Some applications of K-stability and K-energy

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Declaration

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Abstract

In this thesis we apply the notion of K-stability introduced by S. K. Donaldson and the K-energy functional introduced by T. Mabuchi to study Kähler metrics of constant scalar curvature on smooth complex projective varieties and more generally Kähler manifolds.

Our methods are taken both from algebraic and differential geometry.

A general formula for the behaviour of K-stability under blowing up is proved. The theorem of Arezzo-Pacard on blowing up constant scalar curvature Kähler metrics is restated in algebro-geometric terms, and a converse is given.

One of the main results in the field is a theorem of Donaldson stating that a smooth complex projective variety with a constant scalar curvature Kähler metric must be K-semistable. Under the assumption that the group of projective automorphisms is discrete we strengthen this conclusion to prove K-stability (as predicted by Donaldson).

One of the first applications of K-semistability was a beautiful "slope inequality" for subschemes due to Ross-Thomas. Here we show how, at least for divisors, the slope inequality can be recovered in a completely different way via the K-energy, thus making it applicable to more general Kähler, non-projective manifolds.

Many new and concrete examples are scattered throughout the text.

Keywords: algebraic geometry, Kähler geometry, Geometric invariant theory, stability of varieties, canonical metrics.

Contents

1 Introduction

In this thesis we study the constant scalar curvature equation on projective and Kähler manifolds. Let X be a compact, connected Kähler manifold of complex dimension n. For a given Kähler cohomology class $\Omega \in H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$ and a Kähler metric g whose associated $(1,1)$ form $\omega = \omega_g$ lies in Ω we let $s(\omega)$ denote the scalar curvature,

$$s(\omega) = -g^{i\bar{j}}\partial_i\partial_{\bar{j}}\log(\det(g_{k\bar{l}})) \tag{1.1}$$

in local coordinates. The average of $s(\omega)$ with respect to the volume form $d\mu_\omega = \frac{\omega^n}{n!}$ is a fixed topological quantity

$$\widehat{s} = \int_X s(\omega)\, d\mu_\omega = \frac{n\int_X c_1(X) \cup \Omega^{n-1}}{\int_X \Omega^n},$$

where $c_1(X) = c_1(TX)$ denotes the first Chern class of X (i.e. of its holomorphic tangent bundle). Thus we are concerned with the nonlinear problem

$$s(\omega) = \widehat{s}, \quad [\omega] \in \Omega. \tag{1.2}$$

We call this the *cscK equation*. If it is solvable we will say that the metric ω (or even the class Ω) is *cscK*.

Equation 1.2 is the natural higher dimensional analogue of the constant Gaussian curvature equation on a Riemann surface. Establishing necessary and sufficient conditions for the solvability of the cscK equation when $n > 1$ is regarded as one of the main open questions in Kähler geometry, starting with the work of E. Calabi in the 1970's.

There are in fact a number of effective obstructions to the solvability of the cscK equation, many of which will be recalled later in this thesis. On the other hand deep and recent results of S. K. Donaldson [14] and X. X. Chen and G. Tian [6] imply uniquess of solutions modulo the action of the group of complex automorphisms $\mathrm{Aut}(X)$. This situation should be contrasted with the Riemannian analogue for a fixed conformal class $\mathcal{C}$,

$$s(g) = \text{const}, \quad g \in \mathcal{C} \tag{1.3}$$

(the average is no longer topological of course). Equation 1.3 definitely enjoys existence thanks to the celebrated solution of the Yamabe problem (see e.g. [26]) but there is no uniqueness even if we require the constant in 1.3 to be minimal. From this point of view adding the constraint $\nabla_g J = 0$ for an integrable almost complex structure J compatible with ω and parametrising by Kähler potentials $\omega = \omega_0 + i\partial\overline{\partial}\phi$ instead of conformal factors $g = e^u g_0$ changes the problem dramatically (except for complex dimension one, where the two problems essentially coincide).

Indeed suppose that we are in the projective case, so $\Omega \in H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z})$ is now the first Chern class of an ample line bundle L. In this case S. K. Donaldson [11] developed a beautiful theory that relates the solvability of equation 1.2 to algebro-geometric properties of the variety X with the ample line bundle L, a *polarised manifold* (X, L). We say that (X, L) *is cscK* if the class $c_1(L)$ is cscK. Let $\mathrm{Aut}(X, L)$ denote the group of projective automorphisms.

Theorem 1.1 (Donaldson [11] 2001) *If the polarised manifold* (X, L) *is cscK and* $\mathrm{Aut}(X, L)$ *is discrete (hence finite) then* (X, L) *is* asymptotically Chow stable *(this is a classical algebro-geometric notion recalled in Section 2.5).*

Thus differential geometry can be used to grasp a subtle algebro-geometric property, (asymptotic) Chow stability. Later T. Mabuchi was able to remove the assumption on $\mathrm{Aut}(X, L)$ under a non-degeneracy condition [20].

Even more importantly for us Donaldson [12] (inspired by previous work of Tian) defined a conjectural analogue of Mumford-Takemoto stability for sheaves in the category of projective varieties. This notion is known as *(algebraic) K-stability*. It will be defined in Section 2.1, but in essence for any equivariant one-parameter degeneration (X_0, L_0) of (X, L) one computes a weight $F(X_0, L_0)$ (the *Donaldson-Futaki invariant, or weight*). We say (X, L) is K-(semi)stable if F is always nonnegative (respectively positive for non-trivial degenerations). Donaldson then went on to prove the following fundamental fact (see Chapter 2).

Theorem 2.16 (Donaldson [13] 2005) *If* (X, L) *is cscK then it is K-semistable (see Definition 2.7).*

One of the main results in this thesis is a refinement of Theorem 2.16. It was conjectured by Donaldson as part of [12], p. 290.

Theorem 4.1 (S. [28] 2008) *If* (X, L) *is cscK and* $\mathrm{Aut}(X, L)$ *is discrete (hence finite) then* (X, L) *is K-stable (see Definition 2.7).*

The proof of Theorem 4.1 will be given in Chapter 4. The methods we use are essentially algebro-geometric. Perhaps surprisingly they rest on the behaviour of the Donaldson-Futaki weight F and of a given cscK metric when we *blow up* a finite number of points, as we now discuss.

Topologically the blowup $\mathrm{Bl}_{p_1,\dots,p_m} X$ is a connected sum along small balls around the points

$$\mathrm{Bl}_{p_1,\dots,p_m} X = X\#_{p_1}\overline{\mathbb{CP}^n}\dots\#_{p_m}\overline{\mathbb{CP}^n}$$

with negatively oriented complex projective spaces $\mathbb{CP}^n$. It is well-known that $\mathrm{Bl}_{p_1,\dots,p_m} X$ has a canonical complex structure depending only on the points $p_1,\dots,p_m$. If Ω is a Kähler class on X then there are Kähler classes on the blowup which coincide with Ω outside a compact neighborhood of each p_i. When Ω is cscK it is natural to ask if these classes on $\widehat{X}$ are cscK too.

From the algebro-geometric point of view let (X, L) be a polarised manifold. Let $\widehat{X}$ be the blowup of X along a finite collection of points $p_1,\dots,p_m$ with the projection $\pi : \widehat{X} \to X$ and exceptional divisors $\pi^{-1}(p_i) \subset \widehat{X}$ (see e.g. [17] II Definition following 7.12). Then the line bundle on $\widehat{X}$

$$\widehat{L} = \pi^* L^{\gamma} \otimes_{i=1}^{m} \mathscr{O}(\pi^{-1}(p_i))^{-a_i}$$

is ample for all positive integers $a_1,\dots,a_m$ and any γ sufficiently large (depending on $a_1,\dots,a_m$). Suppose that $c_1(L)$ is represented by a cscK metric. Then we ask if the same is true for $c_1(\widehat{L})$. More precisely we ask for which $m-$tuples of points $p_1,\dots,p_m$ and weights $a_1,\dots,a_m$ the class $c_1(\widehat{L})$ is represented by a cscK metric *for any sufficiently large* γ. We will prove that some stability constraint must hold for the positions and weights of the blown-up points (the precise statement and proof will be given in Chapter 3).

Theorem 3.16 (S. [27] 2007) *Suppose that the Futaki character of* (X, L) *vanishes (see Remark 2.6). If the* 0*-dimensional cycle* $\sum_{i=1}^{m} a_i^{n-1} p_i$ *on* X *is* unstable *with respect to the natural linearisation of the* $\mathrm{Aut}(X, L)$*-action on effective* 0*-cycles then the blowup* $(\widehat{X}, \widehat{L})$ *is* K-unstable *for all* $\gamma \gg 0$*. Thus the first Chern class of the line bundle* $\widehat{L}$ *is* not *representable by a cscK metric for any sufficiently large* γ*.*

To prove this we will develop a formula for the behaviour of the Donaldson-Futaki weight F under blowup, Theorem 3.14. As we mentioned this is also a key ingredient in the proof of Theorem 4.1. But Theorem 3.16 also has independent interest as a converse to a well-known result proved by C. Arezzo and F. Pacard in 2005. This is first discussed in Chapter 2 in the original formulation given by Arezzo and Pacard in [2] and then recast in the following algebraic form in Chapter 3 (see there for the precise statement).

Theorem 3.15 (Algebraic Arezzo-Pacard) *If* (X, L) *is* cscK, *the cycle* $\sum_{i=1}^{m} a_i^{n-1} p_i$ *is* stable *for the natural linearisation of the* $\mathrm{Aut}(X, L)$*-action on effective* 0*-cycles and a further nondegeneracy condition holds for the points* $p_1 \ldots, p_m$ *then the first Chern class of the line bundle* $\pi^* L^{\gamma} \otimes_{i=1}^{m} \mathscr{O}(\pi^{-1}(p_i))^{-a_i}$ can *be represented by a cscK metric for any* γ *sufficiently large.*

The Futaki character and its relation to K-stability are briefly discussed in Remark 2.6. It is well known that for cscK Kähler manifolds the Futaki character vanishes (in the algebraic case we will see this is part of K-semistability).

Going back to Donaldson's Theorem 2.16 an important question is to use K-(semi)stability to obtain explicit cohomological conditions that the class $c_1(L)$ must satisfy if it can be represented by a cscK metric. Much work on this problem has been done by R. Thomas and J. Ross [25], [24].

For any closed subscheme $Z \subset X$ Ross and Thomas defined a quantity $\mu(Z, L)$ (the *slope*), which can be computed intrinsically on X, and proved that K-semistability implies the following.

Theorem 5.3 (Ross-Thomas [24] 2006) *If* (X, L) *is cscK then for all closed subschemes* $Z \subset X$ *the* slope-semistability condition *holds*

$$\mu(Z, L) \leq \mu(X, L)$$

where $\mu(X, L)$ *is the slope of the polarised mailfold* (X, L)*.*

Precise definitions and statements will be given in Chapter 5.

Now in the case of an effective divisor $D \subset X$ the Hirzebruch-Riemann-Roch Theorem computes the slope $\mu(D, L)$ as an explicit rational function of the Chern classes $c_1(X), c_1(L), c_1(\mathscr{O}(D))$. For example when X is an algebraic surface, Theorem 5.3 becomes the interesting Chern number inequality (depending on a positive parameter c)

$$\frac{3 \int_X (c_1(L) \cup c_1(\mathscr{O}(D)) - c[c_1(\mathscr{O}(D))^2 - c_1(X) \cup c_1(\mathscr{O}(D))])}{2c \int_X (3c_1(L) \cup c_1(\mathscr{O}(D)) - c_1(\mathscr{O}(D))^2)} \geq \frac{\int_X c_1(X) \cup c_1(L)}{\int_X c_1(L)^2}$$

which must hold as long as $L - cD$ is ample. This inequality has important applications to cscK metrics on complex surfaces, see e.g. Ross-Panov [23].

In general it is then tempting to replace $c_1(L)$ by any Kähler class Ω on a Kähler manifold (not necessarily projective) and ask if Theorem 5.3 still holds, at least in the case of divisors. This was first conjectured by Ross-Thomas as part of [24] Section 4.4. We will offer a proof in Chapter 5.

Theorem 5.5 (S. [29] 2008) *Let X be a Kähler (not necessarily projective) manifold. If a Kähler class Ω on X is cscK then X is slope-semistable with respect to Ω and all effective divisors.*

In Section 5.3 we will give an example of a slope-unstable Kähler manifold which cannot be deformed to a projective one.

Our proof also elucidates the differential-geometric meaning of the slope-stability condition. It will turn out that slope-stability is just the condition that the well-known K-energy functional of Mabuchi remains bounded from below as the Kähler form concentrates along the divisor D to some extent dictated by positivity. This functional and its relationship with cscK metrics are discussed at the end of the next Chapter. For now we only say that in the non-projective case the role of test configurations is played by rays in the space of Kähler potentials with respect to a reference metric, while the Futaki invariant is replaced by the asymptotics of the K-energy.

Finally, this Introduction would not be complete without mentioning the famous conjecture of S. T. Yau, G. Tian and S. Donaldson.

YTD Conjecture *A polarised manifold (X, L) is cscK if and only if it is K-polystable (see Definition 2.10).*

Thus a proof of the YTD Conjecture would give the analogue (for projective varieties) of the Hitchin-Kobayashi Correspondence between Mumford-Takemoto polystable vector bundles and Hermite-Einstein connections.

There is not much progress on this general statement to the present day except in the case of toric surfaces thanks to the work of Donaldson [12], [15].

The Theorem of Arezzo-Pacard and our converse may be seen as infinitesimal or perturbation manifestations of the YTD correspondence.

Note however that it is now widely believed that a stronger notion of stability (yet to be discovered) may be needed in the statement of the YTD Conjecture. Indeed we will present a new argument in favour of this belief in Remark 3.21.

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Some less standard notations

$s, \widehat{s}$	scalar curvature and its average, i
cscK	"constant scalar curvature Kähler", i
$\pi^{-1}(p_i)$	exceptional divisor over p_i, iii
$L, L^{\otimes r}, L^r$ etc.	invertible sheaf and its tensor powers, ii–iii
$\mathbf{G}_m$	the group of nonzero complex numbers, 1
$(\mathscr{X}, \mathscr{L})$	test configuration for a polarised manifold (X, L), 1
$(\mathscr{X}_0, \mathscr{L}_0)$	central fibre of a test configuration, 1
$A_k, \mathrm{tr}(A_k)$	infinitesimal generator and its trace, 2
H^0, h^0	space of global sections and its dimension, 2
$F(\mathscr{X})$	Donaldson-Futaki weight of a test configuration, 2
$\mathbb{P}(E)$	the projective bundle of lines in a vector bundle, 4
$\eta(X)$	"real" holomorphic vector fields, 6
$\mathfrak{kill}(g)$	Killing vector fields, 6
$\mathfrak{ham}$	Hamiltonian holomorphic vector fields, 7
m	moment map, 7
$L(x)$	fibre of L over $x, L_x/\mathfrak{m}_x$, 9
$\mu^L(x, \alpha)$	Hilbert-Mumford weight, 9
$ch^L(\sum_i n_i x_i, \alpha)$	Chow weight, 12
$\delta\mathscr{M}_\omega$	first variation of K-energy, 13
$\mathscr{H}$	space of Kähler potentials, 14
$\mathrm{Ric}(\omega)$	Ricci curvature (1,1) form, 15
$I, J, \int_X \log\det$	int. by parts components of $\mathscr{M}_\omega$, 15
$(\mathbf{G}_m q)^-$	closure of the $\mathbf{G}_m$-orbit, 16
$i^{-1}\mathscr{I} \cdot \mathscr{O}$	inverse image ideal sheaf, 19
len	length of a finitely generated module, 21
ρ_r	sequence of base changes, 24
$\mathscr{X}_0^{\mathrm{red}}$	central fibre with reduced structure, 29
$F_{(Z,L)}(c)$	D.-F. invariant of degeneration to normal cone 36
$\mu_c(\mathscr{O}_Z, L)$	Ross & Thomas quotient slope, 37

$\mu(X, L)$	ambient slope (half $\widehat{s}$), 37
h	a metric on $\mathscr{O}(D)$, 38
Θ	curvature form of h, 38
$[D]$	the current of an effective divisor, 39
$\varphi, \widetilde{\varphi}$	weight for h (resp. for $h\|_D$, locally extended trivially), 40
r	uniformly $O(\|w\|)$ quantity, 40
$F_I, F_J, F_{\log}$	int. by parts components of $F_{(D,\Omega)}$, 42–43.

2 General theory

In this Chapter we recall a few fundamental facts from the theory of K-stability, cscK metrics, and finite dimensional Geometric invariant theory.

While we follow the definition of algebraic K-semistability given by Donaldson [12], we emphasise that our definition of K-stability differs from the original one of Donaldson in a way which is by now quite standard in the literature (see e.g. [25]). Namely we call K-polystable what Donaldson would simply call stable, and reserve the word K-stable for K-polystable polarised manifolds with no nontrivial holomorphic vector fields lifting to the polarisation (this is closer to Geometric invariant theory, but complicates the terminology a bit).

It is understood that we work over the complex numbers. The multiplicative group of nonzero complex numbers will be denoted by $\mathbf{G}_m$ throughout. We use the standard abbreviation "1-PS" for "1-parameter subgroup".

2.1 Test configurations and K-(semi, poly)stability

The version of K-(semi)stability presented in this Section is due to Donaldson [12]. Previously a more analytic notion was introduced by Tian. Its relationship to what we call K-(semi)stability is the object of current research, and we will not need to take up the matter again in this thesis.

Let X be a connected smooth complex projective variety with complex dimension n. If L is any ample line bundle on X we will say that the pair (X, L) is a *polarised manifold*.

Definition 2.1 (Test configuration) A *test configuration* $(\mathscr{X}, \mathscr{L})$ for a polarised manifold (X, L) is a flat family $\mathscr{X} \to \mathbb{A}^1$ together with a line bundle $\mathscr{L}$ on the total space $\mathscr{X}$ such that

. the line bundle $\mathscr{L}$ is relatively ample;

. the total space $\mathscr{X}$ is endowed with an action of $\mathbf{G}_m$ which covers the natural action of $\mathbf{G}_m$ on the base $\mathbb{A}^1$;

. the $\mathbf{G}_m$-action on $\mathscr{X}$ admits a natural lift to $\mathscr{L}$, i.e. it is $\mathscr{L}$-linearised;

. the fibre $(\mathscr{X}_1, \mathscr{L}_1)$ is isomorphic as a polarised manifold to (X, L^r) for some positive integer r (it follows that the general fibre $(\mathscr{X}_{t\in\mathbf{G}_m}, \mathscr{L}_{t\in\mathbf{G}_m})$ is isomorphic to (X, L^r), though not canonically).

The integer r is called the *exponent* of the test configuration.

Since the action of $\mathbf{G}_m$ on $\mathscr{X}$ covers the natural action $t \cdot z = tz$ for $z \in \mathbb{A}^1$ there is a natural induced action of $\mathbf{G}_m$ on the *central fibre* $\mathscr{X}_0$.

Remark 2.2 While the general fibre $\mathscr{X}_t$, $t \neq 0$ is a smooth subscheme of $\mathscr{X}$ the central fibre $\mathscr{X}_0$ may be in general singular, reducible and nonreduced. However being the flat limit of a one-parameter family of smooth varieties imposes many restrictions on the singularities $\mathscr{X}_0$ can have (e.g. isolated singularities must be invariant under an action of $\mathbf{G}_m$ and they must be smoothable). We do not investigate this further in this thesis.

