

## ANALYSIS OF A MODEL FOR WEALTH REDISTRIBUTION

DANIEL MATTHES

GIUSEPPE TOSCANI

Department of Mathematics at the University of Pavia  
via Ferrata 1, 27100 Pavia, Italy.

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ABSTRACT. A recent application of the kinetic theory for many particle systems is the description of the redistribution of wealth among trading agents in a simple market economy. This paper provides an analytical investigation of the particular model with *quenched saving propensities*, which has been introduced by Chakrabarti, Chatterjee and Manna [11]. We prove uniqueness and dynamical stability of the stationary solution to the underlying Boltzmann equation, and provide estimates on the rate of equilibration. As one main result, we obtain that realistic steady wealth distributions with Pareto tail are only algebraically stable in this framework.

### 1. Introduction.

**1.1. Motivation: econophysical models.** In the past decade, various kinetic models have been proposed in order to describe the redistribution of wealth in a simple market economy. In an *ad hoc* manner, molecules and momenta are replaced by agents and their respective wealth, and binary collisions translate into trade interactions. Just as the classical Boltzmann equation is defined by prescription of the collision kernel for the microscopic particle interactions, the econophysical model is defined in terms of exchange rules for wealth in trades. The paradigm behind such a description is that the laws of statistical mechanics for particle systems also govern the features of a market composed of a sufficiently large number of interacting agents. Though the “microscopic” modelling of the individuals and their interactions is over-simplified, the “macroscopic” observables like the shape of the wealth distribution curve are often in good agreement with empirical data.

The probably most widely accepted requirement on a realistic econophysical model is that it predicts the formation of a rich “upper class” of agents. Denoting by  $P_\infty(w)$  the density<sup>1</sup> of agents with wealth  $w > 0$  in the system’s steady state, then  $P_\infty(w)$  is expected to possess a fat tail,

$$\int_w^\infty P_\infty(w') dw' \propto w^{-\nu}, \quad \text{for } w \gg 1. \quad (1)$$

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<sup>1</sup>Here we adopt the unfortunate choice to denote by  $P$  the *density*, not the *distribution* function; this abuse of notation is common in the econophysics literature.

The exponent  $\nu$  is referred to as *Pareto index*, named after the economist Vilfredo Pareto, who proposed a law of the form (1) at the end of the 19th century [22], see also [18]. Pareto tails (1) appear in various economic contexts. For instance, they fit very precisely the income distribution among the richest 10% of the population in an arbitrary western country. The respective index  $\nu$  ranges between 1.5 and 2.5.

Up to now, there is little *a priori* knowledge about how to design a kinetic trade model satisfying Pareto's law. Hence, the value of a specific model is usually estimated *a posteriori* by comparing its predictions for  $P_\infty$  to (1).

Although there is a huge amount of freedom in the design of econophysical models, certain features are shared by almost all of them. One is *randomness*, meaning that the amount of money changing hands is non-deterministic. Another is *saving*, meaning that each agent is guaranteed to retain at least a certain minimal fraction of his initial wealth at the end of the trade. A third – and very decisive – ingredient for the vast majority of models is the *pointwise conservation of wealth* in individual trades; this corresponds to the strict conservation of momentum in collisions in ideal gases. Studies on this particular class of models resulted in dozens of publications within the last few years; see the book [12] for an overview on recent research.

Numerous numerical simulations for pointwise conservative models have been carried out, with different mechanisms for saving and varying degree of randomness, leading to very similar results. It was found that the particular realization of randomness has little influence on the shape of  $P_\infty$ , and can usually be substituted by a deterministic process [9, 10]. Furthermore, for “generic choices” of the model parameters, one observes a prevalence of two extremal cases: In part of the experiments,  $P_\infty$  has an exponentially small tail [1, 5, 6, 15]; in the other part, Pareto tails are found [10, 11, 17], but are of critical index  $\nu = 1$ , corresponding to an unrealistic steady distribution of infinite mean wealth.

For definiteness, we restrict attention to one typical example from now on, the model with *quenched saving propensity* introduced by Chatterjee, Chakrabarti and Manna [11], called CCM-model in the following. Each agent carries a personal spending propensity<sup>2</sup> – a parameter  $s \in (0, \frac{1}{2})$  – which varies from agent to agent and does not change with time. In a binary trade between agents  $i$  and  $j$ , money changes hands according to the rule

$$\begin{aligned} w_i^* &= (1 - 2s_i)w_i + 2\epsilon(s_i w_i + s_j w_j) \\ w_j^* &= (1 - 2s_j)w_j + 2(1 - \epsilon)(s_i w_i + s_j w_j). \end{aligned} \quad (2)$$

Here  $w_i$ ,  $w_j$  and  $w_i^*$ ,  $w_j^*$  denote the agents' wealth before and after the trade, respectively, and  $s_i$ ,  $s_j$  are the respective personal spending propensities. This model is clearly pointwise conservative,  $w_i^* + w_j^* = w_i + w_j$ . The quantity  $\epsilon \in [0, 1]$  is a random variable originally, but we limit ourselves to the deterministic case  $\epsilon \equiv \frac{1}{2}$ .

The defining parameter for the CCM-model is the probability density  $\rho$  on  $(0, \frac{1}{2})$ , according to which the spending propensities are distributed among the agents. It has been argued by Mohanty [20] that steady wealth distributions with arbitrary tail can be created if the density  $\rho$  is suitably chosen: Obviously, equilibrium is achieved in (2) if there exists a constant  $M > 0$  such that  $s_i w_i = M$  for all agents.

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<sup>2</sup>The original model was formulated in terms of the *saving propensities*  $\lambda = 1 - 2s$ .

This corresponds to the wealth distribution

$$P_*(w) = \frac{M}{w^2} \rho\left(\frac{M}{w}\right). \quad (3)$$

The tail of  $P_*$  is completely determined by the behavior of  $\rho(s)$  in an arbitrarily small neighborhood of  $s = 0$ .

Apart from the aforementioned numerical investigations, there are numerous theoretical studies on the kinetic dynamics of the CCM-model, e.g. [8, 10, 14, 17]. However, these publications deal with appropriate limits, simplifications or finite-size approximations of the kinetic equation. To our knowledge, the results presented here are the first rigorous ones for the full Boltzmann equation corresponding to the CCM-model. We stress that there are relations to various “non-standard” Boltzmann models of Maxwell type. The latter have attracted a lot of attention in the kinetic community during the last decade: We refer to [3] and references therein for first rigorous investigations on the relaxation behavior of inelastic Maxwell molecules, and to [4] for some more recent results and generalizations. Although the CCM-model does not fit into the framework of these papers, we could adapt techniques developed therein.

Our investigation is performed in two ways. First, we analyse the steady states of the kinetic model with *finitely many agents*, as it appears in the context of Monte Carlo simulations. Second, we investigate the Boltzmann equation associated to the kinetic model in the limit of an *infinite number of agents*. We prove under natural conditions on  $\rho$  that the wealth distribution  $P_*$  in (3) indeed constitutes the unique steady state for the infinite-agent model. And we estimate the rates at which solutions converge to this steady state. These rates are directly related to the tail shape of the wealth distribution: For compactly supported  $P_*$  or exponentially small tails, the steady state is attained at an exponential or super-algebraic rate, respectively. If  $P_*$  is of Pareto type with Pareto index  $\nu > 1$ , then the system equilibrates on an algebraic time scale only, with rate  $t^{-(\nu-1)}$ . The convergence behavior we find is in agreement with recent numerical experiments and semi-analytic considerations, see e.g. [8] and references therein.

We close with a remark concerning the applicability of the CCM-model. On one side, we prove convergence towards the prescribed money distribution  $P_*$  from (3) in a variety of cases. On the other side, our results reveal an undesirable property of the model: the appearance of Pareto tails is a fragile phenomenon. Not only is the complete structure of the tail of  $P_*$  destroyed by an arbitrarily small variation of the model parameter  $\rho(s)$  near  $s = 0$ . But also the slow algebraic convergence rate towards the steady money distribution means that the stationary state will become unstable under perturbative effects.

In comparison, Pareto tails of adjustable index appear naturally once one breaks with the paradigm of pointwise conservation of money. In several models for open economies, where wealth increases due to trade interactions, the money distribution curve attains a self-similar profile with a Pareto tail, see e.g. [13, 21, 23]. Likewise, a combination of trades with risky investments, which preserve money in the statistical mean, but not in individual trades, are able to generate Pareto behavior [19]. In all mentioned models, the Pareto index depends in a robust way on the parameters for the microscopic interaction, and the steady state is exponentially attracting.

**1.2. Analysis of Monte Carlo simulations.** In section 2, we analyse the kinetic model with exchange rules (2) for *finitely many agents*. This is the typical situation encountered in Monte-Carlo simulations, which have been described in detail e.g. in [7]. A particular realization of the finite-agent model is defined by the number  $N \geq 2$  of participating agents and the individual spending propensities  $s_i \in (0, \frac{1}{2})$ , which are sampled in accordance to the given probability density  $\rho$ . The state of the system at a particular time corresponds to the  $N$  numbers  $w_i \geq 0$ , representing the current wealth of the agents.

We prove that each realization possesses a unique steady state, and that the  $w_i$  converge towards their respective limits  $W_i = M/s_i$  as the number of successive trade interactions goes to infinity. The key tool is the use of the entropy functional  $H$  defined in (9), which monotonically decreases with the number of trades. Subsequently, we estimate the speed of convergence by relating the entropy production, i.e. the average change of  $H$  by any particular trade, to  $H$  itself. We find that the convergence is exponential in the number of trades, but with a rate inverse proportional to the smallest appearing saving propensity  $s_i > 0$ . Thus, the relaxation time strongly depends on the specific realization.

Next, we introduce the *configurational average*  $P_N : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , which is the average over the stationary wealth distributions for *all possible realizations* of the  $N$ -agent model with fixed parameter  $\rho$ . The average of the results from a sufficiently large number of Monte-Carlo simulations is an approximation to  $P_N$ . For the limit of infinitely many agents,  $N \rightarrow \infty$  (keeping the average wealth  $m > 0$  per agent fixed) we prove that if

$$\int_0^{\frac{1}{2}} s^{-1} \rho(s) ds < \infty, \quad (4)$$

then  $P_N$  converges uniformly to the steady state  $P_*$  given in (3). If (4) is violated, then  $P_N$  converges (weakly in the sense of measures) to a Dirac distribution at  $w = 0$ . We remark that (4) is violated for a uniform measure  $\rho \equiv 2 \cdot \mathbf{1}_{(0,1/2)}$ , which is frequently used in the CCM-model.

Finally, we calculate the tail shape of  $P_N$  explicitly for finite  $N$ , when  $\rho(s)$  coincides with a power function in an arbitrarily small neighborhood of  $s = 0$ .  $P_N$  displays the finite-size correction of the Pareto tail of  $P_*$ , which forms in the limit  $N \rightarrow \infty$ . In particular, we obtain a formula for the finite-size Pareto tail of critical index one, which has been the object of extensive studies, [7, 10, 17].

**1.3. The Boltzmann equation.** To free the model from finite-size effects, we pass to the related homogeneous Boltzmann equation (21) in section 3. The latter is the evolution equation for the density  $f(t; w, s)$  of agents with wealth  $w \geq 0$  and spending propensity  $s \in (0, \frac{1}{2})$  at time  $t \geq 0$ . The collision kernel is defined by means of the rules (2).

