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Some applications of K-stability and K-energy

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This is a short summary of the main results in the papers [11], [12], [13] and thesis [14]. In brief we propose several applications of the notion of K-stability introduced by S. K. Donaldson and the K-energy functional of T. Mabuchi to the study of Kähler metrics of constant scalar curvature on smooth complex projective varieties and more generally Kähler manifolds.

1 Introduction

In the papers [11], [12], [13] and thesis [14] the author studies the constant scalar curvature equation on projective and Kähler manifolds. In the present paper we provide some background on the problem and then proceed to state the main results obtained in the aforementioned works.

Let X be a compact, connected Kähler manifold of complex dimension n. For a given Kähler cohomology class $\Omega \in H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{R})$ and a Kähler metric g whose associated $(1,1)$ form $\omega = \omega_g$ lies in Ω we let $s(\omega)$ denote the scalar curvature,

$$s(\omega) = -g^{i\bar{j}}\partial_i\partial_{\bar{j}}\log(\det(g_{k\bar{l}})) \tag{1}$$

in local coordinates. The average of $s(\omega)$ with respect to the volume form $d\mu_\omega = \frac{\omega^n}{n!}$ is a fixed topological quantity

$$\widehat{s} = \int_X s(\omega)\,d\mu_\omega = \frac{n\int_X c_1(X)\cup\Omega^{n-1}}{\int_X \Omega^n},$$

where $c_1(X) = c_1(TX)$ denotes the first Chern class of X (i.e. of its holomorphic tangent bundle). Thus we are concerned with the nonlinear problem

$$s(\omega) = \widehat{s}, \;\; [\omega] \in \Omega. \tag{2}$$

We call this the *cscK equation*. If it is solvable we will say that the metric ω (or even the class Ω) is *cscK*.

Equation 2 is the natural higher dimensional analogue of the constant Gaussian curvature equation on a Riemann surface. Establishing necessary and sufficient conditions for the solvability of the cscK equation when $n > 1$ is regarded as one of the main open questions in Kähler geometry, starting with the work of E. Calabi in the 1970's.

There are in fact a number of effective obstructions to the solvability of the cscK equation, starting with those of Matsushima and Futaki. On the other hand deep and recent results of S. K. Donaldson [5] and X. X. Chen and G. Tian [2] imply uniquess of solutions modulo the action of the group of complex automorphisms $\mathrm{Aut}(X)$. This situation should be contrasted with the Riemannian analogue for a fixed conformal class $\mathcal{C}$,

$$s(g) = \text{const}, \;\; g \in \mathcal{C} \tag{3}$$

(the average is no longer topological of course). Equation 3 definitely enjoys existence thanks to the celebrated solution of the Yamabe problem (see e.g. [10]) but there is no uniqueness even if we require the constant in 3 to be minimal. From this point of view adding the constraint $\nabla_g J = 0$ for an integrable almost complex structure J compatible with ω and parametrising by Kähler potentials $\omega = \omega_0 + i\partial\overline{\partial}\phi$ instead of conformal factors $g = e^u g_0$ changes the problem dramatically (except for complex dimension one, where the two problems essentially coincide).

2 Main results

Indeed suppose that we are in the projective case, so $\Omega \in H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z})$ is now the first Chern class of an ample line bundle L. In this case S. K. Donaldson [3] developed a beautiful theory that relates the solvability of equation 2 to algebro-geometric properties of the variety X with the ample line bundle L, a *polarised manifold* (X, L). We say that (X, L) *is cscK* if the class $c_1(L)$ is cscK. Let $\mathrm{Aut}(X, L)$ denote the group of projective automorphisms.

Theorem 2.1 *(Donaldson* [3] *2001) If the polarised manifold* (X, L) *is cscK and* $\mathrm{Aut}(X, L)$ *is discrete (hence finite) then* (X, L) *is asymptotically Chow stable.*

Thus differential geometry can be used to grasp a subtle algebro-geometric property, (asymptotic) Chow stability.

