Block-Göttsche invariants from wall-crossing

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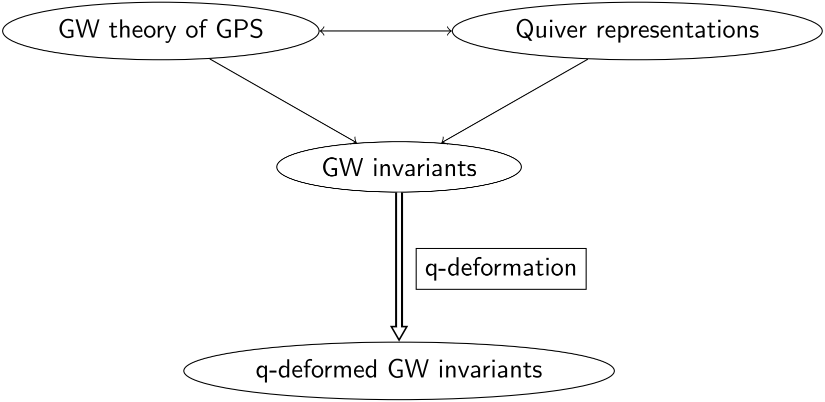
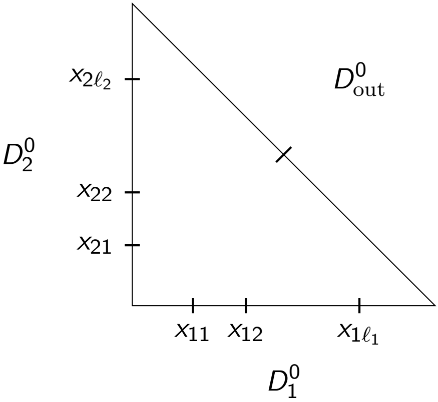
Outline


GW theory of GPS
Quiver representations
GW invariants
q-deformation
q-deformed GW invariants

The GW theory of GPS

- $\mathbb{P}(a,b,1) = \mathbb{C}^3 \backslash \{0,0,0\} / (\mathbb{C}^*_{a,b,1}) =$ weighted projective plane
- Assume: $gcd(a,b) = 1$
- Toric picture:



Fix partitions $\mathbf{P}_1, \mathbf{P}_2$; $\mathbf{P}_i = (p_{ij})$.
Suppose $gcd(\mathbf{P}_1, \mathbf{P}_2) = 1$.

Invariants:

$$N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)] = \sharp^{vir} \left\{ \begin{array}{l} \text{rational curves with } x_{ij} \text{ prescribed} \\ \text{singularities with multiplicities } p_{ij} \text{ and} \\ \text{tangent to order } k \text{ to } D_{\mathrm{out}} \text{ at some point} \end{array} \right\}$$

Examples

1. $N_{(1,3)}[(1, 1+1+1)] = 1$ given by

$$(u:v) \mapsto (u : -\frac{y_1}{x_1x_2x_3}(u-x_1v)(u-x_2v)(u-x_3v) : v)$$

2. $N_{(1,1)}[(1+1, 1+1)] = 2$ given by

$$(u:v) \mapsto (u(u-v) : (u-2v)(u-4v) : v^2)$$

$$(u:v) \mapsto (u(u-\frac{5}{\sqrt{3}}v) : -(u-2\sqrt{3}v)(u+\frac{4}{\sqrt{3}}v) : v^2)$$

Conjectural BPS structure

Define a series

$$N_{\mathbb{P}(a,b,1)} := \sum_{k=1}^{\infty} N_{(a,b)}[(k\mathbf{P}_1, k\mathbf{P}_2)]\tau^k$$

where $gcd(|\mathbf{P}_1|, |\mathbf{P}_2|) = 1$ (start with coprime pair of partitions). Then rewrite formally

$$N_{\mathbb{P}(a,b,1)} := \sum_{k=1}^{\infty} n_{(a,b)}[(k\mathbf{P}_1, k\mathbf{P}_2)] \sum_{d=1}^{\infty} \frac{1}{d^2} \binom{d(k-1)-1}{d-1} \tau^{dk}$$

The $n_{(a,b)}[(k\mathbf{P}_1, k\mathbf{P}_2)]$ are the BPS invariants underlying the GW invariants $N_{(a,b)}[(k\mathbf{P}_1, k\mathbf{P}_2)]$.

