A degeneration formula for quiver moduli and its GW equivalent

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19 June 2012

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Quivers

Quiver Q: an oriented graph; vertices Q_0, arrows Q_1 $(\alpha : i \to j)$.

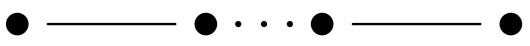
Representations of Q: vector spaces M_i $(i \in Q_0)$, linear maps M_α $(\alpha \in Q_1)$. Same as modules over the path algebra kQ.

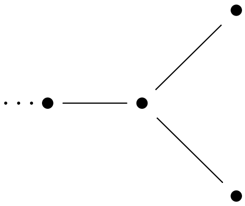
Lattice of Q: $\Lambda = \mathbb{Z}Q_0$; dimension vectors $\Lambda^+ = \mathbb{N}Q_0$ $(d = \sum_{i \in Q_0} d_i i)$.

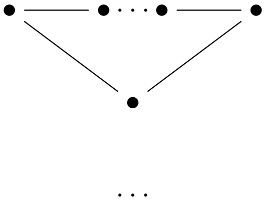
Euler form: $\langle d, e \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha : i \to j} d_i d_j$; $\{d, e\} = \langle d, e \rangle - \langle e, d \rangle$. It's the Euler form of category mod$-kQ$: Ext^i vanish for $i > 1$.

Classification

Problem: classify rep's of Q modulo isomorphism. Find a normal form for each given representation (like Jordan form). Solved only for quivers with support a Dynkin diagram A_n, D_n, E_6, E_7, E_8 or an extended Dynkin diagram $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, e.g.

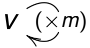




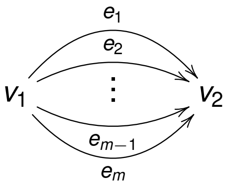


Wild quivers: examples

m-loop:



m-Kronecker:



Moduli

Constructed by A. King.

Parameter space: $R_d(Q) = \bigoplus_{\alpha:i\to j} Hom(M_i, M_j)$.

Gauge group: $G_d = \prod_{i\in Q_0} GL(M_i)$. $G_d \curvearrowright R_d$ via base change: $(g_i)_i \cdot (M_\alpha)_\alpha = (g_j M_\alpha g_i^{-1})_{\alpha:i\to j}$.

Moduli space: quotient "R_d/G_d".

(Affine) GIT

Reduced gauge group: $PG_d = G_d/\mathrm{scalars}$ (to have finite stab's).

Linearization: $\Theta : \Lambda \to \mathbb{Z}$; $\chi_\Theta((g_i)_i) = \prod_{i \in Q_0} \det(g_i)^{\Theta(d) - \dim(d)\theta_i}$.

(Semi)stables: $R_d^{sst}(Q) = R_d^{\Theta-sst}(Q) = (R_d(Q))^{\chi_\Theta - sst}$;
$R_d^{st}(Q) = R_d^{\Theta-st}(Q) = (R_d(Q))^{\chi_\Theta - st}$.

Moduli: $M_d^{\Theta-st}(Q) = R_d^{\Theta-st}(Q)/PG_d$: iso. classes of stables;
$M_d^{\Theta-sst}(Q) = R_d^{\Theta-sst}(Q)//PG_d$: equiv. classes of polystables.

GIT properties

- $R_d^{\Theta-st}(Q) \subset R_d^{\Theta-sst}(Q)$ open inclusion;
- $M_d^{\Theta-st}(Q)$ smooth;
- $R_d^{\Theta-st}(Q) \to M_d^{\Theta-st}(Q)$ principal PG_d-bundle;
- If $M_d^{\Theta-st}(Q) \neq \emptyset$ then $\dim = 1 - \langle d, d \rangle$;
- No oriented cycles $\Rightarrow M_d^{\Theta-sst}(Q)$ projective;
- Slope: $\mu(d) = \frac{\Theta(d)}{\dim(d)}$. (Semi)stable iff $\mu(U) < (\leq)\mu(M)$ for all nontrivial subrep's $U \subset M$;
- If d is Θ-coprime $\Rightarrow M_d^{\Theta-st}(Q) = M_d^{\Theta-sst}(Q)$ is smooth, projective.

