A NOTE ON FROBENIUS TYPE AND CV-STRUCTURES IN DONALDSON-THOMAS THEORY

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Abstract. We rephrase some well-known results in Donaldson-Thomas theory in terms of Frobenius type and CV-structures on an infinite-dimensional bundle, and discuss some applications.

Contents

1. Introduction

This is an expanded version of a talk given at the "Current Developments of Mirror Symmetry" session of the TSIMF 2014 Conference, organised by Professors M. Kontsevich, Y. Soibelman and S. T. Yau. I am grateful to them for a very stimulating conference, as well as for the beautiful "Master Lectures in Mathematics" series. This note is based on joint work with A. Barbieri [BS] and with S. A. Filippini and M. Garcia-Fernandez [FGS], motivated by certain aspects of the fundamental work of Bridgeland-Toledano Laredo [BT], Joyce [J], Kontsevich-Soibelman [KS] and of the equally important physical work of Gaiotto, Moore and Neitzke [GMN]. The style is informal and the proofs are omitted or sketched. I am very grateful to my coauthors and to Andy Neitzke and Tom Sutherland for helpful discussions related to this note.

1.1. **Categorical vs formal setup.** This note is written from the point of view of Donaldson-Thomas theory for three-dimensional Calabi-Yau triangulated (3CY) categories, because it is the one we believe a potential reader will find more interesting. However the results we shall describe are not deep enough to really depend on the finer aspects of the theory, and everything could be stated simply

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement no. 307119.

in terms of a collection of functions on a lattice Γ times its vector space of "central charges", $\mathrm{DT}\colon \Gamma \times \mathrm{Hom}(\Gamma, \mathbb{C}) \to \mathbb{Q}$. These functions $\mathrm{DT}(\alpha, Z)$ should be locally constant in strata of $\mathrm{Hom}(\Gamma, \mathbb{C})$, they should satisfy the wall-crossing formulae of [JS, KS] across different strata, and moreover they should enjoy the symmetry $\mathrm{DT}(\alpha, Z) = \mathrm{DT}(-\alpha, Z)$ (induced by the shift functor [1] in the categorical case). In particular our results can be stated independently of the (partially open) foundational problems of Donaldson-Thomas theory for 3CY categories, by replacing every statement about Donaldson-Thomas invariants with a statement about such a collection of functions. The concerned reader should certainly do so.

1.2. **Motivation and summary of results.** Let $\mathcal{D}$ denote a suitable 3CY category. There is a basic observation (made first in [BT, KS, GMN] and pionereed by Reineke in [R]) that the Kontsevich-Soibelman wall-crossing formula for the Donaldson-Thomas type invariants of $\mathcal{D}$ (see [KS]) has precisely the same form as the constant generalised monodromy condition (i.e. isomonodromy, expressed in terms of Stokes factors) for a family of irregular meromorphic connections on $\mathbb{P}^1$, with respect to a certain infinite-dimensional structure group. In the finite-dimensional situation one would construct some interesting differential geometry from this, e.g. a Frobenius manifold or a tt^*-structure.

In the case of the Donaldson-Thomas theory of $\mathcal{D}$ the Stokes factors are in fact certain infinite products of birational automorphisms of the algebraic torus $(K(\mathcal{D}) \otimes \mathbb{C})^*$ (where $K(\mathcal{D})$ is a finite dimensional quotient of the Grothendieck group). One can still proceed formally and write down a corresponding Frobenius type structure on an infinite-dimensional bundle, in the sense of Hertling [H]. This is done in Proposition 2.2 below, which is just a rephrasing of some results in [BT, J, KS]. This Frobenius type structure is expressed in terms of the Joyce holomorphic generating function for Donaldson-Thomas invariants $f(Z)$ introduced in [J]. However $f(Z)$ is an ill-defined formal infinite sum, and nothing is known about its convergence or even in general how to regard it as a formal power series. A possible solution is to replace $\mathcal{D}$ with a finite length abelian category $\mathcal{A}$, for which all sums become finite, and indeed one can even work fully at the motivic level (see [BT]).

A different approach, which is naturally suggested by the physical work [GMN], is to embed the Frobenius type structure of Proposition 2.2 in a richer structure introduced by Hertling [H] and called a CV-structure (after Cecotti-Vafa). This is done in Proposition 2.7. Indeed the passage from a Frobenius type to a CV-structure is quite natural from the viewpoint of [H].

By passing to a CV-structure the Joyce holomorphic generating function $f(Z)$ is naturally deformed to an operator $\mathcal{Q}(Z)$. In general this construction is still purely formal, and $\mathcal{Q}(Z)$ involves ill-defined formal infinite sums. But in Propositions 3.1 and 3.2 we show that at least locally on the space of stability conditions, and under very restrictive assumptions on the category $\mathcal{D}$, it is possible to lift canonically the ill-defined structures of Propositions 2.2, 2.7 to well-defined formal families of Frobenius type and CV-structures on a bundle, parametrised by a formal neighbourhood of $0 \in \mathbb{C}^n$. In particular both the graded components of

$f(Z)$ and the matrix elements of $\mathcal{Q}(Z)$ become well-defined formal power series with values in $\mathbb{C}[K(\mathcal{D})]$, denoted by $f_{\mathbf{s}}(Z)$ and $\mathcal{Q}_{\mathbf{s}}(Z)$, where $\mathbf{s}$ is the vector of formal parameters $s_1, \ldots, s_n$.

Assuming moreover that the Donaldson-Thomas invariants at a fixed stability condition $(\mathcal{A}, Z_0)$ do not grow too fast (at most exponentially) we show in Theorem 4.2 that for all $\rho > 0$ there is a $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ the formal power series given by the matrix elements of $\mathcal{Q}_{\mathbf{s}}(\lambda Z)$ all converge in the ball $||\mathbf{s}|| < \rho$. If the exponential bound is uniform for $Z \in \operatorname{Hom}(K(\mathcal{A}), \mathbb{C})$ the matrix elements of $\mathcal{Q}_{\mathbf{s}}(\lambda Z)$ for $\lambda > \lambda_0$ evaluated say at $\mathbf{s} = (1, \ldots, 1)$ give well-defined real-analytic functions on every open subset of $\operatorname{Hom}(K(\mathcal{A}), \mathbb{C})$ which is bounded away from 0.

In section 5 we study the $\lambda \to \infty$ scaling limit of $\mathcal{Q}(\lambda Z)$ from a different point of view. This turns out to be closely related to the enumerative geometry of rational tropical curves immersed in $\mathbb{R}^2$ (see Theorems 5.1 and 5.3). These results also admit a natural q-deformation or "quantisation" (see section 5.3).

2. Formal infinite-dimensional picture

2.1. **Frobenius type structure.** We recall the notion of a Frobenius type structure on a general vector bundle (not necessarily the tangent bundle), due to Hertling ([H] Definition 5.6 (c)).