The corresponding wealth distribution  $P(t; w)$  is given by the  $w$ -marginal of  $f$ , and the model parameter  $\rho$  constitutes the time-independent  $s$ -marginal. A consequence of our results is that the marginal  $P(t)$  converges weakly (with explicit rates) to  $P_*$  given in (3), under mild hypotheses on the initial condition and provided that (4) holds. If (4) is violated, then  $P(t)$  converges weakly to a Dirac distribution at  $w = 0$ .

Probably more interesting than the convergence of  $P(t)$  is the relaxation behavior of the solution to the underlying kinetic equation. Define the mean wealth  $W(t; s)$

of agents with a certain spending propensity  $s \in (0, \frac{1}{2})$ ,

$$W(t; s) = \rho(s)^{-1} \cdot \int_{\mathbb{R}_+} w f(t; w, s) dw.$$

This quantity satisfies the evolution equation

$$\frac{d}{dt} W(t; s) = -sW(t; s) + \int_0^{\frac{1}{2}} \sigma W(t; \sigma) d\rho(\sigma). \quad (5)$$

The unique steady state of the Boltzmann equation is the singular density  $f_\infty$ , concentrated on the graph

$$\mathcal{M}_\infty = \{(w, s) | w = W_\infty(s)\} \subset \mathbb{R}_+ \times (0, \frac{1}{2}),$$

where  $W_\infty$  is a stationary solution to (5). More precisely, if (4) holds, then  $W_\infty(s) = Ms^{-1}$ , where  $M > 0$  is related to the average wealth. If (4) is violated, then  $W_\infty \equiv 0$ .

The main result, formulated in Theorem 3.8, is that under mild conditions on the initial data and the parameters,  $f(t)$  concentrates on the graph

$$\mathcal{M}(t) = \{(w, s) | w = W(t; s)\},$$

which itself converges to  $\mathcal{M}_\infty$ . Moreover, in the regime corresponding to Pareto tails of index  $1 < \nu < 2$  for  $P_*$ , one finds a clear separation of time scales: In the first stage,  $f(t)$  concentrates on  $\mathcal{M}(t)$ . In the second stage, the concentrated measure moves with the graph  $\mathcal{M}(t)$  towards the limit  $f_\infty$ . For a Pareto index  $\nu$  close to one, the two time scales are approximately separated by a factor  $t^{(1-\nu)/2}$ .

Correspondingly, the proof consists of two steps. First, the speed of convergence of  $W(t)$  towards its limit  $W_\infty$  is estimated, using suitable entropy functionals that decay along solutions of (5). As expected, the convergence rate is directly connected to the vanishing behavior of  $\rho(s)$  near  $s = 0$ ; for a compactly supported  $P_*$ , we prove exponential convergence, for  $P_*$  with a Pareto tail, the rate is algebraic. Second, an evolution equation for the  $w$ -variance of  $f(t)$  with respect to the mean  $W(t)$  is derived. The solution of the latter equation can be estimated from the results of the first step.

We stress that, despite its simple appearance, equation (5) is not completely trivial to analyse. As a by-product of our results, we obtain an asymptotic expansion of its solutions in Proposition 1 in terms of  $s^{-1}$ , with rigorous error estimates. In fact, our results for (5) carry over – word by word – to the following generalization of the *radiative transfer equation* [16],

$$\frac{d}{dt} W(t; \omega) = -s(\omega)W(t; \omega) + \int_{\Omega} s(\omega')W(t; \omega')d\mu(\omega'),$$

in which  $\mu$  is a given probability measure on the domain  $\Omega$ , and  $s : \Omega \rightarrow \mathbb{R}_+$  is some prescribed function. The speed of convergence of  $W(t)$  to the steady state  $W_\infty(\omega) \propto s(\omega)^{-1}$  depends on if and how  $s$  tends to zero at the boundary of  $\Omega$ .

**1.4. Notations.** For us, the set  $\mathbb{R}_+ = [0, +\infty)$  includes zero. By abuse of notation, we shall frequently abbreviate  $\int_0^{\frac{1}{2}} \phi(s) \cdot \rho(s) ds$  by  $\int \phi(s) d\rho(s)$ . Even worse, we shall occasionally write  $\int \phi(w, s) \cdot f(w, s) dw ds$  in cases where  $f$  is not a function but merely symbolizes the density of a measure  $f(w, s) dw ds$  on  $\mathbb{R}_+ \times (0, \frac{1}{2})$ , possibly with concentrations. Finally, in proofs,  $C$  denotes a generic (positive and finite) constant, which may vary from one line to the next.

## 2. Analysis of Monte Carlo simulations.

**2.1. Convergence to the steady state.** There is a fixed number  $N$  of agents participating in the simulated game. Denote by  $s_i$  the *spending propensity* of the  $i$ th agent, with  $0 < s_i < \frac{1}{2}$ , so that  $\lambda_i = 1 - 2s_i$  is the saving propensity in the original CCM-model [11]. For simplicity of notation, assume that the parameters  $s_i$  are ordered,  $s_1 \leq s_2 \leq \dots \leq s_N$ . The state of the system at an instant of time is completely described by  $N$  non-negative real numbers  $w_i$ , which denote the respective wealth of the  $i$ th agent.

Some initial distribution of the money is given, i.e. there are  $N$  numbers  $w_i \geq 0$  prescribed. We assume that the average wealth per agent  $m > 0$  is a fixed parameter, so that  $N \cdot m = \sum_{k=1}^N w_k$ . A time step consists of the following:

- Two distinct indices  $i \neq j$  between one and  $N$  are randomly picked.
- The numbers  $w_i$  and  $w_j$  change according to the collision rules (2). In other words, replace  $w_i$  and  $w_j$ , respectively, by

$$w_i^* = (1 - s_i)w_i + s_j w_j, \quad (6)$$

$$w_j^* = s_i w_i + (1 - s_j)w_j. \quad (7)$$

- All other  $w_k$  are left unchanged.

**Lemma 2.1.** *The wealth distribution with*

$$W_i = M/s_i, \quad M := N \cdot m \cdot \left( \sum_{k=1}^N 1/s_k \right)^{-1} \quad (8)$$

*constitutes a steady state for the dynamics.*

*Proof.* Simply consider rules (6)&(7) for two arbitrary agents  $i$  and  $j$ . One finds that

$$W_i^* = (1 - s_i)W_i + s_j W_j = W_i - s_i(m/s_i) + s_j(m/s_j) = W_i,$$

and likewise for  $W_j$ . The normalization constant  $M$  is chosen so that the mean wealth is  $m$ .  $\square$

In order to study the convergence to equilibrium, introduce the deviations  $\Delta_i = w_i - W_i$ . Obviously  $\sum \Delta_k = 0$ , and  $\Delta_1 = \dots = \Delta_N = 0$  iff  $w_i = W_i$  for all  $i = 1, \dots, N$ . To measure the distance of a given wealth distribution from the associated steady state, define

$$H := \sum_{k=1}^N (s_k \Delta_k^2). \quad (9)$$

This quantity constitutes a Lyapunov functional for the dynamics.

**Lemma 2.2.** *Denote by  $H$  and  $H^*$  the respective values at two consecutive time steps. Then,*

$$H^* \leq H - (2 - s_i - s_j) \cdot (s_i \Delta_i - s_j \Delta_j)^2, \quad (10)$$

*where  $i$  and  $j$  are the indices of the agents which have interacted in trading, and the quantities  $\Delta_i$  and  $\Delta_j$  denote the values before the trade took place.*

*Proof.* We only need to investigate the changes of  $s_i\Delta_i^2$  and  $s_j\Delta_j^2$  due to the trade. By linearity of (6)&(7), and since the  $W_k$  constitute a steady state, the differences  $\Delta_i^*$  and  $\Delta_j^*$  after the trade satisfy

$$\Delta_i^* = (1 - s_i)\Delta_i + s_j\Delta_j, \quad \Delta_j^* = s_i\Delta_i + (1 - s_j)\Delta_j.$$

But then,

$$\begin{aligned} & s_i(\Delta_i^*)^2 + s_j(\Delta_j^*)^2 \\ &= s_i[(1 - 2s_i + s_i^2)\Delta_i^2 + 2(s_j - s_i s_j)\Delta_i\Delta_j + s_j^2\Delta_j^2] \\ &\quad + s_j[s_i^2\Delta_i^2 + 2(s_i - s_i s_j)\Delta_i\Delta_j + (1 - s_j + s_j^2)\Delta_j^2] \\ &= s_i\Delta_i^2 + s_j\Delta_j^2 \\ &\quad + (s_i + s_j - 2)s_i^2\Delta_i^2 + 2(2 - s_i - s_j)s_i s_j\Delta_i\Delta_j + (s_i + s_j - 2)s_j^2\Delta_j^2 \\ &= s_i\Delta_i^2 + s_j\Delta_j^2 - (2 - s_i - s_j)(s_i\Delta_i - s_j\Delta_j)^2. \end{aligned}$$

All the other components  $\Delta_k$  remain unaltered. This gives (10).  $\square$

By (10), the “distance”  $H$  cannot increase. Moreover, since  $s_k < \frac{1}{2}$  by assumption,  $H$  is *strictly* reduced by the trade unless  $s_i\Delta_i = s_j\Delta_j$ . Recall that  $\sum \Delta_k = 0$ ; hence  $s_1\Delta_1 = \dots = s_N\Delta_N$  implies that  $\Delta_1 = \dots = \Delta_N = 0$ . In conclusion, as long as the system is not in equilibrium, there is always a pair of agents whose “collision” on the market will strictly reduce  $H$ .

**Theorem 2.3.** *The steady state given in (8) is the unique one. Moreover, as the number of collisions goes to infinity, each  $w_i$  converges to the respective  $W_i$ , for  $i = 1, \dots, N$ , almost surely. This is independent of the initial wealth distribution.*

**2.2. Rate of convergence.** Below, we provide an estimate on the rate at which the wealth distribution approaches its steady state. The key ingredient are the estimate (10) above and the following

**Lemma 2.4.** *Let  $\Delta_i$  for  $i = 1, \dots, N$  be arbitrary real numbers satisfying  $\sum_k \Delta_k = 0$ . Then*

$$\sum_k (s_k \Delta_k)^2 - \frac{1}{N} \left( \sum_k s_k \Delta_k \right)^2 \geq s_1 \cdot \left( \sum_k s_k \Delta_k^2 \right), \quad (11)$$

where  $s_i > 0$  are the spending propensities.