Note that $\mathscr{X}$ is connected (since its generic fibre over $\mathbb{A}^1$ is). By definition the relatively ample line bundle $\mathscr{L}$ on $\mathscr{X}$ gives a linearisation for the $\mathbf{G}_m$-action, i.e. the isomorphism class $[\mathscr{L}]$ is an element of $\mathrm{Pic}^{\mathbf{G}_m}(\mathscr{X})$. For any reductive algebraic group G acting on a complete connected scheme Y there is an exact sequence (see [22] I, Section 3)

$$0 \to \chi(G) \to \mathrm{Pic}^G(Y) \to \mathrm{Pic}(Y) \to 0$$

where $\chi(G)$ is the group of characters of G. Thus is our case choosing a lift of the $\mathbf{G}_m$-action to $\mathscr{L}$ is the same as the choice of an element $\xi \in \chi(\mathbf{G}_m) = \mathrm{Hom}(\mathbf{G}_m, \mathbb{C})$. For any ξ there are induced $\mathbf{G}_m$-actions on the vector spaces of sections $H^0(\mathscr{L}_0^k)$ for all $k \geq 0$. We will write A_k for the infinitesimal generators of these actions. Note that if we choose a different lift ξ' the infinitesimal generators transform in a simple way,

$$A'_k = A_k + k(\xi' - \xi)I_k \tag{2.1}$$

where $\xi' - \xi$ is regarded as an element of $\mathbb{Z}$ in a natural way and I_k denotes the identity $1 \in \mathrm{End}(H^0(\mathscr{L}_0^k))$.

Now by the Hirzebruch-Riemann-Roch Theorem the dimension $h^0(\mathscr{L}_0^k)$ is a degree n rational polynomial in k for k sufficiently large. Similarly the equivariant version of Hirzebruch-Riemann-Roch shows that for any choice of a lift $\xi \in \chi(\mathbf{G}_m)$ the quantity $\mathrm{tr}(A_k)$ is a degree $n+1$ rational polynomial in k for k sufficiently large. Thus there is an asymptotic expansion as $k \to \infty$

$$\frac{\mathrm{tr}(A_k)}{kh^0(\mathscr{L}_0^k)} = F_0 - F_1\frac{1}{k} + O(k^{-2}). \tag{2.2}$$

We claim that the coefficient F_1 does not depend on the choice of ξ. Indeed using the transformation rule 2.1 we see that

$$\frac{\mathrm{tr}(A'_k)}{kh^0(\mathscr{L}_0^k)} = \xi' - \xi + \frac{\mathrm{tr}(A_k)}{kh^0(\mathscr{L}_0^k)},$$

so F_1 is unchanged.

Definition 2.3 The Donaldson-Futaki invariant (or weight) of a test configuration $(\mathscr{X}, \mathscr{L})$ is the rational number F_1 in the expansion 2.2. We will denote this quantity by $F(\mathscr{X})$ in general.

To unravel the definition of $F(\mathscr{X})$ we write down the Riemann-Roch expansions

$$h^0(\mathscr{L}_0^k) = b_0 k^n + b_1 k^{n-1} + O(k^{n-2}),$$
$$\mathrm{tr}(A_k) = a_0 k^{n+1} + a_1 k^n + O(k^{n-1}).$$

We find the explicit formula

$$F(\mathscr{X}) = b_0^{-2}(a_0 b_1 - b_0 a_1). \tag{2.3}$$

Definition 2.4 A test configuration is *trivial* if there is an isomorphism $\mathscr{X} \cong X \times \mathbb{A}^1$ which is $\mathbf{G}_m$-equivariant, where $\mathbf{G}_m$ acts on $X \times \mathbb{A}^1$ by its natural action on the second factor.

In particular it is not enough to have just any isomorphism $\mathscr{X} \cong X \times \mathbb{A}^1$. Indeed suppose there is a given nontrivial $\mathbf{G}_m$-action on X. We may act on $X \times \mathbb{A}^1$ by $t \cdot (x, z) = (t \cdot x, tz)$. The product $X \times \mathbb{A}^1$ endowed with this induced action is *not* a trivial test configuration.

Definition 2.5 A test configuration is a *product* if it is induced by an action of $\mathbf{G}_m$ on X. If the action is nontrivial then the test configuration is nontrivial.

Remark 2.6 Suppose that a holomorphic vector field v on X generates an action of $\mathbf{G}_m$, so that we get a product test configuration $\mathscr{X}_v$. It is shown in [12] (computing with the Cartan model for equivariant cohomology) that in this case $F(\mathscr{X}_v)$ equals ($(2\pi)^{2n}$ times) the classical differential-geometric Futaki invariant $F(v)$ as defined for example in [31] Chapter 3. Here we only recall that for any Kähler class Ω on a Kähler manifold X with Lie algebra of holomorphic vector fields $\eta(X)$ there exists a Lie algebra character $F_\Omega \colon \eta(X) \to \mathbb{C}$ which must vanish identically if Ω can be represented by a cscK metric.

Definition 2.7 A polarised manifold (X, L) is *K-semistable* if for all test configurations $(\mathscr{X}, \mathscr{L})$ the Donaldson-Futaki invariant is nonnegative. It is *K-stable* if the strict inequality $F(\mathscr{X}) > 0$ holds for all nontrivial test configurations.

Example 2.8 As a first example of a K-stable polarised manifold consider any Riemann surface C with genus $g \geq 1$ polarised by any ample line bundle

L. By the classical Uniformisation Theorem the class $c_1(L)$ can be represented by a Hermitian (thus Kähler by dimension) metric of constant Gaussian curvature. Since the Gaussian curvature is twice the scalar curvature, the solution also gives a cscK metric. When $g \geq 2$ the group $\mathrm{Aut}(X)$ itself is finite, so our refinement of Donaldson's Theorem, Theorem 4.1, implies K-stability. The same is true for $g = 1$ since then $\mathrm{Aut}(C, L)$ is finite. It is also possible (and desirable) to give a direct algebraic proof of the K-semistability of (C, L), see Ross-Thomas [25] Section 6.

Remark 2.9 A K-stable polarised manifold (X, L) must have discrete (hence finite) automorphism group $\mathrm{Aut}(X, L)$. Indeed if $\alpha \colon \mathbf{G}_m \hookrightarrow \mathrm{Aut}(X, L)$ is a nontrivial 1-PS in $\mathrm{Aut}(X, L)$ inducing a test configuration with weight $F > 0$ then then the inverse α^{-1} yields a test configuration with $F < 0$. This should be compared with the fact that a (Gieseker or Mumford-Takemoto) stable sheaf has only scalar automorphisms.

In light of the above remark a different notion is needed to take into account continous automorphisms.

Definition 2.10 A polarised manifold (X, L) is *K-polystable* if it is K-semistable and moreover $F(\mathscr{X}) = 0$ for a test configuration $(\mathscr{X}, \mathscr{L})$ implies that $(\mathscr{X}, \mathscr{L})$ is a product.

Example 2.11 Let $0 \to F' \to F \to F'' \to 0$ by a nontrivial extension over a Riemann surface with F', F'' line bundles. Suppose F', F'' have the same slope (so F is Mumford-Takemoto semistable, but not stable). If the extension is given by $e \in \mathrm{Ext}^1(F'', F')$ (so $e \neq 0$) the line $\langle e \rangle \subset \mathrm{Ext}^1(F'', F')$ induces a flat family $\mathscr{F}$ of vector bundles over $\mathbb{A}^1$ with general fibre isomorphic to F while the central fibre is $F' \oplus F''$. The total space $\mathscr{F}$ has a natural $\mathbf{G}_m$-action induced by $t \cdot t'e = tt'e$. The projectivisation $\mathbb{P}(\mathscr{F})$ has natural ample linearisations for this action, namely $\pi^* L \otimes \mathscr{O}_{\mathbb{P}(\mathscr{F})}(1)$ where $L \in \mathrm{Pic}(C)$ is sufficiently positive. Thus $(\mathbb{P}(\mathscr{F}), \pi^* L \otimes \mathscr{O}_{\mathbb{P}(\mathscr{F})}(1))$ gives a test configuration for $(\mathbb{P}(F), L \otimes \mathscr{O}_{\mathbb{P}(F)}(1))$. It is a theorem of Ross-Thomas (see [25] Theorem 5.13 and Remark 5.14) that this test configuration has vanishing Futaki invariant, yet it is clearly not a product configuration. Thus $(\mathbb{P}(F), L \otimes \mathscr{O}_{\mathbb{P}(F)}(1))$ is not K-polystable. We conjecture that it is actually K-semistable. This would be interesting as it is known that it is not cscK (this was proved by Apostolov and Tønnesen-Friedman in [1] by a deformation argument).

It will be useful to introduce one more piece of terminology for K-semistability.

Definition 2.12 A polarised manifold (X, L) is *properly K-semistable* if it is K-semistable but not K-stable, namely there exists a nonproduct test configuration with vanishing Donaldson-Futaki weight.

There are two natural operations defined on test configurations. The first one is to take tensor powers of the line bundle $\mathscr{L}$, i.e. passing from $(\mathscr{X}, \mathscr{L})$ to $(\mathscr{X}, \mathscr{L}^s)$ for some positive integer s. Since the Donaldson-Futaki invariant is defined by the asymptotic behaviour of $\operatorname{tr}(A_k)$ as $k \to \infty$ this operation does not change the value of $F(\mathscr{X})$. Thus we obtain the following:

Lemma 2.13 *To check K-(semi, poly)stability we can always assume that the line bundle $\mathscr{L}$ of a test configuration is* relatively very ample. *In particular we can assume that there is an embedding $\mathscr{X}_0 \hookrightarrow \mathbb{P}(H^0(\mathscr{L}_0)^*)$.*

In this case the induced $\mathbf{G}_m$-action on $(\mathscr{X}_0, \mathscr{L}_0)$ acts through projective transformations, that is it corresponds to a 1-PS $\alpha \hookrightarrow \mathrm{GL}(H^0(\mathscr{L}_0)^*)$. In general it will be useful to know if α factors through $\mathrm{SL}(H^0(\mathscr{L}_0)^*)$. This can always be achieved by performing the second operation, namely pulling back $(\mathscr{X}, \mathscr{L})$ by a ramified cover of $\mathbb{A}^1$ given by $z \to z^d$. This base change gives a new test configuration $(\widetilde{\mathscr{X}}, \widetilde{\mathscr{L}})$ with $\widetilde{\mathscr{X}_0} \cong \mathscr{X}_0$ (equivariantly), but such that

$$\operatorname{tr}(\widetilde{A}_k) = d \cdot \operatorname{tr}(A_k).$$

Therefore the new Futaki invariant is simply $\widetilde{F}(\mathscr{X}) = d \cdot F(\mathscr{X})$, while we can solve the equation

$$d \cdot \operatorname{tr}(A_1) + (\xi' - \xi)h^0(\mathscr{L}_0) = 0$$

e.g. by choosing $d = h^0(\mathscr{L}_0)$, $\xi' - \xi = -\operatorname{tr}(A_1)$ to ensure that the new induced action on $H^0(\mathscr{L}_0)^*$ is special linear.

Lemma 2.14 *To check K-(semi, poly)stability we can always assume that the induced action on the vector space $H^0(\mathscr{X}_0, \mathscr{L}_0)$ is special linear.*

The following observation is sometimes useful to check K-(semi, poly)stability. It was stated explicitely for the first time in [25] 3.7 although it is implicit in the work of Tian and Donaldson.

Lemma 2.15 *To check K-(semi, poly) stability for (X, L) it is enough to consider test configurations induced by a projective embedding $i_{L^r} \colon X \hookrightarrow \mathbb{P}^N$ and a 1-PS in $\mathrm{GL}(N+1)$ for all r sufficiently large (the central fibre $\mathscr{X}_0$ is then the flat limit of X under the $\mathbf{G}_m$-action on $\mathbb{P}^N$).*

There is also an important notion of the *norm* of a test configuration. Once again this is defined by a Riemann-Roch expansion,

$$\mathrm{tr}(A_k^2) - \frac{\mathrm{tr}(A_k)^2}{h^0(\mathscr{L}_0^k)} = \|\mathscr{X}\|^2 k^{n+2} + O(k^{n+1}). \tag{2.4}$$

This will only be needed in Theorem 2.16. See [13] for further details.

2.2 Donaldson's lower bound on the Calabi functional

In the 1970's E. Calabi introduced a functional which has since been known as the *Calabi functional*. For a Kähler cohomology class Ω on a Kähler manifold X it is defined on metrics representing Ω by

$$\mathrm{Cal}(\omega) = \int_X (s(\omega) - \widehat{s})^2 d\mu_\omega.$$

In [13] Donaldson proved the following fundamental fact.

Theorem 2.16 (Donaldson) *Let (X, L) be a polarised manifold. Then*

$$\inf_{[\omega]\in c_1(L)} \mathrm{Cal}(\omega) \geq \sup_{(\mathscr{X},\mathscr{L})} -\frac{F(\mathscr{X})}{\|\mathscr{X}\|^2}$$

where the infimum is taken over all test configurations for (X, L).

Thus if $c_1(L)$ can be represented by metrics which are arbitrarily L^2-close to a cscK metric then (X, L) is K-semistable. In particular this must hold if (X, L) is cscK.

Remark 2.17 *Uniform K-stability.* It is important to point out that the Donaldson-Futaki invariant of a K-stable polarised manifold (X, L) can *never* be bounded away from 0. For example this can be seen by considering the test configuration given by degeneration to the normal cone with parameter c and letting $c \to 0$ (see Chapter 5 for definitions). In other words it makes no sense to ask that F is uniformly bounded away from 0.

In this connection a refinement of K-stability was proposed by G. Székelyhidi. If $\omega \in c_1(L)$ is cscK there should be a strictly positive lower bound for a suitable normalisation of F over all nonproduct test configurations. This condition is called *uniform K-polystability*. In [30] Section 3.1.1 it is shown that the correct normalisation in the case of algebraic surfaces coincides with that of Theorem 2.16, $\frac{F(\mathscr{X})}{\|\mathscr{X}\|}$. For toric surfaces K-polystability implies uniform K-polystability with respect to torus-invariant test configurations; this is shown in [30] Section 4.2.

2.3 The Theorem of Arezzo-Pacard

There is a well-known result in the theory of cscK metrics that will play a key role in this thesis: the Arezzo-Pacard Theorem on blowing up cscK metrics. For its statement we need to say a little more about the group of complex automorphisms of a cscK manifold.

For any complex manifold let $\eta(X)$ denote the Lie algebra of holomorphic vector fields of X. This is defined as the subalgebra of the Lie algebra of vector fields $\mathfrak{X}(X)$ given by fields v such that the Lie derivative of the complex structure $\mathcal{L}_v J$ vanishes.

Remark 2.18 There is an isomorphism of complex vector spaces from $\eta(X)$ to $H^0(TX)$ sending v to $V = \frac{1}{2}(v - iJv)$. Its inverse maps V to $V + \overline{V}$.

Suppose now ω is (the $(1,1)$ form of) a Kähler metric on X, with underlying Riemannian metric g. The infinitesimal isometries of g form the Lie subalgebra of Killing vector fields $\mathfrak{kill}(g) \subset \mathfrak{X}(X)$. One can prove that $\mathfrak{kill}(g)$ is really a subalgebra of $\eta(X)$ (and in fact $v \in \eta(X)$ is contained in $\mathfrak{kill}(g)$ if and only if $\mathrm{div}_g(v) = 0$). The Killing fields have a distinguished abelian subalgebra $\mathfrak{a}$ given by parallel holomorphic vector fields, i.e. $v \in \eta(X)$ with $\nabla_g v = 0$.

When X has a cscK metric the Lie algebra $\eta(X)$ has a very special structure.

Theorem 2.19 (Matsushima) *Suppose X has a constant scalar curvature Kähler metric ω with underlying Riemannian metric g. Then the Lie algebra $\eta(X)$ decomposes as a direct sum*

$$\eta(X) = \eta_0(X) \oplus \mathfrak{a}$$

where $\eta_0(X)$ is the complexification of the quotient Lie algebra

$$\mathfrak{ham} = \mathfrak{kill}(g)/\mathfrak{a},$$

i.e. the Lie algebra of Hamiltonian holomorphic fields. In particular the identity component $\mathrm{Aut}(X)_0$ *of the complex automorphisms of X is a reductive group.*

It turns out that the Lie algebra $\mathfrak{ham}$ is the Lie algebra of the group $\mathrm{Ham}(\omega, J)$ of Hamiltonian isometries of g, i.e. isometries which are Hamiltonian with respect to ω. It is also well known that any $v \in \mathfrak{ham}$ admits a unique holomorphy potential f. This is the unique real solution f_v of the equation

$$-\overline{\partial} f_v = \frac{1}{2} i_V \omega$$

such that $\int_X f_v d\mu_\omega = 0$ (as above V is the image of v in $H^0(TX)$). This identifies $\mathfrak{ham}$ with the Lie algebra given by zero mean real smooth functions on X together with the Poisson bracket $\{f, g\} = \int_X \omega(\nabla f, \nabla g)\, d\mu_\omega$. An important feature of the action of $\mathrm{Ham}(\omega, J)$ on X is that it always admits an equivariant moment map

$$m \colon X \to \mathfrak{ham}^*$$

given by evaluation,

$$\langle m(x), f\rangle = f(x) \tag{2.5}$$

for any $f \in \mathfrak{ham} \cong C^\infty(X, \mathbb{R})_0$.

Theorem 2.20 (Arezzo-Pacard) *Let X be a Kähler manifold with a cscK metric ω. Let $p_1, \dots, p_m$ be a m-tuple of distinct points in X and $a_1, \dots a_m$ a m-tuple of positive real numbers. Suppose that the following conditions are satisfied:*

1. *there is no $v \in \mathfrak{ham}$ such that $v(p_i) = 0$ for $i = 1, \dots, m$;*
2. *the images $m(p_1), \dots, m(p_m)$ under the moment map 2.5 span the real vector space $\mathfrak{ham}^*$;*
3. *$\sum_{i=1}^m a_i^{n-1} m(p_i) = 0$ in $\mathfrak{ham}^*$.*

Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the blowup

$$\pi \colon \mathrm{Bl}_{p_1, \dots, p_m} X \to X$$

admits a (unique) cscK metric ω_ε in the class

$$\pi^*[\omega] - \varepsilon^2 \sum_{i=1}^m a_i c_1(\mathscr{O}(\pi^{-1}(p_i))).$$

Moreover as $\varepsilon \to 0$ the metric ω_ε converges to ω in C^∞ norm away from $p_1 \dots, p_m$.

Let us briefly discuss the role of the three assumptions in Theorem 2.20.

First we require that no hamiltonian holomorphic vector field vanishes at every p_i, $i = 1, \dots, m$. This is not really essential and without this assumption the Theorem holds in a slightly modified form. Namely if we only drop the first assumption we can conclude that there exist weights $a_{i,\varepsilon}$, $i = 1, \dots, m$ such that $\pi^*[\omega] - \varepsilon^2 \sum_{i=1}^m a_{i,\varepsilon} c_1(\mathscr{O}(\pi^{-1}(p_i)))$ can be represented by a cscK metric and moreover $|a_{i,\varepsilon} - a_i| = O(\varepsilon)$.

The second condition is that the images of our m-tuple of points under the moment map span the (dual of) the Lie algebra of Hamiltonian isometries. This condition is of a technical nature and it is better understood in the light of the proof of Theorem 2.20. Since we do not want to enter the details here we only say that it is needed in applying the implicit function theorem, that is to control the kernel of the relevant linearised operator. Notice that this condition is generic in $p_1, \dots, p_m$.

