Notes on the extremal field of odd symplectic Grassmannians

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Abstract

The aim of these notes is to compute the extremal field of a Kähler manifold in a nontrivial example.

Thus we compute the Futaki character for the family of odd symplectic Grassmannians $\mathbb{G}_\omega(2, 2n+1)$, i.e. Grassmannians of "Lagrangian" 2-planes in a complex vector space of dimension $2n+1$ endowed with a skew 2-form of maximal rank ω. These are Fano manifolds and more precisely generic elements of the Plücker linear system on the Grassmannian $\mathbb{G}(2, 2n+1)$. The character is nonvanishing (so our varieties are not Kähler-Einstein) and we identify the extremal field in the sense of Calabi. This turns out to be the generator of the centre of $\mathfrak{Lie}\,\mathrm{Aut}(X)$. This computation is crucial if one wants to investigate the existence of an extremal metric on $\mathbb{G}_\omega(2, 2n+1)$.

1 Introduction

The problem of the existence of a Kähler-Einstein metric on a compact complex manifold X was completely solved by Yau [15] more than 25 years ago in case X has vanishing or ample canonical bundle. When X is a Fano manifold, on the other hand, the problem is definitely more subtle. During the years many obstructions to the existence of a Kähler-Einstein metric on a Fano manifold have been found, their algebro-geometric nature becoming increasingly deep and difficult to control. The first such an obstruction was found by Matsushima in 1957: if a Fano X supports a Kähler-Einstein metric then $\mathrm{Aut}(X)$ is reductive group. The second type of obstruction is a character F_X on $\mathfrak{Lie}\,\mathrm{Aut}(X)$ discovered by Futaki in 1982 and bearing his name. If X admits a Kähler-Einstein metric this character vanishes. These two obstructions deal with automorphisms of X, so are trivially satisfied when $\mathrm{Aut}(X) = \{1\}$. Other obstructions relate the existence of a Kähler-Einstein metric to the stability of Chow and Hilbert points of $(X, -mK_X)$ and more generally to other stability properties of the polarised manifold $(X, -K_X)$. The first of these obstructions was found by Tian in 1997 [14]. Other stability obstructions were later found in [2], [3], [10] [11] . Notwithstanding the great progress towards the final solution to the Calabi conjecture in the Fano case, there are still very few nontrivial examples were the various obstructions can be explicitely computed. The purpose of this note is to carry out the computation of the Futaki invariants for a hyperplane section X of the Grassmannian $\mathbb{G}(2, 2n+1)$ embedded by Plücker. X is a Fano manifold with $c_1(X) = 2n \det(S^*)$, where S is the universal bundle on $\mathbb{G}$ and so $\det(S^*)$ is the line bundle associated to the Plücker embedding. The group $\mathrm{Aut}(X)$ is not reductive (see Theorem 2.4) so X admits no Kähler-Einstein metric. Hwang and Mabuchi conjectured [7] that the automorphism group of a Fano is either reductive or has nontrivial center. In fact $\mathrm{Aut}(X)$ has a 1-dimensional center and F_X can be nonzero only on the center. These manifolds have been studied recently also by Ion Mihai in his 2005 PhD thesis at Grenoble, in connection with representation theory.

The computation is based on the localization formula due to Futaki, see Theorem 3.3 below.

The organization of these notes is as follows: in section 2 we determine the automorphism group of the hyperplane section. In section 3 we study the normal bundle to the two components of the zero locus of w, the central element of $\mathfrak{Lie}\,\mathrm{Aut}(X)$. In section 4 we compute the Futaki invariant of this field by applying the localization formula. In our case this becomes a problem in Schubert calculus.

Although the resulting formula (4.2) is completely explicit and can be used to compute effectively $F_X(w)$ for particular values of n, it is quite hard to show that the invariant is nonzero in general. In the final section, we show that $F_X(w) < 0$, at least for large n.

Acknowledgements. These notes are based on some old work (2005) under the direction of G. P. Pirola and A. Ghigi, to whom I am very grateful.

2 The automorphism group

By $\mathbb{G}(2, k)$ we denote the Grassmannian of 2-planes in $\mathbb{C}^k$. S and Q denote respectively the universal subbundle and the quotient bundle. The Plücker embedding corresponds to the line bundle $\mathcal{O}_{\mathbb{G}}(1) = \det S^* = \Lambda^2 S^*$. Recall that $H^0(\mathbb{G}(2, k), S^*) = (\mathbb{C}^k)^*$ and $H^0(\mathbb{G}(2, k), \Lambda^2 S^*) = \Lambda^2(\mathbb{C}^k)^*$. Therefore the Plücker embedding maps $\mathbb{G}(2, k)$ to $\mathbb{P}(\Lambda^2 \mathbb{C}^k)$ and the hyperplane section

X_α corresponding to $\alpha \in H^0(\mathbb{G}(2,k),\mathcal{O}(1)) = \Lambda^2(\mathbb{C}^k)^*$ is simply the locus of 2-planes u such that $\alpha_{|u} = 0$. By a suitable choice of basis $\{e_i\}$ of $\mathbb{C}^k$, the 2-form α can always be put in the form $\alpha = \sum_{k=0}^{r-1} e^*_{2k+1} \wedge e^*_{2k+2}$, where $2r$ is the rank of α. Therefore given two forms α and β there is an element $g \in \mathrm{Gl}(k,\mathbb{C})$ such that $g^*\alpha = \beta$ if and only if α and β have the same rank. It is easy to see that the action of $\mathrm{Gl}(k,\mathbb{C})$ on hyperplanes is just the pullback of the corresponding $2-$forms. This yields the following.

Lemma 2.1 *Hyperplane sections of $\mathbb{G}(2,k)$ in the Plücker embedding split into $\lfloor k/2 \rfloor$ classes of projective isomorphism indexed by the rank of the corresponding $2-$form.*

It is a simple matter to decide which classes contain smooth sections.