*Proof.* Since the sum of the  $\Delta_i$  is zero, we obtain

$$\left( \frac{1}{N} \sum_k s_k \Delta_k \right)^2 = \left( \frac{1}{N} \sum_k (s_k - s_1) \Delta_k \right)^2 \leq \frac{1}{N} \sum_k (s_k - s_1)^2 \Delta_k^2,$$

where the estimate follows from Hölder’s inequality for sums. Consequently,

$$\sum_k s_k^2 \Delta_k^2 - \frac{1}{N} \left( \sum_k s_k \Delta_k \right)^2 \geq 2s_1 \sum_k s_k \Delta_k^2 - s_1^2 \sum_k \Delta_k^2,$$

and the claim (11) follows since  $s_1$  is the smallest of the  $s_k$ .  $\square$

Denote by  $\langle H^* \rangle$  the expectation value of  $H$  after one trade between two randomly chosen agents  $i$  and  $j$ . According to (10),

$$\begin{aligned} \langle H^* \rangle - H &\geq \langle (s_i \Delta_i - s_j \Delta_j)^2 \rangle = \frac{1}{N(N-1)} \sum_{i,j} (s_i \Delta_i - s_j \Delta_j)^2 \\ &= \frac{2}{N-1} \left( \sum_k (s_k \Delta_k)^2 - \frac{1}{N} \left( \sum_k s_k \Delta_k \right)^2 \right). \end{aligned}$$

Estimate (11) now yields the relation

$$\langle H^* \rangle \leq \left( 1 - \frac{2s_1}{N-1} \right) \cdot H.$$

Iterations lead to an estimate for  $H_T$ , the value of  $H$  after  $T$  consecutive trades:

$$\langle H_T \rangle \leq H_0 \cdot \exp \left( -2s_1 \cdot \frac{T}{N-1} \right). \quad (12)$$

Define  $t = T/N$  as the time variable of the simulations, so that the unit time interval corresponds to one trade per agent on the average. By definition of  $H$ , one obtains exponential relaxation of  $w$  in  $L^2$  in the statistical mean,

$$\left\langle \sum_{k=1}^N (w_k(t) - W_k)^2 \right\rangle \leq \frac{H_0}{s_1} \cdot e^{-s_1 t}. \quad (13)$$

Although (13) constitutes only a lower bound on the speed of convergence, numerical experiments indicate [8] that the temporal rate in this estimate is sharp. The size of  $s_1 > 0$  – which depends on the specific realization of the experiment – is seemingly decisive for the relaxation process. This is disturbing in particular since the corresponding average relaxation time  $(s_1)^{-1}$  depends on the system size  $N$  in general, or may even be infinite.

**2.3. Shape of the tails.** This section provides a calculation of the shape of the ensemble-averaged density  $P$ . Let us recall how  $P(w)$  is calculated from the numerical simulations [7]: a series of  $K > 1$  experiments is carried out, in which the number of participating agents  $N$  and the average wealth per agent  $m$  is fixed. The spending propensities  $s_i \in (0, \frac{1}{2})$  are assigned to agents according to a fixed law with continuous density  $\rho(s)$  at the beginning of each experiment. By the previous considerations, there is a unique steady state for the discrete game, given by (8), and it is approached exponentially fast. To the steady state of the  $k$ th game, the empirical distribution

$$P_N^{(k)}(w) = \frac{1}{N} \sum_{j=1}^N \delta_{w_j}(w)$$

is associated,  $\delta_{w_j}(w)$  being the Dirac distribution centered at  $w = w_j$ . Finally, the ensemble density  $P_N$  is calculated as the average of these single distributions,

$$P_N(w) = \frac{1}{K} \sum_{k=1}^K P_N^{(k)}.$$

We propose the following definition of  $P_N$ :

**Definition 2.5.** Let  $N$  independent, identically distributed (according to  $\rho(s)ds$ ) random variables  $s_i$  on  $(0, \frac{1}{2})$  be given. To these, associate the normalization constant

$$M_N = m \cdot N \cdot \left( \sum_{j=1}^N s_j^{-1} \right)^{-1}.$$

Then  $P_N$  is the probability density of  $w_1 = M_N/s_1$ , the wealth of the first agent in the steady state.

Notice that, since we do *not* assume ordering of the  $s_i$  in this definition, the first agent is not distinguished from the others. Our main interest is the convergence of the  $P_N$  towards the limit distribution as introduced by Mohanty [20],

$$P_*(w) = \frac{m}{\Gamma w^2} \rho\left(\frac{m}{\Gamma w}\right), \quad (14)$$

where  $\Gamma$  is the expectation value of  $s_1^{-1}$ ,

$$\Gamma := \int_0^{1/2} s^{-1} d\rho(s). \quad (15)$$

But the latter might be infinite, so that (14) is not well-defined. We obtain the following:

**Theorem 2.6.** *Let the density function  $\rho : (0, \frac{1}{2}) \rightarrow \mathbb{R}_+$  be given.*

- *If  $\rho$  is continuous, with  $\lim_{s \rightarrow 1/2} \rho(s) = 0$ , and such that  $\Gamma < +\infty$ , then each  $P_N(w)$  is a continuous function on  $\mathbb{R}_+$  with support in  $[0, N \cdot m]$ , and  $P_N(0) = 0$ . Moreover,  $P_N$  converges to  $P_\infty$  as  $N \rightarrow \infty$  in the sense that*

$$\sup_{w \in \mathbb{R}_+} \left( w^2 \cdot |P_*(w) - P_N(w)| \right) \rightarrow 0. \quad (16)$$

- *If  $\Gamma = +\infty$ , then  $P_N$  converges weakly towards a Delta distribution at  $w = 0$  as  $N \rightarrow \infty$ .*

The convergence in (16) guarantees a slightly better approximation of the tail of  $P_*$  than just uniform convergence on  $\mathbb{R}_+$ .

*Proof.* In order to derive a formula for  $P_N$ , define yet another random variable

$$S_N = \frac{1}{N} \sum_{j=2}^N s_j^{-1},$$

which has some density  $\eta_N$  on  $\mathbb{R}_+$ , which can in principle be calculated from  $\rho$ . We arrive at an alternative representation of  $w_1$ ,

$$w_1 = m \cdot \left( \frac{1}{N} + s_1 S_N \right)^{-1}. \quad (17)$$

Elementary calculations give

$$\Theta(w) := \mathbb{P}[w_1 \leq w] = \mathbb{P}[m \leq (s_1 S_N + 1/N)w] = \mathbb{P}[s_1 S_N \geq \frac{m - w/N}{w}].$$

Further, by independence of  $s_1$  and  $S_N$ ,

$$\Theta(w) = \int_0^\infty \mathbb{P}[s_1 \geq \frac{m - w/N}{wz}] \cdot \eta_N(z) dz = \int_0^\infty \left( \int_{m/(wz) - 1/(Nz)}^\infty \rho(s) ds \right) \eta_N(z) dz.$$

In summary,

$$P_N(w) = \Theta'(w) = \frac{m}{w^2} \int_1^\infty \rho\left(\frac{m}{wz} - \frac{1}{Nz}\right) \eta_N(z) \frac{dz}{z}, \quad (18)$$

with the convention that  $\rho(s) = 0$  for  $s$  outside the interval  $[0, \frac{1}{2}]$ . In the representation (18), we have used that  $s_i \leq \frac{1}{2}$  and hence  $S_N \geq 2(N-1)/N \geq 1$  to improve the lower limit of the integral.

Now, assume that  $\rho$  is continuous, and  $\Gamma < \infty$ . Continuity of  $P_N$  follows immediately by the representation (18). Furthermore, using that  $\rho(s) = 0$  for  $s \geq 1/2$ ,

$$\begin{aligned} P_N(w) &= \frac{m}{w^2} \int_{2(m/w-1/N)}^\infty \eta_N(z) \rho\left(\frac{m}{wz} - \frac{1}{Nz}\right) \frac{dz}{z} \\ &\leq \frac{mR}{w^2} \int_{2(m/w-1/N)}^\infty \eta_N(z) \frac{dz}{z} \\ &\leq \frac{R}{4m} \int_{2(m/w-1/N)}^\infty z \eta_N(z) dz, \end{aligned}$$

where  $R := \sup \rho < \infty$  by continuity of  $\rho$ . The last expression converges to zero as  $w \searrow 0$ , since  $\eta_N$  is the probability density of  $S_N$ , which has finite expectation value by assumption. This proves that  $P_N(w) \rightarrow 0$  as  $w \rightarrow 0$ .

Let some  $\epsilon > 0$  be given. By the law of large numbers,  $\mathbb{P}[|S_N - \Gamma| < \delta] \rightarrow 0$  as  $N \rightarrow \infty$  for any fixed  $\delta > 0$ . For each  $\delta > 0$ , there exists an  $N_0 \in \mathbb{N}$  such that

$$\int_{\Gamma-\delta}^{\Gamma+\delta} \eta_N(z) dz > 1 - \epsilon$$

for all  $N \geq N_0$ . In combination, we derive

$$\begin{aligned} &\frac{w^2}{m} |P_*(w) - P_N(w)| \\ &= \left| \frac{1}{\Gamma} \rho\left(\frac{m}{w\Gamma}\right) - \int_1^\infty \eta_N(z) \rho\left(\frac{m}{wz} - \frac{1}{Nz}\right) \frac{dz}{z} \right| \\ &\leq \left| \frac{1}{\Gamma} \rho\left(\frac{m}{w\Gamma}\right) - \int_{\sigma-\delta}^{\sigma+\delta} \eta_N(z) \rho\left(\frac{m}{wz} - \frac{1}{Nz}\right) \frac{dz}{z} \right| + \epsilon \cdot R \\ &\leq \int_{\Gamma-\delta}^{\Gamma+\delta} \eta_N(z) \left| \frac{1}{\Gamma} \rho\left(\frac{m}{w\Gamma}\right) - \frac{1}{z} \rho\left(\frac{m}{wz} - \frac{1}{Nz}\right) \right| dz + \epsilon \cdot P_\infty(w) + \epsilon \cdot R \\ &\leq \sup_{|z-\Gamma|<\delta} \left| \frac{1}{\Gamma} \rho\left(\frac{m}{w\Gamma}\right) - \frac{1}{z} \rho\left(\frac{m}{wz} - \frac{1}{Nz}\right) \right| + 2\epsilon \cdot R. \end{aligned}$$

If  $w \leq m/\Gamma$  and  $\delta < \Gamma/2$ , then it is easily checked that both terms inside the supremum are zero. On the other hand, it follows from continuity of  $\rho$  that this supremum can be made smaller than  $R\epsilon$  – uniformly in  $w \geq m/\Gamma$  – by diminishing  $\delta$  and increasing  $N_0$  accordingly. In conclusion, for all  $w > 0$ ,

$$w^2 \cdot |P_*(w) - P_N(w)| \leq 3m\epsilon R.$$

In the case that the integral in (15) diverges, it is another application of the law of large numbers that for each fixed  $\gamma > 0$ , one has  $\mathbb{P}[S_N < \gamma] \rightarrow 0$  as  $N \rightarrow \infty$ . Consequently, for any  $W > 0$ ,

$$\mathbb{P}[w_1 > W] = \mathbb{P}\left[\frac{1}{N} + s_1 S_N < \frac{1}{W}\right] \leq \mathbb{P}[s_1 < \delta] + \mathbb{P}[s_1 \geq \delta] \cdot \mathbb{P}\left[S_N < \frac{1}{\delta} \left(\frac{1}{W} - \frac{1}{N}\right)\right]$$

can be made smaller than any given  $\epsilon > 0$  by choosing  $\delta > 0$  suitably, and having  $N$  large enough. This means that all mass concentrates at  $w = 0$  as  $N \rightarrow \infty$ .  $\square$

If  $\rho(s)$  behaves like a power function near  $s = 0$ , then the tail of  $P_N$  is explicit.

**Theorem 2.7.** *Assume that  $\rho(s) = a \cdot s^\alpha$  with some  $a > 0$ ,  $\alpha \geq 0$  for  $0 \leq s \leq \bar{s}$ . Then there exist constants  $G_N^\rho > 0$  such that the corresponding density functions  $P_N$  satisfy*

$$P_N(w) = \begin{cases} G_N^\rho \cdot \frac{(m-w/N)^\alpha}{w^{2+\alpha}} & \text{if } \frac{m}{2\bar{s}} \leq w \leq N \cdot m, \\ 0 & \text{if } w > N \cdot m. \end{cases} \quad (19)$$

If  $\alpha > 0$ , then  $G_N^\rho \rightarrow G_\infty^\rho > 0$  as  $N \rightarrow \infty$ , whereas  $G_N^\rho = O(1/\log N)$  if  $\alpha = 0$ .