The heart of Theorem 2.20 is the third condition, which we write down by itself and call the *balanced condition*

$$\sum_{i=1}^{m} a_i^{n-1} m(p_i) = 0. \tag{2.6}$$

Indeed suppose that we are in the projective case $[\omega] = c_1(L)$. It will be clear to the reader familiar with the Kempf-Ness Theorem (recalled below) that in this case condition 2.6 is closely related to the stability of the point $\sum_{i=1}^{m} a_i^{n-1} p_i$ in the symmetric product $X^{(\sum_{i=1}^{m} a_i^{n-1})}$ under the induced action of the complexification of $\mathrm{Ham}(\omega, J)$ with its natural linearisation.

This complexification turns out to be the group $\mathrm{Aut}(X, L)$ of complex automorphisms lifting to L.

In Chapter 3 we will give a different statement of Theorem 2.20 in the projective case, Theorem 3.15, and then proceed to prove a converse via a formula for the Donaldson-Futaki weight of a blowup, Theorem 3.14.

2.4 Finite dimensional G.I.T. and 0-cycles

In this Section we recall the few definitions and facts from finite dimensional Geometric invariant theory that we will need. The standard reference for G.I.T. is [22].

Definition 2.21 (G.I.T. stability) Let G be a reductive algebraic group acting on a scheme (X, L) with a linearisation on an invertible sheaf L. We say a closed point $x \in X$ is *semistable* if there exists some k and an invariant global section s of L^k which does not vanish at x. A semistable point x is *polystable* if the induced action on the invariant affine open subscheme $U_s = \{s \neq 0\}$ is closed (this means that regular functions on U_s separate orbits of G in U_s together with their tangent spaces). If moreover the stabiliser of x is finite we say that x is *stable*.

Definition 2.22 (Hilbert-Mumford weight) Let G be a reductive algebraic group acting on a scheme X with a linearisation on an invertible sheaf

L. For any 1-PS $\alpha : \mathbf{G}_m \hookrightarrow G$ we define the Hilbert-Mumford weight of a closed point $x \in X$ with respect to α, denoted by $\mu^L(x, \alpha)$, to be the weight of the induced action of $\mathbf{G}_m$ on the line $L(x_0)$, the fibre of L over the specialisation $x_0 = \lim_{t\to 0} t \cdot x$.

Suppose that X is a projective scheme with a fixed very ample line bundle L, so $X \subset \mathbb{P}(H^0(L)^*)$. If a reductive algebraic group G acts on X via a monomorphism $G \hookrightarrow \mathrm{SL}(H^0(L)^*)$ then the line bundle L has a natural linearisation, see [22] 32–33. In this case we can find a more explicit formula for the Hilbert-Mumford weight.

Lemma 2.23 *Let α be a 1-PS of $\mathrm{SL}(N+1)$, inducing a $\mathbf{G}_m$-action on $\mathbb{P}^N$. Choose projective coordinates $[x_0 : ... : x_N]$ such that α is given by $\mathrm{Diag}(\lambda^{m_0}, ...\lambda^{m_N})$. The Hilbert-Mumford weight of a closed point $q \in \mathbb{P}^N$ is given by*

$$\mu^{\mathscr{O}(1)}(q, \alpha) = -\min\{m_i : q_i \neq 0\}.$$

Note that this coincides with the weight of the induced action on the fibre of the hyperplane line bundle $\mathscr{O}(1)$ over the specialisation $\lim_{\lambda\to 0} \lambda \cdot q$.

When the linearisation L for the action of G is ample there is a well-known numerical criterion for (semi)stability.

Theorem 2.24 (Hilbert-Mumford Criterion) *Let (X, L) be a polarised scheme. Then*

- *x is semistable if and only if $\mu^L(x, \alpha) \leq 0$ for all 1-PS $\alpha\colon \mathbf{G}_m \hookrightarrow G$;*
- *x is stable if and only if $\mu^L(x, \alpha) < 0$ for all 1-PS $\alpha\colon \mathbf{G}_m \hookrightarrow G$.*

While proving the converse of the Theorem of Arezzo-Pacard we will also need a well-known result which links G.I.T. with symplectic geometry. For this suppose that G acts on $X \subset \mathbb{P}^N$ by a monomorphism $G \hookrightarrow \mathrm{SL}(N+1)$, and that its real form $K \subset G$ acts through the real form $\mathrm{SU}(N+1)$. If $\mathfrak{k}$ denotes the Lie algebra of K we obtain a Lie algebra representation

$$\rho\colon \mathfrak{k} \to \mathfrak{su}(N+1).$$

For any $x \in X$ we write $\widehat{x}$ for any choice of a lift to $\mathbb{C}^{N+1}$. Then the prescription

$$\langle m(x), k\rangle = -i\frac{\overline{\widehat{x}^t}\rho(k)\,\widehat{x}}{\|\widehat{x}\|^2}$$

for $x \in X$, $k \in \mathfrak{k}$ defines an equivariant moment map

$$m\colon X \to \mathfrak{k}^*.$$

Theorem 2.25 (Kempf-Ness) *In the situation above a point $x \in X$ is polystable if and only if its G-orbit contains a zero of the moment map m.*

Finally we specialise our discussion of stability to the special case of effective 0-cycles on a polarised manifold (X, L). Let the symmetric group Σ_d on d letters acts on the d-fold product X^d. The symmetric product $X^{(d)} = X^d/\Sigma_d$ is a projective variety. The points of $X^{(d)}$ are the orbits of the d-tuples of points of X under permutation and so can be identified with effective 0-cycles $\sum n_i x_i$ with $x_i \in X$, $n_i > 0$ and $\sum n_i = d$. This shows that $X^{(d)}$ is actually the Chow variety of length d effective 0-cycles on X. The construction of an $\mathrm{Aut}(X, L)$-linearised ample line on $X^{(d)}$ can be made very explicit as follows. Let $V = H^0(X, L^r)^*$ for some large γ and embed $X \hookrightarrow \mathbb{P}(V)$. Denote by $\mathbb{P}(V^*)$ the projective space of hyperplanes in $\mathbb{P}(V)$, and by $\mathrm{Div}^d(\mathbb{P}(V^*))$ the projective space of effective divisors of degree d in $\mathbb{P}(V^*)$. For any $p \in \mathbb{P}(V)$ consider the hyperplane in $\mathbb{P}(V^*)$ given by

$$H_p := \{l \in \mathbb{P}(V^*) : p \in l\}.$$

Define a morphism $ch : (\mathbb{P}(V))^d \to \mathrm{Div}^d(\mathbb{P}(V^*))$ by

$$ch(x_1, ..., x_d) := \sum_i H_{x_i}.$$

This is the *Chow form* of $\{x_1, ..., x_d\}$: the divisor of hyperplanes whose intersection with $\{x_1, ..., x_d\}$ is nonempty. As for X^d we have the composition

$$ch : X^d \hookrightarrow \mathbb{P}(V)^d \hookrightarrow \mathrm{Div}^d(\mathbb{P}(V^*))$$

induced by the product line $(L^r)^{\boxtimes d}$. Now ch is Σ_d equivariant and by the universal property of the geometric quotient factors through $\mathbb{P}(V)^{(d)}$ defining a morphism

$$ch : X^{(d)} \to \mathrm{Div}^d(\mathbb{P}(V^*)).$$

One can check that ch defines an isomorphism on its image. We can thus identify $X^{(d)}$ with its image $ch(X^{(d)})$ in $\mathrm{Div}^d(\mathbb{P}(V^*))$. By the usual identifications

$$\mathrm{Div}^d(\mathbb{P}(V^*)) \cong \mathbb{P}(H^0(\mathbb{P}(V^*), \mathscr{O}(d))) \cong \mathbb{P}(S^d V)$$

we see ch as a map with values in $\mathbb{P}(S^d V)$:

$$ch : X^{(d)} \hookrightarrow \mathbb{P}(S^d V).$$

Under these identifications then ch is the map

$$X^{(d)} \ni x_1 + ... + x_d \mapsto [x_1 \cdot ... \cdot x_d] \tag{2.7}$$

defined via the embedding $X \hookrightarrow \mathbb{P}(V)$.

Remark 2.26 It is important to emphasise that ch is given by the descent of $(L^r)^{\boxtimes d}$ under the action of Σ_d. This means that the Chow line $\mathscr{O}_{\mathrm{Div}^d(\mathbb{P}(V^*))}(1)|_{X^{(d)}}$ pulls back to $(L^r)^{\boxtimes d}$ under the quotient map. This holds because by 2.7 $\mathscr{O}_{\mathrm{Div}^d(\mathbb{P}(V^*))}(1)$ pulls back to the line $\mathscr{O}_{\mathbb{P}(V)}(1)^{\boxtimes d}$ on $\mathbb{P}(V)^d$ under ch and this in turn pulls back to $(L^r)^{\boxtimes d}$.

Now we assume that $\alpha \hookrightarrow \mathrm{Aut}(X, L)$ acts through a 1-PS $\alpha \hookrightarrow \mathrm{SL}(V)$. Then we know that the Hilbert-Mumford weight is given by

$$\mu(x, \alpha) := -\min\{m_i : x_i \neq 0\}.$$

In the same way we can define the Hilbert-Mumford weight for the induced action of α on $X^{(d)}$. The key fact is that $\alpha \hookrightarrow \mathrm{SL}(V)$ naturally induces a 1-PS $\alpha \hookrightarrow \mathrm{SL}(S^d V)$. Thus the embedding ch described above gives a natural linearisation for this action. We will write ch^{L^r} for this Hilbert-Mumford weight (there is no confusion since we the morphism ch above will no longer be needed).

Definition 2.27 (Chow weight of a 0-cycle) Let (X, L) be a polarised scheme with an action of $\mathbf{G}_m$ which acts through $\mathrm{SL}(H^0(L^r)^*)$ for some r for which L^r is very ample. Then the above weight $ch^{L^r}(\sum_i n_i x_i)$ is called the Chow weight of the 0-cycle $\sum_i n_i x_i$ with respect to L^r.

Then by (2.7) we immediately obtain the relation

$$ch^{L^r}(\sum_i n_i x_i, \alpha) = \sum_i n_i \, \mu^{L^r}(x_i, \alpha) \tag{2.8}$$

for any $\sum_i n_i x_i \in X^{(d)}$.

While the numerical value of the Hilbert-Mumford weight depends on the power L^r we take, the fact that a cycle $\sum_i n_i x_i$ is semistable is independent of r. This follows from the definition of stability in terms of invariant sections. Thus we see that

Lemma 2.28 *The sign of $\mu^{L^r}(\sum_i n_i x_i, \alpha)$ does not depend on r whenever μ^{L^r} is well defined. In particular suppose L is very ample and α acts through $\mathrm{SL}(H^0(L)^*)$. Then*

$$ch^{L^r}(\sum_i n_i x_i, \alpha) = c(r)\, ch^L(\sum_i n_i x_i, \alpha)$$

whenever ch^{L^r} is well defined, where $c(r)$ is a positive *constant.*

By the Hilbert-Mumford Criterion then,

Lemma 2.29 *A* 0*-cycle* $\sum_i n_i x_i$ *is (semi)stable with respect to the natural* $\mathrm{Aut}(X, L)$*-linearisation induced by* L *if and only if*

$$\mu^{L^r}(\sum_i n_i x_i, \alpha)\,(\leq) < 0$$

for all 1*-PS* $\alpha \hookrightarrow \mathrm{Aut}(X, L)$ *and all* r *for which the Chow weight is well defined.*

2.5 Asymptotic Chow stability of varieties

It is a classical fact in algebraic geometry that a degree d subvariety

$$X^n \subset \mathbb{P}^N$$

of projective space is identified uniquely by its *Chow form*

$$\Phi_X \in \mathbb{P}(H^0(\mathbf{G}, \mathscr{O}_{\mathbf{G}}(d)))$$

where

$$\mathbf{G} = \mathrm{Grass}(N - n - 1, \mathbb{P}^N).$$

In other words the form Φ_X of degree d cuts out the divisor of $N-n-1$ projective subspaces of $\mathbb{P}^N$ that intersect X. Then $X \longleftrightarrow \Phi_X$ sets up a bijection with degree d *effective algebraic cycles* (effective integral linear combinations of subvarieties). One can show that the image of Φ is a quasi-projective scheme in $\mathbb{P}(H^0(\mathbf{G}, \mathscr{O}_{\mathbf{G}}(d)))$, whose closure is denoted by $\mathbf{Chow}(n, d, N)$. This is a projective scheme with a natural linearised action of $\mathrm{SL}(N + 1)$.

Remark 2.30 When X is a 0-dimensional algebraic cycle we recover the notion of Chow stability for 0-cycles of the previous section.

All of the above applies to a polarised manifold (X, L) together with the Kodaira embedding $X \hookrightarrow \mathbb{P}(H^0(X, L^r)^*)$ whenever L^r is very ample. This embedding defines a closed point $y_r \in \mathbf{Chow}(n, r^n c_1(L)^n, h^0(L^r) - 1)$.

Definition 2.31 A polarised manifold (X, L) is *asymptotically Chow (semi) stable* if for any r sufficiently large the point y_r is (semi)stable for the natural linearised action of $\mathrm{SL}(h^0(L^r))$.

A more refined study of the G.I.T. behind Chow (semi)stability proves the following (see e.g. [25] Theorem 3.9).

Theorem 2.32 *If* (X, L) *is asymptotically Chow semistable then it is K-semistable.*

2.6 The K-energy functional of Mabuchi

The cscK equation 1.2 is the Euler-Lagrange equation for the *K-energy functional* $\mathscr{M}_\omega$ introduced by Mabuchi [19]. Its first variation is given by

$$\delta\mathscr{M}_\omega|_\phi(\delta\phi) = -\int_X \delta\phi\,(s(\omega_\phi) - \widehat{s})\,d\mu_{\omega_\phi} \tag{2.9}$$

Here the Kähler form ω is any base point in $[\omega]$ and we write $\omega_\phi = \omega + i\partial\overline{\partial}\phi$ for any other cohomologous Kähler form (this is possible by the well-known $\partial\overline{\partial}$ Lemma, e.g. [31] Chapter 1). Then $\mathscr{M}_\omega$ should be interpreted as a 1 form in the space of Kähler potentials

$$\mathscr{H} = \{\phi \in C^\infty(X,\mathbb{R}) : \omega_\phi \text{ is Kähler}\}$$

with tangent space $T\mathscr{H} \cong C^\infty(X,\mathbb{R})$ (so the variation $\delta\phi$ is just a smooth real function). The space $\mathscr{H}$ is contractible and one can show that the 1 form $\delta\mathscr{M}_\omega$ is closed, so it defines the functional $\mathscr{M}_\omega$ uniquely once we choose the basepoint ω. Any other choice of basepoint only adds a constant to the K-energy.

Now for any smooth path ϕ_t in $\mathscr{H}$ one can compute following Mabuchi

$$\frac{d^2}{dt^2}\mathscr{M}_\omega(\phi_t) = \|\overline{\partial}\nabla^{(1,0)}\dot{\phi}_t\|^2_{\phi_t} - \int_X (\ddot{\phi}_t - \frac{1}{2}|\nabla^{1,0}\dot{\phi}_t|^2_{\phi_t})(S(\omega_{\phi_t}) - \widehat{s})\,d\mu_{\phi_t} \tag{2.10}$$

where all the metric quantities are computed with respect to ω_{ϕ_t}. This puts intro great evidence the equation

$$\ddot{\phi}_t - \frac{1}{2}|\nabla^{1,0}\dot{\phi}_t|^2_{\phi_t} = 0 \tag{2.11}$$

together with natural boundary conditions $0 = \phi_0, \phi_1 = \phi \in \mathscr{H}$. A solution to this boundary value problem is called a *geodesic segment* joining ω and ω_ϕ, and equation 2.11 itself is called the *geodesic equation.* Uniqueness of geodesic segments is proved in [10] by an ingenious maximum principle argument. The existence and regularity theory on the contrary is very hard and still not completely settled.

The main point for us is that the K-energy is *convex along geodesic segments.* This is clear from the second variation formula 2.10. The same formula shows that $\mathscr{M}_\omega$ is even strictly convex modulo the action of the group $\mathrm{Ham}(\omega, J)$. On the other hand it is well-known that a cscK metric always gives a local minumum of $\mathscr{M}_\omega$. Thus the existence of smooth geodesic segments would have two fundamental consequences, namely

- The uniqueness of cscK metrics modulo the action of $\mathrm{Aut}(X)$,

- the lower boundedness of $\mathscr{M}_\omega$ whenever $[\omega]$ admits a cscK representative.

The approach we outlined so far is due to Donaldson [10]. While as we mentioned a full regularity theory is still not available, Chen [4] and Chen-Tian [6] have developed a partial regularity theory which is sufficient to prove the results above. The lower bound on $\mathscr{M}_\omega$ will play a major role in Chapter 5 of this thesis. A different proof was given by Donaldson [14] in the projective case via the theory of balanced metrics.

Theorem 2.33 (Donaldson, Chen, Chen-Tian) *If $[\omega]$ admits a cscK representative then the K-energy functional $\mathscr{M}_\omega$ is bounded below on the space of Kähler potentials of ω.*

Remark 2.34 We briefly clarify the use of the word *geodesic*: Semmes and Donaldson have shown that 2.11 is indeed the equation of geodesics for the locally symmetric negatively curved space structure on $\mathscr{H}$ induced by the unique torsion free, metric connection for the L^2 metric on $\mathscr{H}$,

$$\langle \delta\phi', \delta\phi'' \rangle_\phi = \int_X \delta\phi' \cdot \delta\phi'' \, d\mu_{\omega_\phi}.$$

Remark 2.35 Suppose that X is Fano, that is the anticanonical bundle $-K_X$ is ample. Bando [3] has shown that if the K-energy is bounded from below in the class $c_1(X)$ then the traceless Ricci tensor can be made arbitrarily small in L^2 (in fact in C^0). Therefore by Theorem 2.16 if the K-energy is bounded from below in $c_1(X)$ then $(X, -K_X)$ is K-semistable.

Another formula that will play an important role in Chapter 5 can be obtained by integrating by parts in the definition of the K-energy, see Chen [5] and Tian [31].

Lemma 2.36 *Let $\mathrm{Ric}(\omega)$ denote the Ricci curvature form of a Kähler metric. There exist functionals I, J on the space $\mathscr{H}$ which are well defined up to constants by*

$$\delta I = \int_X \delta\phi \frac{\omega_\phi^n}{n!},$$

$$\delta J = \int_X \delta\phi \, \mathrm{Ric}(\omega) \wedge \frac{\omega_\phi^{n-1}}{(n-1)!}.$$

Moreover an integration by parts formula holds,

$$\delta \mathscr{M}_\omega|_\phi(\delta\phi) = \delta \int_X \log\left(\frac{\omega_\phi^n}{\omega^n}\right) \frac{\omega_\phi^n}{n!} + \widehat{s}\, \delta I - \delta J.$$

3 The blowup formula and a converse to the Theorem of Arezzo-Pacard

In this Chapter we give new results about the behaviour of the Donaldson-Futaki invariant under blowing up and prove a converse to the Arezzo-Pacard Theorem.

3.1 Blowing up test configurations

We begin by recalling the definition of blowing up in the algebraic category, see [17] II Definition following 7.12. Let Y be a scheme and $Z \subset Y$ a closed subscheme. Then the blowup of Y along Z is defined to be the Proj scheme of the sheaf of graded algebras associated to Z,

$$\operatorname{Bl}_Z Y = \operatorname{Proj} \bigoplus_{k \geq 0} \mathscr{I}_Z^k, \tag{3.1}$$

where by definiton $\mathscr{I}_Z^0 = \mathscr{O}_Y$.