Lemma 2.2 *The smooth hyperplane sections of $\mathbb{G}(2,k)$ are exactly those given by a 2-form of maximal rank.*

Proof. In Plücker coordinates the equation of a hyperplane section of rank r is

$$\{\lambda_{12} + ... + \lambda_{2r-1,2r} = 0\} \cap \mathbb{G}(2,k).$$

Recall that elements $u \in \mathbb{G}(2,k)$ can be parametrised by $2 \times k$ matrices Z of rank 2, whose rows give a basis of u. (These are the so-called *Stiefel coordinates* for $\mathbb{G}(2,l)$, see [5, p. 92].) For such a matrix Z denote by Z_{ij} the 2×2-minor composed of the i-th and the j-th column and let U_{ij} denote the set of $2 \times k$ matrices for which $\det Z_{ij} \neq 0$. The affine coordinates on U_{ij} (or better on its image in $\mathbb{G}(2,k)$) are the (nontrivial) entries of the matrix

$$Z_{ij}^{-1} \cdot Z = \begin{pmatrix} x_{11} & \dots & 1 & 0 & \dots x_{1k} \\ x_{21} & \dots & 0 & 1 & \dots x_{2k} \end{pmatrix}.$$

If $r < \lfloor k/2 \rfloor$ and $\alpha = e_1^* \wedge e_2^* + \cdots + e^*_{2r-1} \wedge e^*_{2r}$, then $X_\alpha \cap U_{k-1,k}$ has equation

$$x_{11}x_{22} - x_{12}x_{21} + \cdots + x_{1,2r-1}x_{2,2r} - x_{1,2r}x_{2,2r-1} = 0.$$

This is a homogeneous affine quadric hypersurface so the origin, which corresponds to the point $u = e_{k-1} \wedge e_k$, is a singular point of X_α. On the other hand if $r = \lfloor k/2 \rfloor$ the equation becomes

$$x_{11}x_{22} - x_{12}x_{21} + \cdots + x_{1,2n-3}x_{2,2n-2} - x_{1,2n-2}x_{2,2n-3} + 1 = 0$$

if $k = 2n$ and

$$x_{11}x_{22} - x_{12}x_{21} + \cdots + x_{1,2n-3}x_{2,2n-2} - x_{1,2n-2}x_{2,2n-3} - x_{2,2n} = 0$$

if $k = 2n+1$. In both cases $X_\alpha \cap U_{k-1,k}$ is smooth. The intersection of X_α with any other U_{ij} behaves in a similar way.

Q.E.D.

The smooth hyperplane section of $\mathbb{G}(2,2n)$ turns out to be a rational homogeneous space. In fact it is the set of isotropic 2-planes in the complex symplectic vector space $(\mathbb{C}^{2n},\alpha)$. The action of $\mathrm{Sp}(2n,\mathbb{C})$ is transitive on such planes, therefore X_α is homogeneous and supports a Kähler-Einstein metric by a theorem of Borel-Hirzebruch. In particular the Futaki invariant vanishes. This is the reason why in these notes we restrict attention to section of $\mathbb{G}(2,k)$ with $k=2n+1$ an odd number.

From now on, we will denote by X the smooth hyperplane section of $G=\mathbb{G}(2,2n+1)$ corresponding to $\alpha=e_1^*\wedge e_2^*+\cdots+e_{2n-1}^*\wedge e_{2n}^*$.

The following is a well-known application of Bott theorem [1]. For an elementary proof see [8].

Theorem 2.3

$$\mathfrak{Lie}\,\mathrm{Aut}(\mathbb{G}(k,n))\cong\mathfrak{sl}(n,\mathbb{C});$$

$$\mathrm{Aut}^0(\mathbb{G}(k,n))\cong\mathbb{P}\mathrm{Sl}(n,\mathbb{C}).$$

The second isomorphism is induced by the action of $A\in\mathrm{Sl}(n,\mathbb{C})$ on k-planes sending π to $A\pi$.

We come back to the problem of finding $\mathfrak{Lie}\,\mathrm{Aut}(X)$. There is one remarkable subgroup of $\mathrm{Aut}(X)$, namely automorphisms of G that leave X fixed. By the above result, we can choose a matrix representative M for any such automorphism. Let A denote the antisymmetric $(2n+1)\times(2n+1)$ matrix associated to the canonical form α. An admissible M must satisfy ${}^tMAM=\mu A$ for some nonzero complex number μ. By a straightforward calculation, M must then be of the form

$$M=\begin{pmatrix}\mu S & 0\\ a & \mu^{-2n}\end{pmatrix}$$

where $S\in\mathrm{Sp}(n,\mathbb{C})$, $\mu\in\mathbb{C}^*$, $a\in\mathbb{C}^{2n}$. This shows that $\mathfrak{Lie}\,\mathrm{Aut}(G)$ contains the semi-direct sum algebra $\mathfrak{g}=\mathfrak{sp}(n,\mathbb{C})\oplus\mathbb{C}\oplus\mathbb{C}^{2n}$. The natural way to think about $\mathfrak{g}$ is to consider matrices of the form

$$\begin{pmatrix}\mu I+H & 0\\ a & -2n\mu\end{pmatrix}$$

where $\mu\in\mathbb{C}$, H is a complex symplectic $2n\times 2n$ matrix and $a\in\mathbb{C}^{2n}$. The main result of this section is the following.

Theorem 2.4

$$\mathfrak{Lie}\,\mathrm{Aut}(X)=\mathfrak{g}.$$

The following result reduces the problem to a (not so simple) dimension count.

Lemma 2.5 *Let M be a compact complex manifold with $h^1(M, \mathcal{O}) = 0$. Let $L \to M$ be an* $\mathrm{Aut}(M)-$*equivariant line bundle. Suppose that the action of* $\mathrm{Aut}(M)$ *on the divisors in $|L|$ has an open orbit U. Finally let D be any smooth divisor in U such that the restriction map*

$$r : H^0(M, T'M) \to H^0(D, T'M_{|D})$$

is an isomorphism. Then

$$h^0(D, T'D) = h^0(M, T'M) - h^0(D, L_{|D}). \tag{2.1}$$

Proof. Let D be given by $s = 0$, for some $s \in H^0(M, L)$. Let v be any holomorphic vector field on M; it generates a 1-parameter subgroup ϕ_t. Since L is $Aut(M)$-equivariant, ϕ_t lifts to a 1-psg ϕ_t^* of L. The derivative $\mathcal{L}_v s = \frac{d}{dt}\phi_t^* s$ is therefore a well defined section of L. Put $E = T_D U$. Since U is an orbit for the action of $\mathrm{Aut}(M)$ on $|L|$, $E = \{\mathcal{L}_v s | v \in H^0(M, T'M)\}$. Thus we get a decomposition $H^0(M, L) = E \oplus \mathbb{C}s$. Now we show that the restriction map $r : E \to H^0(D, L_{|D})$ is an isomorphism. The restriction (to D) short exact sequence induces the exact sequence