Moreover, if  $\rho(s) = 0$  for  $0 \leq s \leq \bar{s}$ , then each  $P_N$  is supported in the interval  $[0, m/(2\bar{s})]$ .

We emphasize that the hypothesis on  $\rho(s)$  is only concerned with its behavior near  $s = 0$ , and the conclusions hold for  $P_N$  on the whole range of large  $w$ . In order to estimate the magnitude of the tail, i.e. the values of  $G_N^\rho$ , one needs additional information on  $\rho$ .

*Proof.* Clearly,  $P_N(w) = 0$  for  $w > N \cdot m$  by Definition 2.5. Assume that  $w \geq \frac{m}{2\bar{s}}$ ; one easily verifies that the argument of  $\rho$  in (18) is always less than  $\bar{s}$  on the whole range on integration. Hence, by the prescribed homogeneity of  $\rho$ ,

$$P(w) = \frac{(m-w/N)^\alpha}{w^{2+\alpha}} \cdot ma \cdot \int_1^\infty \eta_N(z) \cdot z^{-(\alpha+1)} dz.$$

This proves the claim (19) on the shape of the tail of  $P_N$ . For  $\alpha > 0$ , convergence of the  $G_N^\rho$  follows from Theorem 2.6. It remains to prove the decay of  $G_N^\rho$  if  $\alpha = 0$ . Recall that  $\int_0^\infty w dP(w) = m$ , the average wealth  $m$  per agent, which is independent of  $N$ . But by (19),

$$\int_0^\infty w dP(w) \geq G_N^\rho \cdot \int_{m/(2\bar{s})}^{N \cdot m} \frac{dw}{w} = G_N^\rho \cdot \log(2N\bar{s}),$$

which implies  $G_N^\rho \leq m/\log(2N\bar{s})$ .

The proof in the case  $\rho(s) = 0$  is an obvious alternation of the argument above.  $\square$

### 3. Analysis of the Kinetic Model.

**3.1. The Boltzmann equation.** It is one paradigm of econophysics that the evolution of a system of very many interacting agents is modelled by the homogeneous Boltzmann equation. In the model at hand, the Boltzmann equation describes particles which have two characteristics: their (time-dependent) wealth  $w \geq 0$  and their (fixed) spending propensity  $s \in (0, \frac{1}{2})$ . Denote by  $f(t; w, s)$  the probability density of the agents possessing wealth  $w$  and spending propensity  $s$  at time  $t > 0$ . Then  $f$  satisfies

$$\partial_t f = Q_+(f, f) - f, \quad (20)$$

where the *collisional gain operator*  $Q_+$  describes the redistribution of wealth due to the binary trade interactions of agents. With the collision rules defined in (6)&(7),

the weak form of equation (20) reads

$$\frac{d}{dt} \int \Phi(w, s) df(w, s) = \int [\Phi((1-s)w + s'w', s) - \Phi(w, s)] df(w, s) df(w', s'). \quad (21)$$

Here  $\Phi : \mathbb{R}_+ \times (0, \frac{1}{2}) \rightarrow \mathbb{R}$  denotes an arbitrary – non-negative and sufficiently regular – test function. It is a classical result, see e.g. [2], that for a prescribed initial probability density

$$f(0; w, s) = f_0(w, s),$$

there exists a measure-valued solution  $f(t)$  satisfying (21). We are going to study the convergence of  $f(t)$  for large times.

Assume that the  $s$ -marginal of  $f_0$  coincides with  $\rho$ , the prescribed density of the saving propensities,

$$\int_{\mathbb{R}_+} f_0(w, s) dw = \rho(s).$$

By substituting a  $w$ -independent test function  $\Phi(w, s) = \phi(s)$  into (21), one verifies that the respective integral is time-independent,

$$\int_0^{\frac{1}{2}} \phi(s) \left( \int_{\mathbb{R}_+} f(w, s) dw \right) = \int_0^{\frac{1}{2}} \phi(s) d\rho(s).$$

Hence the  $s$ -marginal of  $f(t; w, s)$  is always  $\rho(s)$ , in accordance to the fact that agents do not change their spending propensity in time. Next, introduce the local mean wealth  $W(s)$  for  $s$  in the support of  $\rho$  by

$$W(t; s) := \rho(s)^{-1} \cdot \int_{\mathbb{R}_+} w f(t; w, s) dw.$$

Using test functions of the form  $\Phi(w, s) = w\phi(s)$  in (21), one obtains

$$\frac{d}{dt} \int \phi(s) W(t; s) d\rho(s) = - \int s W(t; s) \phi(s) d\rho(s) + \int \phi(s) d\rho(s) \int s W(t; s) d\rho(s).$$

This is a weak form of

$$\partial_t W(t; s) = -s W(t; s) + \int s' W(t; s') d\rho(s'), \quad t \geq 0, s \in \text{supp}(\rho). \quad (22)$$

The following section 3.2 is devoted to the study of (22). We defer the further analysis of (21) to section 3.3, where we prove that  $f(t; w, s)$  concentrates on the graph defined by  $w = W(t; s)$ .

**3.2. Convergence properties of  $W(t)$ .** The equation (22) is linear,

$$\partial_t W(t) = -A[W(t)], \quad A[f](s) = sf(s) - \int s' f(s') d\rho(s'),$$

and the operator  $A$  is well-defined and bounded on  $L_\rho^1$ . To any given non-negative initial datum  $W_0 \in L_\rho^1$ , there exists a unique solution  $W(t)$ ; this solution is non-negative and preserves the mean wealth,

$$m := \int W_0(s) d\rho(s) = \int W(t; s) d\rho(s).$$

For the following asymptotic analysis, assume the following bound on the initial data  $W_0 \in L^1_\rho$ ,

$$W_0(s) \leq Ks^{-1}, \quad (23)$$

for a suitable finite constant  $K > 0$ . Then, provided that the integral

$$\Gamma := \int s^{-1} d\rho(s) \quad (24)$$

exists,  $W(t)$  converges – pointwise and in  $L^1_\rho$  – to the steady state

$$W_\infty(s) := m/\Gamma s^{-1}. \quad (25)$$

This is the unique positive steady state for (22) with given total mean wealth  $m > 0$ . If the integral in (24) diverges, then  $W(t)$  converges to  $W_\infty \equiv 0$  pointwise, but not in  $L^1_\rho$ . The rates of convergence are now studied in detail, using a hierarchy of Lyapunov functionals.

Introduce functions  $J_k(s)$  for  $k = 0, 1, 2, \dots$  recursively by

$$J_0 = W, \quad J_{k+1} = A[J_k].$$

Clearly, each  $J_k$  satisfies the same linear equation as  $W$  itself,

$$\frac{d}{dt} J_k(t; s) = -J_{k+1}(t; s) = -sJ_k(t; s) + \int \sigma J_k(t; \sigma) d\rho(\sigma). \quad (26)$$

Further, define the associated entropy functionals

$$F_k = \int J_k(s)^2 d\rho(s) \geq 0, \quad G_k = \int sJ_k(s)^2 d\rho(s) \geq 0.$$

Note that  $F_k$  is the variance of  $sJ_{k-1}(s)$  and thus measures the deviation of  $W(s)$  from a steady state.

**Lemma 3.1.** *Under the hypothesis (23), the entropies  $F_1, F_2, \dots$  and  $G_0, G_1, \dots$  are finite, and decay as follows:*

$$\frac{d}{dt} F_k = -2G_k, \quad \frac{d}{dt} G_{k-1} = -2F_k \quad (27)$$

for each  $k = 1, 2, \dots$

*Proof.* The bound (23) implies that  $W(t; s) \leq Ks^{-1}$  at any positive time  $t > 0$ . This follows easily since  $Ks^{-1}$  constitutes a steady state of the equation (22), which imposes a maximum principle on its solutions. Thus  $G_0 = \int sW(t; s)^2 d\rho(s) \leq K \int W(t; s) d\rho(s) = K \cdot m$ . Furthermore,  $J_1(s) \leq sW \leq K$  is bounded, and so are all  $J_k$  with  $k \geq 1$ . This proves finiteness of  $F_k$  and  $G_k$ .

Next, observe that  $dJ_k/dt = -A[J_k] = J_{k+1}$  by definition. Hence,

$$\begin{aligned} \frac{d}{dt} F_k &= 2 \int J_k(s) \cdot \frac{d}{dt} J_k(s) d\rho(s) \\ &= 2 \int \left( -sJ_k(s)^2 + J_k(s) \int s' J_k(s') d\rho(s') \right) d\rho(s) \\ &= -2G_k + \int J_k(s) d\rho(s) \cdot \int s' J_k(s') d\rho(s'). \end{aligned}$$

By definition of  $J_k(s)$ , the integral  $\int J_k(s) d\rho(s)$  vanishes for  $k \geq 1$ ; this yields the first relation in (27). Furthermore,

$$\begin{aligned} \frac{d}{dt} G_{k-1} &= 2 \int s J_{k-1}(s) \cdot \frac{d}{dt} J_{k-1}(s) d\rho(s) \\ &= 2 \int s J_{k-1} J_k d\rho(s) \\ &= -2F_k + 2 \int J_k(s) d\rho(s) \cdot \int s' J_{k-1}(s') d\rho(s'). \end{aligned}$$

Again,  $\int J_k(s) d\rho(s)$  vanishes for  $k \geq 1$ , producing the second relation in (27).  $\square$

Hence all  $F_k$  and  $G_k$  are non-increasing in time, except for  $F_0 = \int W(s)^2 d\rho(s)$ . In fact, they converge to zero as  $t \rightarrow \infty$ , except for  $F_0$  and possibly  $G_0$ . From the structure of the differential equation hierachy, it is immediate to draw conclusions on their decay behavior.

It is natural to distinguish two cases, namely the one where the spending propensity of all agents has a positive lower bound, and the one where the infimum of the spending propensities is zero. In terms of the linear operator  $A$ , this corresponds to the existence and the absence of a spectral gap at zero, respectively.

3.2.1. *Finite spectral gap.* Assume the existence of a spectral gap of size  $\epsilon > 0$ , i.e.

$$\rho(s) = 0 \quad \text{for all } 0 \leq s \leq \epsilon. \quad (28)$$

In particular,  $\Gamma$  is well-defined by (24).

**Lemma 3.2.** *Under the assumption (28), the entropies  $F_k$  with  $k \geq 1$  decay exponentially fast,*

$$F_k(t) \leq F_k(0) \exp(-2\epsilon t). \quad (29)$$

*Proof.* From (27), one has

$$\frac{d}{dt} \int_{\epsilon}^{\frac{1}{2}} J_k(s)^2 d\rho(s) = -2 \int_{\epsilon}^{\frac{1}{2}} s J_k(s)^2 d\rho(s) \leq -2\epsilon \int_{\epsilon}^{\frac{1}{2}} J_k(s)^2 d\rho(s).$$

The Gronwall Lemma now yields estimate (29).  $\square$

**Theorem 3.3.** *Under the assumption (28), there exists some  $C > 0$  – depending only on  $\epsilon$  and on  $F_1(0)$  – for which*

$$\|W(t) - W_{\infty}\|_{L^1_{\rho}} \leq C \cdot \exp(-\epsilon t), \quad (30)$$

$$\sup_{s \geq \epsilon} |W(t; s) - W_{\infty}(s)| \leq C \cdot \exp(-\epsilon t). \quad (31)$$

It is not hard to conclude that the given temporal relaxation rates cannot be improved without further restrictions on  $\rho$  or the initial condition  $W_0$ .