Definition 2.1. A *Frobenius type structure* on a holomorphic vector bundle $K \to M$ is a collection of holomorphic objects $(\nabla^r, C, \mathcal{U}, \mathcal{V}, g)$, with values in the bundle K, where

- ∇^r is a flat connection,
- C is a Higgs field, that is a 1-form with values in the endomorphisms,
- $\mathcal{U}, \mathcal{V}$ are endomorphisms,
- g is a symmetric nondegenerate bilinear form,

satisfying the conditions

$$\begin{aligned} \nabla^r(C) &= 0, \\ [C, \mathcal{U}] &= 0, \\ \nabla^r(\mathcal{V}) &= 0, \\ \nabla^r(\mathcal{U}) - [C, \mathcal{V}] + C &= 0 \end{aligned} \tag{2.1}$$

plus the conditions on the "metric" g

$$\begin{aligned} \nabla^r(g) &= 0, \\ g(C_X a, b) &= g(a, C_X b), \\ g(\mathcal{U}a, b) &= g(a, \mathcal{U}b), \\ g(\mathcal{V}a, b) &= -g(a, \mathcal{V}b). \end{aligned} \tag{2.2}$$

One can use Donaldson-Thomas theory to attach to a three-dimensional Calabi-Yau (3CY) triangulated category a formal Frobenius type structure on an infinite-dimensional bundle over the space of stability conditions. This is essentially a rephrasing of results in [BT, J, KS].

To explain this let $\mathcal{D}$ denote a 3CY triangulated category for which there is a well-defined *numerical* (as opposed to *motivic*) Donaldson-Thomas type theory virtually enumerating semistable objects (basic references include [JS, KS]). We denote by $K(\mathcal{D})$ the Grothendieck group of $\mathcal{D}$ and assume it has finite rank n. We denote by $\operatorname{Stab}(\mathcal{D})$ the space of locally finite Bridgeland stability conditions on $\mathcal{D}$ (introduced in [B]). It is an n-dimensional complex manifold with a local biholomorphism $\operatorname{Stab}(\mathcal{D}) \to \operatorname{Hom}(K(\mathcal{D}), \mathbb{C})$. Indeed points of $\operatorname{Stab}(\mathcal{D})$ are given by pairs $(\mathcal{A}, Z)$ where $\mathcal{A} \subset \mathcal{D}$ is the heart of a bounded t-structure and $Z\colon K(\mathcal{A}) \to \mathbb{C}$ is a group homomorphism, known as the central charge. The local biholomorphism $\operatorname{Stab}(\mathcal{D}) \to \operatorname{Hom}(K(\mathcal{D}), \mathbb{C})$ is the forgetful map $(\mathcal{A}, Z) \mapsto Z$. Abusing notation we will sometimes denote stability conditions simply by their central charge Z. The inclusion induces a canonical homomorphism $K(\mathcal{A}) \cong K(\mathcal{D})$ and so we have canonically $Z \in \operatorname{Hom}(K(\mathcal{D}), \mathbb{C})$. The lattice $K(\mathcal{D})$ comes with a natural integral skew-symmetric bilinear pairing $\langle -, - \rangle$ induced by the Euler form of the category.

The group-algebra $\mathbb{C}[K(\mathcal{D})]$ endowed with the Lie bracket induced by $\langle -, - \rangle$ becomes a Poisson algebra, known as the Kontsevich-Soibelman algebra. It is generated by monomials $x_{\alpha}, \alpha \in K(\mathcal{D})$ with bracket $[x_{\alpha}, x_{\beta}] = (-1)^{\langle \alpha, \beta \rangle} \langle \alpha, \beta \rangle x_{\alpha+\beta}$.

Joyce [J] introduced a holomorphic generating function for Donaldson-Thomas invariants. It is a formal infinite sum $f(Z)$ of elements of $\mathbb{C}[K(\mathcal{D})]$. Morally it defines a holomorphic function on $\operatorname{Stab}(\mathcal{D})$ with values in $\prod_{\alpha} \mathbb{C} x_{\alpha}$, encoding the Donaldson-Thomas invariants which enumerate semistable objects in $\mathcal{D}$. One can reinterpret this formal construction as giving a Frobenius type structure in the sense of Definition 2.1 on a trivial infinite-dimensional vector bundle over $\operatorname{Stab}(\mathcal{D})$.

Proposition 2.2. *Let $K \to \operatorname{Stab}(\mathcal{D})$ be the trivial infinite-dimensional vector bundle with fibre $\prod_{\alpha} \mathbb{C} x_{\alpha}$. Work with formal infinite sums and ignore convergence issues. Fix a constant $g_0 \in \mathbb{C}^*$. Then there is a Frobenius type structure on K given by*

$$\begin{aligned}
\nabla^r &= d - \sum_{\alpha} \operatorname{ad} f^{\alpha}(Z) \frac{dZ(\alpha)}{Z(\alpha)}, \\
C &= -dZ, \\
\mathcal{U} &= Z, \\
\mathcal{V} &= -\operatorname{ad} f(Z), \\
g(x_{\alpha}, x_{\beta}) &= g_0 \delta_{\alpha\beta}.
\end{aligned}$$

Notice that here we use the Lie algebra structure on $\mathbb{C}[K(\mathcal{D})] \subset \prod_{\alpha} x_{\alpha}$ just to describe endomorphims of K, i.e. we work with a vector bundle not a principal bundle.

Proof (sketch). This can be checked by direct computation. Flatness of ∇^r and covariant constancy of the endomorphism $\mathcal{V}$ both follow from the PDE

$$df^{\alpha}(Z) = \sum_{\beta+\gamma=\alpha} [f^{\alpha}(Z), f^{\gamma}(Z)] \frac{dZ(\beta)}{Z(\beta)}$$

proved in [J]. ∎

There is a standard construction of a "first structure" flat connection from a Frobenius type structure. In the Donaldson-Thomas case this has a further scale invariance property.

Lemma 2.3. *Let $p\colon \mathrm{Stab}(\mathcal{D}) \times \mathbb{P}^1_t \to \mathrm{Stab}(\mathcal{D})$ denote the projection. Let $\lambda \in \mathbb{R}^+$ denote a scaling parameter. The meromorphic connection on $p^* \mathrm{Stab}(\mathcal{D}) \times \mathbb{P}^1_t$ given by*

$$\nabla^r + \frac{C}{t} + \left(\frac{1}{t^2}\mathcal{U} - \frac{1}{t}\mathcal{V}\right) dt$$

is flat and invariant under the rescaling $Z \mapsto \lambda Z$, $t \mapsto \lambda t$. In particular the Joyce function $f(Z)$ has the "conformal invariance" property $f(\lambda Z) = f(Z)$.

The $K(\mathcal{D})$-graded components of $f(Z) \in \prod_\alpha \mathbb{C} x_\alpha$ can be described explicitly. Let $(U\mathbb{C}[K(\mathcal{D})], \otimes)$ denote the universal enveloping algebra of $(\mathbb{C}[K(\mathcal{D})], \langle -, - \rangle)$. There are explicit formulae for the product $\otimes$, and one has in particular

$$x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_k} = c(\alpha_1, \cdots, \alpha_k) x_{\alpha_1 + \cdots + \alpha_k}$$

where $c(\alpha_1, \cdots, \alpha_k) \in \mathbb{Q}$ depends only on the classes $\alpha_i \in K(\mathcal{D})$ and is given by a (complicated) weighted sum over graphs.