Let $(\mathscr{X}, \mathscr{L})$ be a test configuration for the polarised manifold (X, L). Let $q \in \mathscr{X}_1$ be a closed point in the fibre over $1 \in \mathbb{A}^1$, which can be identified with X. The orbit of q under the action of $\mathbf{G}_m$ is a locally closed subscheme $\mathbf{G}_m q \subset \mathscr{X}$, and we denote its scheme-theoretic closure by $(\mathbf{G}_m q)^-$. We would like to define a test configuration for the blowup manifold $\operatorname{Bl}_q X$ by blowing up the total space $\mathscr{X}$ along $(\mathbf{G}_m q)^-$. There are two issues we should consider in this connection

1. check if the composition $\operatorname{Bl}_{(\mathbf{G}_m q)^-} \mathscr{X} \to \mathscr{X} \to \mathbb{A}^1$ is flat;
2. choose a suitable polarisation for $\operatorname{Bl}_q X$ and a suitable relatively ample linearisation for $\operatorname{Bl}_{(\mathbf{G}_m q)^-} \mathscr{X}$.

We would also like to consider the general case when the orbit of a 0-cycle on X is blown up. More precisely let $Z \subset \mathscr{X}_1 \cong X$ denote the 0-dimensional scheme $Z = \cup_i a_i p_i$ for closed points $\{p_i\} \subset \mathscr{X}_1 \cong X$ (i.e. the closed subscheme $a_i p_i$ is defined by the ideal sheaf $\mathscr{I}_{p_i}^{a_i}$). Once again the orbit of Z under the $\mathbf{G}_m$-action on $\mathscr{X}$ is a locally closed subscheme $\mathbf{G}_m Z \subset \mathscr{X}$ and we take its scheme-theoretic closure $(\mathbf{G}_m Z)^-$. There is a natural projection $\pi : \operatorname{Bl}_{(\mathbf{G}_m Z)^-} \mathscr{X} \to \mathscr{X}$ and composing with $\mathscr{X} \to \mathbb{A}^1$ we obtain a family $\operatorname{Bl}_{(\mathbf{G}_m Z)^-} \mathscr{X} \to \mathbb{A}^1$.

Lemma 3.1 *The family* $\operatorname{Bl}_{(\mathbf{G}_m Z)^-} \mathscr{X} \to \mathbb{A}^1$ *is flat.*

Proof. We use the flatness criterion [17] III Proposition 9.7. Thus we need to prove that all the associated points of $\mathrm{Bl}_{(\mathbf{G}_m Z)^-}\mathscr{X}$ (i.e. embedded subschemes) map to the generic point of $\mathbb{A}^1$. By flatness this is true for the morphism $\mathscr{X} \to \mathbb{A}^1$, and roughly speaking blowing up $(\mathbf{G}_m Z)^-$ does not contribute new associated points, only the Cartier exceptional divisor $\pi^{-1}(\mathbf{G}_m Z)^-$.

To make this precise let $y \in \mathrm{Bl}_{(\mathbf{G}_m Z)^-}\mathscr{X}$ be an associated point of the blowup and denote by $\{y\}^-$ be its closure, a closed subscheme of $\mathrm{Bl}_{(\mathbf{G}_m Z)^-}\mathscr{X}$. Write $E = \pi^{-1}((\mathbf{G}_m Z)^-)$ for the exceptional divisor.

Suppose that $\{y\}^- \subseteq E$. This would imply that the maximal ideal $\mathfrak{m}_y$ of the local ring $\mathscr{O}_{\mathrm{Bl}_{(\mathbf{G}_m Z)^-}\mathscr{X}, y}$ contains the image of a local defining equation f for E. But y being an associated point means precisely that every element of $\mathfrak{m}_y$ is a zero divisor (see the Definition following [17] III Corollary 9.6). Since E is a Cartier divisor f is not a zero divisor, a contradiction.

Therefore $\{y\}^- \nsubseteq E$, or in other words $\mathfrak{m}_y$ contains some nontrivial zero divisor orthogonal to E. Thus the projection $\pi(y)$ is an associated point of $\mathscr{X}$ which in turn maps to the generic point of $\mathbb{A}^1$ by flatness of $\mathscr{X} \to \mathbb{A}^1$.

Q.E.D.

Lemma 3.2 *For any γ sufficiently large the family*

$$\widehat{\mathscr{X}} = \{\mathrm{Bl}_{(\mathbf{G}_m Z)^-}\mathscr{X} \to \mathbb{A}^1\}$$

together with the invertible sheaf

$$\widehat{\mathscr{L}} = \pi^*\mathscr{L}^\gamma \otimes \mathscr{O}(\pi^{-1}(\mathbf{G}_m Z)^-)^{-1}$$

is a test configuration for the polarised manifold

$$(\mathrm{Bl}_{p_1,\dots,p_m} X, \pi^* L^\gamma \otimes_{i=1}^m \mathscr{O}(\pi^{-1}(p_i))^{-a_i}).$$

Proof. We already know that $\widehat{\mathscr{X}} \to \mathbb{A}^1$ is flat. Since $\widehat{\mathscr{X}}$ is the blowup of $\mathscr{X}$ along an invariant closed subscheme it carries a natural induced action of $\mathbf{G}_m$ for which $\widehat{\mathscr{X}} \to \mathbb{A}^1$ is equivariant. It is a general fact about blowups that the invertible sheaf $\widehat{\mathscr{L}}$ is ample for $\gamma \gg 0$ (see [17] II Proposition 7.10 (b)). Since $(\mathbf{G}_m Z)^-$ is invariant the invertible sheaf $\mathscr{O}(\pi^{-1}(\mathbf{G}_m Z)^-)$ gives an element of $\mathrm{Pic}^{\mathbf{G}_m}\widehat{\mathscr{X}}$ and so $\widehat{\mathscr{L}}$ also gives a natural (relatively ample) linearisation for the action of $\mathbf{G}_m$ on $\widehat{\mathscr{X}}$. Finally the polarised fibre $(\widehat{\mathscr{X}_1}, \widehat{\mathscr{L}_1})$ is isomorphic to $(\mathrm{Bl}_Z X, \pi^* L^\gamma \otimes \mathscr{O}(\pi^{-1} Z)^{-1})$. The isomorphisms of schemes $\mathrm{Bl}_Z X \cong \mathrm{Bl}_{p_1,\dots,p_m} X$ and of (linearised) invertible sheaves $\mathscr{O}(\pi^{-1} Z))^{-1} \cong \otimes_{i=1}^m \mathscr{O}(\pi^{-1}(p_i))^{-a_i}$ follow directly from the definition of blowup, see [17] II Exercise 7.11.

Q.E.D.

It is natural to ask what is the central fibre $(\widehat{\mathscr{X}_0}, \widehat{\mathscr{L}_0})$ for the test configuration of Lemma 3.2. Each of our points $p_i \in \mathscr{X}_1 \cong X$ has a specialisation $p_{i,0} = \lim_{t\to 0} t \cdot p_i$ which is a closed point of $\widehat{\mathscr{X}_0}$. We define the closed subscheme $Z_0 \subset \widehat{\mathscr{X}_0}$ as the central fibre of the flat family $(\mathbf{G}_m Z)^- \to \mathbb{A}^1$ (i.e. the flat limit of $Z \subset \mathscr{X}$ under the action of $\mathbf{G}_m$). Note that $Z_0^{\mathrm{red}} = \cup_i p_{i,0}$. A naive guess would be that $\widehat{\mathscr{X}_0}$ is isomorphic to $\mathrm{Bl}_{Z_0} \mathscr{X}_0$ (or in words that the flat limit of blowups is the blowup along the flat limit). But a little reflection suggests that this is not true in general. Indeed while the family of subschemes $(\mathbf{G}_m Z)^- \to \mathbb{A}^1$ is flat this may well not be true for the families of thickenings $k\,(\mathbf{G}_m Z)^- \to \mathbb{A}^1$ defined by the ideal sheaves $\mathscr{I}^k_{(\mathbf{G}_m Z)^-}$, i.e. the strict inequality $\mathrm{len}(kZ) < \mathrm{len}(kZ_0)$ may hold for infinitely many values of k. Therefore the Hilbert polynomials of $\mathrm{Proj} \bigoplus_k \mathscr{I}^k_Z$ and $\mathrm{Proj} \bigoplus_k \mathscr{I}^k_{Z_0}$ are different. Since the family $\widehat{\mathscr{X}} \to \mathbb{A}^1$ is flat by 3.1 we see that $\widehat{\mathscr{X}_0}$ is *not* isomorphic to $\mathrm{Bl}_{Z_0} \mathscr{X}_0$ in general. Indeed if we use the flatness criterion by Hilbert polynomials [17] III Theorem 9.9 we get

Lemma 3.3 *The central fibre* $\widehat{\mathscr{X}_0}$ *is isomorphic to* $\mathrm{Bl}_{Z_0} \mathscr{X}_0$ *if and only if the family of closed subschemes* $k\,(\mathbf{G}_m Z)^- \to \mathbb{A}^1$ *is flat for all* k *large enough.*

Example 3.4 *To show that* $\widehat{\mathscr{X}_0} \ncong \mathrm{Bl}_{Z_0} \mathscr{X}_0$ *in general.* As a local affine example consider the trivial "test configuration" $\mathscr{X} = \mathbb{A}^2 \times \mathbb{A}^1$. The ideal

$$\mathscr{I}_{(\mathbf{G}_m Z)^-} = (x(x-t), xy, y(y-t)) \subset \mathbb{C}[x, y, t]$$

describes 3 points colliding along 2 orthogonal directions in $\mathbb{A}^2$. One can show that $\widehat{\mathscr{X}}$ is the closed subscheme of $\mathrm{Spec}\,\mathbb{C}[x, y, t] \times \mathrm{Proj}\,\mathbb{C}[\xi_0, \xi_1, \xi_2]$ defined by

$$\mathscr{I}_{\widehat{\mathscr{X}}} = ((x-t)\xi_1 - y\xi_0, (y-t)\xi_1 - x\xi_2)$$

with central fibre (sitting inside $\mathrm{Spec}\,\mathbb{C}[x, y] \times \mathrm{Proj}\,\mathbb{C}[\xi_0, \xi_1, \xi_2]$) cut out by

$$\mathscr{I}_{\widehat{\mathscr{X}_0}} = (x\xi_1 - y\xi_0, y\xi_1 - x\xi_2).$$

In this case $\mathrm{Bl}_{Z_0} \mathscr{X}_0 \cong \mathrm{Bl}_{Z_0} \mathbb{A}^2 \cong \mathrm{Bl}_{(x,y)^2} \mathbb{A}^2$ is the closed subscheme of $\widehat{\mathscr{X}_0}$ given by

$$\mathscr{I}_{\mathrm{Bl}_{Z_0} \mathscr{X}_0} = (x\xi_1 - y\xi_0, y\xi_1 - x\xi_2, \xi_1^2 - \xi_1\xi_2).$$

This is an irreducible component of $\widehat{\mathscr{X}_0}$, but there is an extra irreducible component $P = \mathrm{Proj}\,\mathbb{C}[\xi_0, \xi_1, \xi_2] \cong \mathbb{P}^2$. The intersection $\mathrm{Bl}_{Z_0} \widehat{\mathscr{X}_0} \cap P$ is the quadric $\{\xi_1^2 - \xi_1\xi_2 = 0\}$ sitting in $\mathbb{P}^2$.

This example suggests that $\mathrm{Bl}_{Z_0}\mathscr{X}_0$ is a union of irreducible components of $\widehat{\mathscr{X}_0}$ in general.

Lemma 3.5 *The inclusion* $i : \mathscr{X}_0 \hookrightarrow \mathscr{X}$ *induces a closed immersion*

$$\widehat{i} : \mathrm{Bl}_{Z_0}\mathscr{X}_0 \hookrightarrow \widehat{\mathscr{X}_0}.$$

Since $\dim \mathrm{Bl}_{Z_0}\mathscr{X}_0 = \dim \widehat{\mathscr{X}_0}$ *it follows that* $\mathrm{Bl}_{Z_0}\mathscr{X}_0$ *is a union of irreducible components of* $\widehat{\mathscr{X}_0}$*.*

Proof. Let $\mathscr{I} = i^{-1}\mathscr{I}_{(\mathbf{G}_m Z)^-}\cdot \mathscr{O}_{\mathscr{X}_0}$ denote the inverse image ideal sheaf of the ideal sheaf $\mathscr{I}_{(\mathbf{G}_m Z)^-}$ (see [17] II Definition and Caution 7.12.2). By [17] II Corollary 7.15 there is an induced closed immersion

$$\widehat{i} : \mathrm{Bl}_{i^{-1}\mathscr{I}_{(\mathbf{G}_m Z)^-}\cdot \mathscr{O}_{\mathscr{X}_0}}\mathscr{X}_0 \hookrightarrow \widehat{\mathscr{X}}$$

which factors through $\widehat{\mathscr{X}_0}$. The result follows since $\mathscr{I} = \mathscr{I}_{Z_0}$ directly from the definition (no powers are involved).

Q.E.D.

Definition 3.6 For a blowup test configuration $(\widehat{\mathscr{X}}, \widehat{\mathscr{L}})$ we let P denote the closed subscheme of the central fibre $\widehat{\mathscr{X}_0}$ given by $(\widehat{\mathscr{X}_0} \setminus \mathrm{Bl}_{Z_0}\mathscr{X}_0)^-$. By Lemma 3.5 P is a union of irreducible components of $\widehat{\mathscr{X}_0}$.

The restriction of $\widehat{\mathscr{L}_0}$ to $\mathrm{Bl}_{Z_0}\mathscr{X}_0$ is the expected one.

Lemma 3.7

$$\widehat{\mathscr{L}_0}|_{\mathrm{Bl}_{Z_0}\mathscr{X}_0} \cong \pi^*\mathscr{L}_0^{\gamma} \otimes \mathscr{O}(\pi^{-1}Z_0).$$

3.2 The blowup formula

In this section we will prove an asymptotic expansion for the Donaldson-Futaki weight of the test configuration $(\widehat{\mathscr{X}}, \widehat{\mathscr{L}})$ constructed in Lemma 3.2 as the parameter $\gamma \to \infty$. When $\mathbf{G}_m$ acts on the central fibre $\mathscr{X}_0$ through $\mathrm{SL}(H^0(\mathscr{L}_0)^*)$ (which we can assume without loss of generality) we will give an interpretation of this formula which involves Chow weights and relate it to a stability condition, Theorem 3.14.

According to the definition of Donaldson-Futaki invariant we need to compute (for $k \gg 0$):

- the dimension of the space of sections $h^0(\widehat{\mathscr{L}_0}^k)$;

· the trace of the induced action of $\mathbf{G}_m$ on the space of sections $H^0(\widehat{\mathscr{L}_0^k})$.

A word of warning is in order before we delve in the problem. The computations leading to Theorem 3.14 involve two different parameters k, γ. These have two very different natures: while γ^{-1} measures the volume of the exceptional divisors on the generic fibres, k is the scale parameter which appears in the definition of the Donaldson-Futaki invariant. Also we often use the vanishing of higher cohomology groups for large powers of an invertible sheaf. This will always hold for $k \gg 0$ independently of $\gamma \geq \gamma_0$ (since increasing γ only makes our invertible sheaves more positive).

Lemma 3.8

$$h^0(\widehat{\mathscr{L}_0^k}) = h^0(\mathscr{L}_0^{\gamma k}) - \left(\sum_i a_i^n\right)\frac{k^n}{n!} - \left(\sum_i a_i^{n-1}\right)\frac{k^{n-1}}{2(n-2)!} + O(k^{n-2}).$$

Proof. By flatness

$$h^0(\widehat{\mathscr{L}_0^k}) = h^0(\widehat{\mathscr{L}_1^k}) = h^0(\mathrm{Bl}_Z X, \pi^* L^{\gamma k} \otimes_i \mathscr{O}(\pi^{-1}(p_i))^{-k}).$$

Since $\mathscr{X}_1 \cong X$ is smooth we can use the asymptotic Riemann-Roch expansion

$$\begin{aligned} h^0(\widehat{\mathscr{L}_1^k}) &= \int_X [\,\pi^* c_1(L^\gamma) - \sum_i a_i c_1(\mathscr{O}(\pi^{-1}(p_i)))\,]^n \frac{k^n}{n!} \\ &- \int_X [c_1(X) - (n-1)\sum_i c_1(\mathscr{O}(\pi^{-1}(p_i))] \\ &\cup [\,\pi^* c_1(L^\gamma) - \sum_i a_i c_1(\mathscr{O}(\pi^{-1}(p_i)))\,]^{n-1} \frac{k^{n-1}}{2(n-1)!} + O(k^{n-2}). \end{aligned}$$

The result now follows from a straighforward computation of intersection numbers.

Q.E.D.

Now we need to compute the trace of the induced actions of $\mathbf{G}_m$ on all spaces of sections, $\mathrm{tr}(H^0(\widehat{\mathscr{L}_0^k}))$ for $k \gg 0$. There is an equivariant exact sequence of $\mathbf{G}_m$-representations induced by restriction

$$0 \to H^0_P(\mathscr{I}^k_{\pi^{-1}(Z_0)}\widehat{\mathscr{L}_0^k}|_P) \to H^0_{\widehat{\mathscr{X}_0}}(\widehat{\mathscr{L}_0^k}) \to H^0_{\mathrm{Bl}_{Z_0}\mathscr{X}_0}(\pi^*\mathscr{L}_0^{\gamma k} \otimes \mathscr{O}(\pi^{-1}(Z_0))^{-k}) \to 0 \tag{3.2}$$

So for $k \gg 0$

$$\mathrm{tr}(H^0(\widehat{\mathscr{L}_0^k})) = \mathrm{tr}(H^0_{\mathrm{Bl}_{Z_0}\mathscr{X}_0}(\pi^*\mathscr{L}_0^{\gamma k} \otimes \mathscr{O}(\pi^{-1}(Z_0))^{-k}) + \mathrm{tr}(H^0_P(\mathscr{I}^k_{\pi^{-1}(Z_0)}\widehat{\mathscr{L}_0^k}|_P)). \tag{3.3}$$

To compute the first trace we recall that for $k \gg 0$ there is a natural isomorphism

$$H^0_{\mathrm{Bl}_{Z_0}\mathscr{X}_0}(\pi^*\mathscr{L}_0^{\gamma k} \otimes \mathscr{O}(\pi^{-1}(Z_0))^{-k}) \cong H^0_{\mathscr{X}_0}(\mathscr{I}^k_{Z_0}\mathscr{L}_0^{\gamma k})$$

induced by $\pi_*\mathscr{O}(\pi^{-1}Z_0)^{-k} \cong \mathscr{I}^k_{Z_0}$ for $k \gg 0$. The exact sheaf sequence of $\mathbf{G}_m$-linearised sheaves on $\mathscr{X}_0$

$$0 \to \mathscr{I}^k_{Z_0}\mathscr{L}_0^{\gamma k} \to \mathscr{L}_0^{\gamma k} \to \mathscr{O}_{kZ_0} \otimes_{\mathscr{O}_{Z_0^{\mathrm{red}}}} \mathscr{L}^{\gamma k}|_{Z_0} \to 0$$

is equivariant and induces an exact sequence of sections for $k \gg 0$

$$0 \to H^0_{\mathscr{X}_0}(\mathscr{I}^k_{Z_0}\mathscr{L}_0^{\gamma k}) \to H^0_{\mathscr{X}_0}(\mathscr{L}_0^{\gamma k}) \to \mathscr{O}_{kZ_0} \otimes_{\mathscr{O}_{Z_0^{\mathrm{red}}}} \mathscr{L}_0^{\gamma k}|_{Z_0} \to 0, \tag{3.4}$$

where we have used that $\mathscr{O}_{kZ_0}$ is a 0-dimensional torsion sheaf and $\mathscr{L}_0$ is a locally free sheaf. Thus

$$\mathrm{tr}(H^0_{\mathrm{Bl}_{Z_0}\mathscr{X}_0}(\pi^*\mathscr{L}_0^{\gamma k}\otimes\mathscr{O}(\pi^{-1}(Z_0))^{-k}) = \mathrm{tr}(H^0_{\mathscr{X}_0}(\mathscr{L}_0^{\gamma k})) - \sum_q \mathrm{tr}(\mathscr{O}_{kZ_{0,q}}\otimes_{\mathbb{C}}\mathscr{L}_0^{\gamma k}(q)), \tag{3.5}$$

where the sum is over closed points $q \in Z_0^{\mathrm{red}}$. Also we can write

$$\mathrm{tr}(H^0_P(\mathscr{I}^k_{\pi^{-1}(Z_0)}\widehat{\mathscr{L}_0}^k|_P)) = \sum_q \mathrm{tr}(H^0_{P_q}(\mathscr{I}^k_{\pi^{-1}(Z_{0,q})}\widehat{\mathscr{L}_0}^k|_{P_q})) \tag{3.6}$$

simply by summing over closed points $q \in Z_0^{\mathrm{red}}$ and letting P_q denote the closed subscheme of P which lies over q in the induced map $\widehat{\mathscr{X}_0} \to \mathscr{X}_0$.