Topology

From now on work over $\mathbb{C}$. Let $d^* = (d^1, \ldots, d^s)$ a tuple of dim vectors, with

- $d = d^1 + \ldots + d^s$;
- $d^k \neq 0$ for $k = 1, \ldots, s$;
- $\mu(d^1 + \cdots + d^k) > \mu(d)$ for $k < s$.

Define

$$P_d(q) = \sum_{d^*} (-1)^{s-1} q^{-\sum_{k \leq l} \langle d^l, d^k \rangle} \prod_{k=1}^{s} \prod_{i \in Q_0} \prod_{j=1}^{d_i^k} (1 - q^{-j})^{-1}.$$

Theorem (Reineke)

If d is Θ-coprime, then

$$(q-1) P_d(q) = \sum_i \dim H^i(M_d^{\Theta-st}, \mathbb{C}) q^{i/2}.$$

Corollary: no odd cohomology.

Drawbacks

- All summands are only rational function of $q \Rightarrow$ cannot specialize to $q = 1$ to get χ;
- It's not a "positive" formula: summands have signs.

MPS formula

Discovered by physicists J. Manschot, B. Pioline and Sen motivated by string-theoretic arguments.

Terminology: *abelian quivers* have $d_i \leq 1$ for all i. *Bose-Fermi statistics:* $P(t)$ or χ of non-abelian quivers. *Maxwell-Boltzmann statistics:* $P(t)$ or χ of abelian quivers.

Physical slogan: can trade Bose-Fermi statistics for Maxwell-Boltzmann, provided BPS state count $\Omega(\gamma)$ (roughly χ) is replaced by "effective index" $\bar{\Omega} = \sum_{m|\gamma} \Omega(\gamma/m)/m^2$. Same holds for refined BPS counts $\Omega^{ref}(\gamma, t)$ (roughly $P(t)$).

MPS formula: maths

MPS allows to express $P(t)$ or even motives of nonabelian moduli spaces in terms of abelian ones, at the cost of increasing arrow and twisting the stability condition.

- Fix $i \in Q_0$.
- "Blow i up": define $\widehat{Q}_0 = Q_0 \setminus \{i\} \cup \{i_{k,l} \,:\, k, l \geq 1\}$.
- $\alpha : i \to j$ (resp. $\alpha : j \to i$) in Q for $j \neq i$ induce arrows $\alpha_p : i_{k,l} \to j$ (resp. $\alpha_p : j \to i_{k,l}$) for $k, l \geq 1$ and $p = 1, \ldots, l$.
- Loop $\alpha : i \to i$ in Q induce arrows $\alpha_{p,q} : i_{k,l} \to i_{k',l'}$ for $k, l, k', l' \geq 1$ and $p = 1, \ldots, l$, $q = 1, \ldots, l'$ in $\widehat{Q}$.

Induced dim vectors and slopes

Pick $d \in \mathbb{N}Q_0$. *Multiplicity vector* $m_* \vdash d_i$: a partition $\sum_l l m_l = d_i$, $m_l \geq 0$.
Induced dim vector:

$$\widehat{d}(m_*)_{i_{k,l}} = \begin{cases} 1 & , \quad k \leq m_l \\ 0 & , \quad k > m_l. \end{cases}$$

Level function: $\ell(i_{k,l}) = l$.
Induced slope: $\widehat{\Theta}_{i_{k,l}} = l\Theta_i$.
Twist: $\kappa(d) = \sum_j \ell(j) d_j$.
Twisted slope: $\hat{\mu} = \frac{\hat{\Theta}(d)}{\kappa(d)}$.

MPS formula

Grothendieck ring of varieties: work in
$K = (K_0(Var/\mathbb{C}) \otimes \mathbb{Q})[[\mathbf{L}]^{-1}, ([\mathbf{L}]^n - 1)^{-1}]_{n \geq 1}$.

Theorem (motivic MPS)

For arbitrary Q, d, Θ and i as above, the following identity holds in $\mathcal{K}$:

$$[\mathbf{L}]^{\binom{d_i}{2}} \frac{[R^{\rm sst}_d(Q)]}{[G_d]} = \sum_{m_* \vdash d_i} \prod_{l \geq 1} \frac{1}{m_l!} \left(\frac{(-1)^{l-1}}{l[\mathbb{P}^{l-1}]} \right)^{m_l} \frac{[R^{\rm sst}_{\widehat{d}(m_*)}(\widehat{Q})]}{[G_{\widehat{d}(m_*)}]}.$$

Coprime case

Recall $P(X,t) := \sum_i \dim H^i(X,\mathbb{Q})t^i$. Recall "$q$ numbers" $[n]_q := \frac{q^n-1}{q-1}$.