Then Joyce proves that there exist holomorphic functions with branch-cuts

$$J_n\colon (\mathbb{C}^*)^n \to \mathbb{C}$$

such that

$$f^\alpha(Z) = \sum_{\alpha_1 + \cdots + \alpha_k = \alpha, Z(\alpha_i) \neq 0} J_n(Z(\alpha_1), \ldots, Z(\alpha_k)) \prod_i \mathrm{DT}(\alpha_i, Z) x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_k}.$$

and so one has

$$f^\alpha(Z) = \tilde{f}^\alpha(Z) x_\alpha$$

where the holomorphic "function" $\tilde{f}^\alpha(Z)$ is given by the formal infinite sum

$$\tilde{f}^\alpha(Z) = \sum_{\alpha_1 + \cdots + \alpha_k = \alpha, Z(\alpha_i) \neq 0} c(\alpha_1, \ldots, \alpha_k) J_n(Z(\alpha_1), \ldots, Z(\alpha_k)) \prod_i \mathrm{DT}(\alpha_i, Z). \tag{2.3}$$

The crucial point is that the jumps of the functions $J_n(z_1, \ldots, z_n)$ across their branch-cuts can be chosen to cancel out the jumps of the Donaldson-Thomas invariants $\mathrm{DT}(\alpha_i, Z)$ across walls in $\mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$. The functions $J_n(z_1, \ldots, z_n)$ are universal, i.e. they do not depend on the underlying category $\mathcal{D}$.

However there is not a single example where (2.3) is actually known to converge. Indeed even the convergence question is ill-posed since no specific summation order has been fixed. The convergence problem for $f(Z)$ seems especially hard because of the conformal invariance property of Lemma 2.3.

2.2. **CV-structure.** The Frobenius type structure of Proposition 2.2 is part of a more complicated (formal) structure of a type called CV-structure (after Cecotti and Vafa) in [H]. This point of view is also suggested naturally by [GMN]. To discuss it we introduce the preliminary notion of a $DC\tilde{C}$-structure, also due to Hertling ([H] Definition 2.9).

Definition 2.4. A $(DC\widetilde{C})$-structure on a C^{∞} complex vector bundle $K \to M$ is the collection of C^{∞} objects $(D, C, \widetilde{C})$ with values in K where

- D is a connection,
- C is a $(1,0)$-form with values in endomorphisms of K,
- $\widetilde{C}$ is a $(0,1)$-form with values in endomorphisms of K;

satisfying the conditions

$$\begin{aligned}(D'' + C)^2 &= 0,\\ (D' + \widetilde{C})^2 &= 0,\\ D'(C) &= 0,\\ D''(\widetilde{C}) &= 0,\\ D'D'' + D''D' &= -(C\widetilde{C} + \widetilde{C}C) \end{aligned} \tag{2.4}$$

where D' and D'' are the $(1,0)$ and $(0,1)$ parts of D respectively.

Lemma 2.5. *Let $K \to \mathrm{Stab}(\mathcal{D})$ be the trivial infinite-dimensional vector bundle with fibre $\prod_{\alpha} \mathbb{C}x_{\alpha}$. Work with formal infinite sums and ignore convergence issues. Then there is a $(DC\widetilde{C})$-structure on K given by*

$$\begin{aligned} D &= \nabla^r,\\ C &= -dZ,\\ \widetilde{C} &= d\bar{Z}. \end{aligned}$$

Proof. Let $\bar{\partial}_K$ denote our fixed (trivial) complex structure on K, with $\bar{\partial}_K(x_{\alpha}) = 0$. The condition $(D'' + C)^2 = 0$ says that K is holomorphic and C is a holomorphic Higgs bundle on it, which we know already from Proposition 2.2. Then $D'(C) = 0$ says that C is flat with respect to ∇^r, which we also know already. The condition $(D' + \widetilde{C})^2 = 0$ says that ∇^r is flat (known), $(d\bar{Z})^2 = 0$ and $\nabla^r(d\bar{Z}) = 0$ (easily checked). The condition $D''(\widetilde{C}) = 0$ becomes $\bar{\partial}_K(d\bar{Z}) = 0$ and can be checked e.g. in local coordinates on $\mathrm{Stab}(\mathcal{D})$ given by $z_k = Z(\alpha_k)$ where $\alpha_1, \ldots, \alpha_k$ is a basis for $K(\mathcal{D})$. Finally in our case one checks that we have separately $C\widetilde{C} + \widetilde{C}C = 0$ and $D'D'' + D''D' = 0$. □

We can now recall the notion of a CV structure introduced in [H] Definition 2.16.

Definition 2.6. A CV-structure on a C^{∞} complex bundle $K \to M$ is a collection of C^{∞} objects $(D, C, \widetilde{C}, \kappa, h, \mathcal{U}, \mathcal{Q})$ with values in K where

- $(D, C, \widetilde{C})$ is a $(DC\widetilde{C})$-structure,
- κ is an antilinear involution with $D(\kappa) = 0$ which intertwines C and $\widetilde{C}$, $\kappa C \kappa = \widetilde{C}$,
- h is a hermitian (not necessarily positive) metric, which satisfies $D(h) = 0$, $h(C_X a, b) = h(a, \widetilde{C}_{\bar{X}} b)$ for $(1,0)$ fields X and which is real-valued on the real subbundle $K_{\mathbb{R}} \subset K$ defined by κ,

- $\mathcal{U}$ and $\mathcal{Q}$ are endomorphisms,

satisfying the conditions

$$\begin{aligned}
[C, \mathcal{U}] &= 0, \\
D'(\mathcal{U}) - [C, \mathcal{Q}] + C &= 0, \\
D''(\mathcal{U}) &= 0, \\
D'(\mathcal{Q}) + [C, \kappa\mathcal{U}\kappa] &= 0, \\
\mathcal{Q} + \kappa\mathcal{Q}\kappa &= 0, \\
h(\mathcal{U}a, b) &= h(a, \kappa\mathcal{U}\kappa b), \\
h(\mathcal{Q}a, b) &= h(a, \mathcal{Q}b).
\end{aligned} \tag{2.5}$$

Let us go back to the case when $K \to \mathrm{Stab}(\mathcal{D})$ is the trivial infinite-dimensional vector bundle with fibre $\prod_\alpha \mathbb{C}x_\alpha$ (with the usual fixed trivial complex structure $\bar{\partial}_K$). Let ι denote the involution of K acting as complex conjugation combined with $x_\alpha \mapsto x_{-\alpha}$. Let ψ be a fixed endomorphism of K. Then we can make the following ansatz on the data of a CV-structure on K:

- κ is the conjugate involution $\mathrm{Ad}_\psi(\iota)$,
- the pseudo-hermitian metric h is given by $h(a, b) = g(a, \kappa(b))$ where g is the quadratic form in Proposition 2.2,
- D is the Chern connection of $h, \bar{\partial}_K$,
- the Higgs fields are the conjugates $C = \mathrm{Ad}_\psi(-dZ)$, $\widetilde{C} = \mathrm{Ad}_\psi(d\bar{Z})$ of the fields in Lemma 2.5,
- $\mathcal{U}$ is the endomorphism Z as in Proposition 2.2.