Now recall that for any closed point $q \in Z_0^{\mathrm{red}}$ one has the Hilbert-Mumford weight $\mu^{\mathscr{L}_0}(q)$ for the induced $\mathbf{G}_m$-action on $\mathscr{X}_0$ linearised by $\mathscr{L}_0$. Moreover there is a simple relation

$$\mu^{\mathscr{L}_0^m}(q) = m\mu^{\mathscr{L}_0}(q)$$

for $m > 0$. Also by definition $\mu^{\mathscr{L}_0}(q)$ depends on the choice of a lift $\xi \in \chi(\mathbf{G}_m)$, and there is a simple relation

$$\mu^{\mathscr{L}_0}_{\xi'}(q) = \mu^{\mathscr{L}_0}_{\xi}(q) + \xi' - \xi$$

for any other $\xi' \in \chi(\mathbf{G}_m)$.

Note that we can roughly estimate the traces appearing in 3.5, 3.6 as follows.

Lemma 3.9

$$\mathrm{tr}(\mathscr{O}_{kZ_{0,q}} \otimes_{\mathbb{C}} \mathscr{L}_0^{\gamma k}(q)) = \gamma k\, \mu^{\mathscr{L}_0}(q)\, \mathrm{len}(kZ_{0,q}) + O(\gamma^0 k^{n+1}),$$

$$\mathrm{tr}(H^0_{P_q}(\mathscr{I}^k_{\pi^{-1}(Z_{0,q})}\widehat{\mathscr{L}_0}^k|_{P_q})) = \gamma k\, \mu^{\mathscr{L}_0}(q) h^0_{P_q}(\mathscr{I}^k_{\pi^{-1}(Z_{0,q})}\widehat{\mathscr{L}_0}^k) + O(\gamma^0 k^{n+1}).$$

Proof. For the first trace we only need to note that the action of $\mathbf{G}_m$ on the $\mathbb{C}$ vector space $\mathscr{O}_{kZ_{0,q}}$ (with dimension $\mathrm{len}(kZ_{0,q})$) does not depend on the parameter γ. For the second trace we use the definition

$$\mathscr{I}^{k}_{\pi^{-1}(Z_{0,q})}\widehat{\mathscr{L}_0^k}|_{P_q} \cong \pi^*\mathscr{L}_0^{\gamma k}|_q \otimes \mathscr{I}^{k}_{\pi^{-1}(Z_{0,q})}\mathscr{O}(\pi^{-1}(Z_{0,q}))^{-k}.$$

The induced $\mathbf{G}_m$-action on the sheaf $\mathscr{I}^{k}_{\pi^{-1}(Z_{0,q})}\mathscr{O}(\pi^{-1}(Z_{0,q}))^{-k}$ does not depend on γ, while $\pi^*\mathscr{L}_0^{\gamma k}|_q$ is the trivial invertible sheaf on P_q (since it is pulled back from the closed point q) with an action of $\mathbf{G}_m$ of weight $\gamma k\,\mu^{\mathscr{L}_0}(q)$. Thus the only term in the trace involving γ is $\gamma k\,\mu^{\mathscr{L}_0}(q)$ times the dimension of the space of sections.

Q.E.D.

We can group together our points $\{p_i\}$ according to their specialisation, that is we define

$$A_q = \{p_i : p_{i,0} = q\}.$$

Lemma 3.10

$$\begin{aligned}\mathrm{len}(kZ_{0,q}) &= h^0_{P_q}(\mathscr{I}^{k}_{\pi^{-1}(Z_{0,q})}\widehat{\mathscr{L}_0^k}|_{P_q}) \\ &\quad + \left(\sum_{p_i\in A_q} a_i^n\right)\frac{k^n}{n!} + \left(\sum_{p_i\in A_q} a_i^{n-1}\right)\frac{k^{n-1}}{2(n-2)!} + O(k^{n-2}).\end{aligned}$$

Proof. By the local analogue of 3.4 and 3.2 around q in the analytic topology

$$\begin{aligned}\mathrm{len}(kZ_{0,q}) &= h^0_{\mathscr{X}_0}(\mathscr{L}_0^{\gamma k}) - h^0_{\mathscr{X}_0}(\mathscr{I}^{k}_{Z_{0,q}}\mathscr{L}_0^{\gamma k}) \\ &= h^0_{\mathscr{X}_0}(\mathscr{L}_0^{\gamma k}) - h^0_{\mathrm{Bl}_{Z_{0,q}}\mathscr{X}_0}(\pi^*\mathscr{L}_0^{\gamma k}) \\ &= h^0_{\mathscr{X}_0}(\mathscr{L}_0^{\gamma k}) - h^0_{\widehat{\mathscr{X}_0}}(\widehat{\mathscr{L}_0^k}) + h^0_{P_q}(\mathscr{I}^{k}_{\pi^{-1}(Z_{0,q})}\widehat{\mathscr{L}_0^k}|_{P_q}).\end{aligned}$$

The difference $h^0_{\mathscr{X}_0}(\mathscr{L}_0^{\gamma k})-h^0_{\widehat{\mathscr{X}_0}}(\widehat{\mathscr{L}_0^k})$ can be computed by flatness as in Lemma 3.8.

Q.E.D.

Plugging the results of Lemma 3.9 and Lemma 3.10 into 3.3 we see that

Lemma 3.11

$$\begin{aligned}\operatorname{tr}(H^0(\widehat{\mathscr{L}_0^k})) &= \operatorname{tr}(H^0_{\mathscr{X}_0}(\mathscr{L}_0^{\gamma k})) \\ &\quad - \gamma \sum_q \left(\sum_{p_i \in A_q} a_i^n \right) \mu^{\mathscr{L}_0}(q) \frac{k^{n+1}}{n!} \\ &\quad - \gamma \sum_q \left(\sum_{p_i \in A_q} a_i^{n-1} \right) \mu^{\mathscr{L}_0}(q) \frac{k^n}{2(n-2)!} \\ &\quad + O(\gamma^0 k^{n+1}).\end{aligned}$$

We restate this result is a form which is more useful for computing the Donaldson-Futaki weight. For this expand as usual

$$h^0_{\mathscr{X}_0}(\mathscr{L}_0^{\gamma k}) = d_0 \gamma^n k^n + d_1 \gamma^{n-1} k^{n-1} + O(\gamma^{n-2} k^{n-2}),$$

$$\operatorname{tr}(H^0_{\mathscr{X}_0}(\mathscr{L}_0^{\gamma k})) = c_0 \gamma^{n+1} k^{n+1} + c_1 \gamma^{n-1} k^{n-1} + O(\gamma^{n-1} k^{n-1}),$$

$$h^0_{\widehat{\mathscr{X}_0}}(\widehat{\mathscr{L}_0^k}) = d_0'(\gamma) k^n + d_1'(\gamma) k^{n-1} + O(\gamma^{n-2} k^{n-2}),$$

$$\operatorname{tr}(H^0_{\widehat{\mathscr{X}_0}}(\widehat{\mathscr{L}_0^k})) = c_0'(\gamma) k^{n+1} + c_1'(\gamma) k^{n-1} + O(\gamma^{n-1} k^{n-1}).$$

Corollary 3.12

$$\begin{aligned} d_0' &= d_0 \gamma^n - \frac{1}{n!} \sum_i a_i^n, \\ d_1' &= d_1 \gamma^{n-1} - \frac{1}{2(n-2)!} \sum_i a_i^{n-1}, \\ c_0'(\gamma) &= c_0 \gamma^{n+1} - \sum_q \left(\sum_{p_i \in A_q} a_i^n \right) \mu^{\mathscr{L}_0}(q) \frac{\gamma}{n!} + O(1), \\ c_1'(\gamma) &= c_1 \gamma^n - \sum_q \left(\sum_{p_i \in A_q} a_i^{n-1} \right) \mu^{\mathscr{L}_0}(q) \frac{\gamma}{2(n-2)!} + O(1). \end{aligned}$$

Corollary 3.13

$$F(\widehat{\mathscr{X}}) = F(\mathscr{X}) - \sum_q \sum_{p_i \in A_q} a_i^{n-1} \left(\frac{c_0}{d_0} - \mu^{\mathscr{L}_0}(q) \right) \frac{\gamma^{1-n}}{2(n-2)! \deg(X)} + O(\gamma^{1-n}).$$

Proof. We compute

$$
\begin{aligned}
c_0'd_1' &= c_0 d_1 \gamma^{2n} - c_0 \sum_i a_i^{n-1} \frac{\gamma^{n+1}}{2(n-2)!} + O(\gamma^n), \\
c_1'd_0' &= c_1'd_0'\gamma^{2n} - d_0 \sum_q \mu^{\mathscr{L}_0}(q) \sum_{p_i \in A_q} a_i^{n-1} \frac{\gamma^{n+1}}{2(n-2)!} + O(\gamma^n), \\
(d_0')^2 &= d_0^2 \gamma^{2n} + O(\gamma^n),
\end{aligned}
$$

so

$$
\begin{aligned}
F(\widehat{\mathscr{X}}) &= \frac{c_0'd_1' - c_1'd_0'}{(d_0')^2} \\
&= \frac{c_0 d_1 - c_1 d_0}{d_0^2} \\
&\quad - \frac{1}{d_0} \left(\frac{c_0}{d_0} \sum_i a_i^{n-1} - \sum_q \mu^{\mathscr{L}_0}(q) \sum_{p_i \in A_q} a_i^{n-1} \right) \frac{\gamma^{1-n}}{2(n-2)!} \\
&\quad + O(\gamma^{-n}).
\end{aligned}
$$

Note that $d_0 = c_1(L)^n = \deg(X)$ by Riemann-Roch and $d_0^{-2}(c_0 d_1 - c_1 d_0) = F(\mathscr{X})$ by definition. Now we only need to rearrange,

$$
\frac{c_0}{d_0} \sum_i a_i^{n-1} - \sum_q \mu^{\mathscr{L}_0}(q) \sum_{p_i \in A_q} a_i^{n-1} = \sum_q \sum_{p_i \in A_q} a_i^{n-1} \left(\frac{c_0}{d_0} - \mu^{\mathscr{L}_0}(q) \right).
$$

Q.E.D.

Theorem 3.14 Blowup formula. *There exists a sequence of base changes $\rho_r \colon z \mapsto z^{\rho(r)}$ such that*

$$
\begin{aligned}
F(\widehat{\rho_r^* \mathscr{X}})(\gamma) &= F(\rho_r^* \mathscr{X}) \\
&\quad - ch^{\rho_r^* \mathscr{L}_0^r} \rho_r^* \Big(\sum_i a_i^{n-1} p_i \Big)_0 \frac{\gamma^{1-n}}{2(n-2)! \deg(X)} + O(r^{-1}) + O(\gamma^{-n}).
\end{aligned}
$$

Now assume that $\mathscr{L}$ is relatively very ample and the action on $H^0(\mathscr{L}_0)^*$ is special linear. If the Chow weight $ch^{\mathscr{L}_0}(\sum_i a_i^{n-1} p_{i,0})$ is either strictly positive or strictly negative then there exist constants $c(r) > 0$ bounded away from 0 such that

$$
\begin{aligned}
F(\widehat{\rho_r^* \mathscr{X}})(\gamma) &= \rho(r) F(\mathscr{X}) \\
&\quad - ch^{\mathscr{L}_0} \Big(\sum_i a_i^{n-1} p_{i,0} \Big) \frac{c(r)\, \gamma^{1-n}}{2(n-2)! \deg(X)} + O(r^{-1}\gamma^{1-n}) + O(\gamma^{-n}).
\end{aligned}
$$

Proof. For a fixed r the induced $\mathbf{G}_m$-action on the vector space $H^0(\mathscr{L}_0^r)$ is not special linear in general. We can try to make it special linear by solving

$$\mathrm{tr}(H^0(\mathscr{L}_0^r)) + r\lambda_r h^0(\mathscr{L}_0^r) = 0,$$

that is

$$\lambda_r = -\frac{\mathrm{tr}(H^0(\mathscr{L}_0^r))}{rh^0(\mathscr{L}_0^r)} = -\frac{c_0}{d_0} + O(r^{-1}).$$

This need not be an integer, but we can pull back the test configuration $\mathscr{X}$ by a ramified cover $\rho_r\colon z \mapsto z^{\rho(r)}$ of $\mathbb{A}^1$ to get a new test configuration $\rho_r^*\mathscr{X}$ for which the relevant λ_r is an integer. For this test configuration we can substitute

$$\frac{c_0}{d_0} = -\lambda_r + O(r^{-1})$$

into expansion in Corollary 3.13 to get

$$F(\widehat{\rho_r^*\mathscr{X}})(\gamma) = F(\rho_r^*\mathscr{X}) \\ - \sum_q \sum_{p_i\in A_q} a_i^{n-1}\mu_{\lambda_r}^{\rho_r^*\mathscr{L}_0}(q)\frac{\gamma^{1-n}}{2(n-2)!\deg(X)} + O(r^{-1}\gamma^{1-n}) + O(\gamma^{1-n})$$

where $\mu_{\lambda_r}^{\rho_r^*\mathscr{L}_0}(q) = \mu^{\rho_r^*\mathscr{L}_0}(q) + \lambda_r$ is the Hilbert-Mumford weight with respect to the new linearisation making the action special linear on $H^0(\rho_r^*\mathscr{L}_0^r)$. By Definition of Chow weight

$$\sum_q \sum_{p_i\in A_q} a_i^{n-1}\mu_{\lambda_r}^{\rho_r^*\mathscr{L}_0}(q) = ch^{\rho_r^*\mathscr{L}_0^r}\rho_r^*(\sum_i a_i^{n-1}p_i\,)_0,$$

the Chow weight of the limit inside $\rho_r^*\mathscr{X}$, i.e. $\sum_i a_i^{n-1}\lim_{\mathbf{G}_m\cdot\rho_r^*\mathscr{X}}(p_i)$, with respect to the line $\rho_r^*\mathscr{L}_0^r$.

For the second statement we only need to observe that while the value of the integer $ch^{\rho_r^*\mathscr{L}_0^r}\rho_r^*(\sum_i a_i^{n-1}p_{i,0}\,)_0$ depends on r, the (semi, poly, in)stability of the 0-cycle $\rho_r^*(\sum_i a_i^{n-1}p_i)_0$ always equals the (semi, poly, in)stability of the 0-cycle $\sum_i a_i^{n-1}p_{i,0}$ on $\mathscr{X}_0$. This follows from the definition of stability in terms of invariant sections (by pulling back invariant sections under the canonical equivariant isomorphism $(\rho_r^*\mathscr{X})_0 \to \mathscr{X}_0$).

Q.E.D.

3.3 Algebraic Arezzo-Pacard and its converse

We begin by a simple application of the Kempf-Ness Theorem to spell out Theorem 2.20 in more algebraic terms.

Theorem 3.15 Algebraic Arezzo-Pacard. *Let L be very ample. Suppose (X, L) is cscK and* $\mathrm{Aut}(X, L)$ *acts through* $\mathrm{SL}(H^0(L)^*)$. *Let* $\sum_{i=1}^{l} a_i^{n-1} p_i$ *be a* Chow stable 0-*cycle with respect to this embedding such that condition 2 in Theorem 2.20 is satisfied. Then for all $\gamma \gg 0$ the polarised manifold*

$$(\mathrm{Bl}_{p_1, \dots, p_l} X, \pi^* L^\gamma \otimes_{i=1}^{l} \mathscr{O}(\pi^{-1}(p_i))^{-a_i})$$

is cscK. Let ω_γ be the cscK metric. There exists $\phi \in$ $\mathrm{Aut}(X, L)$ *such that $\frac{\omega_\gamma}{\gamma}$ converges to $\phi^*\omega$ in C^∞ away from $p_1, \dots, p_l$.*

Proof. Firstly, condition 1 in Theorem 2.20 is implicit in the assumption that $\sum_{i=1}^{l} a_i^{n-1} p_i$ is stable (it must have finite stabiliser). Now let $m_{\omega_{FS}}$ be the natural moment map with respect to the Fubini-Study metric. Since $\sum_{i=1}^{l} a_i^{n-1} p_i$ is stable by the Kempf-Ness Theorem there exists $\phi \in \mathrm{Aut}(X, L)$ such that

$$\sum_{i=1}^{l} a_i^{n-1} m_{\omega_{FS}}(\phi^{-1}(p_i)) = 0.$$

But we can also leave the cycle fixed and pull back the metric instead,

$$\sum_{i=1}^{l} a_i^{n-1} m_{\phi^*\omega_{FS}}(p_i) = 0.$$

By our assumptions and Theorem 2.20 then we can constructs the desired metrics ω_γ and moreover $\frac{\omega_\gamma}{\gamma} \to \phi^*\omega$ in C^∞ away from $p_1, \dots, p_l$.

Q.E.D.

Theorem 3.16 Converse to Arezzo-Pacard. *Let L be very ample. Suppose that the Futaki character of (X, L) vanishes and* $\mathrm{Aut}(X, L)$ *acts through* $\mathrm{SL}(H^0(X, L)^*)$. *If the* 0-*cycle* $\sum_{i=1}^{l} a_i^{n-1} p_i$ *is* Chow unstable *with respect to this embedding then the polarised manifold*

$$(\mathrm{Bl}_{p_1, \dots, p_l} X, \pi^* L^\gamma \otimes_{i=1}^{l} \mathscr{O}(\pi^{-1}(p_i))^{-a_i})$$

is K-unstable for all $\gamma \gg 0$. Therefore (X, L) does not admit a cscK metric.