Even more importantly for us Donaldson [4] (inspired by previous work of Tian) defined a conjectural analogue of Mumford-Takemoto stability for sheaves in the category of projective varieties. This notion is known as *(algebraic) K-stability*. It will be not be defined rigorously in this note, but in essence for any equivariant one-parameter degeneration (X_0, L_0) of (X, L) one computes a weight $F(X_0, L_0)$ (the *Donaldson-Futaki invariant, or weight*). We say (X, L) is K-(semi)stable if F is always nonnegative (respectively positive for nontrivial degenerations). Donaldson then went on to prove the following fundamental fact.

Theorem 2.2 *(Donaldson* [6] *2005) If* (X, L) *is cscK then it is K-semistable.*

One of our main results in this connection is a refinement of Theorem 2.2. It was conjectured by Donaldson as part of [4], p. 290.

Theorem 2.3 *(S.* [12] *2008) If* (X, L) *is cscK and* $\mathrm{Aut}(X, L)$ *is discrete (hence finite) then* (X, L) *is K-stable.*

The methods we use to prove this are essentially algebro-geometric. Perhaps surprisingly they rest on the behaviour of the Donaldson-Futaki weight F and of a given cscK metric when we *blow up* a finite number of points, as we now discuss.

Topologically the blowup $\mathrm{Bl}_{p_1, \dots, p_m} X$ is a connected sum along small balls around the points

$$\mathrm{Bl}_{p_1, \dots, p_m} X = X \#_{p_1} \overline{\mathbb{CP}^n} \dots \#_{p_m} \overline{\mathbb{CP}^n}$$

with negatively oriented complex projective spaces $\mathbb{CP}^n$. It is well-known that $\mathrm{Bl}_{p_1, \dots, p_m} X$ has a canonical complex structure depending only on the points $p_1, \dots, p_m$. If Ω is a Kähler class on X then there are Kähler classes on the blowup which coincide with Ω outside a compact neighborhood of each p_i. When Ω is cscK it is natural to ask if these classes on $\widehat{X}$ are cscK too.

From the algebro-geometric point of view let (X, L) be a polarised manifold. Let $\widehat{X}$ be the blowup of X along a finite collection of points $p_1, \dots, p_m$ with the projection $\pi : \widehat{X} \to X$ and exceptional divisors $\pi^{-1}(p_i) \subset \widehat{X}$. Then the line bundle on $\widehat{X}$

$$\widehat{L} = \pi^* L^\gamma \otimes_{i=1}^m \mathcal{O}(\pi^{-1}(p_i))^{-a_i}$$

is ample for all positive integers $a_1, \dots, a_m$ and any γ sufficiently large (depending on $a_1, \dots, a_m$). Suppose that $c_1(L)$ is represented by a cscK metric. Then we ask if the same is true for $c_1(\widehat{L})$. More precisely we ask for which $m-$tuples of points $p_1, \dots, p_m$ and weights $a_1, \dots, a_m$ the class $c_1(\widehat{L})$ is represented by a cscK metric *for any sufficiently large* γ. We will prove that some stability constraint must hold for the positions and weights of the blown-up points.

Theorem 2.4 *(S.* [11] *2007) Suppose that the classical Futaki character of* (X, L) *vanishes. If the* 0*-dimensional cycle* $\sum_{i=1}^m a_i^{n-1} p_i$ *on* X *is unstable with respect to the natural linearisation of the* $\mathrm{Aut}(X, L)$*-action on effective* 0*-cycles then the blowup* $(\widehat{X}, \widehat{L})$ *is K-unstable for all* $\gamma \gg 0$. *Thus the first Chern class of the line bundle* $\widehat{L}$ *is not representable by a cscK metric for any sufficiently large* γ.

To prove this we have developed a formula for the behaviour of the Donaldson-Futaki weight F under blowup. As we mentioned this is also a key ingredient in the proof of Theorem 2.3. But Theorem 2.4 also has independent interest as a converse to a well-known result proved by C. Arezzo and F. Pacard, which we have recast in the following algebro-geometric form.

Theorem 2.5 *(Arezzo-Pacard* [1] *2005) If (X, L) is cscK, the cycle $\sum_{i=1}^{m} a_i^{n-1} p_i$ is stable for the natural linearisation of the $\mathrm{Aut}(X, L)$-action on effective 0-cycles and a further nondegeneracy condition holds for the points $p_1 \ldots, p_m$ then the first Chern class of the line bundle $\pi^* L^\gamma \otimes_{i=1}^{m} \mathcal{O}(\pi^{-1}(p_i))^{-a_i}$ can be represented by a cscK metric for any γ sufficiently large.*

For the classical Futaki character see e.g. [15]. It is well known that for cscK Kähler manifolds the Futaki character vanishes (in the algebraic case this is part of K-semistability).