*Proof.* Introduce

$$Q(t) := \int s W(t; s) d\rho(s).$$

By Hölder's inequality, the definition of  $F_1$  and (29),

$$\begin{aligned} \int_{\epsilon}^{\frac{1}{2}} |W(t; s) - s^{-1} Q(t)| d\rho(s) &\leq \left( \int_{\epsilon}^{\frac{1}{2}} (W(t; s) - s^{-1} Q(t))^2 d\rho(s) \right)^{\frac{1}{2}} \\ &\leq \epsilon^{-\frac{1}{2}} F_1^{\frac{1}{2}} \leq C \exp(-\epsilon t). \end{aligned}$$

Moreover, by conservation of the mean wealth,

$$\begin{aligned} |m - \Gamma Q(t)| &= \left| \int W(t; s) d\rho(s) - Q(t) \int s^{-1} d\rho(s) \right| \\ &\leq \int |W(t; s) - s^{-1}Q(t)| d\rho(s) \leq C \exp(-\epsilon t). \end{aligned}$$

In conclusion,

$$\begin{aligned} &\int_{\epsilon}^{\frac{1}{2}} |W(t; s) - W_{\infty}(s)| d\rho(s) \\ &\leq \left( \int_{\epsilon}^{\frac{1}{2}} (W(t; s) - s^{-1}Q(t))^2 d\rho(s) \right)^{\frac{1}{2}} + \sup_{s \geq \epsilon} |s^{-1}Q(t) - W_{\infty}(s)| \leq C \exp(-\epsilon t), \end{aligned}$$

proving (30). The pointwise estimate (31) follows from the integral representation of (26) for  $J_1$ ,

$$J_1(T; s) = e^{-sT} J_1(0; s) + \int_0^T e^{-s(T-t)} \left[ \int \sigma J_1(t; \sigma) d\rho(\sigma) \right] dt.$$

Using (29), it is immediate to estimate the modulus of  $J_1$ ,

$$\begin{aligned} |J_1(T; s)| &\leq e^{-sT} |J_1(0; s)| + \int_0^T e^{-s(T-t)} \left( \int \sigma^2 d\rho(\sigma) \right)^{\frac{1}{2}} \cdot F_1)^{1/2} dt \\ &\leq e^{-\epsilon T} K + C \exp(-\epsilon t). \end{aligned}$$

□

**3.2.2. No spectral gap.** If the spectrum at zero is closed, i.e. (28) is violated for all  $\epsilon > 0$ , one generically finds algebraic decay of the solution instead of exponential one<sup>3</sup>. In the following, let  $\mu \geq -1$  be an exponent such that

$$\int s^{-(1+\mu)} d\rho(s) < \infty. \quad (32)$$

Introduce correspondingly for  $k = 0, 1, 2, \dots$ ,

$$\alpha_k := \begin{cases} k + \mu/2 & \text{if } \mu > 0 \\ k & \text{if } -1 \leq \mu \leq 0 \end{cases} \quad (33)$$

Notice that (32) is *always* satisfied with  $\mu = -1$ . However, the quantity  $\Gamma$  in (24) is well-defined iff  $\mu \geq 0$  is possible in (32). Also, recall that by (3) a Pareto tail of index  $\nu > 1$  in (1) corresponds to a density satisfying  $\rho(s) \propto s^{\nu-1}$  near  $s = 0$ . Correspondingly, one may choose an arbitrary  $\mu > 0$  with  $\mu < \nu - 1$  in (32).

**Lemma 3.4.** *The entropies  $F_k$  and  $G_k$  with  $k \geq 1$  converge monotonically to zero and satisfy*

$$F_k(t) \leq C_k \cdot (1+t)^{-(2\alpha_k-1)}, \quad G_k(t) \leq C_k \cdot (1+t)^{-2\alpha_k} \quad (34)$$

with  $\alpha_k$  from (33) and suitable constants  $C_k$  depending only on  $m$  and  $K$  in (23).

<sup>3</sup>The borderline case of super-algebraic behavior,  $\rho(s) = O(s^{-\infty})$  as  $s \rightarrow 0$ , is not discussed here. By similar arguments, one proves correspondingly super-algebraic rates for equilibration.

*Proof.* First assume that  $\mu \geq 0$  in (32). As in the proof of Lemma 3.1, we remark that  $W(s; t) \leq Ks^{-1}$ , and hence  $|J_1(s; t)| \leq K$  for all  $t \geq 0$ . By Hölder's inequality,

$$\int J_1(s)^2 d\rho(s) \leq \left( \int s^{-(1+\mu)} d\rho(s) \right)^{\frac{1}{2+\mu}} \cdot K^{\frac{2}{2+\mu}} \cdot \left( \int sJ_1(s)^2 d\rho(s) \right)^{\frac{1+\mu}{2+\mu}},$$

and the first integral on the right-hand side is finite by condition (32). In combination with (27) for  $k = 1$ , one thus obtains

$$\frac{d}{dt} \int J_1(s)^2 d\rho(s) = -2 \int sJ_1(s)^2 d\rho(s) \leq -c \left( \int J_1(s)^2 d\rho(s) \right)^{\frac{2+\mu}{1+\mu}}$$

with a positive constant  $c > 0$ . Comparison with the solution of the ODE  $\dot{x} = -cx^{(2+\mu)/(1+\mu)}$  yields estimate (34) for  $F_1$ . All remaining inequalities now follow inductively from (27): since  $G_k$  is positive and monotonically decreasing, one deduces

$$F_k(T) = 2 \int_T^\infty G_k(t) dt \geq 2 \int_T^{2T} G_k(t) dt \geq 2TG_k(2T),$$

so that  $G_k(t) \leq F_k(t/2)/t$ , which eventually yields (34). A similar argument holds for  $F_k$ .

Finally, if  $\mu \leq 0$  in (32), one uses that  $0 \leq G_0(t) \leq K \cdot m$ , and that  $F_1$  is monotonically decreasing in order to conclude

$$(G_0(0) - G_0(T)) = 2 \int_0^T F_1(t) dt \geq 2T \cdot F_1(T).$$

This provides the estimate (34) for  $F_1$ . The remaining estimates now follow inductively, as above.  $\square$

As a consequence of these estimates, we obtain an asymptotic formula for  $W(t)$ .

**Proposition 1.** *Assume (32) with  $\mu \geq -1$ . Denote  $\beta_k = \int sJ_k(s) d\rho(s)$  and define*

$$\mathcal{W}_k(t; s) := \beta_0(t) \cdot s^{-1} + \beta_1(t) \cdot s^{-2} + \cdots + \beta_{k-1}(t) \cdot s^{-k}.$$

*For arbitrary  $k \geq 1$ , and provided that  $\alpha_k > 1$ , there are finite constants  $C_k, C_\alpha$  such that*

$$\int s^{2k+1} [W(t; s) - \mathcal{W}_k(t; s)]^2 d\rho(s) \leq C_k \cdot (1+t)^{-2\alpha_k}, \quad (35)$$

$$|W(t; s) - \mathcal{W}_k(t; s)| \leq C_\alpha s^{-(\alpha+k)} \cdot (1+t)^{-\alpha}, \quad (36)$$

*where  $\alpha$  is arbitrary with the restriction  $1 \leq \alpha \leq \alpha_k$ . If  $\alpha_k = 1$ , then the right-hand side in (36) is replaced by  $Cs^{-2}(1+t)^{-1} \log(1+t)$ .*

*Moreover,  $|\beta_k| \leq B_k(1+t)^{-\alpha_k}$  for  $k \geq 1$  and suitable finite constants  $B_k$ .*

Notice that  $\beta_k(t) = (-1)^{k+1} \partial_t^k W(t; 0)$ , the signed time derivatives of  $W(t; s)$  at  $s = 0$ .

A particular consequence of Proposition 1 is that

$$|W(t; s) - \beta_0(t)s^{-1}| \leq C \cdot \begin{cases} s^{-2}(1+t)^{-1} & \text{if } \mu > 0, \\ s^{-2}(1+t)^{-1} \log(1+t) & \text{if } \mu < 0. \end{cases}, \quad (37)$$

which is obtained from (36) with  $k = 1$  and  $\alpha = 1$ .

*Proof.* It is easy to check by induction that

$$J_k(s) = s^k W(s) - \beta_0(t) \cdot s^{k-1} - \beta_1(t) \cdot s^{k-2} - \dots - \beta_{k-1}(t) \cdot s^0. \quad (38)$$

Formula (35) now follows immediately from the definition of  $G_k$  and the estimate (34). To obtain the pointwise estimate (36), write the solution to (26) at time  $t \geq 0$  in the form

$$J_k(T; s) = e^{-sT} J_k(0; s) + \int_0^T e^{-s(T-t)} \left( \int \sigma J_k(t; \sigma) d\rho(\sigma) \right) dt. \quad (39)$$

Estimate the product under the integral using (34),

$$\begin{aligned} \left| \int \sigma J_k(t; \sigma) d\rho(\sigma) \right| &\leq \left( \int \sigma d\rho(\sigma) \right)^{1/2} \cdot \left( \int \sigma J_k(t; \sigma)^2 d\rho(\sigma) \right)^{1/2} \\ &\leq C \cdot (1+t)^{-\alpha_k}. \end{aligned}$$

Multiply equation (39) by  $s^\alpha$ , with  $\alpha \geq 1$ . To treat the first term on the resulting right-hand side, observe that  $s^\alpha e^{-st} \leq C(1+t)^{-\alpha}$  for all  $0 \leq s \leq 1/2$  and  $t \geq 0$ . The integral is estimated as follows:

$$\begin{aligned} &\int_0^T s^\alpha e^{-s(T-t)} (1+t)^{-\alpha_k} dt \\ &\leq s^\alpha e^{-sT/2} \int_0^{T/2} (1+t)^{-\alpha_k} dt + (1+T/2)^{-\alpha_k} \int_0^{T/2} s^\alpha e^{-st} dt \\ &\leq C \cdot (1+T)^{-\alpha} \int_0^{T/2} (1+t)^{-\alpha_k} dt + C \cdot (1+T)^{-\alpha_k}. \end{aligned} \quad (40)$$

The integral converges for  $T \rightarrow \infty$  if  $\alpha_k > 1$ . Provided that  $\alpha \leq \alpha_k$ , one directly concludes

$$|J_k(t; s)| \leq C s^{-\alpha} (1+t)^{-\alpha},$$

which, in view of (38), proves (36). For  $\alpha_k = 1$ , one picks up an additional factor  $\log(1+t)$  from the integral in (40). Boundedness of the  $\beta_k$  for  $k \geq 1$  is directly concluded by another application of (34).  $\square$

We are now able to establish a rate for the convergence of  $W(t)$  to equilibrium  $W_\infty$  is  $L_\rho^1$ . In view of the previous lemma, the speed of convergence is essentially determined by the behavior of  $\beta_0(t)$ .

