Grad's equations and hydrodynamics for weakly inelastic granular flows

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Abstract

We introduce and discuss Grad's moment equations for dilute granular systems of hard spheres with dissipative collisions and variable coefficient of restitution, under the assumption of weak inelasticity. An important byproduct is that in this way we obtain the hydrodynamic description of a system of nearly elastic particles by a direct procedure from the Boltzmann equation, without resorting to any homogeneous cooling state assumption. Several crucial results of the pertinent literature are recovered in the present physical context in which deviation from elastic scattering is of the same order as the Knudsen number. In particular, the statistical correlation function plays a fundamental role in the decay of the temperature, and the latter is described asymptotically, in space homogeneous conditions, by a corrected Haff's law.

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1 Introduction

The aim of this paper is to discuss some questions connected with the modeling of hydrodynamic equations for granular flows. In the physical literature, rapid granular flows are frequently described at the macroscopic level by the equations for fluid dynamics, modified to account for dissipation due to collisions among particles. These equations are in general derived by studying the behavior of a granular material from a continuum point of view, treating individual grains as the molecules of a granular fluid,[1] and are not derived from a mesoscopic picture (the Boltzmann or Enskog kinetic equations). Thus the equations are in general phenomenological, with unknown transport coefficients and with unknown limit of validity. A deeper understanding of macroscopic equations has its origin in kinetic theory, which is suitable to describe the evolution of materials composed of many small discrete grains, in which the mean free path of the grains is much larger than the typical particle size. Similar as molecular gases, granular gases can be described at a mesoscopic level within the concepts of classical statistical mechanics, by means of methods borrowed from the kinetic theory of rarefied gases.[2] Many recent papers (see Refs. 3–6 and the references therein), consider in fact Boltzmann-like equations for partially inelastic rigid spheres. This choice relies in the physical hypothesis that the grains must be cohesionless, which implies the hard-sphere interaction only, and no long-range forces of any kind. Derivation of the hydrodynamic equations based on the Boltzmann or Enskog equations, modified to account for inelastic two–particle collisions have been considered in recent times.[7–11] All studies enlighten the dependence of the cooling problem on the coefficient of restitution in the microscopic collision, and emphasize the effects of a non–constant restitution coefficient.[4,12] Special attention has been devoted in this respect to a system of viscoelastic spheres, a quite realistic model whose coefficient of restitution has been recently derived.[13] A common assumption which has been at the basis of several recent papers on the matter is that there are only small spatial variations, so that the zero order approximation of the solution (and of any asymptotic expansion) is constituted by the so–called homogeneous cooling state (see for instance Ref. 9 and the references therein). A detailed theory of the homogeneous cooling state for viscoelastic particles in terms of expansion in Sonine polynomials has been recently developped.[14–15] Such spatially homogeneous solution turns out to depend not only on the similarity variable, as it would occur for constant restitution coefficient, but also on time explicitly. In addition, temperature has been shown to decay asymptotically according to a corrected Haff's law (see also Ref. 16). Asymptotic expansions around the homogeneous cooling state have been used then as hydrodynamic closure for the macroscopic equations in order to achieve a Navier–Stokes level via a Chapman–Enskog procedure also for non–constant restitution coefficient.[17] In particular, the complete set of hydrodynamic equations and transport coefficients have been derived in this frame for a granular gas of viscoelastic particles.[18–19] In this paper, we shall follow a slightly different point of view, namely we shall assume a collision dominated regime in the sense of kinetic theory (small dimensionless mean free path, the so called Knudsen number[2]) in the small inelasticity limit, which means that deviations of the coefficient of restitution from unity are taken to be of the same order of magnitude as the Knudsen number. This corresponds to closing the macroscopic equations by means of the usual Maxwellian equilibrium of the elastic Boltzmann equation, leading order solution at the kinetic level. This regime prevents the derivation to be sensible to the strength of spatial gradients; of course there is a price to be paid, which is moved to the limitation of having a small amount of inelasticity, an assumption however that is correct in many applications. This physical situation has been object of some attention in a recent past.[20–21] We shall consider a general power law for the dependence of the restitution coefficient on the impact velocity, which includes viscoelastic spheres as a special case. In this frame hydrodynamic closure has been discussed already in Refs. 8, 10, but only at the level of Euler equations.

In this paper we deal with a weakly inelastic granular gas subject to dissipative collisions with a coefficient of restitution which depends on the relative velocity, enlightening the importance of such dependence at the level of hydrodynamics. The kinetic description will be provided by the Boltzmann equation for dissipative spheres, suitably corrected to take into account statistical correlation among particles. To overcome the enormous amount of computations for the full three-dimensional problem, we treat only situations which are one dimensional in space. We aim at proceeding further beyond the Euler level, in which closure is achieved by simply using the zero order solution (the equilibrium Maxwellian) for the distribution function. In particular, we shall try to widen the region of validity to a suitable neighborhood of equilibrium by resorting to a Grad 13–moment expansion and we shall derive the relevant Grad equations. It is well known that Grad's method works fairly well and provides equations of hyperbolic type in a well defined region surrounding equilibrium.[22] They correspond to a moment truncation strategy which does not obey a maximum entropy principle with respect to the classical Boltzmann H–functional.[23] However, they recover the behavior of the simplest non–trivial moment system of such a kind when velocity distributions lie near local Maxwellians.[24] Indeed, the Boltzmann equation for inelastic gas could have many Lyapunov functionals independent from Boltzann's H–functional. Whether Grad's equations could fulfil a maximum principle with respect to one of them is an interesting open question, that however will not be addressed in this work. Here we only show that a suitable Lyapunov functional, which generalizes the usual Boltzmann relative entropy to the case of varying temperature, can be fruitfully used to prove, at least formally, that for small Knudsen numbers the solution to the inelastic Boltzmann equation, in the case of small inelasticity, is close to a local Maxwellian. The idea of applying Grad's method to inelastic gases goes back to Jenkins and Richman.[11] In this pioneering paper they outline the main ideas of Grad's derivation of hydrodynamics from a kinetic equation, using the Maxwellian distribution to close the hierarchy of transport equations. An important feature of Grad's equations is that they still contain collision terms, and are affected by the same small parameters as the kinetic equations. They lend themselves then to a classical asymptotic procedure of the Chapman–Enskog type, and provide as important byproduct hydrodynamic equations at the Navier–Stokes level. We refer to the recent paper 25 for a detailed treatment of the classical Chapman–Enskog derivation of hydrodynamics given in the framework of Grad's moment equations. Recent application of this method to a gas undergoing chemical reactions has been given in Ref. 26.

We remark that a similar analysis could be performed starting from the (more realistic) kinetic model given by the Enskog equation, which allows to take into account effects due to the radius of grains into Grad's equations. The additional computations, however, are enormously heavy. This matter is being considered for a future publication.

The paper is organized as follows. Section 2 contains the details on the collision dynamics in two–particle interactions, while we will describe the inelastic Boltzmann equation in Section 3. Section 4 is devoted to a semi–formal discussion on the validity of the H–theorem. Section 5 deals with the uneasy task of deriving Grad's equations for our kinetic model. They are given explicitly in Section 6, where also an asymptotic discussion on the relevant small parameters is performed and asymptotically consistent first–order equations are obtained. Finally, Section 7 is devoted to the hydrodynamic description at the Navier–Stokes level as asymptotic limit of the above equations, and to the relevant constitutive relations and temperature decay law. In order to reach an ideal line of reading, the details of the computations are postponed to several Appendices. Some of them report results and manipulations which may be found also elsewhere in the literature,[12,15] but we prefer to keep them listed here in order to make this work more self–consistent.

2 Two–particle dissipative interaction

In a granular gas, the microscopic dynamics of grains is governed by the restitution coefficient e which relates the normal components of the particle velocities before and after a collision. If the grains are identical perfect spheres of diameter $\sigma > 0$, $(\mathbf{x}, \mathbf{v})$ and $(\mathbf{x} - \sigma\hat{\mathbf{n}}, \mathbf{w})$ are their states before a collision, where $\hat{\mathbf{n}} \in S^2$ is the unit vector along the center of both spheres, the post collision velocities $(\mathbf{v}^*, \mathbf{w}^*)$ are such that

$$(\mathbf{v}^* - \mathbf{w}^*) \cdot \hat{\mathbf{n}} = -e\,(\mathbf{v} - \mathbf{w}) \cdot \hat{\mathbf{n}}\,. \tag{1}$$

The conservation of momentum, together with (1), implies a change of velocity for the colliding particles as

$$\mathbf{v}^* = \mathbf{v} - \frac{1}{2}(1+e)\Big((\mathbf{v} - \mathbf{w}) \cdot \hat{\mathbf{n}}\Big)\hat{\mathbf{n}}\,, \qquad \mathbf{w}^* = \mathbf{w} + \frac{1}{2}(1+e)\Big((\mathbf{v} - \mathbf{w}) \cdot \hat{\mathbf{n}}\Big)\hat{\mathbf{n}}. \tag{2}$$

For elastic collisions one has $e = 1$, while for inelastic collisions e decreases with increasing degree of inelasticity. In the literature, it is frequently assumed that the restitution coefficient is a physical constant. In real situations, however, the restitution coefficient may depend on the relative velocity in such a way that collisions with small relative velocity are close to be elastic.[21] A good description of dissipative collisions is based on the assumption that the spheres are composed by viscoelastic material, which is in good agreement with experimental data. The velocity–dependent restitution coefficient for viscoelastic spheres of diameter $\sigma > 0$ and mass m reads

$$e = 1 - C_1 A\, x^{2/5} |(\mathbf{v} - \mathbf{w}) \cdot \hat{\mathbf{n}}|^{1/5} + C_2 A^2 x^{4/5} |(\mathbf{v} - \mathbf{w}) \cdot \hat{\mathbf{n}}|^{2/5} \dots \tag{3}$$

with

$$x = \frac{3\sqrt{3}}{2} \frac{\sqrt{\sigma} Y}{m(1-\nu^2)}\,, \tag{4}$$

where Y is the Young modulus, ν is the Poisson ratio, and A depends on dissipative parameters of the material. The constant C_1 and C_2 can be explicitly computed. It has to be remarked that formula (3) refers to the case of pure viscoelastic interactions, i.e. it holds when the relative velocity $(\mathbf{v} - \mathbf{w}) \cdot \hat{\mathbf{n}}$ belongs to a certain interval $(a, b) \in \mathbb{R}_+$

with a bounded away from zero (to neglect surface effects) and b not too large (to avoid plastic deformations). The impact velocity dependence (3) of the restitution coefficient $e = e((\mathbf{v} - \mathbf{w}) \cdot \hat{\mathbf{n}})$ has been recently obtained by generalizing Hertz's contact problem to viscoelastic spheres. We skip here details that can be found in the literature (see Refs. 4, 27 and the references therein). In what follows, we will assume that the coefficient of restitution satisfies

$$1 - e = 2\beta\,\gamma\left(|(\mathbf{v} - \mathbf{w}) \cdot \hat{\mathbf{n}}|\right), \tag{5}$$

where $\gamma(\cdot)$ is a given function and β is a parameter which is small in presence of small inelasticity. Viscoelastic spheres can be expressed at the leading order as in (5), choosing $\gamma(r) = r^{1/5}$. In the manipulations developed below, we shall assume $\gamma(r) = r^{\delta}$, where δ is a positive parameter.