Proof. By the Hilbert-Mumford Criterion there exists a 1-PS

$$\alpha \hookrightarrow \mathrm{Aut}(X, L) \hookrightarrow \mathrm{SL}(H^0(X, L)^*)$$

such that $ch^L(\sum_{i=1}^{l} a_i^{n-1} p_i, \alpha) > 0$. By assumption the product test configuration given by $X \times \mathbb{A}^1$ with action

$$t \cdot (x, z) = (\alpha(t) \cdot x, tz)$$

has vanishing Futaki invariant. Applying the blowup formula Theorem 3.14 with $\mathscr{X} = X \times \mathbb{A}^1$ and the action induced by α we have that for some $c(r) > 0$ bounded away from 0

$$F(\widehat{\rho_r^* \mathscr{X}})(\gamma) = -ch^L(\sum_{i=1}^{l} a_i^{n-1} p_i, \alpha) \frac{c(r)\gamma^{1-n}}{2(n-2)!} + O(r^{-1}\gamma^{1-n}) + O(\gamma^{-n}).$$

Thus for $r \gg 0$, $\gamma = \gamma(r) \gg 0$ we see that

$$F(\widehat{\rho_r^* \mathscr{X}})(\gamma) < 0,$$

proving K-instability of the blowup.

Q.E.D.

Corollary 3.17 *If (X, L) is asymptotically Chow stable its blowup along a Chow unstable* 0*-cycle is asymptotically Chow unstable, when the exceptional divisors are small enough.*

Proof. As we recalled in Section 2.5 if (X, L) is asymptotically Chow stable then it is K-semistable. In particular the Futaki character of (X, L) vanishes. Now the same argument as above shows that the blowup along the Chow unstable 0-cycle is K-unstable when the exceptional divisors are small enough, and so in turn asymptotically Chow unstable.

Q.E.D.

Example 3.18 *Projective space.* There is a very satisfactory geometric criterion for Chow stability of points in $\mathbb{P}^n$ (see for example [21] 231–235). A cycle $Z = \sum_i m_i p_i$ is Chow *unstable* if and only if for some proper subspace $V \subset \mathbb{P}^n$ one has

$$\frac{|V \cap Z|}{\dim(V) + 1} > \frac{\sum_i m_i}{n + 1}.$$

So already for $X = \mathbb{P}^2$ Theorem 3.16 gives infinitely many new examples of K-unstable classes: it suffices that more than 2/3 of the points (counted with multiplicities) lie on a line to get a K-unstable blowup. This gives a full generalisation of Example 5.30 in [24] where it is shown that when $m_1 \gg m_j$ $(j > 2)$ the blowup is K-unstable (actually *slope unstable* with respect to E_1).

Example 3.19 *Products.* Consider the product $X \times Y$ of (X, L_X) and (Y, L_Y) polarised by $L_X \boxtimes L_Y$. Then for $\alpha : \mathbf{G}_m \hookrightarrow \mathrm{Aut}(X, L_X)$,

$$ch^{L_X \boxtimes L_Y}(\sum_i a_i p_i, \alpha \times 1) = ch^{L_X}(\sum_i a_i \pi_X(p_i), \alpha).$$

Thus we may apply the above geometric criterion to the fibres of a product $\mathbb{P}^n \times Y$. For example if Y is K-polystable the blowup of $\mathbb{P}^n \times Y$ along an unstable cycle supported at a single $\mathbb{P}^n$ fibre will be K-unstable. A special case is the product $\mathbb{P}^n \times \mathbb{P}^m$ polarised by $\mathscr{O}(1) \boxtimes \mathscr{O}(1)$. A 0-cycle will be unstable whenever its projection to one of the two factors is, e.g. in the case of 3 distinct point, when 2 of them lie on a vertical or horizontal fibre. This gives more examples of unstable blowups.

Example 3.20 $\mathbb{P}^1$ *bundles.* Similarly we can consider the projective completion X of some line bundle L over a polarised manifold. In this case the so-called momentum construction yields many examples of cscK metrics, see [18]. Any polarisation $\mathscr{L}$ on X restricts to $\mathscr{O}_{\mathbb{P}^1}(k)$ on all the fibres for some k. There is a natural $\mathbf{G}_m$-action on L given by complex multiplication on the fibres, and this extends to a $\mathbf{G}_m$-action α on X so that points lying on the zero (resp. infinity) section X_0 (X_∞) are fixed, but with weight k (resp. $-k$) on the line above them. By acting with α^{-1} instead if necessary, we conclude that whenever more than half the points lie on X_0 or X_∞, the corresponding 0-cycle is Chow unstable (i.e. its Chow weight is $> ck$ for some positive constant c). We see that the conclusion is really independent of k and so the blowup along such a cycle will be K- and asymptotically Chow-unstable for *any* polarisation on X making the base cscK.

Remark 3.21 Suppose we wish to prove an algebraic analogue of the Arezzo-Pacard theorem. For simplicity consider the following statement.

If (X, L) is K-stable then the blowup of X along any 0-cycle with the now familiar polarisations is K-stable (this should certainly be true in the regime predicted by the YTD Conjecture).

By the blowup formula Theorem 3.14 it is clear then that we would need a uniform lower bound for the Futaki invariant of nontrivial test configurations to prove this algebraically. But as we explained in Remark 2.17 this never holds.

We believe this observation gives an interesting argument in favour of the uniform notion of K-stability advocated by G. Székelyhidi (see again Remark 2.17).

4 A proof of K-stability

In this Chapter we will offer the first proof (to the best of our knowledge) of the K-stability of a cscK polarised manifold with discrete automorphisms, refining the K-semistability proved by Donaldson.

Theorem 4.1 *If (X, L) is cscK and $\mathrm{Aut}(X, L)$ is discrete (hence finite) then (X, L) is K-stable.*

Recall that if (X, L) is K-stable we know a priori that $\mathrm{Aut}(X, L)$ is finite.

Our proof rests on the following fact.

Proposition 4.2 *Let (X, L) be a properly K-semistable polarised manifold. Then there exists a point $q \in X$ such that the polarised blowup*

$$(\mathrm{Bl}_q X, \pi^* L^{\gamma} \otimes \mathscr{O}(\pi^{-1}(q))^{-1})$$

is K-unstable for $\gamma \gg 0$.

In turn this is implied by the following algebro-geometric estimate (whose proof will occupy most of this Chapter).

Proposition 4.3 *Let $(\mathscr{X}, \mathscr{L})$ be a nonproduct test configuration for a polarised manifold (X, L) with nonpositive Donaldson-Futaki invariant. Then there exists $q \in \mathscr{X}_1 \cong X$ such that*

$$ch^{\mathscr{L}_0}(\lim_{t \to 0} t \cdot q) > 0.$$

Proof. By the embedding result for test configurations Lemma 2.15 we reduce to the case of a nontrivial $\mathbf{G}_m$-action on $\mathbb{P}^N$ for some N, of the form

$$\mathrm{diag}(\lambda^{m_0} x_0, \ldots \lambda^{m_N} x_N),$$

ordered by

$$m_0 \leq m_1 \ldots \leq m_N.$$

Let $\{Z_i\}_{i=1}^k$ be the distinct projective weight spaces, where Z_i has weight m_i (i.e. the induced action on Z_i is trivial with weight m_i). Each Z_i is a projective subspace of $\mathbb{P}^N$, and the central fibre with its reduced induced structure $\mathscr{X}_0^{\mathrm{red}}$ is a contained in $\mathrm{Span}(Z_{i_1}, \ldots Z_{i_l})$ for some minimal flag $0 = i_1 < i_2 \ldots < i_l$.

The case $1 < l$. In this case the induced action on closed points of $\mathscr{X}_0$ is nontrivial. Let $q \in \mathscr{X}_1$ be any point with

$$\lim_{\lambda \to 0} \lambda \cdot q = q_0 \in Z_{i_l}.$$

Such a point exists by minimality and because the specialisation of every point must lie in some Z_j. Since the action on $\mathscr{X}_0$ is induced from that on $\mathbb{P}^N$, q_0 belongs to the totally repulsive fixed locus $R = \mathscr{X}_0 \cap Z_{i_l} \subset \mathscr{X}_0$. By this we mean that every closed point in $\mathscr{X}_0 \setminus R$ specialises to a closed point in $\mathscr{X}_0 \setminus R$. In particular the natural birational morphism $\mathscr{X}_0 \dashrightarrow \mathrm{Proj}(\bigoplus_d H^0(\mathscr{X}_0, \mathscr{L}_0^{\otimes d})^{\mathbf{G}_m})$ blows up along R. So $q_0 \in R$ is an unstable point for the $\mathbf{G}_m$-action in the sense of Geometric invariant theory. By the Hilbert-Mumford criterion the weight of the induced action on the line $\mathscr{L}_0|_{q_0}$ must be strictly positive. Since after base-change we may assume that the induced action on $H^0(\mathscr{X}_0, \mathscr{L}_0)^*$ is special linear this weight coincides with the Chow weight, so $ch^{\mathscr{L}_0}(\, q_0\,) > 0$.

Degenerate case. In the rest of the proof we will show that in the degenerate case $\mathscr{X}_0^{\mathrm{red}} \subset Z_0$ the Donaldson-Futaki invariant is strictly positive. Note that since by assumption the original $\mathbf{G}_m$-action on $\mathbb{P}^N$ is nontrivial, $Z_0 \subset \mathbb{P}^N$ is a proper projective subspace.

We digress for a moment to make the following observation: for any $\mathbf{G}_m$-action on $\mathbb{P}^N$ with ordered weights $\{m_i\}$, and a smooth nondegenerate manifold $Y \subset \mathbb{P}^N$, the map $\rho\colon Y \ni y \mapsto y_0 = \lim_{\lambda \to 0} \lambda \cdot y$ is *rational*, defined on the open dense set $\{y \in Y : \mu(y) = m_0\}$ of points with minimal Hilbert-Mumford weight. Indeed, in the above notation, generic points specialise to some point in the lowest fixed locus Z_0. In any case the map ρ blows up exactly along loci where the Hilbert-Mumford weight jumps.

Going back to our discussion of the case $\mathscr{X}_0^{\mathrm{red}} \subset Z_0$, we see that this means precisely that all points of $\mathscr{X}_1$ have minimal Hilbert-Mumford weight m_0, so there is a well defined *morphism*

$$\rho\colon \mathscr{X}_1 \to Z_0.$$

Moreover ρ is a finite map: the pullback of $\mathscr{L}_0$ under ρ is L which is ample, therefore ρ cannot contract a positive dimensional subscheme. The morphism ρ cannot be an isomorphism on its image, since it would then fit in a $\mathbf{G}_m$-equivariant isomorphism $\mathscr{X} \cong X \times \mathbb{A}^1$. Therefore ρ cannot be injective, either on closed points or tangent vectors. If, say, ρ identifies distinct points x_1, x_2, this means that the x_i specialise to the same x under the $\mathbf{G}_m$-action; by flatness then the local ring $\mathscr{O}_{\mathscr{X}_0,x}$ contains a nontrivial nilpotent pointing outwards of Z_0, i.e. the sheaf $\mathscr{I}_{\mathscr{X}_0 \cap Z_0}/\mathscr{I}_{\mathscr{X}_0}$ is nonzero. In other words $\mathscr{X}_0$ is not a closed subscheme of Z_0. The case when ρ annihilates a tangent vector produces the same kind of nilpotent in the local ring of the limit, by specialisation.

To sum up, the central fibre $\mathscr{X}_0$ is nonreduced, containing nontrivial Z_0-orthogonal nilpotents. Equally important, the induced action on the closed subscheme $\mathscr{X}_0 \cap Z_0 \subset \mathscr{X}_0$ is trivial. The proof will be completed by a weight computation.

Donaldson-Futaki invariant. Suppose $Z_0 \subset \mathbb{P}^N$ has projective coordinates $[x_1 : ... : x_r]$, i.e. it is cut out by $\{x_{r+1} = ... = x_N = 0\}$. We change the linearisation by changing the representation of the $\mathbf{G}_m$-action, to make it of the form

$$[x_0 : ... : x_r : x_{r+1} : ... : x_N] \mapsto [x_0 : ... : x_r : \lambda^{m_{r+1}-m_0} x_{r+1} : ... : \lambda^{m_N - m_0} x_N], \tag{4.1}$$

and recall $m_{r+i} > m_0$ for all $i > 0$. It is possible that the induced action on $H^0(\mathscr{X}_0, \mathscr{L}_0)^*$ will not be special linear anymore, however this does not affect the Donaldson-Futaki invariant.

Note that for all large k,

$$H^0(\mathbb{P}^N, \mathscr{O}(k)) \to H^0(\mathscr{X}_0, \mathscr{L}_0^k) \to H^1(\mathscr{I}_{\mathscr{X}_0}(k)) = 0. \tag{4.2}$$

By 4.2, our geometric description of $\mathscr{X}_0$ and the choice of linearisation 4.1 we see that any section $\xi \in H^0(\mathscr{X}_0, \mathscr{L}_0^k)$ has *nonpositive* weight under the induced $\mathbf{G}_m$-action. The section ξ can only have *strictly negative* weight if it is of the form $x_{r+i} \cdot f$ for some $i > 0$ (x_{r+i} is now regarded as a linear form and the sign is opposite to that of the action on $\mathbb{P}^N$ by duality). Moreover we know there exists an integer $a > 0$ such that $x^a_{r+i}|_{\mathscr{X}_0} = 0$ for all $i > 0$. Let $w(k)$ denote the total weight of the action on $H^0(\mathscr{X}_0, \mathscr{L}_0^k)$, i.e. the induced weight on the line $\Lambda^{P(k)} H^0(\mathscr{X}_0, \mathscr{L}_0^k)$, where $P(k) = h^0(\mathscr{X}_0, \mathscr{L}_0^k)$ is the Hilbert polynomial for $k \gg 0$. Our discussion implies the upper bound

$$|w(k)| \leq C(P(k-1) + ... + P(k-a)) \tag{4.3}$$

for some $C > 0$, independent of k. In particular,

$$w(k) = O(k^n). \tag{4.4}$$

On the other hand by the presence of Z_0-orthogonal nilpotents there exists a section x_{r+i}, $i > 0$ with $x_{r+i}|_{\mathscr{X}_0} \neq 0$. Multiplying by $H^0(\mathscr{X}_0^{\mathrm{red}}, \mathscr{L}_0^{k-1}|_{\mathscr{X}_0^{\mathrm{red}}})$ and writing $Q(k) = h^0(\mathscr{X}_0^{\mathrm{red}}, \mathscr{L}_0^k|_{\mathscr{X}_0^{\mathrm{red}}})$ gives the upper bound

$$w(k) \leq -C_1 Q(k-1) \tag{4.5}$$

for some $C_1 > 0$ independent of $k \gg 0$. Then

$$\frac{w(k)}{kP(k)} = \frac{w(k)}{kQ(k)} \frac{Q(k)}{P(k)} \leq -\frac{C_2}{k}. \tag{4.6}$$

holds for $k \gg 0$ and some $C_2 > 0$ independent of k. Together with

$$\frac{w(k)}{kP(k)} = O(k^{-1}) \tag{4.7}$$

which follows from 4.4 the upper bound we have just proved implies

$$\frac{w(k)}{kP(k)} = -\frac{C_3}{k} + O(k^{-2}) \tag{4.8}$$

for some $C_3 > 0$ independent of k.

By definition of the Donaldson-Futaki invariant, this immediately implies

$$F(\mathscr{X}) \geq C_3 > 0,$$

a contradiction.

Q.E.D.

Remark 4.4 If we assume a priori that $\mathscr{X}$ is integral and factorial (e.g. if $\mathscr{X}$ is smooth, see [17] II Proposition 6.11) we can dispense with the second part of the proof ("degenerate case"). In this case the Weil divisor $\mathscr{X}_0^{\mathrm{red}}$ would be automatically Cartier and so $\mathscr{I}_{\mathscr{X}_0 \cap Z_0}/\mathscr{I}_{\mathscr{X}_0}$ would vanish.

Proof of Proposition 4.2. By definition of proper K-semistability there is a test configuration $(\mathscr{X}, \mathscr{L})$ for (X, L) which is not a product and has vanishing Donaldson-Futaki invariant $F(\mathscr{X}) = 0$. By the general theory of the Donaldson-Futaki invariant, we may assume without loss of generality that the exponent is 1, $\mathscr{L}$ is relatively very ample and the induced $\mathbf{G}_m$-action on the central fibre acts through $\mathrm{SL}(H^0(\mathscr{L}_0)^*)$ (i.e. we have twisted the test configuration and base changed but still F vanishes). Also let $q_0 \in \mathscr{X}_0$ denote the specialisation of q under the $\mathbf{G}_m$-action. Then the blowup formula Theorem 3.14 shows that for a sequence of base changes ρ_r and test configurations

$$(\widehat{\rho_r^* \mathscr{X}}, \widehat{\rho_r^* \mathscr{L}}) = (\mathrm{Bl}_{(\mathbf{G}_m q)^-} \rho_r^* \mathscr{X}, \pi^* \rho_r^* \mathscr{L}^\gamma \otimes \mathscr{O}(\pi^{-1}(\mathbf{G}_m q)^-)^{-1})$$

one has

$$F(\widehat{\rho_r^* \mathscr{X}})(\gamma) = -ch^{\mathscr{L}_0}(q_0) \frac{c(r)\, \gamma^{1-n}}{2(n-2)!\deg(X)} + O(r^{-1}\gamma^{1-n}) + O(\gamma^{-n})$$

for positive constants $c(r) > \varepsilon > 0$. By Proposition 4.3 we may choose q such that $ch^{\mathscr{L}_0}(q) > 0$. Thus for $r \gg 0, \gamma = \gamma(r) \gg 0$ we see that

$$F(\rho_r^* \widehat{\mathscr{X}})(\gamma) < 0,$$

proving K-instability of the blowup.

Q.E.D.

Proof of Theorem 4.1. Suppose by contradiction that (X, L) is not K-stable. Since it is K-semistable by Donaldson's Theorem 2.16 this means it must be properly K-semistable. Since $\mathrm{Aut}(X, L)$ is discrete and it is isomorphic to $\mathrm{Ham}(\omega, J)$ we see that $\mathfrak{ham} \cong \{0\}$, so Theorem 2.20 implies that *for any* $q \in X$, for $\gamma \gg 0$ the polarised manifold

$$(\mathrm{Bl}_q X, \pi^* L^\gamma \otimes \mathscr{O}(\pi^{-1}(q))^{-1})$$

is cscK (in other words when $\mathrm{Aut}(X, L)$ is a finite group one has $X^{ss}(L) = X^s(L) = X$ independently of the linearisation). In particular for all q the blowup is K-semistable. But this contradicts Proposition 4.2 for the some special choice of q.

Q.E.D.

Remark 4.5 One can characterise the degenerate case in the proof of Proposition 4.3 much more precisely.

As observed by Ross-Thomas [25] Section 3 a result of Mumford implies that any test configuration $(\mathscr{X}, \mathscr{L})$ for (X, L) is a contraction of some blowup of $X \times \mathbb{A}^1$ in a flag of $\mathbf{G}_m$-invariant closed subschemes supported in some thickening of $X \times \{0\}$.

The existence of the map $\rho : \mathscr{X}_1 \to Z_0$ means precisely that in this Mumford representation of $\mathscr{X}$ no blowup occurs, i.e. $\mathscr{X}$ is a contraction of the product $X \times \mathbb{A}^1$.

Define a map $\nu : X \times \mathbb{A}^1 \to \mathscr{X}$ by $\nu(x, \lambda) = \lambda \cdot x$ away from $X \times \{0\}$, $\nu = \rho$ on $X \times \{0\}$. This is a well defined morphism, and since ρ is finite, ν is precisely the *normalisation* of $\mathscr{X}$ ([17] II Exercise 3.8).

So in the degenerate case $\mathscr{X}_0^{\mathrm{red}} \subset Z_0$ the normalisation of $\mathscr{X}$ is $X \times \mathbb{A}^1$.