Corollary

If d is Θ-coprime, we have

$$t^{d_i(d_i-1)} P(M_d^{\mathrm{sst}}(Q),t) = \sum_{m_* \vdash d_i} \prod_{l\geq 1} \frac{1}{m_l!} \left(\frac{(-1)^{l-1}}{l[l]_{t^2}} \right)^{m_l} P(M^{\mathrm{sst}}_{\widehat{d}(m_*)}(\widehat{Q}),t)$$

and

$$\chi(M_d^{\mathrm{sst}}(Q)) = \sum_{m_* \vdash d_i} \prod_{l\geq 1} \frac{1}{m_l!} \left(\frac{(-1)^{l-1}}{l^2} \right)^{m_l} \chi(M^{\mathrm{sst}}_{\widehat{d}(m_*)}(\widehat{Q})).$$

MPS proof (sketch)

Identification: for all $m_* \vdash d_i$ we have a $G_{\widehat{d}(m_*)}$-equivariant isomorphism between $R_d(Q)$ and $R_{\widehat{d}(m_*)}(\widehat{Q})$. Furthermore, we have $\widehat{\mu}(\widehat{d}(m_*)) = \mu(d)$.

MPS for trivial stability $\Theta = 0$:

$$[\mathbf{L}]^{\binom{d_i}{2}} \frac{[R_d(Q)]}{[G_d]} = \sum_{m_* \vdash d_i} \prod_{l \geq 1} \frac{1}{m_l!} \left(\frac{(-1)^{l-1}}{l[\mathbb{P}^{l-1}]} \right)^{m_l} \frac{[R_{\widehat{d}(m_*)}(\widehat{Q})]}{[G_{\widehat{d}(m_*)}]}.$$

This is a clever rearrangement using the combinatorics of symmetric functions. It makes precise the idea of passing from B.-F. statistics to M.-B. one. But we still have to incorporate stability!

MPS proof (sketch)

Idea: use Harder-Narasimhan stratification of $R_d(Q)$.
Fix decomposition $d = d^1 + \ldots + d^s$ into non-zero dimension vectors with $\mu(d^1) > \ldots > \mu(d^s)$. This is a HN type for d, $d^* = (d^1, \ldots, d^s) \models d$.
Let $R_d^{d^*}(Q) \subset R_d(Q)$ the locus of rep's with HN type d^*. Then

$$R_d^{d^*}(Q) \simeq G_d \times^{P_{d^*}} V_{d^*},$$

where $P_{d^*} < G_d$ parabolic with Levi $\prod_{k=1}^{s} G_{d^k}$ and $V_{d^*} \to \prod_{k=1}^{s} R_{d^k}^{\mathrm{sst}}(Q)$ vector bundle of rank $\sum_{k<l} \sum_{p \to q} d_p^l d_q^k$.

MPS proof (sketch)

So

$$\frac{[R_d(Q)]}{[G_d]} = \sum_{d^* \models d} [\mathbf{L}]^{-\sum_{k<l}\langle d^l, d^k\rangle} \prod_{k=1}^{s} \frac{[R^{\mathrm{sst}}_{d^k}(Q)]}{[G_{d^k}]}.$$

Now induction on $\dim(d)$. If $\dim(d) = 1$ we're done.
Otherwise compute $[\mathbf{L}]^{\binom{d_i}{2}}[R_d(Q)]/[G_d]$ in two ways.