3 Boltzmann equation for dissipative spheres

Following the standard procedures of kinetic theory,[2] the evolution of the distribution function can be described by the Boltzmann–Enskog equation for inelastic hard–spheres, which for the force–free case reads[8,28]

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = G(\rho)\,\bar{Q}(f, f)(\mathbf{x}, \mathbf{v}, t), \tag{6}$$

where $\bar{Q}$ is the so–called granular collision operator, which describes the change in the density function due to creation and annihilation of particles in binary collisions:

$$\bar{Q}(f, f)(\mathbf{v}) = \sigma^2 \int_{\mathbb{R}^3} \int_{S_+} \mathbf{g} \cdot \hat{\mathbf{n}} \Big\{ \chi\, f(\mathbf{v}^{**}) f(\mathbf{w}^{**}) - f(\mathbf{v}) f(\mathbf{w}) \Big\} d\mathbf{w}\, d\hat{\mathbf{n}}. \tag{7}$$

In (6)

$$\rho(\mathbf{x}, t) = \int_{\mathbb{R}^3} f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v}$$

is the density, and the function $G(\rho)$ is the statistical correlation function between particles, which accounts for the increasing collision frequency due to the excluded volume effects. We refer to Ref. 29 for a detailed discussion of the meaning of the function G. In (7), $\mathbf{g} = \mathbf{v} - \mathbf{w}$, and S_+ is the hemisphere corresponding to $\mathbf{g} \cdot \hat{\mathbf{n}} > 0$. The velocities $(\mathbf{v}^{**}, \mathbf{w}^{**})$ are the pre collisional velocities of the so–called inverse collision, which results with $(\mathbf{v}, \mathbf{w})$ as post collisional velocities. They are given by

$$\begin{aligned} \mathbf{v}^{**} &= \mathbf{v} - \frac{1 - \beta\,\gamma(|\mathbf{g} \cdot \hat{\mathbf{n}}|)}{1 - 2\,\beta\,\gamma(|\mathbf{g} \cdot \hat{\mathbf{n}}|)} (\mathbf{g} \cdot \hat{\mathbf{n}})\, \hat{\mathbf{n}}\,, \\ \mathbf{w}^{**} &= \mathbf{w} + \frac{1 - \beta\,\gamma(|\mathbf{g} \cdot \hat{\mathbf{n}}|)}{1 - 2\,\beta\,\gamma(|\mathbf{g} \cdot \hat{\mathbf{n}}|)} (\mathbf{g} \cdot \hat{\mathbf{n}})\, \hat{\mathbf{n}}\,. \end{aligned} \tag{8}$$

The factor χ in the gain term appears respectively from the Jacobian of the transformation $d\mathbf{v}^{**} d\mathbf{w}^{**}$ into $d\mathbf{v}\, d\mathbf{w}$ and from the lengths of the collisional cylinders $e|\mathbf{g}^{**} \cdot \hat{\mathbf{n}}| = |\mathbf{g} \cdot \hat{\mathbf{n}}|$. For a constant restitution coefficient, $\chi = e^{-2}$. To avoid the presence of the function χ, and to study approximations to the granular operator (7), it is extremely convenient to write the operator (7) in weak form. More precisely, let us define with $< \cdot, \cdot >$ the inner product in $L_1(\mathbb{R}^3)$. For all smooth functions $\varphi(\mathbf{v})$, it holds

$$\begin{aligned} < \varphi\,, \bar{Q}(f,f) > &= \sigma^2 \int_{\mathbb{R}^3} \varphi(\mathbf{v}) \bar{Q}(f,f)(\mathbf{v})\, d\mathbf{v} = \\ &= \sigma^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S_+} \mathbf{g}\cdot\hat{\mathbf{n}} \Big(\varphi(\mathbf{v}^*) - \varphi(\mathbf{v})\Big) f(\mathbf{v}) f(\mathbf{w}) d\mathbf{v}\, d\mathbf{w}\, d\hat{\mathbf{n}} = \\ &= \frac{\sigma^2}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} |\mathbf{g}\cdot\hat{\mathbf{n}}| \Big(\varphi(\mathbf{v}^*) - \varphi(\mathbf{v})\Big) f(\mathbf{v}) f(\mathbf{w}) d\mathbf{v}\, d\mathbf{w}\, d\hat{\mathbf{n}}. \end{aligned} \tag{9}$$

The last equality follows since the integral over the hemisphere S_+ can be extended to the entire sphere S^2, provided the factor $1/2$ is inserted in front of the integral itself. In fact changing $\hat{\mathbf{n}}$ into $-\hat{\mathbf{n}}$ does not change the integrand.

For the sake of simplicity we shall consider here only one space dimension (say, z), with axial symmetry about the z-axis. The kinetic equation may be adimensionalized in the usual way, by measuring distances and times in units of typical macroscopic values L and τ, velocities in units of L/τ, and densities in terms of a reference value ρ_0. Easy manipulations single out spontaneously the mean free path $\lambda = (\pi\sigma^2\rho_0)^{-1}$ and the Knudsen number $Kn = \lambda/L$. A typical value G_0 of the correlation function could enter the definition of λ, in which case the correlation function should be measured in units of G_0. In any case, the weak dimensionless form of the inelastic kinetic equation reads as

$$<\varphi, \frac{\partial f}{\partial t}> + <v_z\, \varphi, \frac{\partial f}{\partial z}> = \frac{G(\rho)}{Kn} \frac{1}{2\pi} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} |\mathbf{g}\cdot\hat{\mathbf{n}}| f(\mathbf{v}) f(\mathbf{w}) \Big[\varphi(\mathbf{v}^*) - \varphi(\mathbf{v})\Big] d\mathbf{v} d\mathbf{w} d\hat{\mathbf{n}} \tag{10}$$

where the Knudsen number Kn has to be considered as a small parameter in the hydrodynamic limit we are interested in. Macroscopic parameters like drift velocity $\mathbf{u}$, pressure tensor $\mathbf{P}$, granular temperature T, heat flux $\mathbf{q}$ are defined in the standard way as

$$\begin{aligned} \mathbf{u} &= \frac{1}{\rho} \int_{\mathbb{R}^3} \mathbf{v}\, f(\mathbf{v})\, d\mathbf{v} \\ \mathbf{P} &= \int_{\mathbb{R}^3} \mathbf{c}\otimes\mathbf{c}\, f(\mathbf{v})\, d\mathbf{v} \\ 3\,\rho T &= \int_{\mathbb{R}^3} c^2\, f(\mathbf{v})\, d\mathbf{v} \\ \mathbf{q} &= \frac{1}{2} \int_{\mathbb{R}^3} c^2\, \mathbf{c}\, f(\mathbf{v})\, d\mathbf{v} \end{aligned} \tag{11}$$

where $\mathbf{c} = \mathbf{v} - \mathbf{u}$ denotes the peculiar velocity. In our assumptions, fluid velocity and heat flux vectors take the simple forms

$$\mathbf{u} = (0,0,u), \qquad \mathbf{q} = (0,0,q) \tag{12}$$

and the deviatoric part of the pressure tensor (or viscous stress) $\mathbf{p}$ is diagonal, so that it is also equivalent to a single scalar

$$\mathbf{p} = \begin{pmatrix} -\frac{1}{2}p & 0 & 0 \\ 0 & -\frac{1}{2}p & 0 \\ 0 & 0 & p \end{pmatrix} \tag{13}$$

with $p \equiv p_{zz}$. The whole pressure tensor is then given by $\mathbf{P} = \rho T \, \underline{\mathbb{I}} + \mathbf{p}$, where $\underline{\mathbb{I}}$ is the identity and ρT the scalar pressure. Notice that all diagonal entries of $\mathbf{P}$ must be positive, so that p is subject to the constraint

$$-\rho T < p < 2\,\rho T. \tag{14}$$

It proves convenient splitting the dimensionless Boltzmann collision operator Q in the right hand side of (10) as $Q = Q_{\mathsf{el}} + I$, where Q_{el} is the elastic collision integral, corresponding to the case $\beta = 0$, and I the correction due to inelasticity. The weak inelastic Boltzmann equation may be rewritten as

$$< \varphi, \frac{\partial f}{\partial t} > + < v_z\,\varphi, \frac{\partial f}{\partial z} > = \frac{G(\rho)}{K\!n}\left[< \varphi, Q_{\mathsf{el}} > + < \varphi, I > \right] \tag{15}$$

where

$$< \varphi, Q_{\mathsf{el}} > = \frac{1}{2\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} |\mathbf{g}\cdot\hat{\mathbf{n}}| f(\mathbf{v}) f(\mathbf{w}) \Big[\varphi(\mathbf{v}') - \varphi(\mathbf{v})\Big]\, d\mathbf{v}\, d\mathbf{w}\, d\hat{\mathbf{n}}, \tag{16}$$

with

$$\mathbf{v}' = \mathbf{v} - (\mathbf{g}\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}} \tag{17}$$

standing for the post collisional velocity in the elastic encounter. Consequently we have

$$< \varphi, I > = \frac{1}{2\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} |\mathbf{g}\cdot\hat{\mathbf{n}}| f(\mathbf{v}) f(\mathbf{w}) \Big[\varphi(\mathbf{v}^*) - \varphi(\mathbf{v}')\Big]\, d\mathbf{v}\, d\mathbf{w}\, d\hat{\mathbf{n}} \tag{18}$$

with

$$\mathbf{v}^* = \mathbf{v}' + \beta\,\gamma(|\mathbf{g}\cdot\hat{\mathbf{n}}|)\,(\mathbf{g}\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}}\,. \tag{19}$$

The advantage of such a splitting is that $\mathbf{v}^*$ differs from $\mathbf{v}'$ by a term of order β, and therefore, for any smooth function φ, the integral in (18) is small for small inelasticity, and the dominant role in the evolution is played by elastic scattering. The interplay of the two small parameters $K\!n$ and β is then crucial for the analysis of the following sections.