$$0 \to \mathbb{C} \cong H^0(M, \mathcal{O}_M) \to H^0(M, L) \to H^0(D, L_{|D}) \to 0$$

where we used $\mathcal{O}_M(L)(-D) \cong \mathcal{O}_M$ and $h^1(M, \mathcal{O}) = 0$. This gives surjectivity. To get injectivity (when resctricted to E) just observe that the kernel of the first map in the above sequence can be canonically identified with $\mathbb{C}s$. To conclude, consider the short exact sequence

$$0 \to \mathcal{O}_D(T'D) \to \mathcal{O}_D(T'M_{|D}) \to \mathcal{O}_D(L_{|D}) \to 0$$

where we used the adjunction formula $N_{D|M} \cong [D]_{|D}$. The induced cohomology sequence is

$$0 \to H^0(D, T'D) \to H^0(D, T'M_{|D}) \xrightarrow{j} H^0(D, L_{|D}) \to \dots .$$

We will show that the last map (which is sometimes called the adjunction map) is surjective. For this, lift $v \in H^0(D, T'M_D)$ to $\widetilde{v} \in H^0(M, T'M)$ uniquely (this is possible by hypothesis). It is easy to check that $j(v) = \mathcal{L}_{\widetilde{v}} s$. Therefore the following diagram commutes:

$$\begin{array}{ccc} H^0(M, T'M) & \longrightarrow & E \\ \downarrow & & \downarrow \\ H^0(D, T'M|_D) & \xrightarrow{j} & H^0(D, L|_D) \end{array}$$

where the upper row is the infinitesimal action $v \mapsto \mathcal{L}_v s$ (which is surjective) and the columns are isomorphisms. So the adjunction map j is onto. This proves (2.1).

Q.E.D.

Next we need the following vanishing results for some cohomology groups.

Lemma 2.6 *Let $G = \mathbb{G}(s, l+1)$. If $k \geq l+1-s$, $2p > (l-k)(l-k+1)$ and $q \geq 0$, then*

$$H^p(G, \Omega^q(k)) = 0.$$

In particular

$$H^p(G, \Omega^q(l)) = 0$$

for any q and any positive p.

Proof. See [12], page 176.

Lemma 2.7 *([12]) Let $G = G(s, l+1)$ and $1 \leq k \leq l$. If $q > n-s$ and $G \neq G(2,4)$, then for any p*

$$H^p(G, \Omega^q(k)) = 0.$$

Proof. See [12], page 163.

Corollary 2.8 *For $1 \leq s < l$ we have*

$$H^0(G(s, l+1), T'G(-1)) \cong 0;$$

$$H^1(G(s, l+1), T'G(-1)) \cong 0.$$

Proof. Set $G = G(s, l+1)$, $n = \dim_{\mathbb{C}} G$, $p \in \{0, 1\}$. Applying twice Serre duality we get $H^p(G, T_G(-1)) \cong H^p(G, \Omega^{n-1}(l))$. In the case $p = 1$, we conclude by the Lemma 2.6. When $p = 0$, we apply Lemma 2.7. This works in our case since $k = l$, $q = n-1 > n-s$ whenever $s > 1$, hence when $s = 2$.

Q.E.D.

Proof of 2.4. We show first that 2.5 applies when $M = G = G(2, 2n+1)$, $D = X$, $L = [X] = \det(S^*)$. Everything is straightforward except that the restriction map $H^0(G, T'G) \to H^0(X, T'G|_X)$ is an isomorphism. The restriction short exact sequence on sheaves induces the exact sequence

$$0 \to H^0(G, T'G(-1)) \to H^0(G, T'G) \to H^0(X, T'G|_X) \to$$

$$\to H^1(G, T'G(-1))$$

and we are done by the above vanishing result. So we get

$$\dim(\mathfrak{Lie}\,\mathrm{Aut}(X)) = \dim(\mathfrak{Lie}\,\mathrm{Aut}(G)) - h^0(X, L_{|X}).$$

Now count dimensions: we have $\dim(\mathfrak{g}) = 2n^2 + 3n + 1$, $\dim(\mathfrak{Lie}\,\mathrm{Aut}(G)) = 4n(n+1)$, $\dim(H^0(X, L_{|X})) = 2n^2 + n - 1$, so $\dim(\mathfrak{g}) = \dim(\mathfrak{Lie}\,\mathrm{Aut}(X)) = 2n^2 + 3n + 1$.

Q.E.D.

3 Zero loci and their normal bundles

Now we turn to the computation of the Futaki invariant of a smooth hyperplane section X of $\mathbb{G}(2, 2n+1)$.

We know that $\mathfrak{Lie}\,\mathrm{Aut}(X)$ is the semidirect sum $\mathfrak{sp}(n, \mathbb{C}) \oplus \mathfrak{b} \oplus \mathfrak{c}$ where $\mathfrak{b} \cong \mathbb{C}$ and $\mathfrak{c} \cong \mathbb{C}^{2n}$. Note that $\mathfrak{b}$ is the (1-dimensional) center of $\mathfrak{Lie}\,\mathrm{Aut}(X)$.

It follows from the general theory that the Futaki invariant vanishes on $\mathfrak{sp}(n, \mathbb{C})$ and $\mathfrak{c}$.

Lemma 3.1

$$F_{X|\mathfrak{sp}(n,\mathbb{C})} = 0.$$

Proof. The Lie algebra $\mathfrak{sp}(n, \mathbb{C})$ is semisimple and the Futaki invariant vanishes on brackets, see [4].

Lemma 3.2

$$F_{X|\mathfrak{c}} = 0.$$

Proof. In fact $\mathfrak{c}$ is the Lie algebra of the unipotent radical of $\mathrm{Aut}(X)$ and a theorem of Mabuchi [9] shows that F vanishes on such a subalgebra.