**Theorem 3.5.** *If  $\mu > 0$  in (32), then  $W(t)$  converges to  $W_\infty$  given in (25) in  $L_\rho^1$ ,*

$$\|W(t; s) - W_\infty(s)\|_{L_\rho^1} \leq C t^{-\mu}. \quad (41)$$

*In particular, the coefficient  $\beta_0(t)$  satisfies*

$$|\beta_0(t) - m/\Gamma| \leq C t^{-\mu}. \quad (42)$$

*with  $\Gamma > 0$  defined in (24).*

Observe that one deduces the pointwise estimate

$$|W(t; s) - W_\infty(s)| \leq C \cdot s^{-(1+\mu)} (1+t)^{-\mu} \quad (43)$$

by combining (42) with (36).

*Proof.* We prove (42) first: With  $\epsilon \in (0, \frac{1}{2})$  to be specified below, using (36), one finds

$$\begin{aligned}
|\Gamma\beta_0 - m| &\leq \left| \Gamma - \int_{\epsilon}^{\frac{1}{2}} s^{-1} d\rho(s) \right| \cdot \beta_0 + \left| \int_{\epsilon}^{\frac{1}{2}} s^{-1} \beta_0 d\rho(s) - \int_{\epsilon}^{\frac{1}{2}} W(s) d\rho(s) \right| \\
&\quad + \left| \int_{\epsilon}^{\frac{1}{2}} W(s) d\rho(s) - m \right| \\
&\leq \beta_0 \cdot \int_0^{\epsilon} s^{-1} d\rho(s) + \int_{\epsilon}^{\frac{1}{2}} |W(s) - \frac{\beta_0}{s}| d\rho(s) + \int_0^{\epsilon} W(s) d\rho(s) \\
&\leq \beta_0 \cdot \int s^{-1} d\rho(s) + C_1 \cdot t^{-2} \int_{\epsilon}^{\frac{1}{2}} s^{-2} d\rho(s) + K \cdot \int s^{-1} d\rho(s) \\
&\leq \int s^{-(1+\mu)} d\rho(s) \cdot (\beta\epsilon^{\mu} + C_1 t^{-2} \epsilon^{\mu-1} + K\epsilon^{\mu}).
\end{aligned}$$

The choice  $\epsilon = t^{-1}$  for  $t > 2$  yields the inequality (42). To prove convergence in  $L_{\rho}^1$ , recall that  $W(t; s) \leq Ks^{-1}$ , and write

$$\begin{aligned}
&\int |W(t; s) - \frac{m}{\Gamma s}| d\rho(s) \\
&\leq |\beta_0 - m/\Gamma| + \int_0^{\epsilon} Ks^{-1} d\rho(s) + \int_{\epsilon}^{\frac{1}{2}} |W(t; s) - \frac{\beta_0(t)}{s}| d\rho(s) \\
&\leq C \cdot (1+t)^{-\mu} + \int s^{-(1+\mu)} d\rho(s) \cdot (K\epsilon^{\mu} + C \cdot (1+t)^{-1} \epsilon^{\mu-1}).
\end{aligned}$$

Again, the choice  $\epsilon = t^{-1}$  provides the estimate (41). This finishes the proof.  $\square$

The rates obtained in (41) are sharp in the following sense. Assume that  $\rho$  satisfies, in addition to (32), also

$$\int_0^{\epsilon} s^{-1} d\rho(s) \geq a \cdot \epsilon^{\eta}$$

for small  $\epsilon > 0$  with suitable constants  $a > 0$  and  $0 < \eta < \mu$ . Further, let a bounded initial datum, say  $W_0(s) \leq 1$ , be given, and denote by  $W_{\infty}(s) = Ms^{-1}$  the associated steady state of (22). It follows that  $\int sW(t; s) d\rho(s) \leq 1/2$ , independently of  $t \geq 0$ . Hence  $W(t; s) \leq 1 + t/2$ . Let  $\sigma > 0$  be such that  $1 + t/2 \leq \frac{1}{2}M\sigma^{-1}t$  for all  $t \geq 1$ . For the  $L_{\rho}^1$ -distance of  $W(t)$  to the steady state, one then obtains

$$\|W(t; s) - W_{\infty}(s)\|_{L_{\rho}^1} \geq \frac{1}{2} \int_0^{\sigma/t} Ms^{-1} d\rho(s) \geq c \cdot t^{-\eta},$$

with some  $c > 0$ . For a specific example, consider a density  $\rho$  with  $\rho(s) \propto s^{\nu-1}$  near  $s = 0$ , which generates a Pareto tail of index  $\nu$  in (3). Then estimate (41) holds with arbitrary  $\mu < \nu - 1$ , but is in general false for any  $\mu > \nu - 1$ .

**Theorem 3.6.** *Assume that the integral in (24) diverges. Then  $W(t; s)$  converges pointwise to zero as  $t \rightarrow \infty$ , but does not converge in  $L_{\rho}^1$ .*

*In particular, if*

$$\int_{\epsilon}^{\frac{1}{2}} s^{-1} d\rho(s) \geq a \cdot \epsilon^{-\eta} \tag{44}$$

holds for all small  $\epsilon > 0$  with some  $a > 0$  and  $0 < \eta < 1$ , then

$$\beta_0(t) \leq B_0 \cdot (1+t)^{-\frac{\eta}{1-\eta}} \quad (45)$$

with a suitable finite constant  $B_0 > 0$ . Finally, if the density  $\rho(s)$  is right-continuous at  $s = 0$ , with  $\rho(0) > 0$ , and initially  $W_0(s) \leq \overline{W} < \infty$ , then

$$b_0 \cdot (\log(1+t))^{-1} \leq \beta_0(t) \leq B_0 \cdot (\log(1+t))^{-1}, \quad (46)$$

with suitable constants  $0 < b_0 < B_0 < \infty$ , where  $b_0$  depends on  $\overline{W}$ .

Formula (46) describes the generic situation where  $\rho$  is continuous at zero with  $\rho(0) > 0$ . For instance, the model with uniformly distributed spending propensity falls into this class. Estimate (46) in combination with (37) indicates that a Pareto tail of critical index  $\nu = 1$  seems to form on a time scale  $t^{-1}$  – but collapses to zero on the scale  $(\log t)^{-1}$ .

*Proof.* Define, for small values  $\epsilon > 0$ , the auxiliary quantity

$$\psi(\epsilon) := \int_{\epsilon}^{1/2} s^{-1} d\rho(s),$$

which is non-increasing in  $\epsilon$ , with  $\lim_{\epsilon \searrow 0} \psi(\epsilon) = +\infty$  by assumption. In the spirit of the previous proof, write

$$\begin{aligned} \beta_0(t) \cdot \int_{\epsilon}^{\frac{1}{2}} s^{-1} d\rho(s) &\leq \int_{\epsilon}^{\frac{1}{2}} \left| \frac{\beta_0(t)}{s} - W(t; s) \right| d\rho(s) + \int_{\epsilon}^{\frac{1}{2}} W(t; s) d\rho(s) \\ &\leq \left( \int_{\epsilon}^{\frac{1}{2}} s^{-2} d\rho(s) \right)^{\frac{1}{2}} \cdot \left( \int_{\epsilon}^{\frac{1}{2}} (sW(t; s) - \beta_0(t))^2 d\rho(s) \right)^{\frac{1}{2}} + m \\ &\leq \epsilon^{-\frac{1}{2}} \int_{\epsilon}^{\frac{1}{2}} s^{-1} d\rho(s) \cdot F_1(t)^{1/2} + m. \end{aligned}$$

Divide both sides by  $\psi(\epsilon)$ , and estimate  $F_1(t)$  according to (34), to find

$$\beta_0(t) \leq C \cdot (1+t)^{-\frac{1}{2}} \epsilon^{-\frac{1}{2}} \psi(\epsilon)^{-\frac{1}{2}} + m\psi(\epsilon)^{-1}. \quad (47)$$

Choosing e.g.  $\epsilon = 1/\log(1+t)$ , one observes that  $\beta_0(t) \rightarrow 0$  as  $t \rightarrow \infty$ . The estimate in (36) implies pointwise convergence  $W(t; s) \rightarrow 0$ , which is actually locally uniformly in  $s \in (0, \frac{1}{2}]$ . On the other hand, the  $L^1_\rho$ -norm  $\|W(t)\|_{L^1_\rho} = m$  is preserved in time, so convergence cannot take place in  $L^1_\rho$ .

In order to conclude the refined estimate (45) from (44), choose  $\epsilon = (1+t)^{-1/(1-\eta)}$  in (47). To derive the upper bound in (46), observe that  $\psi(\epsilon) \geq -c \log \epsilon$ ; now choose  $\epsilon = \log(1+t) \cdot (1+t)^{-1}$  in (47).

The lower bound in (46) is obtained in a similar way:

$$\beta_0(t) \cdot \int_{\epsilon}^{\frac{1}{2}} s^{-1} d\rho(s) \geq m - \int_0^{\epsilon} W(t; s) d\rho(s) - \int_{\epsilon}^{\frac{1}{2}} \left| \frac{\beta_0(t)}{s} - W(t; s) \right| d\rho(s),$$

so that, observing that  $\psi(\epsilon) < -C \log \epsilon$  in the case at hand,

$$\beta_0(t) \geq c(-\log \epsilon)^{-1} \left( m - \epsilon \sup_s W(t; s) \right) - C \cdot (1+t)^{-\frac{1}{2}} \epsilon^{-\frac{1}{2}} (-\log \epsilon)^{-\frac{1}{2}}. \quad (48)$$

By the previously proven upper bound on  $\beta_0(t)$ , it is a direct consequence of

$$\frac{d}{dt} W(t; s) = -sW(t; s) + \beta_0(t) \leq B_0 \cdot (\log t)^{-1}$$

that  $W(t; s) \leq \overline{W} + B_0 \log \log t$ . Now put  $\epsilon = \mu / \log \log t$  with a sufficiently small constant  $0 < \mu < m$  in (48). Since

$$\lim_{t \rightarrow \infty} (t^{-1} (\log \log t) (\log (\log \log t))^{-1}) = 0,$$

estimate (48) implies the lower bound in (46).  $\square$

**3.2.3. Relation to the radiative transfer equation.** In the preceding discussion, we performed a detailed asymptotic analysis of solutions to equation (22). Using the infinite hierarchy of Lyapunov functionals  $F_k$  and  $G_k$ , our analysis resulted in the asymptotic expansions stated in Proposition 1. When one is only interested in  $L_\rho^1$ -convergence of  $W(t)$  to  $W_\infty$ , then there is an alternative, more elegant way to obtain (slightly suboptimal) decay estimates.

Equation (22) is formally similar to the radiative transfer equation [16],

$$\frac{d}{dt} u(t; x) = -u(t; x) + \int_{\Omega} u(t; y) dm(y) \quad (49)$$

for  $u : \Omega \rightarrow \mathbb{R}_{\geq 0}$  on a domain  $\Omega$  with probability measure  $m$ . All entropy functionals of the form

$$E = \int \varphi(u) dm - \varphi \left( \int u dm \right), \quad (50)$$

with arbitrary convex functions  $\varphi$ , decay exponentially in time. The corresponding entropy-entropy production estimate reads [16]:

$$E \leq \frac{1}{2} \int \int [\varphi'(u(x)) - \varphi'(u(y))] \cdot [u(x) - u(y)] dm(x) dm(y). \quad (51)$$

This estimate can be applied to (22), leading to

**Theorem 3.7.** *Let a convex function  $\varphi$  be given and define*

$$E(t) = \int \varphi(sW(t; s)) \frac{d\rho(s)}{s\Gamma} - \varphi \left( \int W(t; s) \frac{d\rho}{\Gamma} \right), \quad (52)$$

where  $W(t)$  satisfies (22), and  $\Gamma$  is given in (24).