4 Low inelasticity and the H-theorem. A semi–formal discussion

As in classical elastic kinetic theory, Grad's expansion method needs to be justified. If the Boltzmann equation for elastic collisions is considered, the well–known Boltzmann H-theorem guarantees that, at any fixed positive time, the solution is close to the Maxwellian equilibrium provided the Knudsen number is small enough. In recent years, many efforts have been done to obtain explicit computable formulas which allow to quantify the space homogeneous time decay of the solution towards the Maxwellian in terms of the time decay of the relative entropy

$$H(f|M) = \int_{\mathbb{R}^3} f(\mathbf{v}) \log\frac{f(\mathbf{v})}{M(\mathbf{v})}\, d\mathbf{v}, \tag{20}$$

where

$$M(\mathbf{v}) = \frac{\rho}{(2\pi T)^{\frac{3}{2}}} \exp\left\{-\frac{|\mathbf{v}-\mathbf{u}|^2}{2T}\right\} \tag{21}$$

is the Maxwellian function with the same constant mass ρ, drift velocity $\mathbf{u}$ and temperature T of f. In particular, lower bounds on the entropy production

$$-D(f) = \int_{\mathbb{R}^3} \log f(\mathbf{v}) Q_{\mathsf{el}}(\mathbf{v}) d\mathbf{v}$$

in terms of the relative entropy have been obtained in Ref. 31.

The main problem here is the lack of a H-theorem for the dissipative Boltzmann equation. Given the solution $f(\mathbf{v},t)$ to the homogeneous dissipative Boltzmann equation (6),

$$\frac{\partial f}{\partial t} = \frac{1}{Kn} \bar{Q}(f,f)(\mathbf{v},t) \tag{22}$$

(since $\rho =$ constant, the factor G may be included into Kn), let us consider the functional

$$H(f)(t) = \int_{\mathbb{R}^3} \left(\frac{|\mathbf{v}-\mathbf{u}|^2}{2T(t)} f(\mathbf{v},t) + f(\mathbf{v},t) \log f(\mathbf{v},t) \right) d\mathbf{v}, \tag{23}$$

where $T(t)$ is the granular temperature. By standard arguments[32] one shows that, on the set of functions with the same constant mass ρ and drift velocity $\mathbf{u}$ of $f(\mathbf{v},t)$,

$$H(f) \geq H(M),$$

where $M(\mathbf{v})$ is the Maxwellian function given in (21), with $T = T(t)$, and where equal sign holds only for $f = M$. Hence, when the temperature is varying with time, the relative entropy to be considered is

$$H(f|M)(t) = H(f)(t) - H(M)(t) = \int_{\mathbb{R}^3} f(\mathbf{v},t) \log f(\mathbf{v},t)\, d\mathbf{v} - \rho \log \rho + \frac{3}{2} \rho \log \left[2\pi \mathrm{e}\, T(t) \right] \tag{24}$$

with $H(f|M) \geq 0$. Since $f(t)$ solves the homogeneous Boltzmann equation (22),

$$\frac{d}{dt} H(f|M) = \frac{d}{dt} \int_{\mathbb{R}^3} f(\mathbf{v},t) \log f(\mathbf{v},t)\, d\mathbf{v} + \frac{3}{2} \rho \frac{T'(t)}{T(t)} =$$

$$= \frac{1}{Kn} \int_{\mathbb{R}^3} \log f(\mathbf{v},t)\, \bar{Q}(f,f)(\mathbf{v},t)\, d\mathbf{v} + \frac{3}{2} \rho \frac{T'(t)}{T(t)} = \frac{1}{Kn} \int_{\mathbb{R}^3} \log f(\mathbf{v},t)\, Q_{\mathsf{el}}(\mathbf{v},t)\, d\mathbf{v}$$

$$+ \frac{3}{2} \rho \frac{T'(t)}{T(t)} + \frac{1}{2\pi Kn} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} |\mathbf{g} \cdot \hat{\mathbf{n}}| f(\mathbf{v},t) f(\mathbf{w},t) \Big[\log f(\mathbf{v}^*,t) - \log f(\mathbf{v}',t) \Big] d\mathbf{v}\, d\mathbf{w}\, d\hat{\mathbf{n}}. \tag{25}$$

We remark that the final formula (25) has been derived from (18), setting $\varphi(\mathbf{v}) = \log f(\mathbf{v})$.

The physical picture is then rather clear. $H(f|M)$ attains its minimum at the Maxwellian M with time dependent temperature T and its derivative along a solution is made up by three contributions, where the dominant role for $Kn \to 0$ is played by the first, which is negative definite and $O(Kn)$. In fact, the second is $O(1)$, and even drives the process in the same direction since $T'(t) < 0$, and also the third one is $O(1)$ in our hypothesis

$\beta = O(Kn)$. Therefore, collisions are pushing any initial distribution towards a local Maxwellian, at the same initial density and momentum, evolving in time according to the granular temperature. Notice that small inelasticity is crucial for the whole reasoning. A rough quantitative estimate under suitable smoothness assumptions may be obtained in several ways. For instance, if the solution $f(\mathbf{v},t)$ is smooth and we assume that

$$\left|\frac{\nabla f(\mathbf{v})}{f(\mathbf{v})}\right| < C_f < +\infty \tag{26}$$

for all $\mathbf{v} \in \mathbb{R}^3$, we get

$$\Big|\log f(\mathbf{v}^*) - \log f(\mathbf{v}')\Big| \le |\mathbf{v}^* - \mathbf{v}'| \left\|\frac{\nabla f(\mathbf{v})}{f(\mathbf{v})}\right\|_{L^\infty} = \beta\,\gamma(|\mathbf{g}\cdot\hat{\mathbf{n}}|)\,|\mathbf{g}\cdot\hat{\mathbf{n}}| \left\|\frac{\nabla f(\mathbf{v})}{f(\mathbf{v})}\right\|_{L^\infty}. \tag{27}$$

Hence

$$\frac{1}{2\pi}\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} |\mathbf{g}\cdot\hat{\mathbf{n}}| f(\mathbf{v}) f(\mathbf{w}) \Big[\log f(\mathbf{v}^*) - \log f(\mathbf{v}')\Big]\, d\mathbf{v}\, d\mathbf{w}\, d\hat{\mathbf{n}}\right| \le$$

$$\le \frac{\beta}{2\pi}\left\|\frac{\nabla f(\mathbf{v})}{f(\mathbf{v})}\right\|_{L^\infty} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} |\mathbf{g}\cdot\hat{\mathbf{n}}|^{2+\delta} f(\mathbf{v}) f(\mathbf{w})\, d\mathbf{v}\, d\mathbf{w}\, d\hat{\mathbf{n}} \tag{28}$$

where also the last integral is finite if f has sufficiently many moments. Assumption (26) could appear strong at first sight, since it is not satisfied when $f(\mathbf{v})$ is a Maxwellian or a perturbation of a Maxwellian. But in such a case the whole proof is useless since $H(f) \simeq H(M)$ and we are already in the regime we want (see next section). The interesting case is when the tails are of power type, and in that case $\nabla f/f$ is actually bounded for large values of $|\mathbf{v}|$, so that assumption (26) is valid.

It has been proven in Ref. 31 that, under suitable smoothness assumptions on f, it holds

$$D(f) = -\int_{\mathbb{R}^3} \log f(\mathbf{v},t)\, Q_{\mathsf{el}}(\mathbf{v},t)\, d\mathbf{v} \ge C_\epsilon(f)[H(f|M)]^{1+\varepsilon},$$

where the constant $C_\varepsilon(f)$ depends on f only through mass, drift velocity and temperature. Let $f(\mathbf{v})$ be a distribution with unit temperature. It is immediate to reckon that, if $f_\sigma(\mathbf{v})$, $\sigma > 0$, denotes the rescaled distribution $\sigma^{-3} f(\mathbf{v}/\sigma)$,

$$H(f|M) = H(f_\sigma|M_\sigma),$$

while

$$D(f_\sigma) = \sigma D(f).$$

Thus, if C_ε is the constant corresponding to distributions with unit temperature, given the solution to the Boltzmann equation at time $t > 0$, that has temperature $T(t)$, we have the lower bound

$$D(f)(t) \ge \sqrt{T(t)}\, C_\varepsilon\, [H(f|M)(t)]^{1+\epsilon}$$

On the other hand, if we denote by C the upper bound of

$$\frac{1}{2\pi}\left\|\frac{\nabla f(\mathbf{v})}{f(\mathbf{v})}\right\|_{L^\infty} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} |\mathbf{g}\cdot\hat{\mathbf{n}}|^{2+\delta} f(\mathbf{v}) f(\mathbf{w})\, d\mathbf{v}\, d\mathbf{w}\, d\hat{\mathbf{n}}$$

corresponding to unit temperature, the same rescaling as before shows that $C\,[T(t)]^{1+\frac{\delta}{2}}$ is the proper upper bound for an arbitrary time dependent temperature. Hence, at least for smooth solutions, the relative entropy satisfies the differential inequality

$$\frac{d}{dt}H(f|M) \le -\frac{\sqrt{T(t)}}{Kn}\,C_\varepsilon\,[H(f|M)]^{1+\varepsilon} + \frac{3}{2}\,\rho\,\frac{T'(t)}{T(t)} + \frac{\beta}{Kn}\,C\,T(t)^{1+\frac{\delta}{2}}, \tag{29}$$

where the second term on the right–hand side is nonpositive, due to the decay of the temperature. If β is of the same order of the Knudsen number Kn, the negative term of order $1/Kn$ prevails as $Kn \to 0$, and this implies $H(f|M) \to 0$. This can be easily seen considering that, in consequence of (29), the relative entropy $z(t) = H(f|M)(t)$ satisfies the differential inequality

$$\frac{dz}{dt} \le \sqrt{T(t)}\left[-\frac{C_\varepsilon}{Kn}z^{1+\varepsilon} + A\right], \tag{30}$$

where we set

$$A = \frac{\beta}{Kn}\,C\,[T(0)]^{\frac{1+\delta}{2}}.$$

Suppose first that the initial relative entropy is such that

$$\frac{C_\varepsilon}{Kn}{z_0}^{1+\epsilon} \le A.$$

Then, since $z(t)$ satisfies (30), at any subsequent time $t > 0$ we have

$$z(t) \le \bar{z} = \left(\frac{A}{C_\varepsilon}Kn\right)^{1/1+\varepsilon}, \tag{31}$$

and $z(t) \to 0$ as $Kn \to 0$. On the other hand, if the initial relative entropy satisfies

$$\frac{C_\varepsilon}{Kn}{z_0}^{1+\varepsilon} > A,$$

thanks to (30), $z(t)$ is decreasing towards the value $\bar{z}$ in (31), and, as long as $z(t) \ge \bar{z}$, it satisfies the (worse) inequality

$$\frac{dz}{dt} \le \sqrt{T(t)}\left[-\frac{C_\epsilon}{Kn}\bar{z}^\epsilon z + A\right] = \sqrt{T(t)}(-Bz + A)$$

where

$$B = A^{\epsilon/(1+\epsilon)}\left(\frac{C_\epsilon}{Kn}\right)^{1/(1+\epsilon)}.$$