Proposition 2.7. *Let $K \to \mathrm{Stab}(\mathcal{D})$ be our fixed trivial bundle. Work with formal infinite sums and ignore convergence issues.*

(a) There exists endomorphisms $\psi(Z)$ and $\mathcal{Q}(Z)$ such that the choices of D, C, $\widetilde{C}$, κ, h, $\mathcal{U}$ above together with $\mathcal{Q}$ give a CV-structure on K. Moreover ψ and $\mathcal{Q}$ induce fibrewise derivations of $\mathbb{C}[K(\mathcal{D})]$ as a commutative algebra.

(b) Fix Z and let $\lambda \in \mathbb{R}^+$ denote a scaling parameter. Then

$$\begin{aligned}
\lim_{\lambda \to 0} \lambda^{-1}\mathcal{Q}(\lambda Z) &= \mathcal{V}, \\
\lim_{\lambda \to 0} \kappa(\lambda Z) &= \iota
\end{aligned}$$

where $\mathcal{V} = -\operatorname{ad} f(Z)$ is the endomorphism of Proposition 2.2 (i.e. essentially the Joyce holomorphic generating function).

Proof (sketch). If we allow ourselves to work with formal infinite sums (ignoring convergence, as in [J] section 5) then the equations (2.5) appear as the generalised constant monodromy condition for the family of connections constructed in [FGS] section 3. The scaling limits follow in turn from the arguments in [FGS] section 4. Notice that this construction follows very closely the original physical proposal in [GMN]. $\square$

In the setup discussed so far the matrix elements $g(x_\alpha, \mathcal{Q}(x_\beta))$, $g(x_\alpha, \kappa(x_\beta))$ are formal infinite sums. There are explicit formulae for the matrix elements which are very similar to (2.3) (see [FGS] section 3.10). They are not known to converge in any example, and indeed the convergence question is a priori ill-posed since no specific summation order has been fixed.

In the light of Proposition 2.7 (b) it is natural to make the following definition.

Definition 2.8. The CV-deformation of the Joyce holomorphic generating function $f(Z)$ is the operator $\mathcal{Q}(Z)$ given by Proposition 2.7 (a).

There is an analogue of Lemma 2.3, which gives a new point of view on the conformal invariance property $f(\lambda Z) = f(Z)$.

Lemma 2.9. *Let $(D, C, \widetilde{C}, \kappa, h, \mathcal{U}, \mathcal{Q})$ be the CV-structure of Proposition 2.7. Let $p\colon \mathrm{Stab}(\mathcal{D}) \times \mathbb{P}^1_z \to \mathrm{Stab}(\mathcal{D})$ denote the projection, and suppose $\lambda \in \mathbb{R}^+$ is a scaling parameter. The meromorphic connection on $p^*K \to \mathrm{Stab}(\mathcal{D}) \times \mathbb{P}^1_z$ given by*

$$D + \frac{C}{z} + z\widetilde{C} + \left(\frac{1}{z^2}\mathcal{U} - \frac{1}{z}\mathcal{Q} - \kappa\mathcal{U}\kappa\right) dz$$

is flat. Under the scaling $Z \mapsto \lambda Z$, $z = \lambda t$, $\lambda \to 0$ it flows to the flat connection of Lemma 2.3.

3. Formal families of Frobenius type and CV-structures

We now describe a very special situation in which the Frobenius type structure of Proposition 2.2 and the CV-structure of Proposition 2.7 can be discussed more rigorously. In this situation, at least locally on an open region of $\mathrm{Stab}(\mathcal{D})$, the ill-defined structures of Propositions 2.2 and 2.7 can be replaced by well-defined formal families of Frobenius type and CV-structures on a bundle, parametrised by a formal neighbourhood of $0 \in \mathbb{C}^n$. The convergence question for matrix elements $g(x_\alpha, \mathcal{V}(x_\beta))$, $g(x_\alpha, \mathcal{Q}(x_\beta))$ and $g(x_\alpha, \kappa(x_\beta))$ becomes well-posed.

The crucial (very restrictive) assumption is that our 3CY triangulated category $\mathcal{D}$ admits a finite length heart of a bounded t-structure $\mathcal{A} \subset \mathcal{D}$. This holds for example in the case of the 3CY category $\mathcal{D}(Q, W)$ constructed from a quiver with potential (Q, W).

We denote the length of $\mathcal{A}$ by n. It also equals the rank of $K(\mathcal{D})$. Let $S_1, \ldots, S_n$ denote the simple objects of $\mathcal{A}$, so the classes $[S_i]$ form a basis of $K(\mathcal{D})$. Let $U(\mathcal{A}) \subset \mathrm{Stab}(\mathcal{D})$ be the *interior* of the locus of stability conditions supported on the heart $\mathcal{A}$.

Let $\mathbb{C}[\![\mathbf{s}]\!]$ denote the ring of formal power series $\mathbb{C}[\![s_1, \ldots, s_n]\!]$. Decompose $\alpha \in K(\mathcal{D})$ as $\sum_i a^i [S_i]$. We define new elements $[\alpha]_\pm \in K(\mathcal{D})$ attached to α by $[\alpha]_\pm = \sum_i [a_i]_\pm [S_i]$ where $[a]_\pm$ denotes the positive (respectively negative) part of an integer.

We "deform" the formal Joyce holomorphic generating function (2.3) to a well-defined formal power series in $\mathbb{C}[\![\mathbf{s}]\!]$ given by

$$\tilde{f}^{\alpha}_{\mathbf{s}}(Z) = \sum_{\alpha_1+\cdots+\alpha_k=\alpha,\, Z(\alpha_i)\neq 0} c(\alpha_1,\ldots,\alpha_k) J(Z(\alpha_1),\ldots,Z(\alpha_k)) \prod_i \mathbf{s}^{[\alpha_i]_+ - [\alpha_i]_-} \operatorname{DT}(\alpha_i, Z). \tag{3.1}$$

We set accordingly

$$f^{\alpha}_{\mathbf{s}}(Z) = \tilde{f}^{\alpha}_{\mathbf{s}}(Z) x_{\alpha},$$
$$f_{\mathbf{s}}(Z) = \sum_{\alpha} f^{\alpha}_{\mathbf{s}}(Z).$$

The following result is a corollary of Proposition 3.2 below, see [BS] section 2.

Proposition 3.1. *Let $K \to U(\mathcal{A})$ be the trivial vector bundle with fibre $\mathbb{C}[K(\mathcal{D})][\![\mathbf{s}]\!]$. Then there is a $\mathbb{C}[\![\mathbf{s}]\!]$-linear Frobenius type structure on K with flat holomorphic connection given by*

$$\nabla^r_{\mathbf{s}} = d - \sum_{\alpha} \operatorname{ad} f^{\alpha}_{\mathbf{s}}(Z) \frac{dZ(\alpha)}{Z(\alpha)},$$

with residue endomorphism

$$\mathcal{V}_{\mathbf{s}} = -\operatorname{ad} f_{\mathbf{s}}(Z)$$

and with $C, \mathcal{U}, g$ extended to $C_{\mathbf{s}}, \mathcal{U}_{\mathbf{s}}, g_{\mathbf{s}}$ by $\mathbb{C}[\![\mathbf{s}]\!]$-linearity. In other words the equations $(\nabla^r_{\mathbf{s}})^2 = 0$ and (2.1) - (2.2) hold as identities of formal power series in $\mathbf{s}$.