Ross-Thomas [25] Proposition 5.1 proved the general result that normalising a test configuration reduces the Donaldson-Futaki invariant. This already implies $F \geq 0$ for the degenerate case, since the induced action on $X \times \mathbb{A}^1$ must have vanishing Futaki invariant. In our special case our direct proof yields the strict inequality we need.

Remark 4.6 The result of Mumford mentioned above states more precisely that any test configuration $(\mathscr{X}, \mathscr{L})$ for (X, L) is a contraction of the blowup of $X \times \mathbb{A}^1$ along some ideal sheaf of the form

$$\mathscr{I}_0 + t\mathscr{I}_1 + ... + t^{r-1}\mathscr{I}_{r-1} + (t^r)$$

where $\mathscr{I}_0 \subseteq ... \subseteq \mathscr{I}_r \subset \mathscr{O}_X$ correspond to a descending flag of closed subschemes $Z_0 \supseteq ... \supseteq Z_{r-1}$. The action $(\mathscr{X}, \mathscr{L})$ is the natural one induced from the trivial action on $X \times \mathbb{A}^1$.

Suppose now that $F(\mathscr{X}) = 0$ and that *no contraction* occurs in Mumford's representation.

Then in Proposition 4.3 we can simply choose any closed point $q \in Z_{r-1}$. This is because the proper transform of $Z_{r-1} \times \mathbb{A}^1$ cuts $\mathscr{X}_0$ in the totally repulsive locus for the induced action, i.e. the action flows every closed point in $\mathscr{X}_0$ outside this locus to the proper transform of $X \times \{0\}$.

Conversely blowing up $q \in X \setminus Z_0$ only increases the Donaldson-Futaki invariant (at least asymptotically).

For example K-stability with respect to test configurations with $r = 1$ and no contraction is known as Ross-Thomas *slope-stability* [25]. It will play a major role in the next Chapter.

5 Stability of Ross-Thomas and its Kähler analogue

5.1 Statement of main result

While theoretically very important, Donaldon's Theorem 2.16 does not a priori give any explicit obstruction to the solvability of the cscK equation 1.2. In other words we are interested in explicit integrability conditions that the class $c_1(L)$ (or perhaps more generally a Kähler class Ω) must satisy if it can be represented by a cscK metric.

Much work in this connection has been done by R. Thomas and J. Ross in the papers [25, 24]. For a polarised manifold (X, L) and a closed subscheme $Z \subset X$ Ross-Thomas studied the $\mathbb{Q}$-test configuration

$$(\mathrm{Bl}_{Z\times\{0\}} X \times \mathbb{A}^1, \pi^* L \otimes \mathscr{O}(\pi^{-1}(Z \times \{0\}))^{-c}) \tag{5.1}$$

for suitable $c \in \mathbb{Q}^+$. This means that we allow our divisors to be $\mathbb{Q}$-divisors; since the construction is invariant under tensor powers, the Donaldson-Futaki invariant and thus K-stability are still well defined, and still obstruct cscK metrics. Geometrically this construction is the *degeneration to the normal cone of* $Z \subset X$ revived by W. Fulton [16] Chapter 5. But metrically the construction also depends on a positive rational parameter c less than the *Seshadri constant* defined by

$$\varepsilon(Z, L) = \sup_{s\in\mathbb{Q}^+} \{\pi^* L \otimes \mathscr{O}(\pi^{-1}(Z))^{-s} \text{ is ample on } \mathrm{Bl}_Z X\}.$$

One can show that for $0 < c < \varepsilon(Z, L)$ the $\mathbb{Q}$-invertible sheaf

$$\mathscr{L}_c = \pi^* L \otimes \mathscr{O}(\pi^{-1}(Z \times \{0\}))^{-c}$$

on

$$\mathscr{X} = \mathrm{Bl}_{Z\times\{0\}} X \times \mathbb{A}^1$$

is relatively ample. Moreover the trivial action of $\mathbf{G}_m$ on the product $X \times \mathbb{A}^1$, i.e. $t \cdot (x, z) = (x, tz)$ gives rise to a nontrivial $\mathbf{G}_m$-action on the blowup $\mathscr{X}$.

In the simplest case when $Z \subset X$ is a smooth irreducible subvariety, the central fibre $\mathscr{X}_0$ is the union of the irreducible components

$$\widehat{X} = \pi^{-1}(X \times \{0\} \setminus Z \times \{0\})^-,$$

the proper transform of $X \times \{0\}$, and

$$P = \mathbb{P}(\nu_{Z\times\{0\}}) \cong \mathbb{P}(\nu_Z \oplus \mathscr{O}_Z),$$

the projective completion of the normal bundle to $Z \times \{0\}$ in $X \times \mathbb{A}^1$. More precisely

$$\mathscr{X}_0 = \widehat{X} \cup_{\mathbb{P}(\nu_Z)} P,$$

glued along the copy of $\mathbb{P}(\nu_Z)$ embedded into $\widehat{X}$ as the exceptional divisor, and into P by the injection $\nu_Z \to \nu_Z \oplus \mathscr{O}_Z$. Then $\mathbf{G}_m$ acts trivially on the $\widehat{X}$ component, while it acts on P by $t \cdot [v : w] = [v : tw]$, $v \in \nu_Z, w \in \mathscr{O}_Z$, thus flowing as $t \to 0$ from the repulsive infinity section $Z \cong \mathbb{P}(\mathscr{O}_Z) \subset P$ to the attractive locus $P(\nu_Z)$.

In [25] Section 4.2 the Donaldson-Futaki weight of degeneration to the normal cone is computed. We only describe the final result. For this consider the Riemann-Roch expansion

$$\chi((\mathscr{I}_Z^{xk}/\mathscr{I}_Z^{xk+1}) \otimes L^{xk}) = \alpha_1(x)k^{n-1} + \alpha_2(x)k^{n-2} + O(k^{n-3})$$

for $k \gg 0, xk \in \mathbb{N}$.

Theorem 5.1 (Ross-Thomas) *The Donaldson-Futaki invariant of degeneration to the normal cone of $Z \subset X$ with paramer c, $0 < c < \varepsilon(Z, L)$, is given by*

$$F_{(Z,L)}(c) = \int_0^c (c - x)\, \alpha_2(x)dx + \frac{c}{2}\alpha_1(0) - \frac{\widehat{s}}{2}\int_0^c (c - x)\, \alpha_1(x)dx. \qquad (5.2)$$

Definition 5.2 (Ross-Thomas) The *quotient slope* of the closed subscheme $Z \subset X$ with respect to the parameter c with $0 < c < \varepsilon(Z, L)$ is defined as

$$\mu_c(\mathscr{O}_Z, L) = \frac{\int_0^c (c - x)\alpha_2(x)dx + \frac{c}{2}\alpha_1(0)}{\int_0^c (c - x)\alpha_1(x)dx}$$

(in particular this is a finite quantity for $0 < c < \varepsilon(Z, L)$).

Corollary 5.3 (Ross-Thomas) *If (X, L) is K-semistable then for any closed subscheme $Z \subset X$ and $0 < c < \varepsilon(Z, L)$ one has*

$$\mu_c(\mathscr{O}_Z, L) \geq \mu(X, L) = \frac{n \int_X c_1(X) \cup c_1(L)^{n-1}}{2 \int_X c_1(L)^n}.$$

One says (X, L) is slope-semistable. *In particular by Donaldson's Theorem 2.16 this must hold if (X, L) is cscK.*

Consider the special case when Z is the closed subscheme associated to an effective divisor D. Then the Hirzebruch-Riemann-Roch Theorem gives

$$\alpha_1(x) = \frac{\int_X c_1(\mathscr{O}(D)) \cup (c_1(L) - xc_1(\mathscr{O}(D))^{n-1}}{(n-1)!}, \tag{5.3}$$

$$\alpha_2(x) = \frac{\int_X c_1(\mathscr{O}(D)) \cup (c_1(X) - c_1(\mathscr{O}(D))) \cup (c_1(L) - xc_1(\mathscr{O}(D)))^{n-2}}{2(n-2)!} \tag{5.4}$$

and the slope inequality becomes an inequality of rational functions in the (integrals of) the Chern classes $c_1(X), c_1(L), c_1(\mathscr{O}(D))$.

Let us now formally replace the Chern class $c_1(L)$ in 5.3 by any Kähler class Ω on a (not necessarily projective) Kähler manifold X, i.e.

$$\alpha_1(x) = \frac{\int_X c_1(\mathscr{O}(D)) \cup (\Omega - xc_1(\mathscr{O}(D))^{n-1}}{(n-1)!}, \tag{5.5}$$

$$\alpha_2(x) = \frac{\int_X c_1(\mathscr{O}(D)) \cup (c_1(X) - c_1(\mathscr{O}(D))) \cup (\Omega - xc_1(\mathscr{O}(D)))^{n-2}}{2(n-2)!}. \tag{5.6}$$

We also define a Seshadri type constant

$$\varepsilon(D, \Omega) = \sup_{x \in \mathbb{R}^+} \{\Omega - xc_1(\mathscr{O}(D)) \text{ is Kähler}\}.$$

Definition 5.4 The *quotient slope* of the effective divisor D in a Kähler, not necessarily projective manifold X, with respect to a Kähler class Ω and a real parameter $0 < c < \varepsilon(D, \Omega)$ is given by the finite quantity

$$\mu_c(\mathscr{O}_D, \Omega) = \frac{\int_0^c (c-x)\alpha_2(x)dx + \frac{c}{2}\alpha_1(0)}{\int_0^c (c-x)\alpha_1(x)dx}$$

where $\alpha_i(x)$, $i = 1, 2$ are defined by 5.5, 5.6. Similarly the formal analogue of the Donaldson-Futaki invariant $F_{(D,\Omega)}(c)$ is defined exactly as in 5.2 but with $\alpha_i(x)$, $i = 1, 2$ defined as in 5.5, 5.6.

In the next Section we will prove the Kähler analogue of Theorem 5.3 in the case of effective divisors. Our result was conjectured by Ross-Thomas as part of [25] Section 4.4.

Theorem 5.5 *Let X be a Kähler (not necessarily projective) manifold and fix a Kähler class Ω. If Ω can be represented by a cscK metric then for any effective divisor D and $0 < c < \varepsilon(D, \Omega)$ one has*

$$\mu_c(\mathscr{O}_D, \Omega) \geq \frac{\widehat{s}}{2} = \frac{n \int_X c_1(X) \cup \Omega^{n-1}}{2 \int_X \Omega^n}.$$

One says that (X, Ω) is slope-semistable with respect to effective divisors.

The crux of the matter is that in the non-projective case we cannot apply Donaldson's Theorem. Thus we replace the test configuration by a metric degeneration inside Ω whose K-energy is asymptotically controlled by the quantity $F_{D,\Omega}(c)$.

Note that while it might be difficult to compute with the definition of $\varepsilon(D,\Omega)$ we can give an alternative numerical definition. We say that a $(1,1)$ class Θ belongs to the *positive cone* $\mathcal{P}$ is Θ evaluates positively on every irreducible analytic subvariety, that is

$$\int_V \Theta^p > 0$$

for all $V \subset X$, p dimensional and irreducible.

Lemma 5.6

$$\varepsilon(D,\Omega) = \sup_{x\in\mathbb{R}^+} \{\Omega - xc_1(\mathscr{O}(D)) \in \mathcal{P}.\}$$

Proof. By the beautiful Kähler Nakai-Moishezon Criterion of Demailly-Paun [9] we know that the Kähler cone is a connected component of $\mathcal{P}$. The result follows since $\Omega - xc_1(\mathscr{O}(D))$ lies in the same connected component of $\mathcal{P}$ as the Kähler form Ω.

Q.E.D.

5.2 K-energy estimates

This Section is devoted to the proof of the following result.

Theorem 5.7 *Let $D \subset X$ be an effective divisor. For any base point $\omega \in \Omega$ and $0 < c < \varepsilon(D,\Omega)$ there exists a family of Kähler forms $\omega_\varepsilon \in \Omega$, $0 < \varepsilon \le 1$ with $\omega_1 = \omega$ and such that as $\varepsilon \to 0$*

$$\mathscr{M}_\omega(\omega_\varepsilon) = -\pi F_{(D,\Omega)}(c)\log(\varepsilon) + l.o.t. \quad (5.7)$$

From this our main result Theorem 5.5 easily follows.
Proof of Theorem 5.5. If the Kähler slope inequality is violated for some effective divisor D and $0 < c < \varepsilon(D,\Omega)$ then the Futaki invariant of the corresponding degeneration to the normal cone is negative,

$$F_{(D,\Omega)}(c) < 0.$$

By Theorem 5.7 then

$$\lim_{\varepsilon\to 0} \mathscr{M}_\omega(\omega_\varepsilon) = -\infty.$$

This contradicts the lower bound on the K-energy, Theorem 2.33.

Q.E.D.

The proof of Theorem 5.7 will be divided in several steps.

Let us first define the Kähler forms ω_ε appearing in the statement of Theorem 5.7. Fix any Hermitian metric h on the fibres of the complex line bundle $\mathscr{O}(D)$ and denote by Θ its curvature form. Given the canonical section $\sigma \in H^0(\mathscr{O}(D))$ we define potentials

$$\psi_\varepsilon = \frac{1}{2}\log(\varepsilon^2 + |\sigma|_h^2)$$

for $0 < \varepsilon \ll 1$. By the Poincaré-Lelong equation (e.g. [7] Theorem 4.4) for any $c \in \mathbb{R}^+$ there is a weak convergence

$$i\partial\overline{\partial}(c\psi_\varepsilon) \rightharpoonup -c\Theta + c[D]$$

where $[D]$ denotes the (closed, positive) current of integration along D. By definition of the Seshadri type constant for $0 < c < \epsilon(D, \Omega)$ we can find a potential u (independent of ε) such that

$$\eta = \omega - c\Theta + i\partial\overline{\partial}u > 0.$$

We define our family ω_ε for $0 < \varepsilon \ll 1$ by

$$\omega_\varepsilon = \omega + i\partial\overline{\partial}u + i\partial\overline{\partial}(c\psi_\varepsilon). \tag{5.8}$$

Since $[D]$ is a positive current the inequality

$$\eta + c[D] > \eta$$

holds in the sense of currents, and by definition the metrics ω_ε converge weakly to $\eta + c[D]$, so the inequality

$$\omega_\varepsilon > \eta$$

holds in the sense of currents for $\varepsilon \ll 1$. In other words $\omega_\varepsilon - \eta$ is a strictly positive $(1,1)$ current for all small ε. But since it is also a smooth $(1,1)$ form, this implies it is actually a Kähler form (more generally a similar pointwise statement holds for a current with locally L^1 coefficients, see [7] Section 3).

Thus for $0 < \varepsilon \ll 1$, ω_ε is a Kähler form. We still need to define the metrics ω_ε away from $\varepsilon = 0$, prescribing the base point to be $\omega_1 = \omega$. This can be achieved by choosing $\psi_\varepsilon \to 0$ as $\varepsilon \to 1$, and making $u = u_\varepsilon$ dependent on ε away from $\varepsilon = 0$ so that $u_\varepsilon \to 0$ as $\varepsilon \to 1$ and $\omega + i\partial\overline{\partial}\psi_\varepsilon + i\partial\overline{\partial}u_\varepsilon$ is always Kähler.

Remark 5.8 In general it is *not* possible to find smooth *positive* $(1,1)$ forms α_ε with $\alpha_\varepsilon \rightharpoonup [D]$ as $\varepsilon \to 0$. Indeed in our case the extra smooth term η appears in the limit of ω_ε as $\varepsilon \to 0$.

Lemma 5.9 *The metrics ω_ε are bounded in C^∞ away from D.*

Proof. This is clear from the above construction of ω_ε.

Q.E.D.

This means that for the sake of proving 5.7 we need only study the behaviour of $\mathscr{M}_\omega$ near points of $\mathrm{Supp}(D)$.

Let us first write down in detail the model case of $x \in \mathrm{Supp}(D)$ near which D is reduced and smooth. The general case will follow from this thanks to standard results in the theory of currents.

So choose coordinates $(z = \{z_i\}_{i=1}^{n-1}, w)$ near x such that w is a local generator for $\mathscr{O}(D)$. Then

$$\psi_\varepsilon = \frac{1}{2}\log(\varepsilon^2 + e^{-2\varphi}|w|^2)$$

near x, where $e^{-\varphi}$ is the weight for h on $\mathscr{O}(D)|_U$ with respect to the Euclidean norm $|\cdot|$. We make the first order expansion in the D-transversal direction

$$e^{-2\varphi} = e^{-2\widetilde{\varphi}}(1 + O(|w|))$$

where $\widetilde{\varphi} = \widetilde{\varphi}(z)$ does not depend on the transversal coordinate w. Moreover from now on we assume that we have chosen normal z coordinates at x so that

$$\widetilde{\varphi} = \partial_{z_i}\widetilde{\varphi} = \partial_{\bar{z}_j}\widetilde{\varphi} = 0.$$

Since $\widetilde{\varphi}$ is independent of w this holds in a small D-transversal slice

$$\{z = z(x)\}.$$

Notation. We introduce a piece of notation that will be very useful for the rest of this Section. Namely we will denote by r any smooth function $r = r(z, w)$ which is $O(|w|)$ uniformly in U, or any smooth differential form whose coefficient functions have the same property in our fixed coordinate system (z, w).

With the above convention in force, direct calculation gives

Lemma 5.10 *Let $x \in \mathrm{Supp}(D)$ be a point where D is reduced and smooth. In the D-transversal slice $\{z = z(x)\}$,*

$$\partial_\varepsilon \psi_\varepsilon = \varepsilon(\varepsilon^2 + (1+r)|w|^2)^{-1};$$
$$\partial_w \partial_{\overline{w}} \psi_\varepsilon = \frac{1}{2}(1+r)\,\varepsilon^2(\varepsilon^2 + (1+r))|w|^2)^{-2};$$
$$\partial_{z_i} \partial_{\overline{w}} \psi_\varepsilon = r;$$
$$\partial_{z_i} \partial_{\overline{z}_j} \psi_\varepsilon = -(1+r)\,\partial_{z_i}\partial_{\overline{z}_j}\widetilde{\varphi}\,|w|^2(\varepsilon^2 + (1+r))|w|^2)^{-1}.$$

Taking exterior powers we find

Lemma 5.11 *Let $x \in \mathrm{supp}(D)$ be a point where D is reduced and smooth. In the D-transversal slice $\{z = z(x)\}$ and for $p \geq 1$,*

$$(i\partial\overline{\partial}\psi_\varepsilon)^p = r + (-1)^p(1+r)^p|w|^{2p}(\varepsilon^2 + (1+r)|w|^2)^{-p}(i\partial\overline{\partial}\widetilde{\varphi})^p +$$
$$\frac{(-1)^{p-1}}{2}(1+r)^{p-1}\varepsilon^2|w|^{2(p-1)}(\varepsilon^2 + (1+r)|w|^2)^{-p-1}p\,(i\partial\overline{\partial}\widetilde{\varphi})^{p-1} \wedge i\,dw \wedge d\overline{w},$$

and so

$$\partial_\varepsilon\psi_\varepsilon(i\partial\overline{\partial}\psi_\varepsilon)^p = r + (-1)^p(1+r)^p\varepsilon|w|^{2p}(\varepsilon^2 + (1+r)|w|^2)^{-p-1}(i\partial\overline{\partial}\widetilde{\varphi})^p +$$
$$\frac{(-1)^{p-1}}{2}(1+r)^{p-1}\varepsilon^3|w|^{2(p-1)}(\varepsilon^2 + (1+r)|w|^2)^{-p-2}p\,(i\partial\overline{\partial}\widetilde{\varphi})^{p-1} \wedge i\,dw \wedge d\overline{w}.$$

These first order expansions are used to prove a global result about the weak convergence of some quantities which we will need later on when computing the K-energy $\mathscr{M}_\omega$.