Let

$$\tau(t) = \int_0^t \sqrt{T(s)}\,ds$$

and $y(\tau) = z(t)$. Then

$$\frac{dy}{d\tau} \le -By(\tau) + A, \tag{32}$$

which can be explicitly solved to give

$$z(t) = y(\tau) \leq y_0 \exp\{-B\tau\} + \frac{A}{B}\left[1 - \exp\{-B\tau\}\right]. \tag{33}$$

Since $t > 0$ implies $\tau > 0$, inequality (33) shows that $z(t) \to 0$ as $Kn \to 0$. This shows that, at any time t strictly greater that zero, the solution to the homogeneous Boltzmann equation, even for a granular gas, is close to a Maxwellian function, in the sense of relative entropy, if the Knudsen number is small enough, provided inelasticity is also small. The classical Csiszar–Kullback inequality

$$\|f - M\|_{L^1}^2 \leq 2H(f|M),$$

then shows that the solution in close to the corresponding local Maxwellian in a L^1–setting, which properly justifies expanding around such distribution in a collision dominated regime.

5 Grad's closure procedure

Grad's expansion method is a well known tool devised in kinetic theory in order to obtain a closed set of balance equations at a macroscopic level.[30] Even in the elastic case the machinery is quite heavy, and, to our knowledge, explicit results in analytical form have been achieved only in the simplified case of Maxwellian molecules. The algorithm has been more recently applied also to granular flows[11] and to chemical kinetics.[33] It consists in a truncated Hermite polynomial expansion which includes the major power moments of the distribution function up to third order. In the present one–dimensional problem the moments of physical interest reduce to five, namely density ρ, mass velocity u, granular temperature T, viscous stress p, and heat flux q. They are moments of f corresponding to the weight functions

$$\varphi = \ 1, \qquad v_z, \qquad \frac{1}{2}c^2, \qquad c_z^2 - \frac{1}{3}c^2, \qquad \frac{1}{2}c^2 c_z, \tag{34}$$

respectively. The relevant balance equations are the weak forms of the Boltzmann equation (15) corresponding to the five options (34) for the test function φ. Fundamental properties of the elastic and inelastic collision terms ensure that

$$< 1, Q_{\mathsf{el}} > = < 1, I > = \ 0\,, \qquad < v_z, Q_{\mathsf{el}} > = < v_z, I > = \ 0\,, \qquad < \frac{1}{2}c^2, Q_{\mathsf{el}} > = \ 0 \tag{35}$$

for any distribution function f. The other contributions can not be evaluated directly in terms of the considered moments, as it would occur for the pseudo–Maxwellian model[8] with constant restitution coefficient, due to the $\mathbf{g}$–dependence of the kernel in the integrands.[34]

A first step in the procedure is the evaluation of the differences $\varphi(\mathbf{v}') - \varphi(\mathbf{v})$ and $\varphi(\mathbf{v}^*) - \varphi(\mathbf{v}')$ for the test functions (34). The differences in the inelastic correction are polynomials in β of degree at most equal to three, vanishing for $\beta \to 0$. All differences exhibit simple dependence on the unit vector $\hat{\mathbf{n}}$, so that all angular integrations may be performed separately and explicitly for both elastic and inelastic interactions. Five types of integrals arise, and they are listed in Appendix A, together with their analytical expressions, involving the quantities

$$J_{nk}(g) = (2k+2)\int_0^1 \mu^{2k+1}\,\gamma^n(g\mu)\,d\mu\,, \qquad\qquad n,k=1,2\,. \tag{36}$$

From now on we shall stick to the form $\gamma(r) = r^\delta$, which incorporates the terms relevant to elastic scattering or constant inelastic restitution (for $\delta = 0$), and of viscoelastic spheres ($\delta = \frac{1}{5}$). Thus we will have simply:

$$J_{nk}(g) = \frac{2k+2}{2k+2+n\delta}\; g^{n\delta}\,. \tag{37}$$

Next, a lengthy and careful algebra shows that the remaining weak forms of the collision terms Q_{el} and I are amenable to only 4 integrals with respect to the variables $\mathbf{v}$ and $\mathbf{w}$, namely

$$\begin{aligned}
A_1(y) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} g^{3+y}\,\tilde f(\mathbf{v})\tilde f(\mathbf{w})\,d\mathbf{v}\,d\mathbf{w}\\
A_2(y) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} g^{1+y}\,g_z^2\,\tilde f(\mathbf{v})\tilde f(\mathbf{w})\,d\mathbf{v}\,d\mathbf{w}\\
A_3(y) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} g^{3+y}\,G_z\,\tilde f(\mathbf{v})\tilde f(\mathbf{w})\,d\mathbf{v}\,d\mathbf{w}\\
A_4(y) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} g^{1+y}\Big(\mathbf{G}\cdot\mathbf{g}\Big)\,g_z\,\tilde f(\mathbf{v})\tilde f(\mathbf{w})\,d\mathbf{v}\,d\mathbf{w}
\end{aligned} \tag{38}$$

where $\tilde f(\mathbf{v}) = f(\mathbf{v}+\mathbf{u})$, and $\mathbf{G} = \frac{1}{2}\,(\mathbf{v}+\mathbf{w})$ is the center of mass velocity. The relevant expressions are given in Appendix B.

At this point, the sought approximate closure for the collision term is achieved by replacing the actual distribution function f in (38) with the Grad distribution function, which, in the spatially one–dimensional case, reads as

$$f_G(\mathbf{v}) = \frac{\rho}{(2\pi T)^{\frac{3}{2}}}\,\mathrm{e}^{-\frac{c^2}{2T}}\left[1+\frac{p}{2\rho T^2}\left(-\frac{1}{2}\,c^2+\frac{3}{2}\,c_z^2\right)+\frac{4}{5}\,\frac{q}{2\rho T^2}\,c_z\left(\frac{c^2}{2T}-\frac{5}{2}\right)\right], \tag{39}$$

and constitutes the weighted polynomial approximation to f sharing the same moments up to q. Consistently with our hypothesis of small Kn and small β, (39) represents a perturbation to a Maxwellian distribution, solution to the elastic problem in the hydrodynamic limit.

Now manipulations become quite cumbersome, though, in principle, straightforward. One has to perform the product $\tilde f_G(\mathbf{v})\,\tilde f_G(\mathbf{w})$, recast $\mathbf{v}$ and $\mathbf{w}$ variables in terms of $\mathbf{G}$ and $\mathbf{g}$, use the latter as integration variables, taking advantage of the fact that $d\mathbf{v}\,d\mathbf{w} = d\mathbf{G}\,d\mathbf{g}$, and solve the final Gaussian–type integrals in terms of Eulerian gamma functions. The long and tedious calculations result in

$$\begin{aligned}
A_1(y) &= \alpha_1\,\rho^2\,T^{\frac{y+3}{2}} + \alpha_2\,\rho\,p\,T^{\frac{y+1}{2}} + \alpha_3\,p^2\,T^{\frac{y-1}{2}} + \alpha_4\,q^2\,T^{\frac{y-3}{2}}\\
A_2(y) &= \beta_1\,\rho^2\,T^{\frac{y+3}{2}} + \beta_2\,\rho\,p\,T^{\frac{y+1}{2}} + \beta_3\,p^2\,T^{\frac{y-1}{2}} + \beta_4\,q^2\,T^{\frac{y-3}{2}}\\
A_3(y) &= \gamma_1\,\rho\,q\,T^{\frac{y+1}{2}} + \gamma_2\,p\,q\,T^{\frac{y-1}{2}}\\
A_4(y) &= \delta_1\,\rho\,q\,T^{\frac{y+1}{2}} + \delta_2\,p\,q\,T^{\frac{y-1}{2}}
\end{aligned} \tag{40}$$

where the y–dependent coefficients α_i, β_i, γ_i, δ_i are reported in Appendix C. Equations (40) show how collision contributions to the moment equations depend on the unknown moments ρ, T, p, q via numerical factors. Since also streaming contributions may be closed by standard methods of kinetic theory (see for instance Refs. 33, 35), Grad's equations may now be written down explicitly.