Notice that in particular the *coefficients* of the formal power series (3.1) in $\mathbf{s}$ are well-defined holomorphic functions on $U(\mathcal{A})$. The deformation (3.1) and the $\mathbb{C}[\![\mathbf{s}]\!]$-linear Frobenius type structure of Proposition 3.1 depend strongly on the choice of a finite length heart $\mathcal{A} \subset \mathcal{D}$ and jump discontinuously across the boundary of $U(\mathcal{A})$ in $\operatorname{Stab}(\mathcal{D})$ where the heart changes (generically by a simple tilt). Notice also that convergence for all the Joyce holomorphic generating formal power series (3.1) is equivalent to convergence for the matrix elements $g(x_{\alpha}, \mathcal{V}(x_{\beta}))$.

The analogue of Proposition 3.1 holds for CV-structures. Let $K \to U(\mathcal{A})$ be the trivial infinite-dimensional vector bundle with fibre $\mathbb{C}[K(\mathcal{D})][\![\mathbf{s}]\!]$ (with trivial complex structure $\bar{\partial}_K$). Let ι denote the involution of K acting as complex conjugation combined with $x_{\alpha} \mapsto x_{-\alpha}$. Note that ι is an anti-linear commutative algebra automorphism. Let $\psi_{\mathbf{s}}$ be a fixed invertible endomorphism of K. Then we can make the following ansatz on the data of a $\mathbb{C}[\![\mathbf{s}]\!]$-linear CV-structure on K:

- $\kappa_{\mathbf{s}}$ is the conjugate involution $\operatorname{Ad}_{\psi_{\mathbf{s}}}(\iota)$,
- the pseudo-hermitian metric $h_{\mathbf{s}}$ is given by $h(a,b) = g_{\mathbf{s}}(a, \kappa_{\mathbf{s}}(b))$ where $g_{\mathbf{s}}$ is the quadratic form in Proposition 3.1,
- $D_{\mathbf{s}}$ is the Chern connection of $h_{\mathbf{s}}, \bar{\partial}_K$,
- the Higgs fields are the conjugates $C_{\mathbf{s}} = \operatorname{Ad}_{\psi_{\mathbf{s}}}(-dZ)$, $\widetilde{C_{\mathbf{s}}} = \operatorname{Ad}_{\psi_{\mathbf{s}}}(d\bar{Z})$ of the fields in Lemma 2.5,
- $\mathcal{U}_{\mathbf{s}}$ is the endomorphism Z extended by $\mathbb{C}[\![\mathbf{s}]\!]$-linearity as in Proposition 3.1.

The following result is proved in [BS].

Proposition 3.2. *Let $K \to U(\mathcal{A})$ be our fixed trivial bundle with fibre $\mathbb{C}[K(\mathcal{D})][\![\mathbf{s}]\!]$.*

(a) There exist $\mathbb{C}[\![\mathbf{s}]\!]$-linear endomorphisms $\psi_{\mathbf{s}}$ and $\mathcal{Q}_{\mathbf{s}}$ of $\mathbb{C}[K(\mathcal{D})][\![\mathbf{s}]\!]$ such that the choices of $D_{\mathbf{s}}$, $C_{\mathbf{s}}$, $\widetilde{C}_{\mathbf{s}}$, $\kappa_{\mathbf{s}}$, $h_{\mathbf{s}}$, $\mathcal{U}_{\mathbf{s}}$ above together with $\mathcal{Q}_{\mathbf{s}}$ give a $\mathbb{C}[\![\mathbf{s}]\!]$-linear CV-structure on K. In other words the equations 2.4 and 2.5 hold as identities of formal power series in $\mathbf{s}$. Moreover $\psi_{\mathbf{s}}$ and $\mathcal{Q}_{\mathbf{s}}$ induce fibrewise $\mathbb{C}[\![\mathbf{s}]\!]$-linear derivations of $\mathbb{C}[K(\mathcal{D})][\![\mathbf{s}]\!]$ as a commutative algebra.

(b) We have

$$\lim_{\lambda \to 0} \lambda^{-1} \mathcal{Q}_{\mathbf{s}}(\lambda Z) = \mathcal{V}_{\mathbf{s}},$$
$$\lim_{\lambda \to 0} \kappa_{\mathbf{s}}(\lambda Z) = \iota$$

where $\mathcal{V}_{\mathbf{s}} = -\operatorname{ad} f_{\mathbf{s}}(Z)$ is the endomorphism of Proposition 3.1 (i.e. essentially the formal power series Joyce holomorphic generating function given by (3.1)).

Definition 3.3. The CV-deformation of the formal power series Joyce holomorphic generating function $f_{\mathbf{s}}(Z)$ is the operator $\mathcal{Q}_{\mathbf{s}}(Z) \in \operatorname{End}_{\mathbb{C}[\![\mathbf{s}]\!]}(\mathbb{C}[K(\mathcal{D})][\![\mathbf{s}]\!])$ given by Proposition 3.2 (a) (in fact this is not just an endomorphism, but a commutative algebra derivation).

The matrix elements $g(x_\alpha, \mathcal{Q}_{\mathbf{s}}(x_\beta))$ are well-defined formal power series in $\mathbb{C}[\![\mathbf{s}]\!]$. Notice that $f_{\mathbf{s}}(Z)$ and its CV-deformation $\mathcal{Q}_{\mathbf{s}}(Z)$ are only defined upon the choice of a finite length heart $\mathcal{A} \subset \mathcal{C}$.

The analogue of Lemma 3.4 also holds.

Lemma 3.4. *Let $(D_{\mathbf{s}}, C_{\mathbf{s}}, \widetilde{C}_{\mathbf{s}}, \kappa_{\mathbf{s}}, h_{\mathbf{s}}, \mathcal{U}_{\mathbf{s}}, \mathcal{Q}_{\mathbf{s}})$ be the formal family of CV-structures of Proposition 3.2. Let $p\colon U(\mathcal{A}) \times \mathbb{P}^1_z \to U(\mathcal{A})$ denote the projection, and suppose $\lambda \in \mathbb{R}^+$ is a scaling parameter. The meromorphic connection on $p^*K \to U(\mathcal{A}) \times \mathbb{P}^1_z$ given by*

$$D_{\mathbf{s}} + \frac{C_{\mathbf{s}}}{z} + z\widetilde{C}_{\mathbf{s}} + \left(\frac{1}{z^2}\mathcal{U}_{\mathbf{s}} - \frac{1}{z}\mathcal{Q}_{\mathbf{s}} - \kappa_{\mathbf{s}}\mathcal{U}_{\mathbf{s}}\kappa_{\mathbf{s}}\right) dz$$

is flat. Under the scaling $Z \mapsto \lambda Z$, $z = \lambda t$, $\lambda \to 0$ it flows to the $\mathbb{C}[\![\mathbf{s}]\!]$-linear flat connection

$$\nabla^r_{\mathbf{s}} - \frac{1}{t} dZ + \left(\frac{1}{t^2}\mathcal{U}_{\mathbf{s}} - \frac{1}{t}\mathcal{V}_{\mathbf{s}}\right) dt.$$

Note that both flat connections actually take values in $\mathbb{C}[\![\mathbf{s}]\!]$-linear commutative algebra derivations.