Conjecture (GPS)

$n_{(a,b)}[(k\mathbf{P}_1, k\mathbf{P}_2)] \in \mathbb{Z}$ *for every* $a, b, k, \mathbf{P}_1, \mathbf{P}_2$.

Remark. When $k = 1$ $n_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)] = N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)]$.

Vague expectation: In great generality people expect BPS invariants to be integers because they are the Euler characteristic χ of some suitable moduli space.

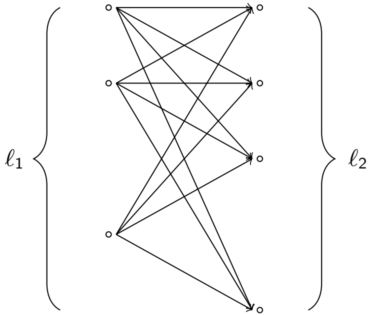
This is true for $N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)]$ in the coprime case!

Reineke–Weist

Theorem

If $\gcd(|\mathbf{P}_1|, |\mathbf{P}_2|) = 1$*, then*

$$N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)] = \chi\,\underbrace{(\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2))}_{\substack{\text{moduli space of stable representations}\\ \text{of complete bipartite quiver}}}$$



Vague expectation: In great generality, BPS invariants should admit a natural q-deformation or quantization.

In our case the Reineke–Weist Theorem provides a natural candidate:

$$\begin{aligned} \widehat{N}'[(\mathbf{P}_1,\mathbf{P}_2)] &= \widehat{P}(\mathcal{M}(\mathbf{P}_1,\mathbf{P}_2))(q) \\ &:= q^{-\frac{1}{2}\dim\mathcal{M}(\mathbf{P}_1,\mathbf{P}_2)}P(\mathcal{M}(\mathbf{P}_1,\mathbf{P}_2))(q), \end{aligned}$$

where $\widehat{P}(\mathcal{M}(\mathbf{P}_1,\mathbf{P}_2))(q)$ is the symmetrized Poincaré polynomial.

Tropical vertex group

Fix integers a, b and a function $f_{(a,b)} \in \mathbb{C}[x, x^{-1}, y, y^{-1}][[t]]$ of the form

$$f_{(a,b)} = 1 + t\, x^a y^b \underbrace{g(x^a y^b, t)}_{g \in \mathbb{C}[z][[t]]}$$

Define $\theta_{(a,b),f_{(a,b)}} \in \mathrm{Aut}_{\mathbb{C}[[t]]}\, \mathbb{C}[x, x^{-1}, y, y^{-1}][[t]]$ by

$$\begin{cases} \theta_{(a,b),f_{(a,b)}}(x) = x\, f_{(a,b)}^{-b}, \\ \theta_{(a,b),f_{(a,b)}}(y) = y\, f_{(a,b)}^{a}. \end{cases}$$

Definition (KS, GS)

The tropical vertex group $\mathbb{V} \subset \mathrm{Aut}_{\mathbb{C}[[t]]} \mathbb{C}[x, x^{-1}, y, y^{-1}][[t]]$ *is the t-adic comletion of the subgroup generated by all* $\theta_{(a,b),f_{(a,b)}}$*.*

Remark. Elements of $\mathbb{V}$ are formal 1-parameter families of holomorphic symplectomorphisms of $\mathbb{C}^* \times \mathbb{C}^*$:
they preserve the form

$$\frac{dx}{x} \wedge \frac{dy}{y}.$$

Example

Fix $\ell_1, \ell_2 \in \mathbb{N}$. *Then*

$$\begin{cases} \theta_{(1,0),(1+tx)^{\ell_1}}(x) & = & x, \\ \theta_{(1,0),(1+tx)^{\ell_1}}(y) & = & y(1+tx)^{\ell_1}. \end{cases}$$

$$\begin{cases} \theta_{(0,1),(1+ty)^{\ell_2}}(x) & = & x(1+tx)^{-\ell_2}, \\ \theta_{(0,1),(1+ty)^{\ell_2}}(y) & = & y. \end{cases}$$

Basic question: compute commutators in $\mathbb{V}$. More precisely, compute

$$[\theta_{(a,b),f}, \theta_{(a',b'),f'}] = \theta^{-1}_{(a',b'),f'}\theta_{(a,b),f}\theta_{(a',b'),f'}\theta^{-1}_{(a,b),f}$$

as some expression involving the generators $\theta_{(a'',b''),f''}$.

Fundamental result: In principle, this is always possible.

