a further subsequence). We note that for all $\phi \in H^1(\Omega)$, $(I - P_h)\phi \to 0$ strongly in $H^1(\Omega)$, so that

$$\lim_{h\to 0}\left(\lim_{\eta\to 0} I_4\right) = \lim_{h\to 0}\left(\int_\Omega (\xi - \bar\varrho \partial_t^2 u_h) \cdot (I - P_h)(\partial_t u^*)\right) = 0.$$

Concerning the fifth term, equation (1.12a) implies

$$I_{5,1} := \int_\Omega \sigma^\eta : \nabla^s (I - P_h)(\psi_{h,\eta}) = \int_\Omega \left(f - \partial_t^2 u^\eta\right) \cdot (I - P_h)(\psi_{h,\eta}),$$

so that, as in I_4,

$$\lim_{h\to 0}\left(\lim_{\eta\to 0} I_{5,1}\right) = \lim_{h\to 0}\left(\int_\Omega \left(f - \partial_t^2 u^*\right) \cdot (I - P_h)(\partial_t u^*)\right) = 0.$$

Finally, owing to Lemma 3.3,

$$I_{5,2} := -\int_\Omega z_{h,\eta} : \nabla^s (I - P_h)(\psi_{h,\eta}) \to -\int_\Omega \left[\int_Y z_h\right] : \nabla (I - P_h)(\partial_t u^*).$$

This expression vanishes in the limit $h \to 0$, since $\partial_t u^*$ is an $H^1(\Omega_T)$-function.

Convergence of the initial data In order to conclude convergence from (3.9), it remains to show smallness, as $h, \eta \to 0$, of

$$\begin{aligned} R_1 &:= \int_\Omega \varrho_\eta |u_1 - \mathcal{P}(u_1^*; U_h)|^2 \\ R_2 &:= \int_\Omega (\sigma_0 - (z_{h,0})_\eta) : C_\eta(\sigma_0 - (z_{h,0})_\eta) \\ R_3 &:= \int_\Omega b_\eta |\varepsilon_{p0}^\eta - (w_{h,0})_\eta|^2. \end{aligned}$$

We show the computation for R_2, for R_1 and R_3 the calculation is analogous.

We consider $\hat u_h := \mathcal{P}(u_0; U_h)$, $\hat\sigma_h := \mathcal{P}(\sigma_0; PC_h(\Omega))$, $\hat v_h := \mathcal{P}(v_0(\cdot, y); PC_h(\Omega))$, and $\hat z_h(., y) := \mathcal{P}(z_0(\cdot, y); PC_h(\Omega))$. The discretization of (1.22) and (1.20) yields

$$\begin{aligned} \nabla^s \hat u_h(x) - C(y)\hat\sigma_h(x) &\overset{(1.22)}{=} w_{h,0}(x, y) \\ &\overset{(1.20)}{=} \nabla^s \hat u_h(x) + \nabla_y^s \hat v_h(x, y) - C(y)\hat z_h(x, y), \end{aligned} \tag{3.10}$$

which implies

$$C(y)\hat\sigma_h(x) = -\nabla_y^s \hat v_h(x, y) + C(y)\hat z_h(x, y), \qquad \text{a.e. in } \Omega \times Y. \tag{3.11}$$

Using (3.11) we compute

$$C_\eta(\sigma_0 - (\hat z_h)_\eta) \;=\; C_\eta(\sigma_0 - \hat\sigma_h) + C_\eta(\hat\sigma_h - (\hat z_h)_\eta) = C_\eta(\sigma_0 - \hat\sigma_h) - \left(\nabla_y^s \hat v_h\right)_\eta.$$

Since the first term on the right-hand side converges strongly to zero in $L^2(\Omega)$, we obtain

$$\lim_{h\to 0}\left(\lim_{\eta\to 0} R_2\right) = \lim_{h\to 0}\left(\lim_{\eta\to 0}\sum_{K\in\mathscr{T}_h^\Omega}\int_K (\sigma_0 - (\hat{z}_h)_\eta)) : \left(\nabla_y^s \hat{v}_h\right)_\eta\right).$$

For all $K \in \mathscr{T}_h^\Omega$, $\forall\, y \in Y$, we have that $x \mapsto \hat{v}_h(x,y)$ is constant on K, and as $\eta \to 0$

$$\left(\nabla_y \hat{v}_h\right)_\eta \rightharpoonup \int_Y \nabla \hat{v}_h(y)\, dy = 0, \quad \text{weakly in } L^2(K),$$

so that

$$\lim_{\eta\to 0}\sum_{K\in\mathscr{T}_h^\Omega}\int_K \sigma_0 : \left(\nabla_y \hat{v}_h\right)_\eta = 0.$$

On the other hand, with an integration by parts, we find

$$\int_K (\hat{z}_h)_\eta : \left(\nabla_y^s \hat{v}_h\right)_\eta = -\int_K (\operatorname{div}_y \hat{z}_h)_\eta \cdot (\hat{v}_h)_\eta + \eta \int_{\partial K} (\hat{z}_h)_\eta \nu \cdot (\hat{v}_h)_\eta.$$

Since $\operatorname{div}_y z_0 = 0$, and the traces of $z_0\nu$ and v_0 are bounded in $L^2(\partial K)$, we conclude

$$\lim_{h\to 0}\left(\lim_{\eta\to 0} R_2\right) = 0.$$

This concludes the proof of Proposition 3.1. □

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