MPS proof (sketch)

First way: MPS for $\Theta = 0$ followed by HN recursion on smaller pieces. Get

$$[\mathbf{L}]^{\binom{d_i}{2}} \times$$

$$\sum_{d^* \models d} [\mathbf{L}]^{-\sum_{k<l} \langle d^l, d^k \rangle} \prod_{k=1}^{s} [\mathbf{L}]^{-\binom{d_i^k}{2}} \sum_{m_* \vdash d_i^k} \prod_{l \geq 1} \frac{1}{m_l!} \left(\frac{(-1)^{l-1}}{l[\mathbb{P}^{l-1}]} \right)^{m_l} \frac{[R^{\mathrm{sst}}_{\widehat{d^k}(m^k_*)}(\widehat{Q})]}{[G_{\widehat{d^k}(m^k_*)}]}$$

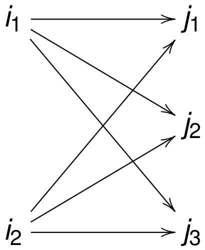
Second way: Just HN. Get

$$[\mathbf{L}]^{\binom{d_i}{2}} \frac{[R_d(Q)]}{[G_d]} = [\mathbf{L}]^{\binom{d_i}{2}} \sum_{d^* \models d} [\mathbf{L}]^{-\sum_{k<l} \langle d^l, d^k \rangle} \prod_{k=1}^{s} \frac{[R^{\mathrm{sst}}_{d^k}(Q)]}{[G_{d^k}]}$$

Summands with $d^* \neq d$ match by induction, so we're done.

MPS and complete bipartite quivers

Q bipartite: $Q_0 = I \cup J$, $I =$ sources, $J =$ sinks.
Natural linear form: $\Theta_i = 1$ for $i \in I$, $\Theta_j = 0$ for $j \in J$.
If we have level $\ell : Q_0 \to \mathbb{N}^+$ can twist as usual, so $\mu = \Theta_\ell/\kappa$.
Complete bip. quivers: $K(l_1, l_2)$. Vertices:
$\{i_1, \dots, i_{l_1}\} \cup \{j_1, \dots, j_{l_2}\}$. Arrows: $\alpha_{k,l} : i_k \to j_l$ for all k, l (just all possible arrows).



Complete bipartite quivers

Dim. vectors: same as $(\mathbf{P}_1, \mathbf{P}_2) = (\sum_{i=1}^{l_1} p_{1i}, \sum_{j=1}^{l_2} p_{2j})$.
From now on assume $|\mathbf{P}_1|, |\mathbf{P}_2|$ *are coprime*. Write

$$M^{\Theta-\mathrm{st}}(\mathbf{P}_1, \mathbf{P}_2) = M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2)}(K(l_1, l_2))$$

Universal MPS quiver (for complete bip): infinite $\mathcal{N}$ with level structure.

$$\mathcal{N}_0 = \{i_{(w,m)} \mid (w, m) \in \mathbb{N}^2\} \cup \{j_{(w,m)} \mid (w, m) \in \mathbb{N}^2\},$$

$$\mathcal{N}_1 = \{\alpha_1, \dots, \alpha_{w \cdot w'} : i_{(w,m)} \to j_{(w',m')}, \forall\, w, w', m, m' \in \mathbb{N}\}.$$

$$l(q_{(w,m)}) = w, \forall\, q \in \{i, j\}, m \in \mathbb{N}$$

Complete bipartite quivers

MPS dim vectors same as *refinements* $(k^1, k^2) = (\{k^1_{wi}\}, \{k^2_{wj}\})$, with $p_{1i} = \sum_w wk^1_{wi}$, $p_{2j} = \sum_w wk^2_{wj}$. Let $m_w(k^i) = \sum_{j=1}^{l_i} k^i_{wj}$.
Induced dim vector:

$$d_{q_{(w,m)}} = \begin{cases} 1 \text{ for } m = 1, \dots, m_w(k^p), \\ 0 \text{ for } m > m_w(k^p), \end{cases}$$

for $q \in \{i, j\}$, and $p = 1, 2$ for $q = i, j$. MPS formula becomes

Lemma (bip MPS)

$$\chi(M^{\Theta-\mathrm{st}}(\mathbf{P}_1, \mathbf{P}_2)) = \sum_{k \vdash \mathbf{P}} \chi(M^{\Theta_l-\mathrm{st}}_{(k^1,k^2)}(\mathcal{N})) \prod_{i=1}^{2} \prod_{j=1}^{l_i} \prod_{w} \frac{(-1)^{k^i_{w,j}(w-1)}}{k^i_{w,j}!\, w^{2k^i_{w,j}}}.$$

Application: χ by summing over trees

Idea: first MPS to get rid of $d_i > 1$ (but increasing arrows). Then use localization wrt $(\mathbb{C}^*)^{\mathcal{N}_1}$ and reduce to just trees. May have to localize many times: theory worked out by Reineke, Weist.