Suppose that $a, b, a', b' \geqslant 0$, and that $\mu(a,b) \leqslant \mu(a',b')$ ((a,b) follows (a',b') in clockwise order).
Then $\exists!$ collection of vectors (a'', b'') with positive entries, and attached functions $f_{(a'',b'')}$ such that

$$[\theta_{(a,b),f}, \theta_{(a',b'),f'}] = \underbrace{\overrightarrow{\prod_{(a'',b'')}} \theta_{(a'',b''),f_{(a'',b'')}}}_{\substack{\text{\underline{decreasing} slopes of rays} \\ \text{(from L to R)}}},$$

with $gcd(a'', b'') = 1$.

Question: How do we compute $\{(a'', b''), f_{(a'',b'')}\}$?

Example

For $\ell_1 = \ell_2 = 2$ a closed formula is known:

$$[\theta_{(1,0),(1+tx)^2}, \theta_{(0,1),(1+ty)^2}] = \prod_k^{\rightarrow} \theta_{(k,k+1),f_{(k,k+1)}} \cdot \theta_{(1,1),f_{(1,1)}} \cdot \theta_{(k+1,k),f_{(k+1,k)}},$$

where

$$\begin{cases} f_{1,1} &= (1-t^2xy)^{-4} \\ f_{k,k+1} &= (1+t^{2k+1}x^k y^{k+1})^2 \\ f_{k+1,k} &= (1+t^{2k+1}x^{k+1}y^k)^2. \end{cases}$$

For now we restrict to the simplest case:

$$[\theta_{(1,0),(1+tx)^{\ell_1}}, \theta_{(0,1),(1+ty)^{\ell_2}}] = \overrightarrow{\prod_{(a,b)}} \theta_{(a,b),f_{(a,b)}}.$$

Even this is already very hard: Closed formulae are not known for $\ell_1\ell_2 > 4$. However, there are very interesting theoretical results on computing $\{(a,b), f_{(a,b)}\}$:

Theorem (Theorem A (GPS '10))

Consider the formal power series

$$\log f_{(a,b)} = \sum_{k\geqslant 0} c_k^{(a,b)} (tx)^{ak}(ty)^{bk}.$$

Then

$$c_k^{(a,b)} = k \sum_{|\mathbf{P}_a|=ka} \sum_{|\mathbf{P}_b|=kb} N_{(a,b)}[(\mathbf{P}_a, \mathbf{P}_b)],$$

where $\mathbf{P}_a, \mathbf{P}_b =$ *ordered partitions, and* $len\mathbf{P}_a = \ell_1$, $len\mathbf{P}_b = \ell_2$.

Tropical significance

The GPS Theorem is based on a <u>tropical computation</u> together with some nice correspondence results. The tropical technique is called *factor/defo* and leads to:

Theorem (Theorem A' (GPS))

$$c_k^{(a,b)} = k \sum_{|\mathbf{P}_a|=ka} \sum_{|\mathbf{P}_b|=kb} \sum_{\mathbf{w}} \prod_{i=1}^{2} \frac{R_{\mathbf{P}_i|\mathbf{w}_i}}{|\operatorname{Aut}(\mathbf{w}_i)|} N_{(a,b)}^{\mathrm{trop}}(\mathbf{w}),$$

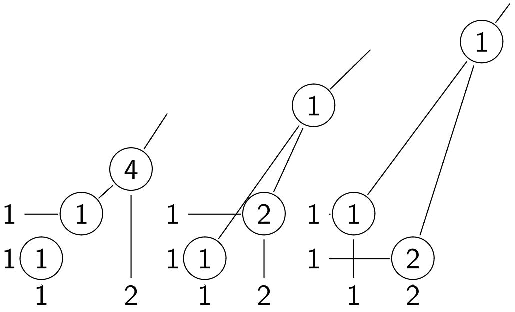
where $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$ *is a pair of weight vectors of arbitrary length parametrizing a family of tropical counts* $\{N_{(a,b)}^{\mathrm{trop}}(\mathbf{w})\}$*.*

$R_{\mathbf{P}_i|\mathbf{w}_i}$, $|\operatorname{Aut}(\mathbf{w}_i)|$ are some ramification and automorphism factors.

Geometric meaning: rational plane tropical curves with $|\mathbf{w}_1| + |\mathbf{w}_2|$ incoming ends and a single outgoing end.

Example

$N^{\mathrm{trop}}((1,1),(1,2)) = 8$



Fact: These counts are well-defined, and depend only on $\mathbf{w}$.