6 Grad's moment equations and asymptotic limit

We first combine and rearrange collision contributions in the equations for T, p, and q. This step is shown, for the reader's convenience, in Appendix D. Then, Grad's moment equations read finally as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho\, u)}{\partial z} = 0\,, \tag{41}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial z}\right) + \frac{\partial (\rho\, T)}{\partial z} + \frac{\partial p}{\partial z} = 0\,, \tag{42}$$

$$\begin{aligned}
&\frac{3}{2}\,\rho\left(\frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial z}\right) + \rho\, T\,\frac{\partial u}{\partial z} + p\,\frac{\partial u}{\partial z} + \frac{\partial q}{\partial z} = \\
= \quad & -\frac{\beta}{Kn}\,\frac{2^{\delta}}{\sqrt{\pi}}\,G(\rho)\left[8\,\rho^2\,T^{\frac{\delta+3}{2}} + \frac{1}{10}\,(\delta+3)\,(\delta+1)\,p^2\,T^{\frac{\delta-1}{2}}\right. \\
&\qquad \left. + \frac{1}{75}\,(\delta+3)\,(\delta+1)\,(1-\delta)\,q^2\,T^{\frac{\delta-3}{2}}\right]\,\Gamma\left(\frac{\delta+4}{2}\right) \\
& + \frac{\beta^2}{Kn}\,\frac{4^{\delta}}{\sqrt{\pi}}\,G(\rho)\left[8\,\rho^2\,T^{\frac{2\delta+3}{2}} + \frac{1}{10}\,(2\,\delta+3)\,(2\,\delta+1)\,p^2\,T^{\frac{2\delta-1}{2}}\right. \\
&\qquad \left. + \frac{1}{75}\,(2\,\delta+3)\,(2\,\delta+1)\,(1-2\,\delta)\,q^2\,T^{\frac{2\delta-3}{2}}\right]\,\Gamma\Big(\delta+2\Big),
\end{aligned} \tag{43}$$

$$\begin{aligned}
&\frac{\partial p}{\partial t} + \frac{\partial (u\, p)}{\partial z} + \frac{4}{3}\,p\,\frac{\partial u}{\partial z} + \rho\, T\,\left(\frac{4}{3}\,\frac{\partial u}{\partial z}\right) + \frac{8}{15}\,\frac{\partial q}{\partial z} = \\
= \quad & -\frac{1}{Kn}\,\frac{1}{\sqrt{\pi}}\,\frac{4}{5}\,G(\rho)\left[4\,\rho\, p\, T^{\frac{1}{2}} + \frac{1}{7}\,p^2\,T^{-\frac{1}{2}} + \frac{2}{75}\,q^2\,T^{-\frac{3}{2}}\right] \\
& - \frac{\beta}{Kn}\,\frac{2^{\delta}}{\sqrt{\pi}}\,\frac{4}{15}\,\delta\, G(\rho)\left[4\,\rho\, p\, T^{\frac{\delta+1}{2}} + \frac{1}{7}\,(\delta+1)\,p^2\,T^{\frac{\delta-1}{2}}\right. \\
&\qquad \left. + \frac{2}{75}\,(\delta+1)\,(1-\delta)\,q^2\,T^{\frac{\delta-3}{2}}\right]\,\Gamma\left(\frac{\delta+4}{2}\right) \\
& + \frac{\beta^2}{Kn}\,\frac{4^{\delta}}{\sqrt{\pi}}\,\frac{4}{15}\,(2\,\delta+3)\,G(\rho)\left[4\,\rho\, p\, T^{\frac{2\delta+1}{2}} + \frac{1}{7}\,(2\delta+1)\,p^2\,T^{\frac{2\,\delta-1}{2}}\right. \\
&\qquad \left. + \frac{2}{75}\,(2\,\delta+1)\,(1-2\,\delta)\,q^2\,T^{\frac{2\delta-3}{2}}\right]\,\Gamma\Big(\delta+2\Big),
\end{aligned} \tag{44}$$

$$
\begin{aligned}
&\frac{\partial q}{\partial t} + \frac{\partial (u\,q)}{\partial z} + \frac{11}{5}\,q\,\frac{\partial u}{\partial z} + \frac{5}{2}\,\rho\,T\,\frac{\partial T}{\partial z} + \frac{5}{2}\,p\,\frac{\partial T}{\partial z} + \rho\,T\,\frac{\partial}{\partial z}\left(\frac{p}{\rho}\right) - \frac{p}{\rho}\,\frac{\partial p}{\partial z} = \\
= \quad & -\frac{1}{Kn}\,\frac{1}{\sqrt{\pi}}\,\frac{8}{5}\,G(\rho)\left[\frac{4}{3}\,\rho\,q\,T^{\frac{1}{2}} + \frac{1}{5}\,p\,q\,T^{-\frac{1}{2}}\right] \\
& - \frac{\beta}{Kn}\,\frac{2^{\delta}}{\sqrt{\pi}}\,\frac{2}{15}\,G(\rho)\left[2\,(25 + 11\,\delta)\,\rho\,q\,T^{\frac{\delta+1}{2}} - \frac{1}{5}\,(\delta+1)\,(\delta+15)\,p\,q\,T^{\frac{\delta-1}{2}}\right]\Gamma\left(\frac{\delta+4}{2}\right) \\
& + \frac{\beta^2}{Kn}\,\frac{4^{\delta}}{\sqrt{\pi}}\,\frac{2}{15}\,(2\,\delta+3)\,G(\rho)\left[22\,\rho\,q\,T^{\frac{2\delta+1}{2}} - \frac{1}{5}\,(2\,\delta+1)\,p\,q\,T^{\frac{2\delta-1}{2}}\right]\Gamma\Big(\delta+2\Big).
\end{aligned}
\tag{45}
$$

These equations have been consistently derived, under the stipulated simplifying assumptions (dilute gas of inelastic spheres in one space dimension for a collision dominated regime with small inelasticity), in a kinetic frame. They represent a direct generalization of the Euler equations established in Ref. 10 for the same physical situation. In particular, the energy dissipation term (corresponding to the Haff law when $\delta = 0$) is recovered as the first addend proportional to β/Kn in the right hand side of (43). On the other hand, the special case $\beta = 0$ corresponds to Grad's equations for elastic spheres, which, to our knowledge, have never been explicitly shown in the literature. The balance equation for energy is affected by additional inelastic terms depending on p and q (the non–hydrodynamic variables for the dominant operator Q_{el}), and vanishing for $p = 0$, $q = 0$. The equations for p and q contain of course elastic collision terms (proportional to $1/Kn$) along with inelastic ones. The former are dominant in the limit of β and Kn both small, and it is worth considering the relevant collision equilibrium p_*, q_*. It is provided by the coupled algebraic equations

$$
\begin{aligned}
&4\,\rho\,p_*\,T^{\frac{1}{2}} + \frac{1}{7}\,p_*^2\,T^{-\frac{1}{2}} + \frac{2}{75}\,q_*^2\,T^{-\frac{3}{2}} = 0 \\
&\frac{4}{3}\,\rho\,q_*\,T^{\frac{1}{2}} + \frac{1}{5}\,p_*\,q_*\,T^{-\frac{1}{2}} = 0
\end{aligned}
\tag{46}
$$

which, for fixed $T > 0$, yield, consistently with our assumptions, the unique solution $p_* = 0,\ q_* = 0$, since the other roots are ruled out by condition (14).

In general, when considering the limiting case of small β and Kn, it is necessary to perform the proper asymptotic analysis and to retain all terms up to the desired order in the chosen small parameter ϵ. We shall put then

$$
Kn = \epsilon \qquad\qquad \beta = \alpha\,\epsilon \tag{47}
$$

with $\alpha = \beta/Kn = O(1)$, and write down asymptotically consistent Grad equations, accurate to order ϵ, disregarding higher order corrections. Since both p and q vanish when $\epsilon \longrightarrow 0$, we put

$$
p = \epsilon\,\bar{p} \qquad\qquad q = \epsilon\,\bar{q} \tag{48}
$$

with $\bar{p}$ and $\bar{q}$ both $O(1)$, and notice that the correction of (39) with respect to the Maxwellian is actually $O(\epsilon)$. In this way one is left with the simpler system of Grad's equations

$$
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho\,u)}{\partial z} = 0\,, \tag{49}
$$

$$\rho \left(\frac{\partial u}{\partial t} + u \, \frac{\partial u}{\partial z} \right) + \frac{\partial (\rho \, T)}{\partial z} + \epsilon \, \frac{\partial \bar{p}}{\partial z} = 0 \,, \tag{50}$$

$$\begin{aligned} & \frac{3}{2} \, \rho \left(\frac{\partial T}{\partial t} + u \, \frac{\partial T}{\partial z} \right) + \rho \, T \, \frac{\partial u}{\partial z} + \epsilon \, \bar{p} \, \frac{\partial u}{\partial z} + \epsilon \, \frac{\partial \bar{q}}{\partial z} = \\ = \quad & - \alpha \, \frac{2^\delta}{\sqrt{\pi}} \; 8 \, \Gamma \left(\frac{\delta + 4}{2} \right) G(\rho) \, \rho^2 \, T^{\frac{\delta+3}{2}} + \epsilon \, \alpha^2 \, \frac{4^\delta}{\sqrt{\pi}} \; 8 \, \Gamma \Big(\delta + 2 \Big) G(\rho) \, \rho^2 \, T^{\frac{2\delta+3}{2}} \,, \end{aligned} \tag{51}$$

$$\begin{aligned} & \epsilon \, \frac{\partial \bar{p}}{\partial t} + \epsilon \, \frac{\partial (u \, \bar{p})}{\partial z} + \epsilon \, \frac{4}{3} \, \bar{p} \, \frac{\partial u}{\partial z} + \rho \, T \, \left(\frac{4}{3} \, \frac{\partial u}{\partial z} \right) + \epsilon \, \frac{8}{15} \, \frac{\partial \bar{q}}{\partial z} = \\ = \quad & - G(\rho) \left[\frac{1}{\sqrt{\pi}} \, \frac{16}{5} \, \rho \, T^{\frac{1}{2}} \, \bar{p} + \epsilon \, \frac{1}{\sqrt{\pi}} \, \frac{4}{35} \, T^{-\frac{1}{2}} \, \bar{p}^2 + \epsilon \, \frac{1}{\sqrt{\pi}} \, \frac{8}{375} \, T^{-\frac{3}{2}} \, \bar{q}^2 \right. \\ & \left. + \epsilon \, \alpha \, \frac{2^\delta}{\sqrt{\pi}} \, \frac{16 \, \delta}{15} \, \Gamma \left(\frac{\delta + 4}{2} \right) \, \rho \, T^{\frac{\delta+1}{2}} \, \bar{p} \right] , \end{aligned} \tag{52}$$

$$\begin{aligned} & \epsilon \, \frac{\partial \bar{q}}{\partial t} + \epsilon \, \frac{\partial (u \, \bar{q})}{\partial z} + \epsilon \, \frac{11}{5} \, \bar{q} \, \frac{\partial u}{\partial z} + \frac{5}{2} \, \rho \, T \, \frac{\partial T}{\partial z} + \epsilon \, \frac{5}{2} \, \bar{p} \, \frac{\partial T}{\partial z} + \epsilon \, \rho \, T \, \frac{\partial}{\partial z} \left(\frac{\bar{p}}{\rho} \right) = \\ = & - G(\rho) \left[\frac{1}{\sqrt{\pi}} \, \frac{32}{15} \, \rho \, T^{\frac{1}{2}} \, \bar{q} + \epsilon \, \frac{1}{\sqrt{\pi}} \, \frac{8}{25} \, T^{-\frac{1}{2}} \, \bar{p} \, \bar{q} + \epsilon \, \alpha \, \frac{2^\delta}{\sqrt{\pi}} \, \frac{4 \, (25 + 11 \, \delta)}{15} \, \Gamma \left(\frac{\delta + 4}{2} \right) \rho \, T^{\frac{\delta+1}{2}} \, \bar{q} \right] \end{aligned} \tag{53}$$

where inelastic contributions are labelled by the parameter $\alpha > 0$. An analysis similar to that described in Ref. 22, not reported here for brevity, shows that there exists a suitable region surrounding the equilibrium $p = 0, \; q = 0$ in which Grad's equations are hyperbolic.