4. A convergence result

We have shown that locally, that is on $U(\mathcal{A})$, there is a natural deformation of the Joyce holomorphic generating function $f(Z)$, namely the CV-deformation $\mathcal{Q}_{\mathbf{s}}(Z) \in \operatorname{End}_{\mathbb{C}[\![\mathbf{s}]\!]}(\mathbb{C}[K(\mathcal{D})][\![\mathbf{s}]\!])$. This deformation preserves some key properties of $f(Z)$ (in particular, its matrix elements are formal power series with real-analytic coefficients), but breaks conformal invariance. So it is possible that $\mathcal{Q}_{\mathbf{s}}(Z)$ has better convergence properties. In [BS] we prove a first result in this direction.

Definition 4.1. The BPS spectrum is the function on $K(\mathcal{D}) \times \mathrm{Stab}(\mathcal{D})$ defined through the equality

$$\mathrm{DT}(\alpha, Z) = \sum_{k>0, k|\alpha} \frac{\Omega(k^{-1}\alpha, Z)}{k^2}.$$

We say that the BPS spectrum grows at most exponentially at Z if fixing a norm $||-||$ on $K(\mathcal{D}) \otimes \mathbb{R}$ there is a constant $C > 0$ such that for all $\alpha \in K(\mathcal{D})$ we have

$$|\Omega(\alpha, Z)| < \exp(-C||\alpha||).$$

The definition does not depend on the particular choice of a norm. This exponential growth condition is very restrictive, but nevertheless holds in some interesting examples, including certain categories $\mathcal{D}(Q, W)$. The following is the main result of [BS].

Theorem 4.2. *Suppose that the BPS spectrum Ω grows at most exponentially at a fixed central charge Z_0. Then for all $\rho > 0$ there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ all the matrix elements*

$$g(x_\alpha, \mathcal{Q}_{\mathbf{s}}(\lambda Z_0) x_\beta) \in \mathbb{C}[\![\mathbf{s}]\!]$$

converge in the open ball $||\mathbf{s}|| < \rho$. Moreover λ_0 depends only on the exponential bound and on a lower bound on the operator norm $||Z|| > \epsilon$, so if these hold uniformly for Z in an open subset, choosing $\rho > 1$ and $\lambda > \lambda_0$ shows that the all matrix elements

$$g(x_\alpha, \mathcal{Q}_{(1,\dots,1)}(\lambda Z_0) x_\beta)$$

are real-analytic functions on the open subset.

The proof of this result in [BS] is very much inspired by the treatment in [GMN], but departs from it in some basic analytic points in order to obtain a uniform result as Z approaches the critical locus in $\mathrm{Hom}(K(\mathcal{A}), \mathbb{C})$ where Donaldson-Thomas invariants jump.

5. Collapse to tropical geometry

Let us go back for a moment to the formal infinite-dimensional setup. According to Proposition 2.7 we have

$$\lim_{\lambda \to 0} \lambda^{-1} \mathcal{Q}(\lambda Z) = -\operatorname{ad} f(Z).$$

One may ask if there are other interesting scaling limits of the operator $\mathcal{Q}$. The physical approach of [GMN] suggests to look at the limit

$$\lim_{\lambda \to \infty} \mathcal{Q}(\lambda Z)$$

where the scaling paramer λ is still positive and real. Note that this time one does not scale the $\mathbb{P}^1$ direction. It turns out that it is more natural to study the $\lambda \to \infty$ scaling limit of the whole meromorphic connection on the trivial bundle on $\mathbb{P}^1$ with fibre $\prod_\alpha \mathbb{C} x_\alpha$ given by

$$\nabla(Z) = d - \left(\frac{1}{z^2} Z - \frac{1}{z}\mathcal{Q} - \kappa\, Z \kappa\right) dz.$$

According to 3.4 this is a deformation of its conformal limit

$$\nabla_0(Z) = d - \left(\frac{1}{t^2}Z + \frac{1}{t}\,\mathrm{ad}\,f(Z)\right) dt.$$

Note that both $\nabla(Z)$ and $\nabla_0(Z)$ take values in commutative algebra derivations. We wish to study the $\lambda \to \infty$ behaviour of $\nabla(\lambda Z)$. This can be done by looking at the $\lambda \to \infty$ behaviour of its flat sections. It turns out that this is a "tropical limit": the behaviour of flat sections of $\nabla(\lambda Z)$ as $\lambda \to \infty$ is naturally related to the geometry of rational tropical curves immersed in $\mathbb{R}^2$.

5.1. **Restriction to abelian categories.** While the results of this section could be stated in the setup of section 3, it turns out that this would not really be more general, and it is equivalent to restrict to the case of a suitable finite length abelian category $\mathcal{A}$. In fact we will soon specialise to the model case when $\mathcal{A}$ is the category of representations of the k-Kronecker quiver Q_k. This is not 3CY of course, but one can still define an analogue of Donaldson-Thomas invariants, and they still satisfy the same wall-crossing formulae, provided the Euler form is replaced by its skew-symmetrisation (see [J] for a discussion).

In the case of such abelian $\mathcal{A}$ everything is graded by the positive cone $K(A)_{>0}$, so there is no need to work with the formal families of section 3. The Joyce function $f^{\alpha}(Z)$, $\alpha \in K(\mathcal{A})_{>0}$ is a well-defined finite sum, and the same holds for the matrix elements $g(x_{\alpha}, \kappa(x_{\beta}))$, $g(x_{\alpha}, \mathcal{Q}(x_{\beta}))$ for $\alpha, \beta \in K(\mathcal{A})_{>0}$.

Since $\nabla(Z)$ is valued in commutative algebra derivations, we can think of flat sections $Y(z, Z)$ of $\nabla(Z)$ as holomorphic functions, with branch-cuts, of the variable $z \in \mathbb{C}^*$ with values in commutative algebra automorphisms $\mathrm{Aut}(\widehat{\mathbb{C}[K_{>0}]})$, where we complete along $J = (x_{[S_1]}, \dots, x_{[S_n]})$. Branch-cuts occur along the rays spanned by the eigenvalues of Z, the Stokes lines. There may be infinitely many branch-cuts, but all that matters is that there are only finitely many when we project to $\mathbb{C}[K_{>0}]/J^N$ for all $N \geq 1$.

There is a canonical way to choose the flat section $Y(z, Z)$ as the central charge Z varies, by prescribing that it is asymptotically close to the identity in each sector between consecutive Stokes lines (modulo J^N, for all $N \geq 1$).