$\mathcal{N}(k^1, k^2) := \mathrm{supp}(k^1, k^2) \subset \mathcal{N}$,
$T(k^1, k^2) := \{\text{connected subtrees of } \mathcal{N}(k^1, k^2)\}$. For $I' \subset T(I)$ (sources) let $\sigma_{I'}(T) = \sum_{j \in N_{I'}} l(j)$, $|I'| = \sum_{i \in I'} l(i)$. Define

$$w(T) = \begin{cases} 1 \text{ if } \sigma_{I'}(T) > \frac{e}{d}|I'| \text{ for all } \emptyset \neq I' \subsetneq T(I) \\ 0 \text{ otherwise} \end{cases}$$

where

$$d := \sum_{i \in T(I)} l(i) \text{ and } e := \sum_{j \in T(J)} l(j)$$

Application: χ by summing over trees

Theorem

$$\chi(M^{\Theta_l-\mathrm{st}}_{(k^1,k^2)}(\mathcal{N})) = \sum_{T\in T(k^1,k^2)} w(T).$$

Corollary

$\chi(M^{\Theta-\mathrm{st}}(\mathbf{P}_1, \mathbf{P}_2))$ *can be computed as a sum over trees.*

Remark

Same method also works for more quivers, e.g. m-Kronecker.

Application: asymptotics

Theorem (Okada)

For a, b coprime, for $m \gg 1$ (depending on a, b), we have

$$\log(\chi(M_{K^m}^{\Theta-st}(a,b))) \sim (a+b-1)\log(m).$$

Theorem (Weist)

For a, b coprime we have

$$\lim_{a\to\infty, b/a\to k} \frac{1}{a}\log(\chi(M_{K^m}^{\Theta-st}(a,b)))$$

$$\leq (k+1)(\log(m)+\log(2)+1)-(k-1)\log(k).$$

Remark

Douglas and Weist have a precise conjecture for the limit.

MPS as a degeneration formula in GW theory

Key idea: χ of moduli spaces of some quivers are very closely related (sometimes **equal**) to certain **Gromov-Witten invariants enumerating rational curves on algebraic surfaces**.

- Gross, Pandharipande, Siebert: "The tropical vertex";
- Kontsevich, Soibelman: "Stability structures...";
- Reineke, "Poisson automorphisms...";
- Gross, Pandharipande: "Quivers, curves...";
- Reineke, Weist: "Refined GW/Kronecker correspondence".

Which quivers? K^m; complete bipartite quivers.
Which surfaces? Weighted projective planes $\mathbb{P}(a, b, 1)$ for coprime a, b.
Which curves? Let D_1, D_2, D_{out} denote toric (i.e. boundary) divisors; D_i^o, D_{out}^o non torus-fixed part. Fix l_i pts on D_i. Fix partitions $(\mathbf{P}_1, \mathbf{P}_2)$ of lenghts l_1, l_2 with $|\mathbf{P}_1|, |\mathbf{P}_2|$ coprime. We look at curves s.t.:

- are rational;
- pass through the l_i pts on D_i^o, with multiplicity p_{ij} (i.e. they are singular pts with prescribed multiplicity);
- pass through a pt on D_{out}^o.

Theorem (GPS)

There is a well-defined virtual count $N[(\mathbf{P}_1, \mathbf{P}_2)]$.

Theorem (Reineke, Weist; refined GW/Kronecker, coprime case)

$$N[(\mathbf{P}_1, \mathbf{P}_2)] = \chi(M^{\Theta-\mathrm{st}}(\mathbf{P}_1, \mathbf{P}_2)).$$

Remark

We're also interested in $|\mathbf{P}_1| = ka, |\mathbf{P}_2| = kb$ *with* a, b *coprime, and curves which are tangent to* D^o_{out} *to order* k*. But the relation to* χ *is much more complicated.*

MPS and GW

Claim: the MPS formula for quiver rep's has a **natural geometric interpretation.** Under identification

$$N[(\mathbf{P}_1, \mathbf{P}_2)] = \chi(M^{\Theta-\mathrm{st}}(\mathbf{P}_1, \mathbf{P}_2))$$