7 Inelastic Navier–Stokes equations

As well known in the kinetic literature,[36] hydrodynamic equations of Navier– Stokes type are obtained also via Grad's equations by a classical Chapman–Enskog algorithm, which is much easier to perform on the Grad level rather than starting directly from the kinetic level. Such an algorithm is in fact straightforward here: one has to close the equations for ρ, u, T by solving the remaining equations for $\bar{p}$ and $\bar{q}$ to leading order in ϵ, after expanding $\bar{p}$ and $\bar{q}$ in powers of ϵ, keeping ρ, u, T unexpanded. This yields very simple algebraic equations, and in particular we get

$$\bar{p}_0 = - \, \frac{5}{16} \, \sqrt{\pi} \, \frac{T^{\frac{1}{2}}}{G(\rho)} \, \left(\frac{4}{3} \, \frac{\partial u}{\partial z} \right) \tag{54}$$

from (52), and

$$\bar{q}_0 = - \, \frac{75}{64} \, \sqrt{\pi} \, \frac{T^{\frac{1}{2}}}{G(\rho)} \, \frac{\partial T}{\partial z} \tag{55}$$

from (53). The factor $\dfrac{4}{3} \, \dfrac{\partial u}{\partial z}$ would appear strange at first glance, but it its simply the $i = 3$, $j = 3$ entry of the symmetrized traceless form of the rate of strain tensor $\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} -$

$\frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}$. Indeed, this is the only entry which is actually needed, because of (13). These are Newtonian constitutive equations, and turn out not to be affected by inelastic corrections, as a consequence of the hypothesis of small inelasticity. A quite important fact however is that viscosity and conduction coefficients are instead affected by the presence of the statistical correlation function in the denominator, so that viscous stress and heat flux are significantly sensitive to the local values of the density ρ. In addition, such coefficients are proportional to $T^{\frac{1}{2}}$ and reproduce the correct results available in the literature for elastically scattering hard spheres[37] in the dilute case $G(\rho) = 1$. The resulting inelastic hydrodynamic equations of Navier–Stokes read as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho\, u)}{\partial z} = 0\,, \tag{56}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial z}\right) + \frac{\partial (\rho\, T)}{\partial z} - \epsilon\,\frac{5}{12}\,\sqrt{\pi}\,\frac{\partial}{\partial z}\left(\frac{T^{\frac{1}{2}}}{G(\rho)}\,\frac{\partial u}{\partial z}\right) = 0\,, \tag{57}$$

$$\begin{aligned}\frac{3}{2}\,\rho\left(\frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial z}\right) + \rho\, T\,\frac{\partial u}{\partial z} - \epsilon\,\frac{5}{12}\,\sqrt{\pi}\,\frac{T^{\frac{1}{2}}}{G(\rho)}\left(\frac{\partial u}{\partial z}\right)^2 - \epsilon\,\frac{75}{64}\,\sqrt{\pi}\,\frac{\partial}{\partial z}\left(\frac{T^{\frac{1}{2}}}{G(\rho)}\,\frac{\partial T}{\partial z}\right) =\\ = \; -\,\alpha\,\frac{2^\delta}{\sqrt{\pi}}\;8\;\Gamma\left(\frac{\delta+4}{2}\right) G(\rho)\,\rho^2\, T^{\frac{\delta+3}{2}} + \epsilon\,\alpha^2\,\frac{4^\delta}{\sqrt{\pi}}\;8\;\Gamma\Big(\delta+2\Big) G(\rho)\,\rho^2\, T^{\frac{2\delta+3}{2}}\,.\end{aligned} \tag{58}$$

Apart from the $O(\epsilon)$ classical corrections to the streaming part, we notice in the collisional part of the energy equation the same $O(1)$ dissipative term of Ref. 10 plus an $O(\epsilon)$ correction, which bears a different T dependence for all non–constant restitution coefficients. In space homogeneous conditions ρ and u are constant in time, whereas the first order differential equation for T can be solved in implicit analytical form in terms of a Gaussian hypergeometric function, as

$$\begin{aligned} T_0^{-\frac{2\delta+1}{2}}\,\Phi\!\left[\frac{\Gamma\left(\frac{\delta}{2}+2\right)}{\epsilon\,\alpha\,2^\delta\,\Gamma(\delta+2)}\,T_0^{-\frac{\delta}{2}}, 1, \frac{2\,\delta+1}{\delta}\right] - T^{-\frac{2\delta+1}{2}}\,\Phi\!\left[\frac{\Gamma\left(\frac{\delta}{2}+2\right)}{\epsilon\,\alpha\,2^\delta\,\Gamma(\delta+2)}\,T^{-\frac{\delta}{2}}, 1, \frac{2\,\delta+1}{\delta}\right] =\\ = \;\epsilon\;\alpha^2\;\delta\;\frac{2^{2\delta+3}}{3\sqrt{\pi}}\;\Gamma(\delta+2)\;G(\rho)\,\rho\;t \end{aligned} \tag{59}$$

where

$$\Phi(z,1,v) = \sum_{n=0}^{\infty}\frac{z^n}{n+v} = \frac{1}{v}\;{}_2F_1(1, v;\, v+1;\, z) \tag{60}$$

is the so–called Lerch transcendent.[38] Indeed, when the exponent δ is the inverse of a natural number, as it occurs for viscoelastic spheres, the last argument of Φ in (59) becomes a positive integer, and consequently Φ itself may be cast in terms of elementary functions of its first argument, such as powers and logarithm. It is easy to check that the present implicit solution predicts, for any value of δ, a $t^{-2/(\delta+1)}$ relaxation to zero for T when $t \to \infty$, in agreement with the generalized Haff's law obtained in Ref. 10 from the Euler equations. However, the discrepancy of that solution with respect to the Navier–Stokes solution expressed by (59), which accounts for $O(\epsilon)$ corrections, is not negligible for intermediate values of t. In addition, the implicit solution (59) can be made more explicit by expanding T as $T^{(0)} + \epsilon\, T^{(1)}$ and equating equal powers of ϵ. It is readily found

$$\frac{T^{(0)}}{T_0} = \left(1 + \frac{t}{\tau_0}\right)^{-\frac{2}{\delta+1}} \tag{61}$$

while $T^{(1)}$ behaves like $t^{-\frac{\delta+2}{\delta+1}}$ for $t \to \infty$, recovering thus the results of Ref. 27, relevant to the same physical situation, obtained in terms of a Sonine polynomial expansion for viscoelastic spheres ($\delta = 1/5$).

It is worth commenting that, in our physical frame, α–dependent correction terms containing the gradients of ρ, u, T would appear in the constitutive equations for p and q only when pushing the analysis to $O(\epsilon^2)$ accuracy (then, at the Burnett level). The relevant manipulations yield

$$\begin{aligned}
\bar{p}_1 &= \frac{25}{192}\,\pi\,\frac{1}{G^2(\rho)}\left[-\frac{T}{\rho^2}\frac{\partial^2\rho}{\partial\, z^2} + \frac{1}{2\,\rho}\frac{\partial^2 T}{\partial\, z^2} + \frac{T}{\rho^3}\left(\frac{\partial\rho}{\partial z}\right)^2 + \frac{87}{128}\,\frac{1}{\rho\, T}\left(\frac{\partial T}{\partial z}\right)^2\right.\\
&\qquad \left. + \left(\frac{20}{21}\,\frac{1}{\rho} + \frac{1}{G(\rho)}\,\frac{\partial G(\rho)}{\partial\rho}\right)\left(\frac{\partial u}{\partial z}\right)^2 - \left(\frac{1}{\rho} + \frac{1}{G(\rho)}\,\frac{\partial G(\rho)}{\partial\rho}\right)\frac{1}{\rho}\,\frac{\partial\rho}{\partial z}\,\frac{\partial T}{\partial z}\right]\\
&+ \alpha\,\frac{2^\delta\sqrt{\pi}}{G(\rho)}\,\frac{25}{72}\left(-1 + \frac{2}{5}\,\delta\right)\Gamma\left(\frac{\delta+4}{2}\right) T^{\frac{\delta+1}{2}}\,\frac{\partial u}{\partial z}
\end{aligned} \tag{62}$$

$$\begin{aligned}
\bar{q}_1 &= \frac{375}{512}\,\pi\,\frac{1}{G^2(\rho)}\left[-\,\frac{7}{30}\,\frac{T}{\rho}\,\frac{\partial^2 u}{\partial\, z^2} - \frac{4}{15}\left(\frac{1}{\rho} + \frac{1}{G(\rho)}\,\frac{\partial G(\rho)}{\partial\rho}\right)\frac{T}{\rho}\,\frac{\partial\rho}{\partial z}\,\frac{\partial u}{\partial z}\right.\\
&\qquad \left. + \left(\frac{8}{5}\,\frac{1}{\rho} + \frac{3}{4}\,\frac{1}{G(\rho)}\,\frac{\partial G(\rho)}{\partial\rho}\right)\frac{\partial u}{\partial z}\,\frac{\partial T}{\partial z}\right]\\
&- \alpha\,\frac{2^\delta\sqrt{\pi}}{G(\rho)}\,\frac{375}{128}\,\Gamma\left(\frac{\delta+4}{2}\right)\left[\left(\frac{1}{G(\rho)}\,\frac{\partial G(\rho)}{\partial\rho} + \frac{1}{\rho}\right) T^{\frac{\delta+3}{2}}\,\frac{\partial\rho}{\partial z} + \frac{1}{20}\,(15-\delta)\, T^{\frac{\delta+1}{2}}\,\frac{\partial T}{\partial z}\right].
\end{aligned} \tag{63}$$

Details on the underlying Chapman–Enskog procedure are given in Appendix E. With reference to the existing literature, this fact clearly follows from our assumption $\beta = O(Kn)$.