There is an explicit formula for these canonical flat sections $Y(z, Z)$ in terms of the central charge and the Donaldson-Thomas invariants, similar to (2.3),

$$Y(z, Z)x_{\alpha} = x_{\alpha} \exp_* \sum_{T} \langle \alpha, W_T(Z)\rangle\, H_T(z, Z), \tag{5.1}$$

where we are summing over rooted trees T with vertices v decorated by elements $\alpha(v) \in K(\mathcal{A})_{>0}$. The weights $W_T(Z) \in K(\mathcal{A})_{>0} \otimes \mathbb{Q}$ are locally constant functions of the central charge, proportional to

$$\prod_{v \in T} \mathrm{DT}(\alpha(v), Z),$$

while we have

$$H_T(z, Z) = \tilde{H}_T(z, Z) x_{\sum_{v\in T} \alpha(v)}$$

where $\tilde{H}_T(z, Z)$ are universal holomorphic functions with branch-cuts, just as the Joyce functions $J_n(z_1, \ldots, z_n)$ (see [FGS] or [FS1] section 3).

Let us now specialise to the case of $\mathcal{A} = \mathbb{C}Q_k$-mod, the category of representations of the k-Kronecker quivers. Notice that according to [KS] section 1.4 this gives a sort of universal local model for wall-crossing formulae. We have $K(\mathcal{A}) \cong \mathbb{Z}^2$ generated by the classes of the two simple objects $S_1 = (0 \to 1)$, $S_2 = (1 \to 0)$. The space of stability conditions $\mathrm{Stab}(\mathcal{A})$ is isomorphic to $\mathbb{H}^2$ ($\mathbb{H} \subset \mathbb{C}$ denoting the upper half-plane) and contains an open chamber $\mathcal{S}$ ("strong coupling" region) such that for $Z \in \mathcal{S}$ the only Z-stable objects are the simple representations S_i, $i = 1, 2$. The chamber $\mathcal{S}$ is cut out by the condition that the phase of $Z([S_1])$ in the upper half-plane is strictly larger than the phase of $Z([S_2])$. In particular for $Z \in \mathcal{S}$ the BPS spectrum Ω on $K(\mathcal{A})_{>0}$ vanishes except for $\Omega([S_i], Z) = 1$, $i = 1, 2$. Let us write $\mathcal{W}$ for the interior of the complement of $\mathcal{S}$ ("weak coupling" region).

5.2. **Tropical formulae.** Fix a reference point $z^* \in -\mathbb{H}$. In the following we assume that all functions of $z \in \mathbb{C}^* \subset \mathbb{P}^1$ are evaluated at $z = z^*$, and drop this reference point from our notation. It turns out that the large λ behaviour of the "building blocks" $H_T(\lambda Z)$ of $Y(\lambda Z)$ is quite easy to understand away from $\partial\mathcal{S}$ (e.g. by stationary phase approximation), but changes across the critical locus $\partial\mathcal{S}$ in an interesting way, governed by the following result.

Theorem 5.1 ([FGS] Theorem 1.1)**.** *Let $Z_{\mathcal{S}} \in \mathcal{S}$, $Z_{\mathcal{W}} \in \mathcal{W}$, and let $H_T(Z_{\mathcal{S}})$ denote an n-vertex function appearing in the expansion* (5.1) *for flat sections of $\nabla(Z)$. There exists an expansion*

$$H_T(\lambda Z_{\mathcal{S}}) = \sum_{T'} \pm H_{T'}(\lambda Z_{\mathcal{W}}) + r(|Z_{\mathcal{S}} - Z_{\mathcal{W}}|, \lambda) \tag{5.2}$$

where $r(|Z_{\mathcal{S}} - Z_{\mathcal{W}}|, \lambda) \to 0$ as $|Z_{\mathcal{S}} - Z_{\mathcal{W}}| \to 0$, and we sum over a finite set of rooted decorated trees T', not necessarily distinct. Let $\alpha = \sum_{v \in T} \alpha(v)$.

The terms corresponding to a single-vertex tree in (5.2) *are labelled by a finite set of graphs C_i containing $n+1$ external* 1*-valent vertices and with* 3*-valent internal vertices. These terms are all equal to $H_\alpha(\lambda Z_{\mathcal{W}})$ up to sign, and differ by a well defined factor $\varepsilon(C_i) = \pm 1$ which is uniquely attached to the graph C_i.*

Moreover the graphs C_i come naturally with extra combinatorial data which endows them with the structure of the combinatorial types of a finite set of rational tropical curves *immersed in the plane* $\mathbb{R}^2$.

Finally the single-vertex terms in (5.2) *are uniquely characterised by their asymptotic behaviour: they are of order*

$$(2|Z_{\mathcal{W}}(\alpha)|\lambda)^{-1} \exp(-2|Z_{\mathcal{W}}(\alpha)|\lambda)$$

as $\lambda \to \infty$, uniformly *as $Z_{\mathcal{W}} \to \partial\mathcal{W}$.*

Example 5.2. The simplest nontrivial example of (5.2) is the expansion

$$H_{[S_1]\to[S_2]}(\lambda Z_{\mathcal{S}}) = H_{[S_1]+[S_2]}(\lambda Z_{\mathcal{W}}) + H_{[S_1]\to[S_2]}(\lambda Z_{\mathcal{W}}) + r(|Z_{\mathcal{S}} - Z_{\mathcal{W}}|, \lambda)$$

which follows from the standard residue theorem of complex analysis. The single-vertex term $H_{[S_1]+[S_2]}(\lambda Z_{\mathcal{W}})$ is characterised by its *uniform* (in $Z_{\mathcal{W}}$) $\lambda \to \infty$ asymptotic behaviour

$$(2|Z_{\mathcal{W}}([S_1]+[S_2])|\lambda)^{-1}\exp(-2|Z_{\mathcal{W}}([S_1]+[S_2])|\lambda).$$

The graph C attached to this single-vertex term has a single 3-valent vertex attached to 3 distinct legs. It is identified naturally with the combinatorial type of a rational tropical curve, in fact simply a tropical line, immersed in $\mathbb{R}^2$: the balancing condition at the single 3-valent vertex reads

$$-[S_1]-[S_2]+[S_1+S_2]=0.$$

Theorem 5.1 says precisely that this simple-minded analysis can be carried out in a similar way for all n-vertex functions $H_T(Z_{\mathcal{S}})$, yielding a finite set of tropical types C_i. The tropical balancing condition at a 3-valent vertex of C_i with incident edges decorated by $\alpha_i \in \Gamma$ always takes the form $-\alpha_1-\alpha_2+\alpha_3=0$, and arises naturally from the residue theorem and the linearity of the central charge Z.