Refinement

We can actually work over $\mathbb{C}[[s_1, \ldots, s_{\ell_1}, t_1, \ldots, t_{\ell_2}]]$, and consider

$$[\prod_{i=1}^{\ell_1} \theta_{(1,0),1+s_i x}, \prod_{j=1}^{\ell_2} \theta_{(0,1),1+t_j y}].$$

Then again

$$[\prod_{i=1}^{\ell_1} \theta_{(1,0),1+s_i x}, \prod_{j=1}^{\ell_2} \theta_{(0,1),1+t_j y}] = \prod_{(a,b)}^{\longrightarrow} \theta_{(a,b),f_{(a,b)}}, \qquad (\star)$$

where

$$\log f_{(a,b)} = k \sum_{|\mathbf{P}_a|=ka} \sum_{|\mathbf{P}_b|=kb} N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)] s^{\mathbf{P}_1} t^{\mathbf{P}_2} x^{ka} y^{kb}.$$

Corollary

The invariants $N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)]$ are determined by the factorization $(\star)$.

Natural q-deformation

Basic idea: Some of the factorizations admit a natural q-deformation. This can be used to q-deform the GW invariants. To see the q-deformation we need a different point of view on the θ's.

Let $(\Gamma, \langle -, - \rangle)$ be a lattice with antisymmetric, bilinear form. Consider the Lie algebra

$$\mathfrak{g} \quad \text{generated by} \quad e_\alpha, \alpha \in \Gamma,$$

with

$$[e_\alpha, e_\beta] = \langle \alpha, \beta \rangle e_{\alpha+\beta},$$

$$e_\alpha e_\beta = e_{\alpha+\beta}.$$

$\Rightarrow \mathfrak{g}$ becomes a Poisson algebra.

Let R be a complete local or Artin $\mathbb{C}$-algebra. Then

$$\hat{\mathfrak{g}} = \mathfrak{g}\,\hat{\otimes}_{\mathbb{C}} R = \varinjlim \mathfrak{g} \otimes_{\mathbb{C}} R/\mathfrak{m}_R^k.$$

Let $f_\alpha \in \hat{\mathfrak{g}}$ be an element of the form

$$f_\alpha \in 1 + \mathfrak{m}_R[e_\alpha]e_\alpha. \tag{1.1}$$

Then we introduce θ_{α,f_α} automorphisms of the R-algebra $\hat{\mathfrak{g}}$ by

$$\theta_{\alpha,f_\alpha}(e_\beta) = e_\beta f_\alpha^{\langle\alpha,\beta\rangle}.$$

Write: $\theta^\Omega_{\alpha,f_\alpha} = \theta_{\alpha,f_\alpha^\Omega}$ for $\Omega \in \mathbb{Q}$.

The Wall-crossing Group

Definition

The wall-crossing group $\widetilde{\mathbb{V}}_{\Gamma,R} \subset \mathbb{V}_{\Gamma,R}$ *is the completion of the subgroup generated by automorphisms of the form* $\theta^{\Omega}_{\alpha,1+\sigma e_\alpha}$ *for* $\alpha \in \Gamma$*,* $\sigma \in \mathfrak{m}_R$ *and* $\Omega \in \mathbb{Q}$*.*

Dilogarithm:

$$\mathsf{Li}_2(\sigma e_\alpha) = \sum_{k \geqslant 1} \frac{\sigma^k e_{k\alpha}}{k^2}.$$

Fact:

$$\theta_{\alpha,1+\sigma e_{m\alpha}} = \exp(\frac{1}{m}\,\mathsf{ad}(\mathsf{Li}_2(-\sigma e_{m\alpha}))).$$

q-deformed algebra

We replace $\mathfrak{g}$ with the associative, noncommutative algebra over $\mathbb{C}(q^{\pm\frac{1}{2}})$:

$$\mathfrak{g}_q \quad \text{generated by} \quad \hat{e}_\alpha, \alpha \in \Gamma,$$

with

$$\hat{e}_\alpha \hat{e}_\beta = q^{\frac{1}{2}\langle\alpha,\beta\rangle} \hat{e}_{\alpha+\beta}.$$

Classical limit: $\displaystyle\lim_{q^{\frac{1}{2}}\to 1} \frac{1}{q-1}[\hat{e}_\alpha, \hat{e}_\beta] = \langle\alpha,\beta\rangle \hat{e}_{\alpha+\beta}.$

Fixing a local complete or Artin $\mathbb{C}$-algebra R as usual, we define

$$\widehat{\mathfrak{g}}_q = \mathfrak{g}_q \,\widehat{\otimes}_{\mathbb{C}} R.$$