- If the finite-gap estimate (28) holds with  $\epsilon > 0$ , then

$$E(t) \leq E(0) \cdot \exp(-\Gamma \epsilon^2 \cdot t). \quad (53)$$

- If the no-gap estimate (32) holds with  $\mu > 0$ , then

$$E(t) \leq C \cdot (1+t)^{-\mu}. \quad (54)$$

In fact, since the function  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\psi(r) = |r - M|$  can be approximated in  $C^0$  by smooth convex functions  $\varphi$ , the estimates (53) and (54), respectively, directly imply convergence of  $W(t; s)$  to  $W_\infty = Ms^{-1}$  in  $L_\rho^1$  at the given rates. Moreover, employing similar techniques as in the previous section, one can derive pointwise bounds from there, like (43). Notice, however, that the decay in (53) is not optimal in general, see (30).

*Proof.* Substitute  $dm(s) = d\rho(s)/(s\Gamma)$  and  $u(s) = sW(s)$  into (50). The corresponding entropy production of (22) amounts to

$$\begin{aligned} P &:= -\frac{d}{dt}E(t) \\ &= -\frac{1}{\Gamma} \int \varphi'(sW(t; s)) \cdot \frac{d}{dt}W(t; s) d\rho(s) \\ &= \frac{1}{\Gamma} \iint \varphi'(sW(t; s)) \cdot (sW(t; s) - s'W(t; s')) d\rho(s) d\rho(s') \\ &= \frac{\Gamma}{2} \iint [\varphi'(sW(t; s)) - \varphi'(s'W(t; s'))] \cdot [sW(t; s) - s'W(t; s')] \frac{d\rho(s)}{\Gamma} \frac{d\rho(s')}{\Gamma}. \end{aligned}$$

In the finite-gap situation (28), one concludes

$$\begin{aligned} P &\geq \frac{\Gamma\epsilon^2}{2} \iint [\varphi'(sW(t; s)) - \varphi'(s'W(t; s'))] \cdot [sW(t; s) - s'W(t; s')] \frac{d\rho(s)}{s\Gamma} \frac{d\rho(s')}{s'\Gamma} \\ &\geq \Gamma\epsilon^2 \cdot E(t), \end{aligned}$$

using inequality (51), with the mentioned substitutions. Estimate (53) is a consequence of Gronwall's inequality.

If instead (32) holds for  $\mu > 0$ , then one starts from (51) and estimates, for arbitrary  $R > 0$ ,

$$\begin{aligned} E(t) &\leq \frac{1}{2} \iint [\varphi'(sW(t; s)) - \varphi'(s'W(t; s'))] \cdot [sW(t; s) - s'W(t; s')] \frac{d\rho(s)}{s\Gamma} \frac{d\rho(s')}{s'\Gamma} \\ &\leq 2 \iint_{ss' \leq R} \overline{\varphi'} \cdot K \cdot \frac{d\rho(s)}{s\Gamma} \frac{d\rho(s')}{s'\Gamma} + (\Gamma R)^{-1} P \\ &\leq \frac{\overline{\varphi'} K}{\Gamma^2} \cdot R^\mu \cdot \left( \int s^{-(1+\mu)} d\rho(s) \right)^2 + (\Gamma R)^{-1} P \\ &\leq C \cdot R^\mu + (\Gamma R)^{-1} \cdot P. \end{aligned}$$

Here we have used that  $W(t; s) \leq Ks^{-1}$ , and defined  $\overline{\varphi'} = \sup_{0 \leq r \leq K} |\varphi'(r)|$ . The choice  $R = P^{1/(\mu+1)}$  yields eventually

$$E(t) \leq C \cdot P^{\mu/(\mu+1)}.$$

Again, Gronwall's inequality yields the desired estimate (54).  $\square$

**3.3. Analysis for the full Boltzmann model.** In this terminal section, we establish the connection between the solution  $W(t; s)$  of the linear equation (22) and the solution  $f(t; w, s)$  of the nonlinear Boltzmann equation (21). Introduce for  $s$  in the support of  $\rho$  the *local variances of wealth*,

$$V(t; s) := \rho(s)^{-1} \cdot \int_{\mathbb{R}_+} (w - W(t; s))^2 f(t; w, s) dw = \rho(s)^{-1} \cdot M_2(t; s) - W(t; s)^2,$$

where  $M_2$  denotes the – local – second moment,

$$M_2(t; s) := \int_{\mathbb{R}_+} w^2 f(t; w, s) dw.$$

We shall assume that the integral over the second moments is finite initially, and hence also

$$\int V(0; s) d\rho(s) < +\infty. \quad (55)$$

The quantity  $V$  measures the (de)concentration of the density  $f(t)$  on the graph

$$\mathcal{M}(t) := \{(s, w) \mid s \in \text{supp}(\rho), w = M(t; s)\} \subset \mathbb{R}_+ \times (0, \frac{1}{2}).$$

In fact, if  $\mu_t$  denotes the measure induced by  $f(t)$  on  $\mathbb{R}_+ \times (0, \frac{1}{2})$ , then by Chebyshev's inequality,

$$\mu_t[\{|w - W(t; s)| \geq \delta s^{-1}\}] \leq \delta^{-2} \int s^2 V(t; s) d\rho(s). \quad (56)$$

Our aim is to estimate the time decay of the integral on the right-hand side.

To have an idea of the concentration behavior of  $f(t)$ , consider the Boltzmann equation (21) in the regime of grazing collisions. The corresponding trade rules read

$$w_i^* = w_i - \varepsilon s_i w_i + \varepsilon s_k w_k, \quad (57)$$

i.e. the spending propensities are scaled by some small parameter  $\varepsilon > 0$ . Assuming that the collision frequency increases accordingly by a factor of  $1/\varepsilon$ , one obtains in the limit  $\varepsilon \rightarrow 0$  for an arbitrary test function  $\Phi$ ,

$$\begin{aligned} & \frac{d}{dt} \int \Phi(w, s) df(w, s) \\ &= \lim_{\varepsilon \rightarrow 0} \int \frac{1}{\varepsilon} [\Phi(w + \varepsilon[-sw + s'w'], s) - \Phi(w, s)] df(w, s) df(w', s') \\ &= \int \partial_w \Phi(w, s) \cdot [-sw + s'w'] df(w, s) df(w', s'). \end{aligned} \quad (58)$$

It is immediately seen that the local mean wealth  $W(t; s)$  evolves according to (22) as in the original model. Moreover, the second moment  $M_2(s)$  satisfies

$$\partial_t M_2(s) = -2sM_2(s) + 2\rho(s)W(s) \int \sigma W(\sigma) d\rho(\sigma).$$

For the evolution of the variances  $V = \rho^{-1}M_2 - W^2$ , this implies

$$\partial_t V(s) = -2sV(s). \quad (59)$$

Hence, for each  $s > 0$ , the wealth of agents with spending propensity  $s > 0$  concentrates exponentially fast around the mean value  $W(t; s)$ . In finite-gap situation of (28), one thus obtains exponential concentration at rate  $\exp(-\varepsilon t)$  in (56). In the no-gap situation (32), one obtains for the integral in (56)

$$\int s^2 V(t; s) d\rho(s) \leq \sup_{0 < s < \frac{1}{2}} (s^2 \exp(-2st)) \cdot \int V(0; s) d\rho(s) \leq C \cdot (1+t)^{-2}.$$

Hence concentration in (56) occurs at least at a rate  $(1+t)^{-2}$ . Now recall that the pointwise convergence of  $W(t)$  to  $W_\infty$  takes place at the rate  $(1+t)^{-\mu}$ , see (43). If this rate is optimal with some  $0 < \mu < 1$ , then concentration of  $f$  on  $\mathcal{M}(t)$  happens faster than convergence of  $\mathcal{M}(t)$  to its limit  $\mathcal{M}_\infty$ . In fact, one may choose  $\delta(t) = \delta_0 \cdot (1+t)^{-1}$  in (56), which shows that – after an intermediate time – an arbitrarily large fraction of the agents is  $\delta(t)$ -close to  $\mathcal{M}(t)$ , whereas  $\mathcal{M}(t)$  and  $\mathcal{M}_\infty$  are apart by approximately  $\delta(t) \cdot (1+t)^{1-\mu}$ . In this situation, one finds a clear separation of time scales for the convergence to equilibrium in the simplified Boltzmann equation (58): In the first phase, agents with a certain spending propensity accumulate at the respective mean wealth  $W(t; s)$ . In the second phase, the wealth distribution closely follows the graph  $\mathcal{M}(t)$  defined by  $W(t)$  to the equilibrium distribution  $W_\infty$  from (25).

In the following, we show inasmuch this qualitative behavior carries over from the grazing collision limit (58) to the full Boltzmann equation (21). We shall restrict ourselves to the no-gap case (32), leaving the – much easier – finite-gap situation (28) to the reader.

**Theorem 3.8.** *Assume that  $\rho$  satisfies condition (32) with  $-1 < \mu < 1$ , and that the total variance is initially bounded, cf. (55). Then, with a suitable finite constant  $C$ , or  $C_\delta$ , respectively,*

$$\int s^2 V(t; s) d\rho(s) \leq \begin{cases} C(1+t)^{-(1+\mu)} & \text{if } \mu > 0 \\ C_\delta(1+t)^{-\delta} & \text{if } \mu \leq 0, \text{ with } \delta < 1 \text{ arbitrary} \end{cases} \quad (60)$$

In particular,  $f(t)$  concentrates on the graph  $\mathcal{M}(t)$  at the given rate.

Notice that separation of scales as described before can only be expected if concentration occurs at quadratic rate with respect to convergence of  $W(t)$  to  $W_\infty$ , i.e. if  $(1+\mu)/2 > \mu$  for  $\mu > 0$ . This allows one to choose  $\delta(t) = \delta_0 \cdot (1+t)^{-(1+\mu)/2} \ll (1+t)^{-\mu}$  in (56), and repeat the argument preceding Theorem 3.8. Consequently, we restrict attention to the range  $\mu < 1$ .

*Proof.* Using test functions of the form  $\Phi(w, s) = w^2 \cdot \phi(s)$  in (21), one derives a weak formulation of an evolution equation for  $M_2$ ; the corresponding strong form reads

$$\frac{d}{dt} M_2(s) = s(s-2)M_2(s) + 2(1-s)\rho(s)W(s) \int \sigma W(\sigma) d\rho(\sigma) + \int \sigma^2 M_2(\sigma) d\rho(\sigma).$$

Taking into account that

$$\frac{d}{dt} W(s)^2 = -2sW(s)^2 + 2W(s) \int \sigma W(\sigma) d\rho(\sigma),$$

we obtain for the wealth variance  $V = \rho^{-1} M_2 - W^2$ ,

$$\begin{aligned} \frac{d}{dt} V(s) &= -2sV(s) + s^2 \rho(s)^{-1} M_2(s) + \int \sigma^2 M_2(\sigma) d\sigma - 2sW(s) \int \sigma W(\sigma) d\rho(\sigma) \\ &= -s(2-s)V(s) + \int \sigma^2 V(\sigma) d\rho(\sigma) + \int [sW(s) - \sigma W(\sigma)]^2 d\rho(\sigma) \\ &= -s(2-s)V(s) + \int \sigma^2 V(\sigma) d\rho(\sigma) + J_1(s)^2 + \int J_1(\sigma)^2 d\rho(\sigma). \end{aligned} \quad (61)$$

In particular, the integrated variance satisfied the equality

$$\frac{d}{dt} \int V(s) d\rho(s) = -2 \int s(1-s)V(s) d\rho(s) + 2F_1. \quad (62)$$

Since  $F_1$  is integrable in time, and the total variance is finite initially by (55), it follows that the latter remains uniformly bounded for all  $t \geq 0$ . As  $V$  is non-negative, it follows further that

$$Z(t) := \int s^2 V(t; s) d\rho(s) \leq \int s(1-s)V(t; s) d\rho(s)$$

is integrable in time.