Theorem 5.1 establishes the large λ tropical behaviour of the building blocks $H_T(\lambda Z)$ of $Y(\lambda Z)$. One might expect that summing the contributions $\varepsilon(C_i)=\pm 1$ over tropical types C_i of Theorem 5.1 may yield an actual tropical enumerative invariant. To state a precise result we first notice that the n external, incoming 1-valent vertices of a graph C_i are necessarily labelled by a collection of positive integer multiples of $[S_1]$ and $[S_2]$, which we denote by $(w_{1i}[S_1], w_{2j}[S_2])$. In the light of Theorem 5.1 it is natural to regard this collection as defining a tropical degree $\mathbf{w}=(w_{1i},w_{2j})$ for plane tropical curves. We denote the length of the partion w_{1i} by l_i, $i=1,2$. The weight vector $\mathbf{w}$ attached to a graph C_i depends only on the decorated tree T from which it originates. We denote by $C_i(T)$ the finite set of tropical types attached to T by Theorem 5.1. The enumerative invariant of rational tropical curves in $\mathbb{R}^2$ of degree $\mathbf{w}$ is denoted by $N^{\mathrm{trop}}(\mathbf{w}) \in \mathbb{N}$ (see [GPS] section 2 for an account). A precise connection with Gromov-Witten theory through relative invariants counting rational curves with tangency conditions is given in [GPS] sections 3 and 4.

Theorem 5.3 ([FGS] Theorem 1.2)**.** *The sum of contributions $\varepsilon(C_i(T))=\pm 1$ over tropical types C_i, weighted by the coefficients W_T in the expansion* (5.1) *for flat sections in $\mathcal{S}$,*

$$\sum_{\deg(T)=\mathbf{w}} W_T(Z_{\mathcal{S}})\sum_i \varepsilon(C_i(T))$$

equals the tropical invariant $N^{\mathrm{trop}}(\mathbf{w})$ enumerating plane rational tropical curves of degree $\mathbf{w}$, times the combinatorial "multi-cover" factor in $K(\mathcal{A})_{>0}\otimes\mathbb{Q}$ given by

$$k^{l_1+l_2}\frac{1}{|\operatorname{Aut}(\mathbf{w})|}\prod_{k,l}\frac{1}{w_{kl}^2}(|\mathbf{w}|_1[S_1]+|\mathbf{w}|_2[S_2]).$$

Our proof uses the beautiful theory of the tropical vertex group developed in [GPS].

5.3. **q-deformation.** Many of the objects we have encountered so far admit natural "quantizations", i.e. q-deformations. A general discussion of q-deformation in the present context may be found in [FS2].

Kontsevich and Soibelman [KS] deform $\mathbb{C}[K(\mathcal{A})_{>0}]$ to an associative, noncommutative algebra $\mathbb{C}[K(\mathcal{A})_{>0}]_q$ over $\mathbb{C}[q^{\pm\frac{1}{2}}]$. The classical product is quantized to

$$\hat{x}_\alpha \hat{x}_\beta = q^{\frac{1}{2}\langle\alpha,\beta\rangle}\hat{x}_{\alpha+\beta}.$$

In the quantization the Lie bracket is the natural one given by the commutator. In other words we are now thinking of the $\hat{x}_\alpha$ as *operators* (as opposed to the classical bracket, which corresponds to a Poisson bracket of the x_α seen as *functions*). Namely we set

$$[\hat{x}_\alpha, \hat{x}_\beta] = (q^{\frac{1}{2}\langle\alpha,\beta\rangle} - q^{-\frac{1}{2}\langle\alpha,\beta\rangle})\hat{x}_{\alpha+\beta}.$$

Since this is the commutator bracket of an associative algebra, $\mathbb{C}[K(\mathcal{A})_{>0}]_q$ is automatically Poisson. We denote by $\mathbb{C}[\widehat{K(\mathcal{A})}_{>0}]_q$ the completion with respect to the two-sided ideal generated by $\hat{x}_{[S_1]}, \dots, \hat{x}_{[S_n]}$.

Let $K_q \to \mathrm{Stab}(\mathcal{A})$ denote the trivial bundle with fibre $\mathbb{C}[\widehat{K(\mathcal{A})}_{>0}]_q$. One can generalise Proposition 2.7 (a) in this case, yielding a CV-structure on $K_q \to \mathrm{Stab}(\mathcal{A})$ (see [FS1] section 4.1). We obtain a corresponding flat meromorphic connection on $p^* K_q \to \mathrm{Stab}(\mathcal{A}) \times \mathbb{P}^1$,

$$D_q + \frac{C_q}{z} + z\widetilde{C}_q + \left(\frac{1}{z^2}Z - \frac{1}{z}\mathcal{Q}_q - \kappa_q Z \kappa_q\right) dz.$$

This "quantised" connection no longer takes values in derivations (i.e. vector fields are quantised just to vector space endomorphisms). In particular we have a family of meromorphic connections on the trivial bundle on $\mathbb{P}^1$ with fibre $\mathbb{C}[\widehat{K(\mathcal{A})}_{>0}]_q$,

$$\nabla_q(Z) = d - \left(\frac{1}{z^2}Z - \frac{1}{z}\mathcal{Q}_q - \kappa_q Z \kappa_q\right) dz$$

with $\mathcal{Q}_q(Z)$, $\kappa_q(Z)$ taking values in vector space endomorphisms $\mathrm{End}(\mathbb{C}[\widehat{K(\mathcal{A})}_{>0}]_q)$. Consider now the restriction $K_q \to \mathcal{S}$ to the "strong coupling" region. Then one can show that an analogue of (5.1) holds: for all $\alpha \in K(\mathcal{A})_{>0}$, there is a flat section σ_α of $\nabla_q(Z)$ of the form

$$\sigma_\alpha(z, Z) = \hat{x}_\alpha \exp \sum_T \langle \alpha, W_{T,q}(Z)\rangle\, H_{T,q}(z, Z) \tag{5.3}$$

where the weights $W_{T,q}(Z) \in K(\mathcal{A})_{>0} \otimes \mathbb{Q}[q^{\pm\frac{1}{2}}]$ are locally constant functions of the central charge, proportional to

$$\prod_{v\in T} \mathrm{DT}_q(\alpha(v), Z),$$

while we have

$$H_T(z, Z) = \tilde{H}_T(z, Z) \overrightarrow{\prod_{v\in T}} \hat{x}_{\alpha(v)},$$

a suitable ordered product using the natural orientation of the rooted tree T. The numbers $\mathrm{DT}_q(\alpha(v), Z) \in \mathbb{Q}[q^{\pm\frac{1}{2}}]$ are a natural q-deformation of $\mathrm{DT}(\alpha(v), Z) \in \mathbb{Q}$ (due to [KS], see [FS2] for a quick introduction). It is important to point out that (5.3) does not seem to generalise readily to flat sections of $\nabla_q(X)$ over the "weak coupling" region $\mathcal{W}$. Note that similar ideas have appeared more recently, and independently, in the physical literature (see [CNV] section 4.2.2).

The analogues of Theorems 5.1 and 5.3 also hold, to wit, the $H_{T,q}(z, Z)$ admit an expansion just like (5.2), with $\varepsilon(C_i)$ replaced by uniquely determined $\varepsilon_q(C_i) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$, and the corresponding weighted sums of contributions $\varepsilon_q(C_i)$ yield q-deformed tropical enumerative invariants (the Block-Göttsche invariants introduced in [BG]). Indeed this result was the original motivation for our study of Block-Göttsche invariants through wall-crossing identities in [FS2].

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