MPS becomes a **standard degeneration formula in GW theory.**

GW degeneration

Deg formula expresses $N[(\mathbf{P}_1, \mathbf{P}_2)]$ in terms of *relative* GW invariants, i.e. with *tangency* conditions.
Weight vectors: $\mathbf{w}_i = (w_{i1}, \ldots, w_{it_i})$ with

$$0 < w_{i1} \leq w_{i2} \leq \cdots \leq w_{it_i}.$$

Relative GW: $N^{\mathrm{rel}}[(\mathbf{w}_1, \mathbf{w}_2)]$, virtually enumerating rational curves in $\mathbb{P}(|\mathbf{w}_1|, |\mathbf{w}_2|, 1)$ which are tangent to D_i at specified points (not fixed by the torus), with order of tangency specified by $\mathbf{w}_i$.
Suppose $|\mathbf{w}_i| = |\mathbf{P}_i|$.
Set partition $I_\bullet$ of $\mathbf{w}_i$: $I_1 \cup \cdots \cup I_{l_i} = \{1, \ldots, t_i\}$, I_i disjoint.
Compatible if for all j, $p_{ij} = \sum_{r \in I_j} w_{ir}$.

Ramification factor: $R_{\mathbf{P}_i|\mathbf{w}_i} = \sum_{l_\bullet} \prod_{j=1}^{t_i} \frac{(-1)^{w_{ij}-1}}{w_{ij}^2}$.

Theorem (very special case of GW deg theory, lots of names)

$$N[(\mathbf{P}_1, \mathbf{P}_2)] = \sum_{(\mathbf{w}_1, \mathbf{w}_2)} N^{\mathrm{rel}}[(\mathbf{w}_1, \mathbf{w}_2)] \prod_{i=1}^{2} \frac{\prod_{j=1}^{t_i} w_{ij}}{|\operatorname{Aut}(\mathbf{w}_i)|} R_{\mathbf{P}_i|\mathbf{w}_i}.$$

Claim: this is **just the same as MPS formula for quivers!**

Comparison

First step. By (rather easy) combinatorics we can rewrite GW deg as a sum over refinements (k^1, k^2) rather than weight vectors $(\mathbf{w}_1, \mathbf{w}_2)$. k^i induces a weight vector $\mathbf{w}(k^i) = (w_{i1}, \dots, w_{it_i})$ of length $t_i = \sum_w m_w(k^i)$, by

$$w_{ij} = w \text{ for all } j = \sum_{r=1}^{w-1} m_r(k^i) + 1, \dots, \sum_{r=1}^{w} m_r(k^i).$$

Then can rewrite GW deg as

$$N[(\mathbf{P}_1, \mathbf{P}_2)] = \sum_{(k_1, k_2) \vdash (\mathbf{P}_1, \mathbf{P}_2)} N^{\mathrm{rel}}[(\mathbf{w}(k^1), \mathbf{w}(k^2))] \prod_{i=1}^{2} \prod_{j=1}^{l_i} \prod_{w} \frac{(-1)^{k^i_{w,j}(w-1)}}{k^i_{w,j}! \, w^{k^i_{w,j}}}.$$

Now compare RHS of MPS

$$\sum_{k \vdash \mathbf{P}} \chi(M^{\Theta_l-\mathrm{st}}_{(k^1,k^2)}(\mathcal{N})) \prod_{i=1}^{2} \prod_{j=1}^{l_i} \prod_{w} \frac{(-1)^{k^i_{w,j}(w-1)}}{k^i_{w,j}! \, w^{2k^i_{w,j}}}$$

and GW degeneration

$$\sum_{(k_1,k_2) \vdash (\mathbf{P}_1,\mathbf{P}_2)} N^{\mathrm{rel}}[(\mathbf{w}(k^1),\mathbf{w}(k^2))] \prod_{i=1}^{2} \prod_{j=1}^{l_i} \prod_{w} \frac{(-1)^{k^i_{w,j}(w-1)}}{k^i_{w,j}! \, w^{k^i_{w,j}}}.$$

We see that the two are equivalent iff
$\chi(M^{\Theta_l-\mathrm{st}}_{(k^1,k^2)}(\mathcal{N})) = N^{\mathrm{rel}}[(\mathbf{w}(k^1),\mathbf{w}(k^2))] \prod_{i=1}^{2} \prod_{j=1}^{l_i} \prod_{w} w^{k^i_{w,j}}$.