It remains to compute $F_{X|\mathfrak{b}}$. As a generator for $\mathfrak{b}$ we choose the field w associated to the matrix $\mathbf{diag}(1, 1, ..., -2n)$. The associated 1-psg is represented by $\phi_t = \mathbf{diag}(\mathrm{e}^t, \mathrm{e}^t, ..., \mathrm{e}^{-2nt})$.

The computation of $F_X(w)$ is based on the folllowing localization formula.

Theorem 3.3 *Let M be an n-dimensional Fano manifold and w a holomorphic vector field on M whose zero locus is the disjoint union of smooth submanifolds Z_λ. Denote by $L_\lambda(x) : N_{Z_\lambda|X} \to N_{Z_\lambda|X}$ the operator $L_\lambda(v) = (\nabla_v w)^\perp$. We say that w is nondegenerate if L_λ is an automorphism of the* *normal bundle. Under such hypotheses the following localization formula holds*

$$F_X(w) = \frac{1}{n+1}\sum_\lambda \int_{Z_\lambda} \frac{(tr(L_\lambda)+c_1(X))^{n+1}}{det(L_\lambda + \frac{\sqrt{-1}}{2\pi}K_\lambda)}$$

where K_λ are the curvature operators of the normal bundles to Z_λ.

See ([4], Theorem 5.2.8 p. 73) and [13] a complete discussion; here we just say that in practice one has to expand the denominator as a formal power series in the Chern classes $c_i(K_\lambda)$ and integrate the terms of degree $\dim(Z_\lambda)$.

The rest of these notes will be devoted to the application of this result in our case.

Lemma 3.4 *The field w has nondegenerate zero locus.*

Proof. We must check that $Z(w)$ contains only smooth connected components Z_λ with nonsingular transverse derivative L_λ. In fact $Z(w) = Z_1 \cup Z_2$ where $Z_1 = \mathbb{G}(2, \mathrm{span}(e_1, ..., e_{2n})) \cap X$ and $Z_2 = \{\Lambda \in X | e_{2n+1} \in \Lambda\} = \{\Lambda \in G(2, 2n+1) | e_{2n+1} \subset \Lambda\}$. Note that Z_1 is the Grassmannian of lagrangian 2-planes in the symplectic space $(\mathrm{span}(e_1, ..., e_{2n}), \alpha)$, while Z_2 is isomorphic to $\mathbb{P}^{2n-1}$ via the map $v \wedge e_{2n+1} \leftrightarrow [v]$. So we have $\dim(Z_1) = 4n-5$ and $\dim(Z_2) = 2n-1$. The normal bundles to Z_1, Z_2 have rank respectively $2, 2(n-1)$. Now checking that $L_i = \nabla^\perp W \in \mathrm{End}(N_{Z_i|X})$ are isomorphisms is a local matter: Z_1 can be covered by the usual charts $U_{ij} \cap Z_1$ with the restriction $1 \le i < j \le 2n$. Let v_k, w_k denote local coordinates for any point in U_{ij}. Parametric equations for its trajectory under ϕ_t in U_{ij} are given by

$$\begin{pmatrix} \mathrm{e}^t v_1 & \mathrm{e}^t v_2 & ... & \mathrm{e}^t & ... & 0 & ... & \mathrm{e}^{-2nt} v_{2n-1} \\ \mathrm{e}^t w_1 & \mathrm{e}^t w_2 & ... & 0 & ... & \mathrm{e}^t & ... & \mathrm{e}^{-2nt} w_{2n-1} \end{pmatrix}.$$

In particular we see that U_{ij} is ϕ_t-invariant and upon multiplying by $\mathbf{diag}(\mathrm{e}^{-t}, \mathrm{e}^{-t})$ we get parametric equations in the coordinates of U_{ij}:

$$\begin{pmatrix} v_1 & v_2 & ... & 1 & ... & 0 & ... & \mathrm{e}^{-(2n+1)t} v_{2n-1} \\ w_1 & w_2 & ... & 0 & ... & 1 & ... & \mathrm{e}^{-(2n+1)t} w_{2n-1} \end{pmatrix}.$$

We see that local equations for Z_1 inside $U_{ij} \cap X$ are $v_{2n-1} = w_{2n-1} = 0$ and thus $\partial_{v_{2n-1}}, \partial w_{2n-1}$ is a basis for the normal bundle in all of U_{ij}. It is now easy to conclude that $w = -(2n+1)\left(\partial_{v_{2n-1}} + \partial w_{2n-1}\right)$ and finally that $L_1 = \mathbf{diag}(-2n-1, -2n-1)$. Next cover Z_2 with charts $U_{i,2n+1}$, $1 \le i \le 2n$. A calculation similar to the one above yields $L_2 = \mathbf{diag}(2n+1, ..., 2n+1)$ in the basis $\partial_{w_1}, ..., \widehat{\partial_{w_i}}, ..., \partial_{w_{2n-1}}$ of $N_{Z_2|X_1}$ over $U_{i,2n+1}$ (remember that L_2 must be a $(2n-2)\times(2n-2)$ matrix).

Q.E.D.

In the following we use the following notation: $N_i = N_{Z_i|X}$ and I_i denotes the integrand over Z_i in the localization formula. So that

$$F_X(w) = \int_{Z_1} I_1 + \int_{Z_2} I_2. \tag{3.1}$$

The following useful facts were obtained as a byproduct of the above proof.

Lemma 3.5

$$\begin{array}{cc} \operatorname{rank} N_1 = 2 & \operatorname{rank} N_2 = 2(2n-1) \\ \operatorname{tr} L_1 = -4n-2 & \operatorname{tr} L_2 = (2n+1)(2n-2) \\ \det L_1 = (2n+1)^2 & \det L_2 = (2n+1)^{2n-2} \end{array} \tag{3.2}$$

The most important ingredients for the Futaki localization formula are the Chern classes $c_k(N_i)$. We will actually determine the normal bundles up to C^∞ isomorphism.