Going back to Donaldson's Theorem 2.2 an important question is to use K-(semi)stability to obtain explicit cohomological conditions that the class $c_1(L)$ must satisfy if it can be represented by a cscK metric. Much work on this problem has been done by J. Ross and R. Thomas [8], [9].

For any closed subscheme $Z \subset X$ Ross-Thomas defined a quantity $\mu(Z, L)$ (the *slope*), which can be computed intrinsically on X, and proved that K-semistability implies the following.

Theorem 2.6 *(Ross-Thomas* [8] *2006) If (X, L) is cscK then for all closed subschemes $Z \subset X$ the slope-semistability condition holds,*

$$\mu(Z, L) \leqslant \mu(X, L)$$

where $\mu(X, L)$ is the slope of the polarised mailfold (X, L).

For the general definition of slope see [9]. In the case of an effective divisor $D \subset X$ the Hirzebruch-Riemann-Roch Theorem computes the slope $\mu(D, L)$ as an explicit rational function of the Chern classes $c_1(X), c_1(L), c_1(\mathcal{O}(D))$. For example when X is an algebraic surface, Theorem 2.6 becomes the interesting Chern number inequality (depending on a positive parameter c)

$$\frac{3\int_X \left(c_1(L) \cup c_1(\mathcal{O}(D)) - c[c_1(\mathcal{O}(D))^2 - c_1(X) \cup c_1(\mathcal{O}(D))]\right)}{2c\int_X \left(3c_1(L) \cup c_1(\mathcal{O}(D)) - c_1(\mathcal{O}(D))^2\right)} \geqslant \frac{\int_X c_1(X) \cup c_1(L)}{\int_X c_1(L)^2}$$

which must hold as long as $L - cD$ is ample. This inequality has important applications to cscK metrics on complex surfaces, see e.g. Ross-Panov [7].

In general it is then tempting to replace $c_1(L)$ by any Kähler class Ω on a Kähler manifold (not necessarily projective) and ask if Theorem 2.6 still holds, at least in the case of divisors (this is the most important case in for cscK metrics). This was first conjectured by Ross-Thomas as part of [8] Section 4.4.

Theorem 2.7 *(S.* [13] *2008) Let X be a Kähler (not necessarily projective) manifold. If a Kähler class Ω on X is cscK then X is slope-semistable with respect to Ω and all effective divisors.*

In the same paper we give an example of a slope-unstable Kähler manifold which cannot be deformed to a projective one.

Our proof also elucidates the differential-geometric meaning of the slope-stability condition. It turns out that slope-stability is just the condition that the well-known K-energy functional of Mabuchi remains bounded from below as the Kähler form concentrates along the divisor D to some extent dictated by positivity. We cannot discuss this functional and its relationship with cscK metrics here. We only mention that in the non-projective case the role of test configurations is played by rays in the space of Kähler potentials with respect to a reference metric, while the Futaki invariant is replaced by the asymptotics of the K-energy.

Finally we cannot fail to mention the main conjecture underpinning the whole theory. It is due to S. T. Yau, G. Tian and S. Donaldson (see e.g. [4]).

YTD Conjecture *A polarised manifold (X, L) is cscK if and only if it is K-polystable.*

Roughly speaking a polystable object differs from a stable one only up to automorphisms. A proof of the YTD Conjecture would give the analogue (for projective varieties) of the Hitchin-Kobayashi Correspondence between Mumford-Takemoto polystable vector bundles and Hermite-Einstein connections.

Not much progress has been made on this general statement to the present day except in the case of toric surfaces where Donaldson has recently solved the problem completely.

The Arezzo-Pacard Theorem and our converse may be seen as infinitesimal or perturbation manifestations of the YTD correspondence.

Note however that it is now widely believed that a stronger notion of stability (yet to be discovered) may be needed in the statement of the YTD Conjecture. Indeed we present a new argument in favour of this belief in [14].

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