8 Conclusions

In this paper we introduced and discussed Grad's moment equations for a dilute granular system of hard–spheres with dissipative collisions. We focused our analysis on dilute systems, driven at a mesoscopic scale by the Boltzmann equation, to emphasize the role of a relative velocity dependent coefficient of restitution at the level of hydrodynamic equations. The assumptions on the variable coefficient of restitution are general enough to include in our analysis the treatment of a system of viscoelastic spheres. It is shown in particular that the dominant term in the large–time behavior of the granular temperature is given explicitly by a generalized version of the classical Haff's law, which was originally obtained in correspondence to a constant coefficient of restitution. However, the discrepancy of the actual trend with respect to such explicit law is not negligible for intermediate values of time. In addition, it is shown that the decay of temperature, and then of scalar pressure, strongly depends on the local value of density, due to the presence of the statistical correlation function. In particular, higher density implies a larger value of $G(\rho)$, and then a faster relaxation to zero, as predicted by the inelastic collision terms in (58).

Of course, other typical corrections of the pertinent literature, due to the size of grains, do not appear here at the Navier-Stokes level, since we are starting from a Boltzmann rather than from an Enskog kinetic description, which provides a correction to the Boltzmann equation including the space shifts $\mathbf{x} \pm \sigma\hat{\mathbf{n}}$. For instance, it is easy to see that our algorithm, when applied to the Enskog equation, upon resorting to a suitable first–order expansion, gives rise to a proper correction to the scalar pressure.[7] This effect appears already at the Euler level, if σ is of the same order of the mean free path. This and other problems, however, will be matter of future investigation.

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Appendix A

We list here the angular integrations appearing in the Grad equations for the test functions (34). The factor γ is defined in (5) and describes the dependence of the restitution coefficient on the relative motion. Integrals appearing in the elastic part are recovered by simply substituting 1 for γ.

$$\int_{S^2} \gamma(|\mathbf{g}\cdot\hat{\mathbf{n}}|)\,|\mathbf{g}\cdot\hat{\mathbf{n}}|^3\,d\hat{\mathbf{n}} = \pi g^3 J_{11}(g) \tag{A.1}$$

$$\int_{S^2} \gamma^2(|\mathbf{g}\cdot\hat{\mathbf{n}}|)\,|\mathbf{g}\cdot\hat{\mathbf{n}}|^3\,d\hat{\mathbf{n}} = \pi g^3 J_{21}(g) \tag{A.2}$$

$$\int_{S^2} \gamma(|\mathbf{g}\cdot\hat{\mathbf{n}}|)\,|\mathbf{g}\cdot\hat{\mathbf{n}}|\,(\mathbf{g}\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}}\,d\hat{\mathbf{n}} = \pi g J_{11}(g)\,\mathbf{g} \tag{A.3}$$

$$\begin{aligned}\int_{S^2} \gamma(|\mathbf{g}\cdot\hat{\mathbf{n}}|)\,|\mathbf{g}\cdot\hat{\mathbf{n}}|\,(\mathbf{g}\cdot\hat{\mathbf{n}})^2\,\hat{\mathbf{n}}\otimes\hat{\mathbf{n}}\,d\hat{\mathbf{n}} &= \\ = \pi g^3\left[\frac{1}{2}\,J_{11}(g) - \frac{1}{3}\,J_{12}(g)\right]\underline{\mathrm{I}} &- \pi g\left[\frac{1}{2}\,J_{11}(g) - J_{12}(g)\right]\mathbf{g}\otimes\mathbf{g}\end{aligned} \tag{A.4}$$

$$\begin{aligned}\int_{S^2} \gamma^2(|\mathbf{g}\cdot\hat{\mathbf{n}}|)\,|\mathbf{g}\cdot\hat{\mathbf{n}}|\,(\mathbf{g}\cdot\hat{\mathbf{n}})^2\,\hat{\mathbf{n}}\otimes\hat{\mathbf{n}}\,d\hat{\mathbf{n}} &= \\ = \pi g^3\left[\frac{1}{2}\,J_{21}(g) - \frac{1}{3}\,J_{22}(g)\right]\underline{\mathrm{I}} &- \pi g\left[\frac{1}{2}\,J_{21}(g) - J_{22}(g)\right]\mathbf{g}\otimes\mathbf{g}\end{aligned} \tag{A.5}$$

The functions $J_{nk}(g)$ are defined by Eq. (36) in the text.

Appendix B

Higher order moments of elastic and inelastic collision operators expressed in terms of the integrals (38):

$$< \frac{1}{2}\, c^2, I >= -\, \frac{\beta}{\mathit{Kn}}\, \frac{1}{(4+\delta)}\, A_1(\delta) + \frac{\beta^2}{\mathit{Kn}}\, \frac{1}{(4+2\,\delta)}\, A_1(2\,\delta)\,, \tag{B.1}$$

$$< c_z^2, Q_{\mathsf{el}} >= \frac{1}{\mathit{Kn}}\, \frac{1}{12} \Big\{ A_1(0) - 3\, A_2(0) \Big\} \tag{B.2}$$

$$\begin{aligned} < c_z^2, I > \;&=\; -\, \frac{\beta}{\mathit{Kn}}\, \frac{1}{(4+\delta)(6+\delta)} \Big\{ 4\, A_1(\delta) + (2\,\delta)\, A_2(\delta) \Big\} \\ &\quad + \frac{\beta^2}{\mathit{Kn}}\, \frac{1}{(4+2\,\delta)(6+2\,\delta)} \Big\{ 2\, A_1(2\,\delta) + (6+4\,\delta)\, A_2(2\,\delta) \Big\} \end{aligned} \tag{B.3}$$

$$< \frac{1}{2}\, c^2 c_z, Q_{\mathsf{el}} >= \frac{1}{\mathit{Kn}}\, \frac{1}{12} \Big\{ A_3(0) - 3\, A_4(0) \Big\} \tag{B.4}$$

$$\begin{aligned} < \frac{1}{2}\, c^2 c_z, I > \;&=\; -\, \frac{\beta}{\mathit{Kn}}\, \frac{1}{(4+\delta)(6+\delta)} \Big\{ (10+\delta)\, A_3(\delta) + (2\,\delta)\, A_4(\delta) \Big\} \\ &\quad + \frac{\beta^2}{\mathit{Kn}}\, \frac{1}{(4+2\,\delta)(6+2\,\delta)} \Big\{ (8+2\,\delta)\, A_3(2\,\delta) + (6+4\,\delta)\, A_4(2\,\delta) \Big\} \end{aligned} \tag{B.5}$$

Notice that the addend of power β^3 in (B.5) is missing because the relevant coefficient vanishes in the integration.

Appendix C

Coefficients α_i, β_i, γ_i, δ_i can be computed using polar coordinates for $\mathbf{G}$ and $\mathbf{g}$, recalling

$$\begin{aligned} \int_{\mathbb{R}^3} x_i^2\, F(x)\, d\mathbf{x} \;&=\; \frac{1}{3} \int_{\mathbb{R}^3} x^2\, F(x)\, d\mathbf{x} \\ \int_{\mathbb{R}^3} x_i^4\, F(x)\, d\mathbf{x} \;&=\; \frac{1}{5} \int_{\mathbb{R}^3} x^4\, F(x)\, d\mathbf{x} \\ \int_{\mathbb{R}^3} x_i^2\, x_j^2\, F(x)\, d\mathbf{x} \;&=\; \frac{1}{15} \int_{\mathbb{R}^3} x^4\, F(x)\, d\mathbf{x} \qquad i \neq j \end{aligned}$$

and bearing in mind that

$$\begin{aligned} \int_{\mathbb{R}^3} g^k\, \mathrm{e}^{-\frac{g^2}{4}}\, d\mathbf{g} \;&=\; 2^{k+4}\, \pi\, \Gamma\left(\frac{k+3}{2}\right) \\ \int_{\mathbb{R}^3} G^k\, \mathrm{e}^{-G^2}\, d\mathbf{G} \;&=\; 2\, \pi\, \Gamma\left(\frac{k+3}{2}\right) \end{aligned}$$

where Γ denotes gamma function.[39] The final result reads as

$$\begin{aligned}
\alpha_1(y) &= \frac{2^y}{\sqrt{\pi}}\, 16\; \Gamma\left(\frac{y+6}{2}\right) \\
\alpha_2(y) &= 0 \\
\alpha_3(y) &= \frac{2^y}{\sqrt{\pi}}\,\frac{1}{5}\,(y+3)\,(y+1)\; \Gamma\left(\frac{y+6}{2}\right) \\
\alpha_4(y) &= \frac{2^y}{\sqrt{\pi}}\,\frac{2}{75}\,(y+3)\,(y+1)\,(1-y)\; \Gamma\left(\frac{y+6}{2}\right)
\end{aligned} \tag{C.1}$$

$$\begin{aligned}
\beta_1(y) &= \frac{2^y}{\sqrt{\pi}}\,\frac{16}{3}\; \Gamma\left(\frac{y+6}{2}\right) \\
\beta_2(y) &= \frac{2^y}{\sqrt{\pi}}\,\frac{16}{15}\,(y+6)\; \Gamma\left(\frac{y+6}{2}\right) \\
\beta_3(y) &= \frac{2^y}{\sqrt{\pi}}\,\frac{1}{105}\,(y+1)\,(11y+45)\; \Gamma\left(\frac{y+6}{2}\right) \\
\beta_4(y) &= \frac{2^y}{\sqrt{\pi}}\,\frac{2}{375}\,(y+1)\,(1-y)\,(3y+13)\; \Gamma\left(\frac{y+6}{2}\right)
\end{aligned} \tag{C.2}$$

$$\begin{aligned}
\gamma_1(y) &= \frac{2^y}{\sqrt{\pi}}\,\frac{8}{3}\,(y+3)\; \Gamma\left(\frac{y+6}{2}\right) \\
\gamma_2(y) &= -\frac{2^y}{\sqrt{\pi}}\,\frac{4}{25}\,(y+3)\,(y+1)\; \Gamma\left(\frac{y+6}{2}\right)
\end{aligned} \tag{C.3}$$

$$\begin{aligned}
\delta_1(y) &= \frac{2^y}{\sqrt{\pi}}\,\frac{8}{15}\,(3y+13)\; \Gamma\left(\frac{y+6}{2}\right) \\
\delta_2(y) &= \frac{2^y}{\sqrt{\pi}}\,\frac{4}{75}\,(y+1)\,(y+9)\; \Gamma\left(\frac{y+6}{2}\right)
\end{aligned} \tag{C.4}$$