Lemma 3.6 *Let S_{2n} be the universal subbundle on $\mathbb{G}(2, 2n)$ and $S_{\mathbb{P}^{2n-1}}$ and $Q_{\mathbb{P}^{2n-1}}$ be respectively the universal and quotient bundles on $\mathbb{P}^{2n-1}$. Then there are isomorphisms as C^∞ vector bundles:*

$$N_1 \cong S^*_{2n|Z_1}$$

$$N_2 \cong Q_{\mathbb{P}^{2n-1}}/S^*_{\mathbb{P}^{2n-1}}$$

Proof. We prove only the first statement; the second one is similar. Note that in the C^∞ category there are splittings $N_{Z_1|X} \oplus N_{X|G_{2n+1}|Z_1} = N_{Z_1|G_{2n+1}}$ and $N_{Z_1|G_{2n+1}} = N_{Z_1|G_{2n}} \oplus N_{G_{2n}|G_{2n+1}|Z_1}$. Also Z_1 is a hypersurface in G_{2n} so $N_{Z_1|G_{2n}} = [Z_1]_{|Z_1} = \mathcal{O}(1)_{Z_1}$. On the other hand

$$N_{G_{2n}|G_{2n+1}|Z_1} = T'G_{2n+1}/T'G_{2n|Z_1} = (S^*_{2n+1} \otimes Q_{2n+1})/(S^*_{2n} \otimes Q_{2n})_{|Z_1}.$$

Here we just used the usual representation of the holomorphic tangent space to a Grassmannian. But

$$Q_{2n+1|Z_1} = \mathbb{C}^{2n+1}/S_{2n+1|Z_1} = \mathbb{C} \oplus Q_{2n}$$

and so

$$N_{G_{2n}|G_{2n+1}|Z_1} = ((S^*_{2n} \otimes \mathbb{C}) \oplus (S^*_{2n} \otimes Q_{2n}))/(S^*_{2n} \otimes Q_{2n|Z_1}) \cong S^*_{2n|Z_1}.$$

Substituting we find

$$N_{Z_1|X} \oplus N_{X|G_{2n+1}|Z_1} = \mathcal{O}(1)_{|Z_1} \oplus S^*_{2n|Z_1}.$$

Obviously $N_{X|G_{2n+1}} = \mathcal{O}(1)_X$ hence we get $N_{Z_1|X} \cong S^*_{2n|Z_1}$.

Q.E.D.

In view of their use in the localization formula, we will express the Chern classes $c_k(N_i)$ as Poincaré duals of certain Schubert cycles. Thus we digress for a moment to establish the notation and some relevant facts about Schubert cycles of the complex Grassmannian $\mathbb{G}(k,m)$. Let $V_1 \subsetneq V_2 ... \subsetneq V_m \subsetneq \mathbb{C}^{m+1}$ be a flag of subspaces. For any multiindex $a = (a_1, ..., a_k)$ set $W_a = \{\Lambda \in G(k,m) | \dim(\Lambda \cap V_{n-k+i-a_i}) = i,\ 1 \le i \le k\}$. Note that W_a is empty unless the a_i are noninscreasing and $a_i \le n-k$. For any a the closure $\overline{W_a} = \{\Lambda | \dim(\Lambda \cap V_{n-k+i-a_i}) \ge i\}$ is an analytic subvariety of $G(k,m)$. Let $\sigma_{a_1,\dots,a_k} = \left[\,\overline{W_{a_1,\dots,a_k}}\,\right]$ denote the corresponding homology class in $H^*(G(k,m),\mathbb{Z})$ (so that the *real* codimension is $2\sum_i^k a_i$). It is a classical fact that the homology of $G(k,m)$ is freely generated by these classes, that is, Schubert cycles. The following result is sometimes called *Chern-Gauss-Bonnet* theorem. For a proof see [6], p.410.

Theorem 3.7

$$c_r(S) = (-1)^r \sigma^*_{1,\dots,1}$$

where $(\ )^*$ *stands for Poincare duality and the multiindex* $\{1, ..., 1\}$ *has lenght* r.

Note as a particular case that $c_1(\mathcal{O}(1)_G) = \sigma_1^*$. Now we come back to $c_k(N_i)$. Recall that Chern classes satisfy $f^*(c_k(E)) = c_k(f^*(E))$ and $c_k(E^*) = (-1)^k c_k(E)$. This immediately yields

Lemma 3.8

$$c_1(N_1) = i^*\sigma_1^*$$
$$c_2(N_1) = i^*\sigma_{1,1}^*.$$

where $i : Z_1 \hookrightarrow G(2,2n)$ *is the inclusion.*

We could have taken the cycles above to live in $G(2,2n+1)$ as well; this is immaterial because of the compatibility of inclusions. This remark holds for any similar situation we will encounter. The situation for Z_2 is even simpler since this is just a projective space $Z_2 \cong \mathbb{P}^{2n-1}$. So if we let σ denote the homology class of a hyperplane, the cohomology is generated by $\sigma^* \in H^2(\mathbb{P}^{2n-1},\mathbb{Z})$. Write S, Q for the universal and quotient on $\mathbb{P}^{2n-1}$. Then obviously the only nonzero Chern class of S^* is $c_1(S^*) = \sigma^*$ while the classes of Q can be worked out by applying Whitney formula to the relation $S \oplus Q = \mathbb{C}^{2n}$. We get $c_r(Q) = (\sigma^*)^r$, $1 \le r \le 2n-1$. Applying Whitney formula once more to the relation $N_2 \oplus S^* = Q$ and equating we obtain

$$(1+\sigma^*)(1 + c_1(N_2) + ... + c_{2n-2}(N_2)) = 1 + \sigma^* + ... + (\sigma^*)^{2n-1} =$$

$$= (1+\sigma^*)(1+(\sigma^*)^2+...+(\sigma^*)^{2n-2}).$$

Lemma 3.9 *It is*

$$c_{2k+1}(N_2) = 0\,,$$
$$c_{2k}(N_2) = (\sigma^*)^{2k}$$

for $0 \le k \le n-1$.