Comparison: tropical counts

Claim: the RHS $N^{\mathrm{rel}}[(\mathbf{w}(k^1), \mathbf{w}(k^2))] \prod_{i=1}^{2} \prod_{j=1}^{l_i} \prod_w w^{k^i_{w,j}}$ has a nutural geometric meaning as a *tropical count*.
Parametrized tropical curves in $\mathbb{R}^2$*:* proper maps $h : \Gamma \to \mathbb{R}^2$ where Γ is weighted $3-$valent graph with unbounded ends, such that

- restriction of h to an edge is an embedding whose image is contained in an affine line of rational slope, and
- *balancing condition* $\sum_{i=1}^{3} w_\Gamma(E_i) m_i = 0$.

Tropical curves: equivalence classes under reparametrization.
Multiplicities: $\mathrm{Mult}_V(h) = w_\Gamma(E_1) w_\Gamma(E_2) |m_1 \wedge m_2|$,
$\mathrm{Mult}(h) = \prod_V \mathrm{Mult}_V(h)$.

Comparison: tropical counts

Weight vectors $(\mathbf{w}_1, \mathbf{w}_2)$ encode a tropical count of curves such that:

- the unbounded edges of Γ are E_{ij} for $1 \leq i \leq 2, 1 \leq j \leq t_i$, plus a single “outgoing" edge E_{out}. We require that $h(E_{ij})$ is contained in a line $e_{ij} + \mathbb{R}e_i$ for some *prescribed* vectors e_{ij}, and its unbounded direction is $-e_i$,
- $w_\Gamma(E_{ij}) = w_{ij}$.

Counting with multiplicity we get a well-defined invariant (Mikhalkin, ...)

Theorem (GPS)

$N^{\mathrm{trop}}[(\mathbf{w}_1, \mathbf{w}_2)] = N^{\mathrm{rel}}[(\mathbf{w}_1, \mathbf{w}_2)] \prod_{i=1}^{2} \prod_{j=1}^{t_i} w_{ij}$.

So we are left with proving,

Theorem

$N^{\text{trop}}[(\mathbf{w}(k^1), \mathbf{w}(k^2))] = \chi(M^{\Theta_l-\text{st}}_{(k^1,k^2)}(\mathcal{N})).$

The proof is based on the results of Gross, Pandharipande, Siebert and Reineke on the *tropical vertex group*.
Work on $Q = \mathcal{N}(k^1, k^2)$. Introduce Poisson algebra

$$R = \mathbb{C}[[x_{j_{(w',m')}}, y_{i_{(w,m)}} \mid w, w', m, m' \in \mathbb{N}]]$$

with bracket

$$\{x_{j_{(w',m')}}, y_{i_{(w,m)}}\} = \{j_{(w',m')}, i_{(w,m)}\}\, x_{j_{(w',m')}} \cdot y_{i_{(w,m)}} = ww' x_{j_{(w',m')}} \cdot y_{i_{(w,m)}}.$$

Kontsevich-Soibelman Poisson automorphisms:

$$T_{j_{(w,m)}}(x_{j_{(w',m')}}) = x_{j_{(w',m')}},$$
$$T_{j_{(w,m)}}(y_{i_{(w',m')}}) = y_{i_{(w',m')}} \left(1 + x_{j_{(w,m)}}\right)^{ww'},$$

and

$$T_{i_{(w,m)}}(x_{j_{(w',m')}}) = x_{j_{(w',m')}} \left(1 + y_{i_{(w,m)}}\right)^{-ww'},$$
$$T_{i_{(w,m)}}(y_{i_{(w',m')}}) = y_{i_{(w',m')}}.$$

Idea: compute the product

$$\prod_{j_{(w,m)}\in Q_0} T_{j_{(w,m)}} \cdot \prod_{i_{(w',m')}\in Q_0} T_{i_{(w',m')}}$$

in two different ways.
First way: Reineke's theorem. The above product equals $\prod^{\leftarrow}_{\mu\in\mathbb{Q}} T_\mu$, where e.g.

$$T_\mu(y_{i_{(w,m)}}) = y_{i_{(w,m)}} \prod_{j_{(w',m')}\in Q_0} (Q_{\mu,j_{(w',m')}})^{ww'},$$

and $Q_{\mu,j_{(w',m')}}$ is generating series of Euler characteristics for moduli spaces of stable representations of Q with slope μ and a 1-dimensional framing at $j_{(w',m')}$.