Appendix D

We have

$$\begin{aligned}
< \frac{1}{2}c^2, Q > &= -\frac{\beta}{K\!n}\left[\mathcal{A}_{11}(\delta)\;\rho^2\,T^{\frac{\delta+3}{2}} + \mathcal{A}_{13}(\delta)\;p^2\,T^{\frac{\delta-1}{2}} + \mathcal{A}_{14}(\delta)\;q^2\,T^{\frac{\delta-3}{2}}\right] \\
&\quad + \frac{\beta^2}{K\!n}\left[\mathcal{A}_{21}(\delta)\;\rho^2\,T^{\frac{2\delta+3}{2}} + \mathcal{A}_{23}(\delta)\;p^2\,T^{\frac{2\delta-1}{2}} + \mathcal{A}_{24}(\delta)\;q^2\,T^{\frac{2\delta-3}{2}}\right]
\end{aligned} \tag{D.1}$$

where

$$\mathcal{A}_{1k}(\delta) = \frac{\alpha_k(\delta)}{4+\delta} \qquad\qquad \mathcal{A}_{2k}(\delta) = \mathcal{A}_{1k}(2\,\delta)\,; \tag{D.2}$$

$$
\begin{aligned}
&< c_z^2 - \frac{1}{3}c^2, Q >= \\
&= -\frac{1}{Kn}\left[\mathcal{B}_{01}(0)\ \rho^2\, T^{\frac{3}{2}} + \mathcal{B}_{02}(0)\ \rho\, p\, T^{\frac{1}{2}} + \mathcal{B}_{03}(0)\ p^2\, T^{-\frac{1}{2}} + \mathcal{B}_{04}(0)\ q^2\, T^{-\frac{3}{2}}\right] \\
&\quad - \frac{\beta}{Kn}\left[\mathcal{B}_{11}(\delta)\ \rho^2\, T^{\frac{\delta+3}{2}} + \mathcal{B}_{12}(\delta)\ \rho\, p\, T^{\frac{\delta+1}{2}} + \mathcal{B}_{13}(\delta)\ p^2\, T^{\frac{\delta-1}{2}} + \mathcal{B}_{14}(\delta)\ q^2\, T^{\frac{\delta-3}{2}}\right] \\
&\quad + \frac{\beta^2}{Kn}\left[\mathcal{B}_{21}(\delta)\ \rho^2\, T^{\frac{2\delta+3}{2}} + \mathcal{B}_{22}(\delta)\ \rho\, p\, T^{\frac{2\delta+1}{2}} + \mathcal{B}_{23}(\delta)\ p^2\, T^{\frac{2\delta-1}{2}} + \mathcal{B}_{24}(\delta)\ q^2\, T^{\frac{2\delta-3}{2}}\right]
\end{aligned}
\tag{D.3}
$$

with

$$
\begin{aligned}
\mathcal{B}_{0k}(0) &= \frac{1}{4}\left[\beta_k(0) - \frac{1}{3}\,\alpha_k(0)\right] \\
\mathcal{B}_{1k}(\delta) &= \frac{2\,\delta}{(4+\delta)(6+\delta)}\left[\beta_k(\delta) - \frac{1}{3}\,\alpha_k(\delta)\right] \\
\mathcal{B}_{2k}(\delta) &= \frac{6+4\,\delta}{(4+2\,\delta)(6+2\,\delta)}\left[\beta_k(2\,\delta) - \frac{1}{3}\,\alpha_k(2\,\delta)\right]\ ;
\end{aligned}
\tag{D.4}
$$

$$
\begin{aligned}
< \frac{1}{2}\,c^2 c_z, Q > &= -\frac{1}{Kn}\left[\mathcal{C}_{01}(0)\ \rho\, q\, T^{\frac{1}{2}} + \mathcal{C}_{02}(0)\ p\, q\, T^{-\frac{1}{2}}\right] \\
&\quad - \frac{\beta}{Kn}\left[\mathcal{C}_{11}(\delta)\ \rho\, q\, T^{\frac{\delta+1}{2}} + \mathcal{C}_{12}(\delta)\ p\, q\, T^{\frac{\delta-1}{2}}\right] \\
&\quad + \frac{\beta^2}{Kn}\left[\mathcal{C}_{21}(\delta)\ \rho\, q\, T^{\frac{2\delta+1}{2}} + \mathcal{C}_{22}(\delta)\ p\, q\, T^{\frac{2\delta-1}{2}}\right]
\end{aligned}
\tag{D.5}
$$

where

$$
\begin{aligned}
\mathcal{C}_{0k}(0) &= \frac{1}{4}\left[\delta_k(0) - \frac{1}{3}\,\gamma_k(0)\right] \\
\mathcal{C}_{1k}(\delta) &= \frac{1}{(4+\delta)(6+\delta)}\left[(10+\delta)\,\gamma_k(\delta) + (2\,\delta)\,\delta_k(\delta)\right] \\
\mathcal{C}_{2k}(\delta) &= \frac{1}{(4+2\,\delta)(6+2\,\delta)}\left[(8+2\,\delta)\,\gamma_k(2\,\delta) + (6+4\,\delta)\,\delta_k(2\,\delta)\right].
\end{aligned}
\tag{D.6}
$$

Notice that collision contributions to the equation for q involve either q or pq. Analogously, collision contributions to the equation for p involve either p, or p^2, or q^2, since it is easy to check that $\mathcal{B}_{01}(0) = \mathcal{B}_{11}(\delta) = \mathcal{B}_{21}(\delta) = 0$.

Appendix E

In order to obtain the second order contributions $\bar{p}_1$ and $\bar{q}_1$ in the asymptotic series

$$
\bar{p} = \sum_{n=0}^{\infty} \bar{p}_n\,\epsilon^n \qquad\qquad \bar{q} = \sum_{n=0}^{\infty} \bar{q}_n\,\epsilon^n
\tag{E.1}
$$

one has to expand the coefficient β as $\sum_{n=0}^{\infty} \alpha_n \epsilon^n$, $\alpha_1 = \alpha$, insert everything into (52) and (53), and equate equal powers of ϵ, bearing in mind that, according to Chapman and Enskog, also the time derivative operator must be formally expanded.[29] To leading order, we reproduce of course (54) and (55). To next order we get the algebraic equations

$$\frac{\partial_0 \bar{p}_0}{\partial t} + \frac{\partial(u\,\bar{p}_0)}{\partial z} + \frac{4}{3}\bar{p}_0\frac{\partial u}{\partial z} + \frac{8}{15}\frac{\partial \bar{q}_0}{\partial z} =$$

$$= -\frac{G(\rho)}{\sqrt{\pi}}\frac{4}{5}\left[4\,\rho\,T^{\frac{1}{2}}\,\bar{p}_1 + \frac{1}{7}\bar{p}_0^2\,T^{-\frac{1}{2}} + \frac{2}{75}\,\bar{q}_0^2\,T^{-\frac{3}{2}}\right] - \alpha\,\frac{G(\rho)}{\sqrt{\pi}}\,\frac{2^{\delta+4}}{15}\,\delta\,\rho\,\bar{p}_0\,T^{\frac{\delta+1}{2}}\Gamma\left(\frac{\delta+4}{2}\right) \tag{E.2}$$

$$\frac{\partial_0 \bar{q}_0}{\partial t} + \frac{\partial(u\,\bar{q}_0)}{\partial z} + \frac{11}{5}\,\bar{q}_0\frac{\partial u}{\partial z} + \frac{5}{2}\bar{p}_0\frac{\partial T}{\partial z} + \rho\,T\frac{\partial}{\partial z}\left(\frac{\bar{p}_0}{\rho}\right) =$$

$$= -\frac{G(\rho)}{\sqrt{\pi}}\frac{8}{5}\left[\frac{4}{3}\,\rho\,T^{\frac{1}{2}}\,\bar{q}_1 + \frac{1}{5}\,\bar{p}_0\,\bar{q}_0\,T^{-\frac{1}{2}}\right] - \alpha\,\frac{G(\rho)}{\sqrt{\pi}}\,\frac{2^{\delta+2}}{15}\,(25+11\,\delta)\,\rho\,\bar{q}_0\,T^{\frac{\delta+1}{2}}\Gamma\left(\frac{\delta+4}{2}\right) \tag{E.3}$$

where the derivatives $\frac{\partial_0 \bar{p}_0}{\partial t}$, $\frac{\partial_0 \bar{q}_0}{\partial t}$ may be obtained by applying the zero–th order time derivative to (54) and (55). This implies the evaluation of the same type of derivatives for ρ, u, T, which may be obtained from (49), (50), (51) as

$$\begin{aligned}
\frac{\partial_0 \rho}{\partial t} &= -\,u\,\frac{\partial \rho}{\partial z} - \rho\,\frac{\partial u}{\partial z} \\
\frac{\partial_0 u}{\partial t} &= -\,u\,\frac{\partial u}{\partial z} - \frac{\partial T}{\partial z} - \frac{T}{\rho}\,\frac{\partial \rho}{\partial z} \\
\frac{\partial_0 T}{\partial t} &= -\,u\,\frac{\partial T}{\partial z} - \frac{2}{3}\,T\,\frac{\partial u}{\partial z} - \alpha\,\frac{2^{\delta}}{\sqrt{\pi}}\,\frac{16}{3}\,G(\rho)\,\rho\,T^{\frac{\delta+3}{2}}\,\Gamma\left(\frac{\delta+4}{2}\right)
\end{aligned} \tag{E.4}$$

At this point, achieving (62), (63) is only matter of heavy algebra.

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