4 Localization formula and Schubert calculus

Lemma 4.1 *The formal power series for* $\det(I + \frac{\sqrt{-1}}{2\pi} L_1^{-1} K_1)^{-1}$ *is*

$$1 + (y-z) + (y-z)^2 + ...$$

where $y = (2n+1)^{-1} c_1(N_1)$, $z = (2n+1)^{-2} c_2(N_1)$. *The contribution of degree* $2k$ *in this series is*

$$\sum_{m=\lceil k/2 \rceil}^{k} \binom{m}{2m-k} (-1)^{k-m} y^{2m-k} z^{k-m}.$$

Proof. Remember that for a C^∞ complex bundle $E \to M$ with curvature matrix K it is $\det(I + t\frac{\sqrt{-1}}{2\pi}K) = 1 + tc_1(E) + t^2 c_2(E) + ...$ where $c_k(E) \in H_d^{2k}(M)$ are the Chern classes. In our case this gives simply $\det\left(I + \left(-\frac{1}{2n+1}\right)\frac{\sqrt{-1}}{2\pi}K_1\right) = 1 - ((2n+1)^{-1}c_1(N_1) - (2n+1)^{-2}c_2(N_1))$. If we set $y = (2n+1)^{-1}c_1(N_1)$, $z = (2n+1)^{-2}c_2(N_1)$ we must then elaborate on the series $1+(y-z)+(y-z)^2+...$ where y is a 2-form and z a 4-form. We need to find the term of this series of degree $2k$, for each k. Note that since only forms of *even* degree are involved, we can work like in a commutative ring and use the binomial formula. So $(y-z)^m = \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} y^l z^{m-l}$. The contribution of degree $2k$ is obtained for $2l + 4(m-l) = 2k$ that is $l = 2m-k$ for $0 \le m \le k \le 2m$. Now to conclude just sum over m.

Q.E.D.

Lemma 4.2 *The integrand in the localization formula over* Z_1 *(with all constants taken into account) is*

$$I_1 = \frac{1}{(2n-1)} \sum_{h=3}^{4n-2} \binom{4n-2}{h} 2^{h-1} (2n+1)^{h-2} \sum_{m=\lceil \frac{h-3}{2} \rceil}^{h-3} (-1)^{m+1} \binom{m}{2m-h+3} \times$$
$$x^{4n-2-h} y^{2m-h+3} z^{h-3-m}.$$

where $x = c_1(X), y = (2n+1)^{-1}c_1(N_1)$, $z = (2n+1)^{-2}c_2(N_1)$.

Proof. Since we have to integrate over Z_1 we can consider only the component of degree $\dim_{\mathbb{R}} Z_1$ in the integrand, that is

$$I_1 = \left[\frac{\left(\operatorname{tr} L_1 + c_1(X)\right)^{n+1}}{\det(L_1 + \frac{\sqrt{-1}}{2\pi} K_1)}\right]_{\dim_{\mathbb{R}} Z_1} = \left[\frac{\left(\operatorname{tr} L_1 + c_1(X)\right)^{n+1}}{(2n+1)^2 \det(I + \frac{\sqrt{-1}}{2\pi} L_1^{-1} K_1)}\right]_{\dim_{\mathbb{R}} Z_1}$$

First we write down the numerator of the integrand. Put $x = c_1(X)$ and remember that $\dim(X) = 4n - 3$ to get

$$(-(4n+2) + x)^{4n-2} = \sum_{h=0}^{4n-2} \binom{4n-2}{h} (-2)^h (2n+1)^h x^{4n-2-h}.$$

The form x^{4n-2-h} has degree $8n-4-2h$. Since we must integrate over Z_1, we are concerned only with forms of degree $\dim_{\mathbb{R}}(Z_1) = 2(4n-5) = 8n-10$, so we are only interested in the product with a term of degree $2h-6 = 2(h-3)$ coming from the power series expansion of the denominator. Such a term exists exactly when $h \geq 3$, and we have already found what it looks like. Sum over h and reorder to get the result.

Q.E.D.

The above lemma shows that calculating $\int_{Z_1} I_1$ reduces to evaluating the cap products $< [x^{4n-2-h} y^{2m-h+3} z^{h-3-m}], [Z_1] >$ for $3 \leq h \leq 4m-2$ and $\lceil \frac{h-3}{2} \rceil \leq m \leq h-3$.

Lemma 4.3

$$< [x^{4n-2-h} y^{2m-h+3} z^{h-3-m}], [Z_1] >=$$
$$= \frac{(2n)^{4n-2-h}}{(2n+1)^{h-3}} < (\sigma_{1,1}^*)^{h-3-m}, (\sigma_1)^{2(2n+m-h+1)} > .$$

Proof. This is an immediate consequence of our Chern classes computations that can be put in the form

$$[x] = 2n\, \sigma_1^*; \qquad [y] = (2n+1)^{-1} \sigma_1^*;$$
$$[z] = (2n+1)^{-2} \sigma_{1,1}^*; \qquad [Z_1] = \sigma_1.$$

The first equality follows from the fact that $x = c_1(X) = c_1(\mathcal{O}(2n)$. The second and the third follow from Lemma 3.8. The last simply follows from the fact that σ_1 is the homology class of a hyperplane section.

Q.E.D.

Our task is then to evaluate a cap product of Schubert cycles of complementary dimensions. This can be done by the basic *incidence relation in complementary dimensions* of Schubert calculus (see [6], p. 198).

Lemma 4.4 *If* $\sum a_i = \sum b_i$ *then*

$$< \sigma^*_{a_1,a_2,...,a_k}, \sigma_{n-k-b_1,...,n-k-b_k} >= \delta_{a_k,b_1} \cdot ... \cdot \delta_{a_1,b_k}.$$

Before we can apply this relation in our case we must work out the powers σ_1^k, $\sigma_{1,1}^k$ for any k. This requires the last result from Schubert calculus that we shall use in our work, namely Pieri formula.

Lemma 4.5 (Pieri formula) *Denote by* $\bullet$ *the intersection product. Let* a *be any multiindex of the form* $\mathrm{a} = (a, 0, ..., 0)$*. Then*

$$\sigma_a \bullet \sigma_b = \sum_{\substack{b_i \le c_i \le b_{i-1} \\ \sum c_i = a + \sum b_i}} \sigma_c \tag{4.1}$$

Lemma 4.6 *For* $1 \le k \le 2n$ *the following relation holds among Schubert cycles of* $\mathbb{G}(2, 2n)$*:*

$$\sigma_1^k = \sum_{0 \le j \le i,\, i+j=k} \gamma_{i,j} \sigma_{i,j}$$

where

$$\gamma_{i,j} = \binom{i+j}{j} - \binom{i+j}{j-1} \quad \textit{and} \quad \gamma_{0,0} = 0.$$

Proof. Any cycle in $G(2, 2n)$ has the form σ_{a_1,a_2}. If we follow the convention that nonadmissible cycles vanish, we have $\sigma_1 \sigma_{a_1,a_2} = \sigma_{a_1+1,\,a_2} + \sigma_{a_1,\,a_2+1}$. This is an immediate application of Pieri formula. If we denote by $\gamma_{i,j}$ the coefficient of $\sigma_{i,j}$ in the power σ_1^{i+j}, we get the identities

$$\gamma_{i+1,j} = \gamma_{i,j} + \gamma_{i+1,j-1}$$

$$\gamma_{i,j+1} = \gamma_{i,j} + \gamma_{i-1,j+1}$$

where the vanishing convention is in force too. Moreover the initial conditions $\gamma_{i,0} = 1$ and $\gamma_{0,j} = 0$ determine $\gamma_{i,j}$ uniquely. To conclude just note that

$$\gamma_{i,j} = \binom{i+j}{j} - \binom{i+j}{j-1}$$

meets all these conditions.