Comparison: tropical vertex

Corollary

The coefficient of the monomial

$$\prod_{j_{(w',m')}\in Q_0} x_{j_{(w',m')}} \cdot \prod_{i_{(w'',m'')}\in Q_0} y_{i_{(w'',m'')}}$$

in the series $y_{i_{(w,m)}}^{-1} T_\mu(y_{i_{(w,m)}})$ *is given by*

$$w \sum_{w'} w' m_{w'}(k^2)\, \chi(M^{\Theta_l-\mathrm{st}}_{(k^1,k^2)}(\mathcal{N}))$$

Comparison: tropical vertex

Second way: scattering diagrams. Sketch: identify $T_{i_{(w,m)}}, T_{j_{(w',m')}}$ with operators acting on the ring

$$R' = \mathbb{C}[x^{\pm 1}, y^{\pm 1}][[\xi_{j_{(w,m)}}, \eta_{i_{(w',m')}} \mid w, w', m, m' \in \mathbb{N}]].$$

Using GPS notation, we identify

$$T_{j_{(w,m)}} = \theta_{(1,0),\left(1+\left(\xi_{j_{(w,m)}} x\right)^{w}\right)^{w}}$$

with a standard element of the tropical vertex group over R', $\mathbb{V}_{R'}$, and similarly

$$T_{i_{(w,m)}} = \theta_{(0,1),\left(1+\left(\eta_{i_{(w,m)}} y\right)^{w}\right)^{w}}.$$

Comparison: tropical vertex

So consider *saturated scattering diagram* for the product

$$\prod_{j_{(w,m)}\in Q_0} \theta_{(1,0),(1+(\xi_{j_{(w,m)}}x)^w)^w} \cdot \prod_{i_{(w',m')}\in Q_0} \theta_{(0,1),(1+(\eta_{i_{(w',m')}}y)^{w'})^{w'}}. \quad (1)$$

(Essentially, add operators along rays to make the whole product around loops trivial). As we are only interested in coeff of $\prod_{j_{(w',m')}\in Q_0} x_{j_{(w',m')}} \cdot \prod_{i_{(w,m)}\in Q_0} y_{i_{(w,m)}}$, work over truncated ring

$$\mathbb{C}[x^{\pm 1}, y^{\pm 1}][[\xi_{j_{(w,m)}}, \eta_{i_{(w',m')}} \mid w, w', m, m' \in \mathbb{N}]]/(\xi^{2w}_{j_{(w,m)}}, \eta^{2w'}_{i_{(w',m')}}).$$

Comparison: tropical vertex

GPS theory: 1-1 correspondence between rays of saturated scattering diagram and rational tropical curves with suitable unbounded edges and weights. With a bit of work, this implies that the coefficient of $\prod_{j_{(w',m')}\in Q_0} x_{j_{(w',m')}} \cdot \prod_{i_{(w,m)}\in Q_0} y_{i_{(w,m)}}$ in the operator attached to ray of slope μ in saturated scattering diagram is

$$w \sum_{w'} w' m_{w'}(k^2) N^{\mathrm{trop}}[(\mathbf{w}(k^1), \mathbf{w}(k^2))].$$

We conclude by essential uniqueness of ordered product factorizations.

Final question

We have proved the equality

$$N^{\mathrm{trop}}[(\mathbf{w}(k^1),\mathbf{w}(k^2))] = \chi(M^{\Theta_l-\mathrm{st}}_{(k^1,k^2)}(\mathcal{N}))$$

But is there actually a natural 1-1 correspondence between curves and representations? At least for generic choices?

In our work we made some progress in this direction, in particular constructing the 1-1 correspondence in some examples. But we don't have a general answer. Personally I believe that in general there is no natural correspondence. Pandharipande observed that this would make the equality of invariants more interesting.