Proposition 5.12 *The following hold in the sense of currents:*

$$\varepsilon\partial_\varepsilon\psi_\varepsilon \rightharpoonup 0 \tag{5.9}$$

and for $p \geq 1$

$$\varepsilon\partial_\varepsilon\psi_\varepsilon(i\partial\overline{\partial}\psi_\varepsilon)^p \rightharpoonup \frac{(-1)^{p+1}\pi}{2(1+p)}\Theta^{p-1} \wedge [D]. \tag{5.10}$$

Proof. Weak convergence can be checked locally (see e.g. [7] Section 2). Consider first the model case of a point $x \in \mathrm{Supp}(D)$ near which D is reduced and smooth. Choosing coordinates at x as in Lemma 5.11 we find that

$$\varepsilon\partial_\varepsilon\psi_\varepsilon = \varepsilon^2(\varepsilon^2 + (1+r)|w|^2)^{-1}.$$

The right hand side is uniformly bounded and converges to 0 uniformly away from $w = 0$, thus it converges to 0 weakly as $\varepsilon \to 0$.

The case when D is smooth but not reduced near x can be handled by a finite ramified cover $w \mapsto w^m$, where m is the local multiplicity. The above weak convegence is unaffected. Thus 5.9 holds away from the set of singular points of $\mathrm{Supp}(D)$.

To prove weak convergence on all of X one uses the Support and Skoda Theorems, as in the proof of Lemma 2.1 of [9] for example.

As for 5.10 multiplying by ε the second expansion in 5.11 we find

$$\varepsilon\partial_\varepsilon\psi_\varepsilon(i\partial\overline{\partial}\psi_\varepsilon)^p = \varepsilon\, r + (-1)^p(1+r)^p\varepsilon^2|w|^{2p}(\varepsilon^2+(1+r)|w|^2)^{-p-1}(i\partial\overline{\partial}\widetilde{\varphi})^p +$$

$$\frac{(-1)^{p-1}}{2}(1+r)^{p-1}\varepsilon^4|w|^{2(p-1)}(\varepsilon^2+(1+r)|w|^2)^{-p-2}p\,\Theta^{p-1}\wedge i\,dw\wedge d\overline{w}.$$

The curvature form Θ appears since $i\partial\overline{\partial}\widetilde{\varphi}$ represents the w-constant extension to U of the pullback of the curvature form Θ to D.

The sequence of functions

$$\varepsilon\, r + (-1)^p(1+r)^p\varepsilon^2|w|^{2p}(\varepsilon^2+(1+r)|w|^2)^{-p-1}$$

is uniformly bounded as $\varepsilon \to 0$ an converges to 0 away from $\{w = 0\}$, thus it gives no contribution to the weak limit.

Therefore 5.10 holds in a neighborhood of x provided the sequence of forms

$$f_\varepsilon = (1+r)^{p-1}\varepsilon^4|w|^{2(p-1)}(\varepsilon^2+(1+r)|w|^2)^{-p-2}dw\wedge d\overline{w}$$

converges to the point mass $\frac{\pi}{p\,(p+1)}\delta_{\{w=0\}}dw\wedge d\overline{w}$ in the sense of currents.

For this note that f_ε is converging uniformly to 0 away from $w = 0$. Moreover the pullback of f_ε under the change of variable $w = \varepsilon w'$ is converging point-wise to $|w'|^{2p-1}(1+|w'|^2)^{-p-2}dw'\wedge\overline{dw'}$, and the transversal integrals

$$\int_{\{z=z(x)\}} f_\varepsilon i\, dw\wedge d\overline{w}$$

are converging to

$$2\pi\int_0^\infty \frac{s^{2p-1}}{(1+s^2)^{p+2}}ds = \frac{\pi}{p(p+1)}$$

by Lebesgue's dominated convergence, proving $f_\varepsilon \rightharpoonup \frac{\pi}{p\,(p+1)}\delta_{\{w=0\}}dw\wedge d\overline{w}$.

Global weak convergence then follows as for 5.9.

Q.E.D.

With these preliminary computations in place we pass to compute the asymptotics of $\mathscr{M}_\omega$ along ω_ε as $\varepsilon \to 0$. By the integration by parts formula Theorem 2.36 the expansion Theorem 5.7 can be split into three separate statements about the I, J and $\int \log\det$ functionals.

Let us begin with an expansion for the J functional.

Definition 5.13 We introduce the cohomological quantity, depending on a positive parameter c,

$$F_J(c) = \sum_{p=1}^{n-1} \frac{(-1)^{p+1}c^{p+1}}{2(p+1)!(n-1-p)!} \int_X c_1(X) \cup \Omega^{n-1-p} \cup c_1^p(\mathscr{O}(D)).$$

Lemma 5.14

$$J(\omega_\varepsilon) = \pi F_J(c) \log(\varepsilon) + l.o.t.$$

Proof. We will actually find the limit of $\varepsilon\partial_\varepsilon J$ as $\varepsilon \to 0$. The integrand for this functional is

$$(n-1)!^{-1}\varepsilon\partial_\varepsilon(c\psi_\varepsilon)\operatorname{Ric}(\omega) \wedge \omega_\varepsilon^{n-1}$$

which by the binomial theorem applied to $(1,1)$ forms can be rewritten as

$$(n-1)!^{-1}\operatorname{Ric}(\omega) \wedge \sum_{p=0}^{n-1} \binom{n-1}{p} (\omega + i\partial\overline{\partial}u)^{n-1-p} \wedge c^{p+1}\varepsilon\partial_\varepsilon\psi_\varepsilon(i\partial\overline{\partial}\psi_\varepsilon)^p.$$

By Corollary 5.12 this converges weakly to

$$\pi(n-1)!^{-1} \sum_{p=1}^{n-1} \binom{n-1}{p} \frac{(-1)^{p+1}c^{p+1}}{2(p+1)} \operatorname{Ric}(\omega) \wedge (\omega + i\partial\overline{\partial}u)^{n-1-p} \wedge \Theta^{p-1} \wedge [D].$$

Integrating over X proves our claim.

Q.E.D.

An identical argument applies to the I functional.

Definition 5.15 Introduce the quantity

$$F_I(c) = \sum_{p=1}^{n} \frac{(-1)^{p+1}c^{p+1}}{2(p+1)!(n-p)!} \int_X \Omega^{n-p} \cup c_1^p(\mathscr{O}(D)).$$

Lemma 5.16

$$I(\omega_\varepsilon) = \pi F_I(c) \log(\varepsilon) + l.o.t.$$

Proof. Same as for Lemma 5.14.

Q.E.D.

Finally we consider the $\int \log\det$ functional. This requires slightly better estimates than for case of I and J.

Definition 5.17 Introduce the quantity

$$F_{\log} = \sum_{p=1}^{n} \frac{(-1)^{p-1}c^p}{p!(n-p)!} \int_X \Omega^{n-p} \cup c_1(\mathscr{O}(D))^p.$$

Lemma 5.18

$$\int_X \log\left(\frac{\omega_\varepsilon^n}{\omega^n}\right) d\mu_{\omega_\varepsilon} = -\pi F_{\log} \log(\varepsilon) + l.o.t.$$

Proof. We study the weak limit on X as $\varepsilon \to 0$ of the sequence of globally well defined (n,n) forms

$$\log(\varepsilon)^{-1} \log\left(\frac{\omega_\varepsilon^n}{\omega^n}\right) d\mu_{\omega_\varepsilon}.$$

Weak convergence can be checked locally, so we can write

$$\log(\varepsilon)^{-1} \log\left(\frac{\omega_\varepsilon^n}{\omega^n}\right) d\mu_{\omega_\varepsilon} = f_\varepsilon - f'_\varepsilon$$

where $f_\varepsilon, f'_\varepsilon$ are the (n,n) forms

$$f_\varepsilon = [\,\log(\varepsilon)^{-1}\log(\det g_\varepsilon)\det g_\varepsilon\,]\, dz \wedge d\overline{z} \wedge dw \wedge d\overline{w},$$
$$f'_\varepsilon = [\,\log(\varepsilon)^{-1}\log(\det g)\det g_\varepsilon\,]\, dz \wedge d\overline{z} \wedge dw \wedge d\overline{w}.$$

Our first claim is that the sequence of (n,n) forms f_ε converges weakly to 0. As in the proof of Lemma 5.12 we can reduce to a good coordinate slice $\{z = z(x)\}$ around a smooth (reduced) point $x \in \mathrm{Supp}(D)$. The sequence certainly converges uniformly to 0 away from $w = 0$. The claim follows if we can show that the integrals in the w direction

$$\int_{\{z=z(x)\}} \log(\det g_\varepsilon)\det g_\varepsilon\, dw \wedge d\overline{w}$$

are uniformly bounded. We can prove this using our previous computation in Lemma 5.11. Namely we make a decomposition

$$\omega_\varepsilon^n = \alpha_\varepsilon + \beta_\varepsilon \tag{5.11}$$

reflecting the decomposition of the powers $(i\partial\overline{\partial}\psi_\varepsilon)^p$ in Lemma 5.11, where

$$\alpha_\varepsilon = r + \sum_{p=0}^{n} \binom{n}{p} (-1)^p (1+r)^p c^p |w|^{2p} (\varepsilon^2 + (1+r)|w|^2)^{-p} (\omega + i\partial\overline{\partial}u)^{n-p} \wedge (i\partial\overline{\partial}\widetilde{\varphi})^p,$$

$$\beta_\varepsilon = \sum_{p=1}^{n} \binom{n}{p} \frac{(-1)^{p-1}}{2} (1+r)^{p-1} c^p \varepsilon^2 |w|^{2(p-1)} (\varepsilon^2 + (1+r)|w|^2)^{-p-1} (\omega + i\partial\overline{\partial}u)^{n-p}$$
$$\wedge\, p\, (i\partial\overline{\partial}\widetilde{\varphi})^{p-1} i\, dw \wedge d\overline{w}.$$

Next we make the change of variable $w = \varepsilon w'$. Under this change of variable α_ε and β_ε pull back to forms defined transversally on $0 \le s = |w'| < \varepsilon^{-1}$. Moreover the pullback of α_ε is uniformly $O(\varepsilon^2)$, while the pullback of β_ε restricted to each slice $\{z = z(x)\}$ is dominated by (in angular coordinates)

$$\frac{C s^{2p-1}}{(1+s^2)^{p+1}}\, ds\, d\theta = O(s^{-3})\, ds\, d\theta,$$

since $p \ge 1$ (where C is a positive uniform constant). It follows that $\int \log(\det g_\varepsilon) \det g_\varepsilon\, dw \wedge d\overline{w}$ is asymptotic in each slice to

$$2\pi \int_0^{\varepsilon^{-1}} \log(O(\varepsilon^2) + O(s^{-3}))(O(\varepsilon^2) + O(s^{-3}))\, ds.$$

By dominated convergence these integrals are uniformly bounded as $\varepsilon \to 0$.

It follows that the (n,n) forms f_ε converge to 0 weakly as $\varepsilon \to 0$ as we claimed.

Let us now compute the weak limit of the (n,n) forms f'_ε. By the decomposition for the volume forms 5.11 above,

$$f'_\varepsilon = \log(\varepsilon)^{-1} \log(\det(g))\, \alpha_\varepsilon + \log(\varepsilon)^{-1} \log(\det(g))\, \beta_\varepsilon.$$

The sequence of forms $\log(\det(g))\, \alpha_\varepsilon$ is uniformly bounded in ε, therefore $\log(\varepsilon)^{-1} \log(\det(g))\, \alpha_\varepsilon$ converges weakly to 0. On the other hand making the change of variable $w = \varepsilon w'$ pulls back the forms $\log(\varepsilon)^{-1} \log(\det(g))\, \beta_\varepsilon$ to

$$\log(\varepsilon)^{-1}(\log(\varepsilon^2) + \log(g(w')))\, \beta_\varepsilon(w') = 2\, \beta_\varepsilon(w') + \log(\varepsilon)^{-1} \log(g(w'))\, \beta_\varepsilon(w').$$

Now $\log(\varepsilon)^{-1} \log(g(w'))\, \beta_\varepsilon(w')$ converges weakly to 0 (say by comparison with the forms f_ε), while $2\, \beta_\varepsilon$ converges weakly to

$$\pi \sum_{p=1}^{n} \frac{(-1)^{p-1} \lambda^p}{2p!(n-p)!} (\omega + i\partial\overline{\partial}u)^{n-p} \wedge (i\partial\overline{\partial}\widetilde{\varphi})^{p-1} \wedge [D].$$

This is proved exactly as in Proposition 5.12 and Lemma 5.14; in other words by the definition of β_ε it is enough to prove that for $p \geq 1$ the sequence of forms

$$(1+r)^{p-1}c^p\varepsilon^2|w|^{2(p-1)}(\varepsilon^2+(1+r)|w|^2)^{-p-1}dw\wedge d\overline{w}$$

converges weakly to $\frac{\pi}{p}\delta_{\{w=0\}}dw\wedge d\overline{w}$. These forms are certainly converging to 0 uniformly away from $\{w=0\}$, and their pullback under the change of variable $w=\varepsilon w'$ converge point-wise to $|w'|^{2p-1}(1+|w'|^2)^{-2p-1}dw'\wedge\overline{dw'}$. Moreover the integrals in the w direction converge to

$$2\pi\int_0^\infty \frac{s^{2p-1}}{(1+s^2)^{2p+1}}ds=\frac{\pi}{p}$$

by Lebesgue's dominated convergence, proving the required weak convergence.

Q.E.D.

Substituting these partial results in the integration by parts formula 2.36 we find

$$\mathscr{M}_\omega(\omega_\varepsilon)=-\pi\left(F_{\log}+F_J-\widehat{s}\,F_I\right)\log(\varepsilon)+l.o.t.$$

The proof of the required asymptotic expansion, Theorem 5.7, is completed by the following combinatorial identity.

Lemma 5.19

$$\int_0^c(c-x)\alpha_2(x)dx+\frac{c}{2}\alpha_1(0)=F_{\log}(c)+F_J(c);$$

$$\int_0^c(c-x)\alpha_1(x)dx=2F_I.$$

Therefore

$$F_{(D,\Omega)}(c)=F_{\log}(c)+F_J(c)-\widehat{s}\,F_I.$$

Proof. The computation is carried out explicitely in [25] Theorem 5.2 when $\Omega=c_1(L)$. The extension to general Kähler classes is purely formal.

Q.E.D.

5.3 Slope-unstable non-projective manifolds

An alternative approach to slope-(in)stability for non-projective manifolds, Theorem 5.5, is by a deformation argument. For example it is a classical result of Kodaira that any compact Kähler surface admits arbitrarily small deformations which are projective. If we could perturb the cscK metric at the same time this would give an alternative proof of Theorem 5.5 for surfaces.

This motivates us to look for a genuinely non-projective example in higher dimensions. This is based on Voisin's manifold, combined with the following result for projective bundles.

Lemma 5.20 *Let $E \to B$ be a vector bundle on the Kähler manifold (B, Ω_B). Let $\mathscr{O}_{\mathbb{P}}(1)$ denote the relative hyperplane line bundle on $\mathbb{P}(E)$. If E is slope-destabilised (i.e. Mumford-Takemoto destabilised) by a corank 1 subbundle $F \subset E$ then $\mathbb{P}(E)$ admits no cscK metrics in the classes*

$$\Omega_r = c_1(\mathscr{O}_{\mathbb{P}}(1)) + r\pi^*\Omega_B$$

for $r \gg 0$.

Proof. This is the Kähler analogue, via Theorem 5.5, of a special case of Ross-Thomas' Theorem [25] 5.12.

Since $\mathbb{P}(F) \subset \mathbb{P}(E)$ is a divisor, all the computions in loc. cit. hold in $NE^1(\mathbb{P}(E))$ and therefore carry over without any change to our situation, with the care of replacing the line bundle $L^{\otimes r}$ there with the form $r\Omega_B$. Note in passing that a large part of the proof in loc. cit. is devoted to prove that $\varepsilon(\mathbb{P}(F), \Omega_r) = 1$ for $r \geq r_0$ where r_0 does not depend on F (using boundedness of quotients). This is required to establish a general correspondence with Mumford-Takemoto stability. However for our weak statement we only need to prove $\varepsilon(\mathbb{P}(F), \Omega_r) = 1$ for $r \geq r_0(F)$. This is trivial since $c_1(\mathscr{O}_E(1)) - x\, c_1(\mathscr{O}_F(1))$ is relatively ample for $0 < x < 1$.

Q.E.D.

Voisin [32] constructed Kähler manifolds in all dimensions ≥ 4 which are not homotopy equivalent to projective ones. The simplest example in dimension 4 is obtained by a torus T with an endomorphism $f : T \to T$. Consider the product $T \times T$ with projections p_i, $i = 1, 2$. There are four sub-tori in $T \times T$ given by the factors $T_i = p_i^* T$, the diagonal T_3 and the graph T_4 of f. Blow up the intersections of all T_k, that is a finite set $Q \subset T \times T$. Then blow up once more along the proper transforms of the T_k's. The result of this process is of course a Kähler manifold M. Voisin proves that for a special choice of (T, f), M is not homotopy equivalent to (and so a fortiori not deformable to) a projective manifold.

For any $q \in Q$, let E_q be the component of the exceptional divisor for

$$\mathrm{Bl}_Q(T \times T) \to T \times T$$

over $q \in Q$. We write $\mathscr{O}_M(-E_q)$ for the pullback of $\mathscr{O}(-E_q)$ to M. The rank 2 vector bundle $\mathscr{O}_M(-E_q) \oplus \mathscr{O}_M$ is Mumford-destabilised by the line bundle $\mathscr{O}_M(-E_q)$. Fix a Kähler class Ω_M on M. By Lemma 5.20 the projective bundle

$$\pi : \mathbb{P}(\mathscr{O}(-E_q) \oplus \mathscr{O}_M) \to M$$

admits no cscK metric in the classes $\mathscr{O}_{\mathbb{P}}(1) + r\pi^*\Omega_M$ for $r \gg 0$.

Note that by taking a projective bundle we have changed to homotopy type of M so Voisin's Theorem does not immediately imply that $\mathbb{P}(\mathscr{O}(-E_q) \oplus \mathscr{O}_M)$ does not have projective deformations. However this is guaranteed by the following result on deformations of projective bundles [8].

Theorem 5.21 (Demailly-Eckl-Peternell) *Let X be a compact complex manifold with a deformation $X \cong \mathcal{X}_0 \hookrightarrow \mathcal{X} \to \Delta$. If $X = \mathbb{P}(E)$ for some holomorphic vector bundle E over a complex manifold Y then $\mathcal{X}_t = \mathbb{P}(\mathcal{E}_t)$ for some deformation $\mathcal{E} \to \mathcal{Y}$ of $E \to Y$.*

Corollary 5.22 *The Kähler manifold $X = \mathbb{P}(\mathscr{O}_M(-E_q) \oplus \mathscr{O}_M)$ is slope unstable (and so has no cscK metrics) with respect to the classes*

$$c_1(\mathscr{O}_{\mathbb{P}}(1)) + r\pi^*\Omega_M$$

for $r \gg 0$. Moreover X has no projective deformations.

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