Q.E.D.

Lemma 4.7 *In $\mathbb{G}(2,2n)$ for $1 \leq k \leq n$ it is*

$$\sigma_{1,1}^k = \sigma_{k,k}.$$

Proof. By Pieri formula we find $\sigma_{1,1} = \sigma_1^2 - \sigma_2$. On the other hand for any cycle σ_{a_1,a_2} it is

$$\sigma_2 \sigma_{a_1,\,a_2} = \sigma_{a_1+2,\,a_2} + \sigma_{a_1+1,\,a_2+1} + \sigma_{a_1,\,a_2+2},$$

$$\sigma_1^2 \sigma_{a_1,\,a_2} = \sigma_{a_1+2,\,a_2} + 2\sigma_{a_1+1,\,a_2+1} + \sigma_{a_1,\,a_2+2}$$

again by Pieri. So we get

$$\sigma_{1,1}\sigma_{a_1,\,a_2} = \sigma_{a_1+1,\,a_2+1}$$

and we are done.

Q.E.D.

We are now in a position to prove

Theorem 4.8

$$\int_{Z_1} I_1 = 2^{4n-3} n^{4n-2} \frac{2n+1}{(2n-1)} \sum_{h=3}^{4n-2} \frac{1}{n^h} \binom{4n-2}{h} \times$$

$$\sum_{m=\lceil \frac{h-3}{2} \rceil}^{h-3} (-1)^{m+1} \binom{m}{2m-h+3} c_{2n+1+m-h}$$

where c_l denotes the l-th Catalan number $c_l = (l+1)^{-1} \binom{2l}{l}$.

Proof. By 4.3 we just need to evaluate $< (\sigma_{1,1}^*)^{h-3-m}, (\sigma_1)^{2(2n+m-h+1)} >$ for all h and by 4.7 this equals $< \sigma_{h-3-m,h-3-m}, (\sigma_1)^{2(2n+m-h+1)} >$. By the incidence relation in complementary dimensions, the result is simply the coefficient of the cycle $\sigma_{2n+1+m-h,2n+1+m-h}$ in the power $\sigma_1^{2(2n+1+m-h)}$. In the notation of Lemma 4.6 this is $\gamma_{2n+1+m-h,2n+1+m-h}$. Substitute this value in the integrand and simplify to get the result, noting that for all i

$$\gamma_{i,i} = \binom{2i}{i} - \binom{2i}{i-1} = \frac{1}{i+1}\binom{2i}{i} = c_i.$$

Q.E.D.

The situation for Z_2 is much simpler because this is a projective space.

Lemma 4.9 *Let I_2 be the integrand of the localization formula over Z_2 (with all constants taken into account). Then*

$$\int_{Z_2} I_2 = \frac{(2n+1)(2n-2)^{2n-1}(2n)^{2n-3}}{4n-2}\times$$

$$\left((2n)^2\binom{4n-2}{2n-1} - (2n-2)^2\binom{4n-2}{2n+1}\right).$$

Proof. The computation is similar to those in Lemmas 4.1 and 4.2. The denominator term in the integrand over Z_2 which we need to expand is

$$\det\left(I + \frac{1}{2n+1}\frac{\sqrt{-1}}{2\pi}K_2\right) = 1 + \frac{c_1(N_2)}{2n+1} + \frac{c_2(N_2)}{(2n+1)^2} + ... + \frac{c_{2n-2}(N_2)}{(2n+1)^{2n-2}}.$$

Setting $y = (2n+1)^{-1}\sigma^*$ ($\sigma^* \in H^2(\mathbb{P}^{2n-1}, \mathbb{Z})$ denoting the generator as usual) and recalling our computation of Chern classes (see Lemma 3.9) we can rewrite this as

$$1 + (y^2) + ... + (y^2)^{n-1} = \frac{1-y^{2n}}{1-y^2}.$$

Thus

$$\det(...)^{-1} = (1 + y^2 + ... + y^{2n-2})^{-1} = (1-y^2)\sum_{k=0}^{\infty} y^{k(2n)}.$$

But now remember that $\dim_{\mathbb{R}} Z_2 = 2(2n-1)$; since y has degree 2, as far as integration over Z_2 is concerned $\sum_{k=0}^{\infty} y^{k(2n)} = 1$. Next we set $x = c_1(X)$ and we write down the numerator in the localization formula as

$$((2n+1)(2n-2)+x)^{4n-2} = \sum_{i=0}^{4n-2}\binom{4n-2}{i}((2n-2)(2n+1))^i x^{4n-2-i}.$$

Since x has degree 2 and since we can complete only with a 1 or $-y^2$ term from the denominator, the only forms in the integral of the right degree $\dim_{\mathbb{R}} Z_2 = 2(2n-1)$ are $x^{2n-1}, -x^{2n-3}y^2$ (up to constants). In other words

$$I_2 = \frac{1}{(4n-2)(2n+1)^{2n-2}}\times$$

$$\left(\binom{4n-2}{2n-1}(2n-2)^{2n-1}(2n+1)^{2n-1}x^{2n-1}\right.$$

$$\left.-\binom{4n-2}{2n+1}(2n-2)^{2n+1}(2n+1)^{2n+1}x^{2n-3}y^2\right).$$

Now it is easy to see that $i^*c_1(X) = 2n\sigma^*$ where $i^* : Z_2 \hookrightarrow X$ denotes inclusion. From this it is straightforward to compute the cap products we need:

$$< [x^{2n-1}], [\mathbb{P}^{2n-1}] >= (2n)^{2n-1};$$

$$< [x^{2n-3}y^2], [\mathbb{P}^{2n-1}] >= \frac{(2n)^{2n-3}}{(2n+1)^2}.$$

Substituting into the localization formula and simplyfing we get the result.