(fundamental case: $\mathfrak{g}_q[[t]]$, where t is a central variable.)

q-dilogarithm:

$$\mathbf{E}(\sigma\hat{e}_\alpha) = \sum_{n\geqslant 0} \frac{(-q^{\frac{1}{2}}\sigma\hat{e}_\alpha)^n}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

For $\Omega \in \mathbb{Q}$ we introduce automorphisms $\hat{\theta}^\Omega[\sigma\hat{e}_\alpha]$ of $\widehat{\mathfrak{g}}_q$ acting by

$$\hat{\theta}^\Omega[\sigma\hat{e}_\alpha](\hat{e}_\beta) = \operatorname{Ad}\mathbf{E}^\Omega(\sigma\hat{e}_\alpha)(\hat{e}_\beta) = \mathbf{E}^\Omega(\sigma\hat{e}_\alpha)\hat{e}_\beta\mathbf{E}^{-\Omega}(\sigma\hat{e}_\alpha).$$

Definition

$\mathbb{U}_{\Gamma,R}$ *is the completion of the subgroup of* $\operatorname{Aut}_{\mathbb{C}(q^{\pm\frac{1}{2}})\otimes_{\mathbb{C}} R}\widehat{\mathfrak{g}}_q$ *generated by automorphisms of the form* $\hat{\theta}^\Omega[(-q^{\frac{1}{2}})^n\sigma\hat{e}_\alpha)]$ *(where* $\alpha \in \Gamma$*,* $\sigma \in \mathfrak{m}_R$*,* $\Omega \in Q$*,* $n \in \mathbb{Z}$*), with respect to the* $\mathfrak{m}_R$*-adic topology.*

Refinement: as in the numerical case, we can work over $\mathbb{C}[[s_1, \ldots, s_{\ell_1}, t_1, \ldots, t_{\ell_2}]]$, and look at

$$[\hat{\theta}^{\ell_1}[\sigma_1 \hat{e}_{\alpha_1}], \hat{\theta}^{\ell_2}[\sigma_2 \hat{e}_{\alpha_2}]]$$

Lemma (Stoppa-F.)

$$\hat{\theta}_{a_1\alpha_1 + a_2\alpha_2} = \mathrm{Ad}\exp\Big(\sum_{|\mathbf{P}_1| = ka_1} \sum_{|\mathbf{P}_2| = ka_2} \sum_{\mathbf{w}} \prod_{i=1}^{2} \frac{\widehat{R}_{\mathbf{P}_i|\mathbf{w}_i}}{|\mathrm{Aut}(\mathbf{w}_i)|} \widehat{N}^{\mathrm{trop}}_{(\alpha_1,\alpha_2)}(\mathbf{w}) s^{\mathbf{P}_1} t^{\mathbf{P}_2} \frac{\hat{e}_{k(a_1\alpha_1 + a_2\alpha_2)}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \Big).$$

where

$$\widehat{R}_{\mathbf{P}_i|\mathbf{w}_i,q} = \prod_j \frac{(-1)^{w_{ij}-1}}{w_{ij}[w_{ij}]_q} \#\{I_{i,\bullet}, \mathbf{P}_i|\mathbf{w}_i\},$$

and $\widehat{N}^{\mathrm{trop}}_{(\alpha_1,\alpha_2)}(\mathbf{w})$ *is the Block-Göttsche invariant obtained by replacing* $m_\Upsilon(V)$ *with* $[m_\Upsilon(V)]_q$ *for every* $V \in \Upsilon$.

Main theorem

Corollary

A natural candidate for the q-deformed GW invariant is

$$\widehat{N}[(\mathbf{P}_1, \mathbf{P}_2)] = \sum_{\mathbf{w}} \prod_{i=1}^{2} \frac{\widehat{R}_{\mathbf{P}_i|\mathbf{w}_i}}{|\operatorname{Aut}(\mathbf{w}_i)|} \widehat{N}^{\mathrm{trop}}_{(\alpha_1,\alpha_2)}(\mathbf{w}).$$

Theorem (Stoppa-F.)

Suppose $(\mathbf{P}_1, \mathbf{P}_2)$ *is primitive. Then the two choices of quantization coincide:*

$$\widehat{N}'[(\mathbf{P}_1, \mathbf{P}_2)] = \widehat{N}[(\mathbf{P}_1, \mathbf{P}_2)].$$