We continue with detailed calculations for the case  $\mu > 0$ , and briefly comment on  $-1 \leq \mu \leq 0$  afterwards. The solution  $V$  to (61) can be represented as

$$V(T; s) = e^{-s(2-s)T} V(0; s) + \int_0^T e^{-s(2-s)(T-t)} (J_1(t; s)^2 + F_1(t) + Z(t)) dt, \quad (63)$$

Multiplication of (63) by  $s^2$  and integration with respect to  $\rho(s)$  yields, after trivial manipulations,

$$\begin{aligned} Z(T) &\leq \int s^2 e^{-sT} V(0; s) d\rho(s) \\ &\quad + \int_0^T \int s^2 e^{-s(T-t)} (J_1(t; s)^2 + F_1(t) + Z(t)) d\rho(s) dt. \end{aligned} \quad (64)$$

We estimate the right-hand side term by term:

$$\begin{aligned} \int s^2 e^{-sT} V(0; s) d\rho(s) &\leq \sup_s (s^2 e^{-sT}) \cdot \int V(0; s) d\rho(s) \\ &\leq C(1+T)^{-2} \leq C(1+T)^{-(1+\mu)}. \end{aligned}$$

Using (36) with  $k = 1$  and  $\alpha = 1$ ,

$$\int s^2 e^{-s(T-t)} J_1(t; s)^2 d\rho(s) \leq C \int s^2 e^{-s(T-t)} (s(1+t))^{-2} d\rho(s) \quad (65)$$

$$\leq C(1+t)^{-2} \sup_s (e^{-s(T-t)} s^{1+\mu}) \cdot \int s^{-(1+\mu)} d\rho(s) \quad (66)$$

$$\leq C(1+t)^{-(1+\mu)} (1+(T-t))^{-(1+\mu)}, \quad (67)$$

since we have assumed  $\mu \leq 1$ . Compute the integral from 0 to  $T$  of the last expression:

$$\begin{aligned} \int_0^T (1+t)^{-(1+\mu)} (1+(T-t))^{-(1+\mu)} dt &\leq 2(1+T/2)^{-(1+\mu)} \int_0^{T/2} (1+t)^{-(1+\mu)} dt \\ &\leq C(1+T)^{-(1+\mu)}. \end{aligned}$$

And with a similar idea, one shows

$$\int_0^T s^2 e^{-s(T-t)} F_1(t) dt \leq C(1+T)^{-(1+\mu)}.$$

We turn to estimate the term involving  $Z$  itself. For  $\eta \in (0, 1)$ , we find the recursion relation

$$\begin{aligned} &\int_0^T \left( \int s^2 e^{-s(T-t)} d\rho(s) \right) \cdot Z(t) dt \\ &\leq \int s^2 e^{-s(1-\eta)T} d\rho(s) \cdot \int_0^{\eta T} Z(t) dt + \sup_{\eta T < t < T} Z(t) \cdot \int_{\eta T}^T \left( \int s^2 e^{-s(T-t)} d\rho(s) \right) dt \\ &\leq \sup_s s^{2+\mu} e^{-s(1-\eta)T} \cdot \int s^{-(1+\mu)} d\rho(s) \cdot \int_0^\infty Z(t) dt \\ &\quad + \sup_{\eta T \leq t \leq T} Z(t) \cdot \int s^2 \left( \int_0^\infty e^{-s\tau} dt \right) d\rho(s) \\ &\leq C(1+T)^{-(2+\mu)} + \frac{1}{2} \sup_{\eta T < t < T} Z(t) \end{aligned}$$

since  $s \leq \frac{1}{2}$ . In conclusion,  $Z(t)$  satisfies, with a suitable constant  $A > 0$ ,

$$Z(T) \leq A \cdot (1+T)^{-(\mu+1)} + \frac{1}{2} \sup_{\eta T < t < T} Z(t). \quad (68)$$

Choose  $\eta \in (0, 1)$  such that  $\eta^{-(\mu+1)} < 2$  and  $B > 0$  such that  $A + \frac{1}{2}\eta^{-(\mu+1)}B < B$ . It follows that for all  $t \geq 0$ ,

$$Z(t) < B \cdot (1+t)^{-(\mu+1)}. \quad (69)$$

Indeed, assume the contrary, and let  $T \geq 0$  be the smallest time, at which (69) is violated. But then (68) immediately implies

$$\begin{aligned} A \cdot (1+T)^{-(\mu+1)} + \frac{1}{2}B \cdot (1+\eta T)^{-(\mu+1)} &\leq \left(A + \frac{1}{2}\eta^{-(\mu+1)}B\right) \cdot (1+T)^{-(\mu+1)} \\ &< B \cdot (1+T)^{-(\mu+1)}, \end{aligned}$$

a contradiction; this finishes the proof for  $\mu > 0$ .

If  $\mu \leq 0$ , then all calculations remain valid *substituting zero for  $\mu$  everywhere* except for the estimates in lines (65)–(67), where one uses the slightly weaker estimate

$$J_1(t; s)^2 \leq C(s^{-1}(1+t)^{-1} \log(1+t))^2 \leq Cs^{-2}(1+t)^{-2\delta},$$

(see Proposition 1), which holds for an arbitrary  $\delta < 1$ .  $\square$

As a consequence of Theorem 3.8 above in combination with Theorem 3.5 or Theorem 3.6, respectively, one can obtain convergence of the wealth distribution function

$$P(t, w) := \int_0^{\frac{1}{2}} f(t; w, s) ds$$

towards the limit  $P_*$  given in (3), or to a Dirac distribution at zero, respectively. The type of (weak) convergence will depend on the regularity of the density  $\rho$ . Assuming that  $\rho$  is a bounded function, one obtains convergence in distribution, with an explicit rate. We prove the corresponding result in the most relevant situation  $\mu > 0$  in (32), leaving the other – slightly easier – cases to the interested reader.

**Corollary 1.** *Assume (32) holds with some  $0 < \mu \leq \frac{1}{2}$ , and that the probability density  $\rho$  is a bounded function. Then, for any  $\hat{w} > 0$ , there exists a suitable  $C > 0$  such that*

$$\left| \int_{\hat{w}}^{\infty} P(t; w) dw - \int_{\hat{w}}^{\infty} P_*(w) dw \right| \leq C(1+t)^{-\mu} \quad (70)$$

*In particular,  $P(t)$  converges weakly in the sense of measures to  $P_*$ .*

*Proof.* The function  $P_*(w)$  coincides with the  $w$ -marginal of the density  $f_\infty(w, s)$ , which is concentrated on the graph

$$\mathcal{M}_\infty = \{(w, s) | w = W_\infty(s)\}, \quad W_\infty(s) = m\Gamma^{-1}s^{-1},$$

and has  $s$ -marginal  $\rho$ . Hence, with  $\hat{s} := m\Gamma^{-1}\hat{w}^{-1}$ ,

$$\int_{\hat{w}}^{\infty} P_*(w) dw = \int_0^{\hat{s}} d\rho(s).$$

Define  $\mu_t$  as the measure on  $\mathbb{R}_+ \times (0, \frac{1}{2})$  associated to  $f(t)$ . Then, for  $\delta > 0$  sufficiently small,

$$\begin{aligned} \int_{\hat{w}}^{\infty} P(t, w) dw &= \mu_t[\{w \geq \hat{w}\}] \\ &\leq \mu_t[\{|w - W_{\infty}(s)| < s^{-1}\delta \ \& \ w \geq \hat{w}\}] + \mu_t[\{|w - W_{\infty}(s)| \geq s^{-1}\delta\}] \\ &\leq \mu_t[\{s\hat{w} \leq m\Gamma^{-1} + \delta\}] \\ &\quad + \mu_t[\{|w - W(t; s)| \geq \frac{1}{2}s^{-1}\delta\}] + \mu_t[\{|W(t; s) - W_{\infty}(s)| \geq \frac{1}{2}s^{-1}\delta\}] \end{aligned}$$

We estimate the last sum term by term.

$$\mu_t[\{s\hat{w} \leq m\Gamma^{-1} + \delta\}] \leq \int_0^{\hat{s}} d\rho(s) + \int_{\hat{s}}^{\hat{s} + \delta/\hat{w}} d\rho(s) \leq \int_{\hat{w}}^{\infty} P_*(w) dw + C\delta,$$

since we have assumed that  $\rho$  is bounded. Further,

$$\mu_t[\{|w - W(t; s)| \geq \frac{1}{2}s^{-1}\delta\}] \leq 4\delta^{-2} \int sV(t; s) d\rho \leq C\delta^{-2}(1+t)^{-(1+\mu)}.$$

Finally,

$$\begin{aligned} \mu_t[\{|W(t; s) - W_{\infty}(s)| \geq \frac{1}{2}s^{-1}\delta\}] \\ \leq \mu_t[\{|J_1(t; s)| \geq \frac{1}{4}\delta\}] + \mu_t[\{|\beta_0(t) - m\Gamma^{-1}| \geq \frac{1}{4}\delta\}] \\ \leq \mu_t[\{s \leq C\delta(1+t)^{-1}\}] \leq C\delta(1+t)^{-1}, \end{aligned}$$

provided that  $t^{-\mu} \leq c\delta$  with a sufficiently small constant  $c > 0$ , implicitly defined in (45). In summary,

$$\int_{\hat{w}}^{\infty} P(t, w) dw = \int_{\hat{w}}^{\infty} P_*(w) dw + C \cdot (\delta + \delta^{-2}(1+t)^{-(1+\mu)} + \delta^{-1}(1+t)^{-1}).$$

Choosing  $\delta = At^{-\mu}$  with  $A > 0$  large enough yields one half of the inequality (70), namely

$$\int_{\hat{w}}^{\infty} P(t; w) dw - \int_{\hat{w}}^{\infty} P_*(w) dw \leq C \cdot (1+t)^{-\frac{\mu}{2}}.$$

The other half is proven in a similar way.  $\square$

**4. Conclusions.** The relaxation behavior of a kinetic equation, which constitutes a model for trading on simple markets, has been analysed. We have shown the existence and uniqueness of an attracting steady state, both for the finite-agent model and for the limiting continuous Boltzmann equation. For solutions to the latter, we obtained explicit rates for the two determining processes of relaxation to equilibrium: the first is concentration of the solution on a lower-dimensional subset of phase space, the second is convergence of the latter subset to its equilibrium position. In a certain regime, these two processes happen (in this order) on separated algebraic time scales. The main analytical tool was the use of suitable Lyapunov functionals.

In our investigations, we restricted attention to a specific model. However, certain features like slow (algebraic) relaxation in the fat tails seem to be typical for the whole class of *pointwise conservative* interactions. Properties of this class will be investigated in further work, in particular in comparison with alternative approaches describing *open economies* or *risky markets*.

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*E-mail address:* giuseppe.toscani@unipv.it