Q.E.D.

Theorem 4.10 *Let X be a smooth hyperplane section of $\mathbb{G}(2, 2n+1)$ and let w be the generator of the center of $\mathfrak{Lie}\,\mathrm{Aut}(X)$. Then*

$$\begin{aligned} F_X(w) = \\ = 2^{4n-3}n^{4n-2}\frac{2n+1}{(2n-1)}\Bigg\{\sum_{h=3}^{4n-2}\frac{1}{n^h}\binom{4n-2}{h} \times \\ \sum_{m=\lceil\frac{h-3}{2}\rceil}^{h-3}(-1)^{m+1}\binom{m}{2m-h+3}c_{2n+1+m-h}\Bigg\}+ \\ +\frac{(2n+1)(2n-2)^{2n-1}(2n)^{2n-3}}{4n-2}\Bigg\{(2n)^2\binom{4n-2}{2n-1}-(2n-2)^2\binom{4n-2}{2n+1}\Bigg\}. \end{aligned} \tag{4.2}$$

As a simple application using a symbolic calculus package we can get the first few values of F_X.

$$n = 1 \Rightarrow f = 0;$$

$$n = 2 \Rightarrow f = -\frac{10240}{3};$$

$$n = 3 \Rightarrow f = -\frac{1693052928}{5};$$

$$n = 4 \Rightarrow f = -\frac{1415071464947712}{7}.$$

5 Final remarks

In this section we sketch a proof of the following lemma. This is achieved by comparing the asymptotic behaviour of the two terms entering in (4.2). The details are elementary but quite tricky.

Lemma 5.1 $F_X(w) < 0$ *for* $n \gg 2$.

Proof. Rearrange $\int I_1$ as a weighted sum of Catalan numbers; that is write

$$\int I_1 = C_n \sum_{l=0}^{2n-3} \gamma_l c_{2n-2-l}$$

where $C_n = 2^{4n-3} n^{4n-2} \frac{2n+1}{(2n-1)}$ and

$$\gamma_l = (-1)^l \sum_{k=3+2l}^{4n-2} (-1)^k n^{-k} \binom{4n-2}{k} \binom{k-3-l}{l}.$$

We will later sketch a proof that γ_l has sign $(-1)^{l+1}$ and that $|\gamma_l|$ is non-increasing for $l \geq 1$, at least for $n \gg 2$. Using this and the fact that the sequence of Catalan numbers is strictly increasing, it is easy to conclude that

$$\int I_1 < C_n(\gamma_0 c_{2n-2} + \gamma_1 c_{2n-3}) = C_n c_{2n-2} \left(\gamma_0 + \gamma_1 \frac{c_{2n-3}}{c_{2n-2}} \right).$$

An explicit calculation shows that $\gamma_0 + \gamma_1 \frac{c_{2n-3}}{c_{2n-2}} \to -\frac{13}{3} + 3\,\mathrm{e}^{-4}$ as $n \to \infty$ so we get $\int I_1 < -4 C_n c_{2n-2}$ for $n \gg 2$. Next rewrite the other contribution as

$$\int I_2 = C_n \left(1 - \frac{1}{n}\right)^{2n-1} \left(\binom{4n-2}{2n-1} - \left(1 - \frac{1}{n}\right)^2 \binom{4n-2}{2n-3} \right).$$

This is asymptotic to $2\mathrm{e}^{-2} C_n (c_{2n-1} + c_{2n-2})$, which in turn is asymptotic to $10\mathrm{e}^{-2} C_n c_{2n-2}$. This shows that $F_X(w) < 0$ for large n. To complete the proof, we note that

$$\gamma_l = (-1)^{l+1} n^{-3-2l} \binom{4n-2}{3+2l} \mathfrak{F} \binom{5+2l-4n, l+1}{2l+4} \left(\frac{1}{n} \right)$$

using Gaussian hypergeometric notation; that is $\mathfrak{F}$ is the unique (finite) sum $\sum_{s \geq 0} a_s^{l,n}$ with first term 1 and such that

$$r(l,s,n) = \frac{a_{s+1}^{l,n}}{a_s^{l,n}} = \frac{(s+5+2l-4n)(s+l+1)}{n(s+1)(s+2l+4)}$$

for $s \geq 0$. A straightforward computation shows that $r(l,s,n) < 0$ for $n \geq 2$, $0 \leq l \leq 2n-3$, $0 \leq s \leq 4n-5-2l$ and $|r(l,s,n)| \leq 1$ for $n \geq 2$, $1 \leq l \leq 2n-3$, $1 \leq s \leq 4n-5-2l$. Thus for each admissible $l \geq 1$, $\mathfrak{F}$ is an alternating sum of an even number of terms, beginning with 1, and nonincreasing in absolute value from the second place. This implies that $\gamma_{l,n} \geq \sum_{s=0}^{6} a_s^{l,n}$ and one can show that this rational function is positive for all admissible values when $n \gg 2$. If this is not so, choose sequences n_k, l_k such that $n_k \to \infty$ and $\gamma_{l_k,n_k} < 0$. In case $l_k/n_k \to 0$, pass to the limit as $n \to \infty$ for l fixed in $\sum_{s=0}^{6} a_s^{l,n}$ and study the resulting rational function of l to get a contradiction. If l_k/n_k is not infinitesimal, choose a subsequence such that $l_k/n_k \to c$ ($0 < c < 2$), pass to the limit as $n \to \infty$ in $\sum_{s=0}^{6} a_s^{c\,n,n}$ and study the resulting rational function of c to get a contradiction. A similar argument works to show that

$$\frac{\gamma_{l+1,n}}{\gamma_{l,n}} \leq n^{-2} \left(\binom{4n-2}{3+2l+2} \Big/ \binom{4n-2}{3+2l} \right) \left(\left(\sum_{s=0}^{7} a_s^{l+1,n} \right) \Big/ \left(\sum_{s=0}^{6} a_s^{l,n} \right) \right) < 1$$

at least for $n \gg 2$.

Q.E.D.

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